

Functional equations of L -functions of varieties over finite fields

By Shuji SAITO

§1. Introduction.

Let X be a projective smooth geometrically irreducible scheme over a finite field \mathbf{F}_q and $\pi_1(X, s)$ its algebraic fundamental group with s a fixed geometric point of X . For an l -adic representation ($l \neq \text{ch}(\mathbf{F}_q)$)

$$\rho : \pi_1(X, s) \longrightarrow \text{GL}_{Q_l}(V),$$

we define its L -function, following E. Artin and A. Grothendieck, by:

$$L_X(\rho, t) = \prod_{x \in X_0} \det(1 - \rho(F_x) t^{d_x} | V)^{-1} \in Q_l[[t]].$$

Here X_0 denotes the set of all closed points of X , $d_x = [\kappa(x) : \mathbf{F}_q]$ ($\kappa(x)$ is the residue field of $x \in X_0$) and F_x is the geometric Frobenius over x (it is defined as a conjugacy class of $\pi_1(X, s)$). For the details, see §2). By the theory of Grothendieck we know the following facts.

(1) $L_X(\rho, t)$ is a rational function over Q_l , more accurately,

$$L_X(\rho, t) = \prod_{i=0}^{\frac{2n}{2}} \det(1 - Ft | H^i(\bar{X}, \mathcal{F}_\rho))^{(-1)^{i+1}},$$

where $n = \dim(X)$, $\bar{X} = X \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_q$, F is the Frobenius map on X , and \mathcal{F}_ρ is the smooth Q_l -sheaf on X corresponding to ρ .

(2) $L_X(\rho, t)$ has the following functional equation

$$L_X(\rho, t) = A \cdot t^B \cdot L_X(\check{\rho}, \frac{1}{q^n t}),$$

where $A \in Q_l^\times$ and $B \in \mathbf{Z}$, and $\check{\rho}$ denotes the dual representation of ρ .

Now, the purpose of this paper is to determine A and B explicitly using the unramified class field theory for X (cf. §3. Main Theorem).

In case $\dim(X)=1$, the problem is classical and solved by A. Weil, R. P. Langlands and P. Deligne ([1]) including the case X is not necessarily proper over \mathbf{F}_q . In this paper, our problem is reduced to the classical results, using Lefschetz pencils, the theory of vanishing cycles and the theory of Chern classes. By the similar method, A. N. Parshin has solved this problem in case $\dim(X)=2$ in his paper [5], which the author found during

writing this paper.

While this paper treats only the case X is projective over F_q , the generalization to the case X is not proper over F_q seems interesting. For example, in case $\dim(X)=2$, it is expected that the class field theory for X including ramifications, which is described in K-S [4], makes some contribution toward this direction.

The author wishes to express his sincere gratitude to Professors K. Kato and T. Shioda for their helpful advice on this subject.

§ 2. Unramified class field theory of schemes over finite fields.

Let X/F_q be as in § 1 and $\pi_1^{\text{ab}}(X)$ its abelian fundamental group. For $x \in X_0$, the natural injection $x \rightarrow X$ defines a homomorphism $i_x : \pi_1^{\text{ab}}(x) \rightarrow \pi_1^{\text{ab}}(X)$. Since $\kappa(x)$ is a finite field, we have an isomorphism $\hat{Z} \cong \pi_1^{\text{ab}}(x)$ which sends the topological generator $1 \in \hat{Z}$ to the Frobenius over $\kappa(x)$. We define *the geometric Frobenius F_x over x* to be the image of $-1 \in \mathbf{Z}$ under i_x (or the conjugacy class in $\pi_1(X, s)$ defined by it). Now F_x 's for all $x \in X_0$ define the homomorphism

$$\tilde{\varphi} : Z_0(X) \stackrel{\text{def}}{=} \bigoplus_{x \in X_0} \mathbf{Z} \longrightarrow \pi_1^{\text{ab}}(X),$$

where $Z_0(X)$ is the group of rational 0-cycles on X .

LEMMA (2.1) (K-S [3] § 8). *The map $\tilde{\varphi}$ annihilates the subgroup of $Z_0(X)$ consisting of 0-cycles rationally equivalent to zero.*

We recall that this is deduced from the classical reciprocity law for curves over finite fields. Now, by (2.1) we obtain the fundamental homomorphism in the unramified class field theory

$$\varphi : CH_0(X) \longrightarrow \pi_1^{\text{ab}}(X),$$

where $CH_0(X)$ denotes the 0-dimensional Chow group of X .

THEOREM (2.2) ([3]). *The map φ is injective and has a dense image. We have the following commutative diagram*

$$\begin{array}{ccc} CH_0(X) & \xrightarrow{\text{deg}} & \mathbf{Z} \\ \downarrow \varphi & & \downarrow \wr \\ \pi_1^{\text{ab}}(X) & \xrightarrow{p} & \text{Gal}(\bar{F}_q/F_q) \cong \hat{Z} \end{array}$$

where the right vertical arrow sends $1 \in \mathbf{Z}$ to $-1 \in \hat{Z}$, deg denotes the

degree map (cf. § 3 (3.2)) and p is the natural map. Moreover, the induced map $\text{Ker}(\text{deg}) \rightarrow \text{Ker}(p)$ is an isomorphism of finite groups.

REMARK. The finiteness of $\text{Ker}(p)$ is known by Katz-Lang [2].

§ 3. Main results.

First we introduce some notations. In general, let X be a projective smooth variety over a field k of dimension n .

DEFINITION (3.1). The canonical zero cycle c_X on X is $c_X = -(\Delta \cdot \Delta) \in CH_0(X)$. Here, Δ denotes the n -cycle on $X \times X$ defined by the diagonal $X \hookrightarrow X \times X$ and $\Delta \cdot \Delta$ is the self-intersection of Δ in the Chow ring of $X \times X$ (This can be naturally viewed as an element of $CH_0(\Delta) = CH_0(X)$).

DEFINITION (3.2). The Euler-Poincaré characteristic χ_X of X is $\chi_X = -\text{deg}(c_X) \in \mathbf{Z}$, where

$$\text{deg} : CH_0(X) \rightarrow \mathbf{Z}; \sum_{x \in X_0} a_x \cdot x \mapsto \sum_{x \in X_0} a_x [\kappa(x) : k]$$

is the degree map.

The following facts are well-known.

(3.3) $\chi_X = \sum_{i=0}^{2n} (-1)^i \dim_{Q_l} H^i(\bar{X}, Q_l)$, where $\bar{X} = X \otimes_k \bar{k}$ and l is any prime number $\neq \text{ch}(k)$.

(3.4) If n is odd, χ_X is even.

DEFINITION (3.5). Assume that $k = F_q$. Then $\kappa_X = \pm 1$ is defined as follows: If n is odd, then $\kappa_X = 1$. If n is even, $\kappa_X = (-1)^\nu$, where ν is the multiplicity of $-q^{n/2}$ in the roots of $P_n(t) = 0$. Here, $P_n(t)$ is the characteristic polynomial of F on $H^n(\bar{X}, Q_l)$.

Now let X/F_q and ρ be as in § 1. We make the following assumption on ρ .

(3.6) There exists an infinite set L of prime numbers λ satisfying

(3.6.1) $L \ni l$, $L \ni p$, and for each $\lambda \in L$ we are given a λ -adic representation

$$\rho_\lambda : \pi_1(X, s) \longrightarrow GL_{Q_\lambda}(V_\lambda)$$

which satisfies

(3.6.2) $\rho_l = \rho$.

(3.6.3) Each ρ_λ is Q -rational and ρ_λ 's are compatible with each other.

Namely, for each $\lambda \in L$ and $x \in X_0$, $P_\lambda(x, t) \stackrel{\text{def}}{=} \det(1 - \rho(F_x)t | V_\lambda)$ has all coefficients in Q , and for $\lambda, \mu \in L$, $P_\lambda(x, t) = P_\mu(x, t)$ ($\in Q[t]$).

The condition (3.6) implies that $L_X(\rho, t) \in \mathcal{O}_L[[t]]$, where \mathcal{O}_L denotes the

subring of \mathbf{Q} consisting of all elements of \mathbf{Q} which are integral at any $\lambda \in L$.

MAIN THEOREM. Under the assumption (3.6), we have

$$(3.7) \quad L_X(\rho, t) = \kappa_X' \det \rho(c_X) (-q^{n/2}t)^{-r_Z X} L_X\left(\check{\rho}, \frac{1}{q^{n/2}t}\right),$$

where $r = \dim(V)$, $\det \rho: \pi_1^{\text{ab}}(X) \rightarrow \mathbf{Q}_l^\times$ is the determinant of ρ and $CH_0(X)$ is considered as a dense subgroup of $\pi_1^{\text{ab}}(X)$ via (2.2).

REMARK. According to (3.4), there is no ambiguity for the notation " $q^{n/2}$ " in (3.7).

COROLLARY (3.8). Let X/\mathbf{F}_q be as before and suppose that $\dim(X)$ is odd. Then there exists $c' \in Z_0(X)$ such that c_X is rationally equivalent to $2c'$.

REMARK. (3.8) in case $\dim(X)=1$ is due to A. Weil, and the following proof is just an imitation of his proof.

PROOF OF (3.8). By (2.2), $CH_0(X)$ is finitely generated, so it suffices to show that for any character of order 2 $\sigma: CH_0(X) \rightarrow \{\pm 1\} \subset \mathbf{Q}_l^\times$, we have $\sigma(c_X)=1$. First suppose that σ factors through $\deg: CH_0(X) \rightarrow \mathbf{Z}$. Then our assertion follows from (3.4). Next we suppose that σ does not factor through the map \deg . By (2.2) we can view σ as a character of $\pi_1^{\text{ab}}(X)$. Let $f: Y \rightarrow X$ be the double covering corresponding to σ . By the assumption, this covering does not contain any constant field extension. From the definition, we can easily see that

$$Z_Y(t) = L_X(\sigma, t) Z_X(t),$$

where $Z_X(t)$ (resp. $Z_Y(t)$) is the zeta function of X (resp. Y). By the functional equation (3.7), we have an equality

$$(-q^{n/2}t)^{-Z_Y} = \sigma(c_X) (-q^{n/2}t)^{-Z_X} \cdot (-q^{n/2}t)^{-Z_X},$$

from which we obtain $\sigma(c_X)=1$ as desired.

For the proof of Main Theorem, we need the following preliminaries (§ 4 and § 5).

§ 4. Canonical zero cycles on fibered varieties.

Let X be a projective smooth scheme over a field k of dimension $n+1$, C a projective smooth curve over k and $f: X \rightarrow C$ a k -morphism satisfying the following conditions. For $v \in C_0$, we put $X_v = X \otimes_{C/k} \kappa(v)$.

(4.1) There exists a finite subset Σ of $C(k)$ satisfying

(4.1.1) If $v \in C_0 - \Sigma$, f is smooth over v .

(4.1.2) If $v \in \Sigma$, X_v has as singularities only a finite number of k -rational isolated singular points $x_{v,1}, \dots, x_{v,r_v}$.

For $v \in \Sigma$ and $1 \leq i \leq r_v$, we define the Milnor number of X/C at $x_{v,i}$

$$\mu_{v,i} = \dim_k \left(\mathcal{O}_{X, x_{v,i}} / \left(\frac{\partial g}{\partial X_1}, \dots, \frac{\partial g}{\partial X_{n+1}} \right) \right),$$

where $\{X_1, \dots, X_{n+1}\}$ is a system of local parameters at $x_{v,i}$ on X , and $g = u \cdot f$ with a local parameter u at v on C . The independence of $\mu_{v,i}$ from the choices of $\{X_1, \dots, X_{n+1}\}$ and u is easily seen.

In addition to (4.1), we make the following assumption.

(4.2) There exists a rational differential form ω on C which satisfies the following conditions. Let $(\omega) = \sum_{v \in C_0} n_v \cdot v$ ($n_v \in \mathbb{Z}$) be the divisor of ω .

(4.2.1) For $v \in \Sigma$, $n_v = 0$.

(4.2.2) If $n_v \neq 0$ for $v \in C_0$, then $v \in C(k)$.

Under the assumptions (4.1) and (4.2), we have the following

LEMMA (4.3). *We have the following equality.*

$$c_X = - \left(\sum_{v \in \Sigma} n_v \cdot i_v(c_{X_v}) \right) + (-1)^n \sum_{v \in \Sigma} \sum_{i=1}^{r_v} \mu_{v,i} \cdot x_{v,i},$$

where $i_v : CH_0(X_v) \rightarrow CH_0(X)$ is the natural homomorphism.

COROLLARY (4.4). *We have the following equality.*

$$\chi_X = - \left(\sum_{v \in \Sigma} n_v \cdot \chi_{X_v} \right) + (-1)^{n+1} \sum_{v \in \Sigma} \sum_{i=1}^{r_v} \mu_{v,i}.$$

Based on the fact that $-c_X$ is equal to the $(n+1)$ -th Chern class of $\tilde{\mathcal{Q}}_{X/k}^1$ (due to Mumford), (4.3) is proved using the theory of Chern classes (cf. the argument of SGA 7 XVI § 2).

§ 5. The theory of local constants.

In this section, we review briefly some results in Deligne [1].

(5.1) Let K be a complete discrete valuation field with finite residue field \mathbf{F}_q , A a field of characteristic $\neq \text{ch}(\mathbf{F}_q)$, $\text{Rep}_A(K)$ the category of all pairs (V, ρ) with a finite dimensional vector space V over A and a continuous homomorphism $\rho : \text{Gal}(\bar{K}/K) \rightarrow \text{GL}_A(V)$ ($\text{GL}_A(V)$ is endowed with discrete topology). We fix a non-trivial additive character $\phi : K \rightarrow A^\times$ and a Haar measure dx on K with values in A (cf. [1] 6.1). Then Deligne defined a

function $\varepsilon_0(*, \phi, dx) : \text{Ob}(\text{Rep}_A(K)) \rightarrow A^\times$. We recall only some properties of $\varepsilon_0(*, \phi, dx)$ necessary for the proof of Main Theorem.

(5.1.1) For any exact sequence $0 \rightarrow V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow 0$ in $\text{Rep}_A(K)$, we have

$$\varepsilon_0(V_2, \phi, dx) = \varepsilon_0(V_1, \phi, dx) \varepsilon_0(V_3, \phi, dx).$$

In particular, $\varepsilon_0(*, \phi, dx)$ can be extended to a homomorphism

$$\varepsilon_0(*, \phi, dx) : R_A(K) \longrightarrow A^\times,$$

where $R_A(K)$ denotes the Grothendieck group on $\text{Rep}_A(K)$.

(5.1.2) If (W, ρ) is unramified, that is, ρ factors through $\text{Gal}(\bar{K}/K) \rightarrow \text{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q)$,

$$\varepsilon_0(W, \phi, dx) = \det(F|W)^{n(\phi)+1} \left(-q^{n(\phi)} \int_{\mathcal{O}_K} dx \right)^{\dim(W)},$$

where $F \in \text{Gal}(\bar{\mathbf{F}}_q/\mathbf{F}_q)$ is the geometric Frobenius over \mathbf{F}_q (=the inverse of the Frobenius over \mathbf{F}_q) and $n(\phi)$ is the maximum integer such that $\phi \equiv 1$ on $\pi^{-n}\mathcal{O}_K$ (π is a prime element of K).

(5.1.3) Let $V, W \in \text{Rep}_A(K)$ and suppose W is unramified, then

$$\varepsilon_0(V \otimes W, \phi, dx) = \det(F|W)^{\text{Sw}(V) + \dim(V)(n(\phi)+1)} \cdot \varepsilon_0(V, \phi, dx)^{\dim(W)},$$

where $\text{Sw}(V)$ is the Swan conductor of V .

(5.2) Let C be a projective smooth geometrically irreducible curve over \mathbf{F}_q , K its function field, K_v ($v \in C_0$) the completion of K at v , $\kappa(v)$ the residue field of v and $F_v \in \text{Gal}(\bar{\kappa}(v)/\kappa(v))$ the geometric Frobenius at v . Let A be a field of characteristic $\neq p = \text{ch}(\mathbf{F}_q)$, and fix the (unique) Haar measure $dx = \bigotimes_{v \in C_0} dx_v$ on the adèle ring A_K of K with values in A such that $\int_{A_K/K} dx = 1$, and a non-trivial character $\phi = (\phi_v)_v : A_K/K \rightarrow A^\times$. We define “the divisor of ϕ ” to be $(\phi) = \sum_v n_v \cdot v$ with $n_v = n(\phi_v)$ (cf. 5.1). Then, it is well-known that there exists a one-to-one correspondence between the set $\{(\phi) | \phi : A_K/K \rightarrow A^\times, \text{non-trivial}\}$ and the set $\{(\omega) | \omega : \text{non-zero rational differential on } C\}$.

For a variable t over A , we define an unramified character $\omega'_v : \text{Gal}(\bar{K}_v/K_v) \rightarrow A(t)^\times$ by $\omega'_v(F_v) = t^{d_v}$ with $d_v = [\kappa(v) : \mathbf{F}_q]$. Then, for a constructible sheaf \mathcal{F} of A -modules on $C_{\text{ét}}$, we define its local constant at v by

$$(5.2.1) \quad \varepsilon(\mathcal{F}_v, \phi_v, dx_v, t) = \varepsilon_0(\mathcal{F}_{\bar{K}_v} \otimes \omega'_v, \phi_v, dx_v) \cdot \det(-F_v t^{d_v} | \mathcal{F}_{\bar{\kappa}(v)})^{-1} \in A(t)^\times$$

where $\mathcal{F}_{\bar{K}_v}$ (resp. $\mathcal{F}_{\bar{\kappa}(v)}$) is the fiber of \mathcal{F} at the geometric point $\text{Spec}(\bar{K}_v) \rightarrow C$ (resp. $\text{Spec}(\bar{\kappa}(v)) \rightarrow C$). Let $K_c^b(C, A)$ be the category of bounded complexes of sheaves of A -modules on $C_{\text{ét}}$ whose cohomologies are constructible. Then, for $\mathcal{K} \in \text{Ob}(K_c^b(C, A))$, we define its local constant at v by

$$\varepsilon(\mathcal{K}_v, \phi_v, dx_v, t) = \prod_{i \in \mathbb{Z}} \varepsilon((\mathcal{F}^i)_v, \phi_v, dx_v, t)^{(-1)^i},$$

where $\mathcal{F}^i = \mathcal{H}^i(\mathcal{K})$. By (5.1.1), this definition makes sense for $\mathcal{K} \in \text{Ob}(\text{D}_c^b(C, A))$, where $\text{D}_c^b(C, A)$ denotes the derived category of $\text{K}_c^b(C, A)$.

Now, let $f : C \rightarrow \text{Spec}(\mathbf{F}_q)$ be the natural morphism. By SGA $4\frac{1}{2}$ Th. de finitude, for $\mathcal{K} \in \text{Ob}(\text{D}_c^b(C, A))$ we have $\mathbf{R}f_*(\mathcal{K}) \in \text{Ob}(\text{D}_c^b(\text{Spec}(\mathbf{F}_q), A))$. Hence, $\mathbf{R}f_*(\mathcal{K})_{\bar{\mathbf{F}}_q}$ is quasi-isomorphic to a bounded complex $(V^i)_{i \in \mathbb{Z}}$ of $A[F]$ -modules V^i which are finite dimensional over A , where F denotes the geometric Frobenius over \mathbf{F}_q . Then we define

$$\sigma_C(\mathcal{K}, t) = \det(-Ft | \mathbf{R}f_*(\mathcal{K})_{\bar{\mathbf{F}}_q})^{-1} \stackrel{\text{def}}{=} \prod_{i \in \mathbb{Z}} \det(-Ft | V^i)^{(-1)^{i+1}}.$$

THEOREM (5.3) ([1], Theorem (7.11)). *For $\mathcal{K} \in \text{Ob}(\text{D}_c^b(C, A))$, we have*

$$\sigma_C(\mathcal{K}, t) = \prod_{v \in C_0} \varepsilon(\mathcal{K}_v, \phi_v, dx_v, t).$$

§ 6. The proof of Main Theorem.

Combining (3.6) with the fact that the map $\mathcal{O}_L \rightarrow \prod_{\lambda \in L} \mathcal{O}_L / \lambda \mathcal{O}_L$ is injective, it suffices to show (3.7) after reduction modulo λ for all $\lambda \in L$. Thus, let A be a field of characteristic $\neq p = \text{ch}(\mathbf{F}_q)$ and $\rho : \pi_1(X, s) \rightarrow \text{GL}_A(V)$ a continuous representation of $\pi_1(X, s)$ on a finite-dimensional vector space V over A ($\text{GL}_A(V)$ is given discrete topology). As in § 1, we define its L -function by:

$$L_X(\rho, t) = \prod_{x \in X_0} \det(1 - \rho(F_x) t^{d_x} | V)^{-1} \in A[[t]].$$

Then, Main Theorem follows from

THEOREM (6.1). *$L_X(\rho, t)$ is a rational function over A and satisfies the functional equation (3.7).*

By the theory of Grothendieck,

$$L_X(\rho, t) = \det(1 - Ft | \mathbf{R}f_*(\mathcal{F}_\rho)_{\bar{\mathbf{F}}_q})^{-1},$$

where \mathcal{F}_ρ is the locally constant sheaf on $C_{\text{ét}}$ corresponding to ρ , $\mathbf{R}f_*(\mathcal{F}_\rho) \in \text{Ob}(\text{D}_c^b(\text{Spec}(\mathbf{F}_q), A))$ is its direct image by the natural morphism $f : X \rightarrow \text{Spec}(\mathbf{F}_q)$ and F the geometric Frobenius over \mathbf{F}_q . Therefore, (6.1) follows from

THEOREM (6.2). *Let $r = \dim(V)$, then*

$$\det(-Ft | \mathbf{R}f_*(\mathcal{F}_\rho)_{\bar{\mathbf{F}}_q})^{-1} = \kappa_X^r \cdot \det \rho(c_X) (-q^{n/2} t)^{-r \chi_X}.$$

We prove (6.2) by the induction on $\dim(X)$. First, in case $\dim(X)=1$, (6.2) follows from (5.3). Assuming that (6.2) is proved for varieties of dimension $\leq n$, we prove (6.2) in case $\dim(X)=n+1$. For this, we first make the following assumptions.

(6.3) There exists a fibration $g: X \rightarrow C$ (C is a projective smooth curve over \mathbf{F}_q) satisfying (4.1.1) and the following (4.1.2)'. (The notations are the same as § 4.)

(4.1.2)' If $v \in \Sigma$, X_v has as singularities the unique \mathbf{F}_q -rational ordinary double point x_v . We denote by μ_v the Milnor number of X/C at x_v (cf. § 4).

(6.4) There exists a rational differential ω on C satisfying (4.2.1) and (4.2.2). We fix a non-trivial character $\psi: \mathbf{A}_K/K \rightarrow A^\times$ such that $(\psi) = (\omega)$ (cf. (5.2)).

Now, let $h: C \rightarrow \text{Spec}(\mathbf{F}_q)$ be the natural morphism ($f = h \circ g$). Then, we have

$$(6.5) \quad \det(-Ft | \mathbf{R}f_*(\mathcal{F}_\rho)_{\overline{\mathbf{F}}_q}) = \det(-Ft | \mathbf{R}h_*(\mathcal{K})_{\overline{\mathbf{F}}_q}),$$

where $\mathcal{K} = \mathbf{R}g_*(\mathcal{F}_\rho) \in \text{Ob}(\text{D}_c^b(C, A))$. We apply (5.3) to \mathcal{K} . Thus, let $dx = \bigotimes_{v \in C_0} dx_v$ be as in (5.2), and for ϕ fixed in (6.4), put $(\phi) = \sum_{v \in C_0} n_v \cdot v$. Then, we compute local constants $\varepsilon_v = \varepsilon(\mathcal{K}_v, \phi_v, dx_v, t)$ using the formulas (5.1.1)~(5.1.3).

Case (i). For $v \in \Sigma$, $\mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\mathbf{K}}_v}$ is unramified and we have an isomorphism of $\text{Gal}(\overline{\kappa(v)}/\kappa(v))$ -modules $\mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\mathbf{K}}_v} \cong \mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}}$. Hence, by (5.2.1) and (5.1.2),

$$\begin{aligned} \varepsilon_v &= \varepsilon_0(\mathcal{K}_{\overline{\mathbf{K}}_v} \otimes \omega_v^t, \phi_v, dx_v) \det(-F_v t^{d_v} | \mathcal{K}_{\overline{\kappa(v)}})^{-1} \\ &= \det(F_v t^{d_v} | \mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}})^{n_v+1} \left(-q^{d_v n_v} \int_{\mathcal{O}_{\overline{\mathbf{K}}_v}} dx_v \right)^{e_v} \det(-F_v t^{d_v} | \mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}})^{-1} \\ &= \det(-F_v t^{d_v} | \mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}})^{n_v} (-1)^{e_v(n_v+1)} \left(-q^{d_v n_v} \int_{\mathcal{O}_{\overline{\mathbf{K}}_v}} dx_v \right)^{e_v}, \end{aligned}$$

where $e_v = \dim_A(\mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}})$. On the other hand, by the proper smooth base change theorem, $\mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}} \cong \mathbf{R}(g_v)_*(\mathcal{F}_\rho|_{X_v})_{\overline{\kappa(v)}}$, with $g_v = g \otimes_C \kappa(v)$. Hence, by the induction we have

$$e_v = r \chi_{X_v} \quad \text{and for } v \in C(\mathbf{F}_q),$$

$$\det(-F_v t | \mathbf{R}g_*(\mathcal{F}_\rho)_{\overline{\kappa(v)}})^{-1} = \kappa_{X_v}^r \det \rho(i_v(c_{X_v})) (-q^{n_v/2} t)^{-r \chi_{X_v}},$$

where $i_v: CH_0(X_v) \rightarrow CH_0(X)$ is the natural map. So we can write

$$(6.6) \quad \varepsilon_v = \det \rho(-n_v i_v(c_{X_v})) \cdot t^{r n_v \chi_{X_v}} \cdot \alpha_v^r,$$

where $\alpha_v \in A^\times$ is independent of ρ . Since $n_v = 0$ for $v \in C(\mathbf{F}_q)$ (cf. (6.4)), (6.6) also holds true for any $v \in C_0 - \Sigma$.

Case (ii). For $v \in \Sigma$, by the theory of vanishing cycles (SGA 7 XV),

we have the following long exact sequence

$$(6.7) \quad \cdots \rightarrow H^i(X_v, \mathcal{F}_\rho) \rightarrow H^i(X_{\bar{K}_v}, \mathcal{F}_\rho) \rightarrow \Phi_v^i(\mathcal{F}_\rho) \rightarrow \cdots$$

where $X_v = X \otimes_{C\kappa(v)} \bar{C}$, $X_{\bar{K}_v} = X \otimes_C \bar{K}_v$ and

$$\Phi_v^i(\mathcal{F}_\rho) = \begin{cases} 0 & \text{if } i \neq n \\ A^{\tau_v}(m-n) \otimes_A (\mathcal{F}_\rho)_{\bar{x}_v} & \text{if } i = n \end{cases}$$

where we put $n=2m$ or $2m+1$ according as n is even or odd, $\tau_v: \text{Gal}(\bar{K}_v/K_v) \rightarrow A^\times$ is a character which is determined independently of ρ , A^{τ_v} is one-dimensional vector space over A on which $\text{Gal}(\bar{K}_v/K_v)$ acts via τ_v , $A^{\tau_v}(m-n)$ is its Tate twist, and $(\mathcal{F}_\rho)_{\bar{x}_v}$ is the geometric fiber of \mathcal{F}_ρ at x_v on which F_v acts as F_{x_v} (=the geometric Frobenius over x_v). By SGA 7 XVI (2.4) (or (1.13)), $\text{Sw}(A^{\tau_v}) = \mu_v - 1$. Hence, by the similar calculation as Case (i) (using (6.7), (5.1.1) and (5.1.3)), we have

$$(6.8) \quad \varepsilon_v = \det((-1)^n \mu_v \cdot x_v) \cdot t^{r\mu_v(-1)^n} \cdot a_v^r,$$

where $a_v \in A^\times$ is independent of ρ .

Lastly, combining (6.6) and (6.8), we obtain

$$\prod_{v \in C_0} \varepsilon_v = \det \rho(\alpha) \cdot t^{r\beta} \cdot A^r,$$

where

$$\alpha = - \left(\sum_{v \in \Sigma} n_v \cdot i_v(c_{x_v}) \right) + (-1)^n \left(\sum_{v \in \Sigma} \mu_v \cdot x_v \right),$$

$$\beta = \sum_{v \in \Sigma} n_v \cdot \chi_{x_v} + (-1)^n \sum_{v \in \Sigma} \mu_v,$$

and $A \in A^\times$ is independent of ρ . By (4.3) and (4.4), we have $\alpha = c_X$ and $\beta = -\chi_X$. Hence, by (5.3) and (6.5), we obtain

$$\det(-Ft | Rf_*(\mathcal{F}_\rho)_{\bar{F}_q})^{-1} = \det \rho(c_X) t^{-r\chi_X} \cdot A^r.$$

Consequently, to complete the proof, it suffices to show the following

LEMMA (6.9). *Let X/\mathbf{F}_q be as before, then we have*

$$\det(-Ft | Rf_*(A)_{\bar{F}_q})^{-1} = \kappa_X(-q^{\dim(X)/2} \cdot t)^{-\chi_X}.$$

This follows from Riemann hypothesis and the Poincaré duality for X .

Now, we prove (6.2) in the general case. By the theory of Lefschetz pencils (cf. SGA 7 XVII), we can see the following fact.

LEMMA (6.10). *We can find relatively prime integers k_1, k_2 such that if we blow up $X_i = X \otimes_{\mathbf{F}_q} \mathbf{F}_q^{k_i}$ ($i=1, 2$) along an appropriate smooth closed subscheme defined over $\mathbf{F}_q^{k_i}$, then the obtained scheme \tilde{X}_i satisfies (as $\mathbf{F}_q^{k_i}$ -scheme) (6.3) and (6.4).*

We have already proved (6.2) for \tilde{X}_i , and comparing the behavior of canonical zero cycles and L -functions under blowing-up, we can verify (6.2) for X_i . Combining this with the following commutative diagram

$$\begin{array}{ccc} CH_0(X_i) & \xrightarrow{N_i} & CH_0(X) \\ \downarrow & & \downarrow \\ \pi_1^{\text{ab}}(X_i) & \longrightarrow & \pi_1^{\text{ab}}(X), \end{array}$$

where N_i is the norm map, we obtain

$$(6.11) \quad \det(-F^{k_i}t^{k_i}|Rf_*(\mathcal{F}_o)_{\bar{F}_q})^{-1} = \kappa_{X_i}^r \det(N_i(c_{X_i}))(-q^{n/2}t)^{-r} \chi_{X_i}^{k_i}.$$

Noting that $N_i(c_{X_i}) = k_i c_X$, (6.2) follows from (6.11) for $i=1, 2$.

Bibliography

- Deligne [1] Deligne, P., Les constantes des equations fonctionnelles des fonctions L , Modular Functions of One Variables II, Lecture Notes in Math. 349, Springer, Berlin-Heidelberg-New York, 1972.
- Katz-Lang [2] Katz, N. and S. Lang, Finiteness theorems in geometric class field theory, L'Enseignement Mathematique 2. S. 27 (1981), 285-314.
- K-S [3] Kato, K. and S. Saito, Unramified class field theory of arithmetical surfaces, Ann. of Math. 118 (1983), 241-275.
- K-S [4] Kato, K. and S. Saito, Two dimensional class field theory, Advanced Studies in Pure Math. Vol. 2, Galois Groups and Their Representations, pp. 103-152.
- Parshin [5] Parshin, A. N., Chern classes, adeles and L -functions, J. Reine Angew. Math. 341 (1983), 174-192.
- SGA 4 ½ Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 ½, Cohomologie Etale, Lecture Notes in Math. 569, Springer, Berlin-Heidelberg-New York, 1977.
- SGA 7 Séminaire de Géométrie Algébrique du Bois-Marie SGA 7 II, Groupes de Monodromie en Géométrie Algébrique, Lecture Notes in Math. 340, Springer, Berlin-Heidelberg-New York, 1973.

(Received December 2, 1983)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan