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Spectral theory on a free group and algebraic curves

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In this note we develop a spectral theory for linear difference operators on a free group, and give the eigenfunction expansions for random walks on a free group by explicit computation of Green functions which turn out to be algebraic. Our method is in some sense a generalization and a unification of the result about spectrums by P. van Moerbeke-D. Mumford on the one hand and the usual Poisson kernel formula on a free group by E. B. Dynkin and others on the other. (See [3], [4], [5], [8], [21] and [25].) Our main results are Theorems 1~4.

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§1. Algebraicity of Green function.

Let Γ be a free group generated by a system of free generators $A=\{a_1, a_2, \cdots, a_g\}$. We consider a random walk on Γ with transition matrix $P=((p_{\gamma,\gamma'}))_{\gamma,\gamma'\in\Gamma}$, $p_{\gamma,\gamma'}\in\boldsymbol{R}$ on $l^2(\Gamma)$

$$\begin{cases}\sum_{\gamma\in\Gamma} p_{\gamma,\gamma'}=1\\ \sum_{\gamma'\in\Gamma} p_{\gamma,\gamma'}=1 \quad \text{and} \quad p_{\gamma,\gamma'}\geqq 0\,.\end{cases} \tag{1.1}$$

Then the matrix P defines a bounded operator on the Hilbert space $l^2(\Gamma)$ as follows: For a fixed $\gamma\in\Gamma$,

$$(Pu)(\gamma)=\sum_{\gamma'\in\Gamma} p_{\gamma,\gamma'}u(\gamma')\,. \tag{1.2}$$

The operator P can also be regarded as a difference operator on the Cayley graph (group complex) $\mathcal{G}$ attached to Γ. We shall denote by $\mathcal{G}_\gamma$ the subgraph consisting of elements $\gamma'\in\Gamma$ such that $\gamma<\gamma'$, where $\gamma<\gamma'$ means that the reduced expression of γ' contains that of γ as its initial part. We say γ' *is greater than* γ.

We shall firstly consider the matrix P under the weaker condition than (1.1):

$$(1.1)' \qquad \sum_{\gamma} |p_{\gamma,\gamma'}|^2 \leqq M,$$

for $\gamma' \in \Gamma$ and a certain positive number M. Assume that

($\mathcal{H}$.1) *P has a finite range, namely for a certain positive integer m,*

$$(1.3) \qquad p_{\gamma,\gamma'} = 0 \quad \textit{for} \quad L(\gamma^{-1}\gamma') > m$$

where $L(\gamma)$ denotes the length function of the reduced expression of γ with respect to A.

($\mathcal{H}$.2) *There exists a subgroup Γ_0 of finite index of Γ, such that P is left-invariant with respect to Γ_0, namely*

$$(1.4) \qquad p_{\gamma_0\gamma,\gamma_0\gamma'} = p_{\gamma,\gamma'}$$

for an arbitrary $\gamma_0 \in \Gamma_0$.

($\mathcal{H}$.3) *We have a regularity condition*

$$(1.5) \qquad p_{\gamma,\gamma'} \neq 0$$

if $L(\gamma'^{-1}\gamma) \leqq m$ and $\gamma' < \gamma$.

In case of $\Gamma = Z$, these conditions satisfy the classical Hamburger's ones related to continued fractions[1]. In the same way we can show that for $z \in C$, $|z| \gg 1$, there exists *the unique Green kernel* $G(\gamma, \gamma'|z)$ for P in $l^2(\Gamma)$ such that

$$(1.6) \qquad (z-P)^{-1}u(\gamma) = \sum_{\gamma' \in \Gamma} G(\gamma, \gamma'|z)u(\gamma')$$

where $G(\gamma, \gamma'|z)$ is holomorphic with respect to z, $|z| \gg 1$ and

$$(1.7) \qquad \sum_{\gamma'} |G(\gamma, \gamma'|z)|^2 < \infty$$

for a fixed $\gamma \in \Gamma$.

Then we can prove

THEOREM 1. *For fixed $\gamma, \omega \in \Gamma$, $G(\gamma, \omega|z)$ can be prolonged algebraically on the whole plane $\boldsymbol{C}$, such that we have the Laurent expansion*

$$(1.8) \qquad \frac{G(\gamma a_i^{\mp 1}, \omega|z)}{G(\gamma, \omega|z)} = \frac{C_{\omega,i}^{\pm}(\gamma)}{z} + \cdots \quad \textit{if} \quad L(\omega^{-1}\gamma) \equiv 0 \mod m$$

$$= C_{\omega,i}^{\pm}(\gamma) + \cdots \quad \textit{if} \quad L(\omega^{-1}\gamma) \not\equiv 0 \mod m$$

at the infinity if $\omega^{-1}\cdot\gamma < \omega^{-1}\cdot\gamma a_i^{\mp}$ respectively, where $C_{\omega,i}^{\pm}(\gamma)$ denote constants different from zero depending on γ, ω and i.

1) T. Carleman, Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala, 1923.

$G(\gamma, \omega|z)$ becomes a meromorphic function of z on an algebraic curve $\mathfrak{C}$ ramified over $\boldsymbol{CP}^1$ and satisfies

$$G(\gamma_0\gamma, \gamma_0\omega|z) = G(\gamma, \omega|z) \tag{1.9}$$

for $\gamma_0 \in \Gamma_0$. The curve $\mathfrak{C}$ is independent of γ, ω.

This is a generalization of a theorem in [21], which has proved the above in the case $g=1$, namely for periodic ordinary linear difference systems. In fact the eigen-function equation

$$zu(\gamma) - \sum_{\gamma' \in \Gamma} p_{\gamma,\gamma'} u(\gamma') = 0 \tag{1.10}$$

can be regarded as a generalization of Toda lattice equation on $\boldsymbol{Z}$ to the case of an arbitrary free group.

To prove the theorem, we put $\alpha^{\pm}_{i,\omega}(\gamma|z)$ to be equal to $\dfrac{G(\gamma a_i^{\pm 1}, \omega|z)}{G(\gamma, \omega|z)}$ according as $\omega^{-1}\cdot\gamma < \omega^{-1}\cdot\gamma a_i$ or $\omega^{-1}\cdot\gamma < \omega^{-1}\cdot\gamma a_i^{-1}$. Then $G(\gamma, \omega|z)$ satisfies the equation (1.10) except for $\gamma=\omega$ namely.

LEMMA 1. *For all $\gamma \in \Gamma$, $\gamma \neq e$, we have*

$$\begin{aligned} z = & \sum_{\gamma < \gamma a_i^{\pm 1}} p_{\gamma,\gamma a_i^{\pm 1}} \alpha^{\pm}_{i,\omega}(\gamma|z) + \sum_{\gamma > \gamma a_i^{\pm 1}} p_{\gamma,\gamma a_i^{\pm 1}} \times \frac{1}{\alpha^{\pm}_{i,\omega}(\gamma a_i^{\mp 1}|z)} \\ & + \sum_{\gamma < \gamma a_i^{\pm 1} < \gamma a_i^{\pm 1} a_j^{\pm 1}} p_{\gamma,\gamma a_i^{\pm 1} a_j^{\pm 1}} \alpha^{\pm}_{j,\omega}(\gamma a_i^{\pm 1}|z) \alpha^{\pm}_{i,\omega}(\gamma|z) \\ & + \sum_{\gamma a_i^{\mp 1} < \gamma, \gamma a_i^{\pm 1} < \gamma a_i^{\pm 1} a_j^{\pm 1}} p_{\gamma,\gamma a_i^{\pm 1} a_j^{\pm 1}} \times \frac{\alpha^{\pm}_{j,\omega}(\gamma a_i^{\pm 1}|z)}{\alpha^{\pm}_{i,\omega}(\gamma a_i^{\pm 1}|z)} \\ & + \sum_{\gamma a_i^{\pm 1} a_j^{\pm 1} < \gamma a_i^{\pm 1} < \gamma} p_{\gamma,\gamma a_i^{\pm 1} a_j^{\pm 1}} \times \frac{1}{\alpha^{\pm}_{i,\omega}(\gamma a_i^{\pm 1}|z) \alpha^{\mp}_{j,\omega}(\gamma a_i^{-1} a_j^{\mp 1}|z)} + \cdots \\ & + \sum_{\gamma > \gamma a_{i_1}^{\pm 1} > \gamma a_{i_1}^{\pm 1} a_{i_2}^{\pm 1} > \cdots > \gamma a_{i_1}^{\pm 1} \cdots a_{i_m}^{\pm 1}} p_{\gamma,\gamma a_{i_1}^{\pm 1} \cdots a_{i_m}^{\pm 1}} \\ & \times \frac{1}{\alpha^{\pm}_{i_1,\omega}(\gamma a_{i_1}^{\mp 1}|z) \alpha^{\pm}_{i_2,\omega}(\gamma a_{i_1}^{\pm 1} a_{i_2}^{\pm 1}|z) \cdots \alpha^{\pm}_{i_m,\omega}(\gamma a_{i_1}^{\mp 1} a_{i_2}^{-1} \cdots a_{i_m}^{\mp 1}|z)}\,. \end{aligned} \tag{1.11}$$

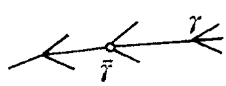
We first want to prove Proposition 1 which shows the basic symmetry property of the Green functions $G(\gamma, \omega|z)$.

For arbitrary elements $\omega, \gamma \in \Gamma$ such that $\omega^{-1}\gamma$ has a reduced expression $\bar{\gamma} a_{i_1}^{\varepsilon_1} \cdots a_{i_m}^{\varepsilon_m}$ $(\bar{\gamma} \in \Gamma)$ we call "γ-twig" of m-th degree with base point ω and denote by $\Delta_\omega^{(m)}(\gamma)$ the set of all elements $\gamma' \in \Gamma$ such that $\bar{\gamma} \leqq \omega^{-1}\gamma'$, $\mathrm{dis}(\gamma', \gamma) \leqq m$ and $\mathrm{dis}(\omega, \gamma') < \mathrm{dis}(\omega, \gamma)$. We also denote $\gamma' \,\overline{\underline{m}}\, \gamma$, if $\gamma' \in \Delta_\omega^{(m)}(\gamma)$. (See the figure.) The number m^* of the elements of $\Delta_\omega^{(m)}(\gamma)$ is given by the following formula:

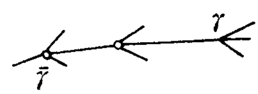
$$
m^*=\begin{cases}\dfrac{(2g-1)^{\bar{m}+1}+(2g-1)^{\bar{m}}-1}{2g-2} & \text{for} \quad m=2\bar{m}+1 \\[2ex] \dfrac{(2g-1)^{\bar{m}}-1}{2g-2} & \text{for} \quad m=2\bar{m}\,. \end{cases} \tag{1.12}
$$

We say that $\Delta_\omega^{(m)}(\gamma_1)$ and $\Delta_\omega^{(m)}(\gamma_2)$ are equivalent with respect to Γ_0 if there exists $\gamma_0\in\Gamma_0$ such that $\gamma_0\gamma_1=\gamma_2$ and $\gamma_0\Delta_\omega^{(m)}(\gamma_1)=\Delta_\omega^{(m)}(\gamma_2)$.

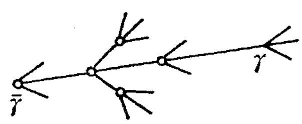
$m=1,\ g=2 \quad m^*=1$



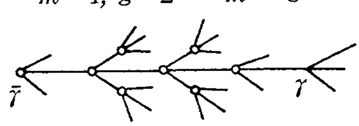
$m=2,\ g=2 \quad m^*=2$



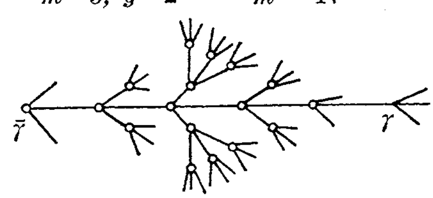
$m=3,\ g=2 \quad m^*=5$



$m=4,\ g=2 \quad m^*=8$



$m=5,\ g=2 \quad m^*=17$



PROPOSITION 1. *There exist m^* algebraic functions of z $\alpha_{\gamma,\omega}(\gamma'|z)$ for $\gamma'\in\Delta_\omega^{(m)}(\gamma)-\{\gamma\}$ depending on ω and $\Delta_\omega^{(m)}(\gamma)$ such that*

$$
G(\omega,\gamma|z)=\sum_{\gamma'\in\Delta_\omega^{(m)}(\gamma)}\alpha_{\gamma,\omega}(\gamma'|z)G(\omega,\gamma'|z) \tag{1.13}
$$

where $\alpha_{\gamma,\omega}(\gamma'|z)$ satisfy

$$
\alpha_{\gamma,\omega}(\gamma'|z)=\frac{a_{\gamma\gamma'}}{z}+O\Big(\frac{1}{z^2}\Big) \quad \text{at the infinity } z=\infty \tag{1.14}
$$

and the periodicity

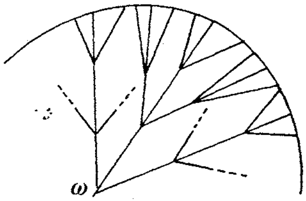
$$
\alpha_{\gamma_0\gamma,\omega}(\gamma_0\gamma'|z)=\alpha_{\gamma,\omega}(\gamma'|z) \tag{1.15}
$$

provided $\Delta_\omega^{(m)}(\gamma')$ and $\Delta_\omega^{(m)}(\gamma_0\gamma')$ are Γ_0-equivalent.

PROOF. We want to prove this proposition first by constructing the Green functions on finite domains and then by its limiting procedure of exhaution to the whole Γ. Let $D_l(\omega)$ be the subdomain consisting of elements $\gamma\in\Gamma$ such that $L(\omega^{-1}\gamma)<l$, for certain $l>m$. We denote by $B_l(\omega)$ the boundary of $D_l(\omega)$. Then we can uniquely construct the Green function $G_{D_l(\omega)}(\gamma,\gamma'|z)$ on $D_l(\omega)$ so that $G_{D_l(\omega)}(\gamma,\gamma'|z)$ vanishes on $B_l(\omega)$ in m-th order:

$$G_{D_l(\omega)}(\gamma,\gamma'|z)=0 \tag{1.16}$$

for $\gamma\in D_l(\omega)$ such that $\mathrm{dis}(\gamma,B_l(\omega))<m$, where $\mathrm{dis}(\gamma,B_l(\omega))$ denotes the distance between γ and $B_l(\omega)$. This Green function can be computed, step by step, from the boundary seeing that Γ is a tree.



Assume that $|z|\gg1$. Recall that we have put $\bar{m}=(1/2)m$ or $(1/2)(m-1)$ according as m is equal to even or odd. Let γ_1 be an arbitrary vertex lying in $B_l(\omega)$ and $l=[\bar{\gamma}_1,\gamma_1]$ be the unique geodesic of length $\bar{m}$ in $\mathcal{G}$ starting from $\bar{\gamma}_1$ and ending in γ_1 for $\bar{\gamma}_1<\gamma_1$. Then we have

$$(\mathcal{E}_{\gamma_1}):\qquad zu(\gamma_1)-\sum_{\gamma\in\Gamma}a_{\gamma_1,\gamma}u(\gamma)=0 \qquad \text{for} \quad u(\gamma)=G_{D_l}(\gamma,\omega|z)\,.$$

We denote by $\gamma_1^{(1)}=\gamma_1,\gamma_1^{(2)},\cdots,\gamma_1^{(g-1)^{\bar{m}}}$ the vertices γ lying in $B_l(\omega)$ such that $\bar{\gamma}_1<\gamma$. Then there exist the $(g-1)^{\bar{m}}-1$ other equations similar to $(\mathcal{E}_{\gamma_1})$, each corresponding to $\gamma_1^{(\nu)}$, $1\leqq\nu\leqq(g-1)^{\bar{m}}$.

$$\mathcal{E}_{\gamma_1^{(\nu)}}:\qquad zu(\gamma_1^{(\nu)})-\sum_{\gamma\in\Gamma}a_{\gamma_1^{(\nu)},\gamma}u(\gamma)=0\,.$$

Since $G_{D_l}(\gamma,\omega|z)=0$ for $L(\gamma)>l$, we can uniquely solve these with respect to $G_{D_l}(\gamma_1^{(\nu)},\omega|z)$ which has the following expression:

$$G_{D_l}(\gamma_1,\omega|z)=\sum_{\omega^{-1}\gamma\,\overline{\overline{m}}\,\omega^{-1}\gamma_1}\alpha_{\gamma_1,\omega}^{(l)}(\gamma|z)G(\gamma,\omega|z) \tag{1.17}$$

where $\alpha_{\gamma_1,\omega}^{(l)}(\gamma|z)$ denote rational functions of z depending on γ_1 and ω such that

$$\alpha_{\gamma_1,\omega}^{(l)}(\gamma|z)=O\left(\frac{1}{z}\right) \qquad \text{for} \quad |z|\gg1\,. \tag{1.18}$$

Similarly as above for an arbitrary $\gamma_1\in D_l$ such that $L(\omega^{-1}\gamma_1)\geqq m$ we can prove by induction decreasing in $L(\omega^{-1}\gamma_1)$ that the following relation holds:

(1.19) $$G(\gamma_1, \omega|z) = \sum_{\omega^{-1}\gamma \overline{\underline{m}} \omega^{-1}\gamma_1} \alpha^{(l)}_{\gamma_1, \omega}(\gamma|z) G(\gamma, \omega|z),$$

$\alpha^{(l)}_{\gamma_1,\omega}(\gamma|z)$ having the same property as above for $\gamma_1 \in \Gamma - \{\omega\}$ such that $L(\omega^{-1}\gamma_1) \leqq m-1$, we have from $(\mathcal{E}_{\gamma_1})$, by substitution of (1.19),

$(\mathcal{E}_{\gamma_1})'$: $$0 = zu(\gamma_1) - \sum_{\gamma} a_{\gamma_1\gamma} u(\gamma)$$
$$= zu(\gamma_1) - \sum_{L(\gamma) \leqq m-1} \tilde{a}_{\gamma_1\gamma} u(\gamma)$$

where $\tilde{a}_{\gamma\gamma'} = a_{\gamma\gamma'} + O(1/z)$. Since the number of vertices $\gamma_1 \in \Gamma - \{\omega\}$ such that $L(\omega^{-1}\gamma_1) \leqq m-1$ is equal to $g(g-1)^{m-2}$, there are $g(g-1)^{m-2}$ such linear equations which are linearly independent.

On the other hand there are $g(g-1)^{m-2}$ unknowns $G(\gamma, \omega|z)$ for $\gamma \in \Gamma - \{\omega\}$ and $L(\omega^{-1}\gamma) \leqq m-1$ in the equations $\mathcal{E}_{\gamma_1}$. Hence

(1.20) $$\begin{cases} \alpha^{(l)+}_{j,\omega}(\gamma|z) \dfrac{G_{D_l}(\gamma a_j, \omega|z)}{G_{D_l}(\gamma, \omega|z)} & \text{if} \quad \omega^{-1}\gamma < \omega^{-1}\gamma a_j \quad \text{or} \\ \alpha^{(l)-}_{j,\omega}(\gamma|z) \dfrac{G_{D_l}(\gamma a_j^{-1}, \omega|z)}{G_{D_l}(\gamma, \omega|z)} & \text{if} \quad \omega^{-1}\gamma < \omega^{-1}\gamma a_j^{-1}, \end{cases}$$

according as $\omega^{-1}\gamma < \omega^{-1}\gamma a_j$ or $\omega^{-1}\gamma < \omega^{-1}\gamma a_j^{-1}$, are completely determined.

On the other hand, the Green functions $G_{D_l}(\gamma, \omega|z)$ and $G(\gamma, \omega|z)$ have the Laurent expansions for $|z| \gg 1$:

(1.21) $$G_{D_l}(\gamma, \omega|z) = \sum_{n=0}^{\infty} \frac{C^{(l)}_n(\gamma, \omega)}{z^{n+1}}$$

(1.22) $$G(\gamma, \omega|z) = \sum_{n=0}^{\infty} \frac{(A^n)_{\gamma,\omega}}{z^{n+1}}.$$

For fixed γ and n, we have

(1.23) $$\lim_{l \to \infty} C^{(l)}_n(\gamma, \omega) = (A^n)_{\gamma,\omega}$$

and

(1.24) $$\lim_{l \to \infty} G_{D_l}(\gamma, \omega|z) = G(\gamma, \omega|z).$$

This implies that the following limit exists:

(1.25) $$\lim_{l \to \infty} \alpha^{(l)}_{\gamma_1,\omega}(\gamma|z) = \alpha_{\gamma_1,\omega}(\gamma|z)$$

for $\omega^{-1}\gamma \overline{\underline{m}} \omega^{-1}\gamma_1$ and $|z| \gg 1$, so that we have (1.13) and (1.14).

For $\gamma_1 \in \Gamma - \{\omega\}$ such that $L(\omega^{-1}\gamma_1) \leqq m-1$, $G(\gamma, \omega|z)$ satisfies the equations $(\mathcal{E}_{\gamma_1})$. These equations, together with (1.13), completely determine the ratio

$$\alpha^{\pm}_{i,\omega}(\gamma|z)=\frac{G(\gamma a_i^{\pm 1},\omega|z)}{G(\gamma,\omega|z)} \tag{1.26}$$

according as $\omega^{-1}\gamma<\omega^{-1}\gamma a_i$ or $\omega^{-1}\gamma<\omega^{-1}\gamma a_i^{-1}$. Hence we have the formulae:

$$\lim_{l\to\infty}\alpha^{(l)\pm}_{i,\omega}(\gamma|z)=\alpha^{\pm}_{i,\omega}(\gamma|z)\,. \tag{1.27}$$

LEMMA 2. *Assume that two Cayley subgraphs $\mathcal{G}_{\omega^{-1}\bar{\gamma}_1}$ and $\mathcal{G}_{\omega^{-1}\bar{\gamma}_2}$ are isomorphic to each other modulo Γ_0, then for two geodesic lines $l_1=[\bar{\gamma}_1,\gamma_1]$ and $l_2=[\bar{\gamma}_2,\gamma_2]$ such that $\omega^{-1}\bar{\gamma}_1<\omega^{-1}\gamma_1$ and $\omega^{-1}\bar{\gamma}_2<\omega^{-1}\gamma_2$, the equality*

$$\alpha_{\gamma_1,\omega}(\gamma|z)=\alpha_{\gamma_2,\omega}(\gamma_0\gamma|z) \tag{1.28}$$

holds if $\gamma_0\gamma_1=\gamma_2$ and $\gamma_0\Delta^{(m)}_{\omega}(\omega^{-1}\gamma_1)=\Delta^{(m)}_{\omega}(\omega^{-1}\gamma_2)$ for some $\gamma_0\in\Gamma_0$.

This implies that among all $\alpha_{\gamma,\omega}(\gamma'|z)$ for $\omega^{-1}\gamma'\overline{\underline{m}}\,\omega^{-1}\gamma$ there are only a finite number of different ones. In fact this follows from the corollary of the following lemma which is elementarily proved.

LEMMA 3. *For an arbitrary reduced expression $a_{i_1}^{\varepsilon_1}\cdots a_{i_l}^{\varepsilon_l}$ and an arbitrary element $\gamma\in\Gamma$, there always exists a reduced expression $\gamma'\in\Gamma_0\gamma$ such that γ' contains $a_{i_1}^{\varepsilon_1}\cdots a_{i_l}^{\varepsilon_l}$ as its final part, namely*

$$\gamma'=\cdots a_{i_1}^{\varepsilon_1}\cdots a_{i_l}^{\varepsilon_l}\,. \tag{1.29}$$

COROLLARY. *We can find a finite set E such that $\Gamma=\Gamma_0\cdot E$ and that the intersection of each coset of $\Gamma_0\backslash\Gamma$ and E contains reduced expression having as their final parts $a_j^{\pm 1}$, $1\leqq j\leqq g$.*

Now for an arbitrary $\gamma\in\Gamma$ such that $L(\omega^{-1}\gamma)\geqq m$, the equation $(\mathcal{E}_\gamma)$ can be expressed by using $\alpha_{\gamma,\omega}(\gamma'|z)$ as follows:

$$z\sum_{\omega^{-1}\gamma'\overline{\underline{m}}\,\omega^{-1}\gamma}\alpha_{\gamma,\omega}(\gamma'|z)G(\gamma',\omega|z) \tag{1.30}$$

$$=\sum_{\substack{\omega^{-1}\gamma'_r\overline{\underline{m}}\,\omega^{-1}\gamma'_{r-1}\overline{\underline{m}}\cdots\overline{\underline{m}}\,\omega^{-1}\gamma'_1\\ \omega^{-1}\gamma'_r\overline{\underline{m}}\,\omega^{-1}\gamma}}\alpha_{\gamma,\gamma'_1}\alpha_{\gamma'_1,\omega}(\gamma'_2|z)\alpha_{\gamma'_2,\omega}(\gamma'_3|z)\cdots\alpha_{\gamma'_{r-1},\omega}(\gamma'_r|z)G(\gamma'_r,\omega|z)$$

$$+\sum_{\substack{\omega^{-1}\gamma=\omega^{-1}\gamma'_r\overline{\underline{m}}\,\omega^{-1}\gamma'_{r-1}\overline{\underline{m}}\cdots\overline{\underline{m}}\,\omega^{-1}\gamma'_1\\ \omega^{-1}\gamma'\overline{\underline{m}}\,\omega^{-1}\gamma}}\alpha_{\gamma,\gamma'_1}\alpha_{\gamma'_1,\omega}(\gamma'_2|z)\cdots\alpha_{\gamma'_{r-1},\omega}(\gamma'_r|z)\alpha_{\gamma,\omega}(\gamma'|z)G(\gamma',\omega|z).$$

Since $G(\gamma',\omega|z)$ for $\omega^{-1}\gamma'\overline{\underline{m}}\,\omega^{-1}\gamma$ are linearly independent we have the equalities (This also follows from the limiting procedure of $\alpha^{(l)}_{\gamma,\omega}(\gamma'|z)$.): For $\omega^{-1}\gamma'\overline{\underline{m}}\,\omega^{-1}\gamma$,

$$(1.31)\qquad \begin{aligned} z\alpha_{\gamma,\omega}(\gamma'|z) &= \sum_{\substack{\omega^{-1}\gamma'=\omega^{-1}\gamma'_r\boxed{m}\omega^{-1}\gamma'_{r-1}\boxed{m}\cdots\boxed{m}\omega^{-1}\gamma'_1 \\ \omega^{-1}\gamma'_r\boxed{m}\omega^{-1}\gamma}} a_{\gamma,\gamma'_1}\alpha_{\gamma'_1,\omega}(\gamma'_2|z)\cdots\alpha_{\gamma'_{r-1},\omega}(\gamma'_r|z) \\ &\quad + \sum_{\substack{\omega^{-1}\gamma=\omega^{-1}\gamma'_r\boxed{m}\cdots\boxed{m}\omega^{-1}\gamma'_1 \\ \omega^{-1}\gamma'\boxed{m}\omega^{-1}\gamma}} a_{\gamma,\gamma'_1}\alpha_{\gamma'_1,\omega}(\gamma'_2|z)\cdots\alpha_{\gamma'_{r-1},\omega}(\gamma'_r|z)\alpha_{\gamma,\omega}(\gamma'|z) \\ &= a_{\gamma,\gamma'} + (\text{higher degree terms of } \alpha_{\gamma',\omega}(\gamma''|z) \text{ or } \alpha_{\gamma,\omega}(\gamma'|z)). \end{aligned}$$

Hence these algebraic equations have the unique solutions $\alpha_{\gamma,\omega}(\gamma'|z)$ which are algebraic functions z such that (1.14) holds. The proposition has completely been proved.

PROOF OF THE THEOREM. The Green function $G(\gamma,\omega|z)$ satisfies the $g(g-1)^{m-2}$ equations $(\mathscr{E}_\gamma)$ for $L(\omega^{-1}\gamma)\leqq m-1$ and

$$(\mathscr{E}_\omega)\qquad zG(\omega,\omega|z)-\sum_{\gamma\in\Gamma}a_{\omega,\gamma}G(\gamma,\omega|z)=1.$$

On the other hand owing to the relations (1.19) the above $1+g(g-1)^{m-2}$ equations are reduced to the linear equations with respect to the unknowns $G(\gamma,\omega|z)$ for $L(\omega^{-1}\gamma)\leqq m$, which have exactly $1+g(g-1)^{m-2}$. Hence we can solve them in a unique way such that

$$(1.32)\qquad \begin{cases} G(\omega,\omega|z)=\dfrac{1}{z}+O\left(\dfrac{1}{z^2}\right) \\ G(\gamma,\omega|z)=\dfrac{a_{\omega,\gamma}}{z^2}+O\left(\dfrac{1}{z^3}\right) \qquad \text{for} \quad L(\omega^{-1}\gamma)\leqq m-1. \end{cases}$$

This shows Theorem 1.

REMARK 1. It seems probable that Theorem 1 still holds without $(\mathscr{H}.3)$.

Let $\bar\gamma_0=e,\bar\gamma_1,\cdots,\bar\gamma_{h-1}$ be a system of representatives of left cosets of Γ by Γ_0. We denote by $\beta_{\gamma'}(\gamma)$ the quotient $\dfrac{G(\gamma'\gamma,\gamma'|z)}{G(\gamma',\gamma'|z)}$ for $\gamma\in\Gamma_0$. Then (1.10) are equivalent to the following:

$$(1.33)\qquad z\beta_{\gamma'}(\bar\gamma^{-1}\gamma)u(\bar\gamma)-\sum_{\gamma'}p_{\gamma,\gamma'}\beta_{\gamma'}(\bar\gamma''^{-1}\gamma'')u(\bar\gamma'')=0$$

if $\gamma\neq e$, where $\bar\gamma$ and $\bar\gamma''$ denote some of $\{\bar\gamma_j\}_{0\leqq j\leqq h-1}$ in the same cosets as γ and γ'' respectively. To solve the equations (1.33) we have only to consider them for a finite set of $\{\gamma\}$. For the existence of the solutions $u(\bar\gamma),\beta_{\gamma'}(\gamma_0)$ $(\gamma_0\in\Gamma_0)$ must satisfy certain system of polynomial equations, which define the same algebraic curve $\mathfrak{C}$. Therefore each $u(\bar\gamma)$ is a meromorphic function on $\mathfrak{C}$. As in the one dimensional case we can make the following conjecture:

Conjecture. Has each $\beta_{\gamma'}(\gamma_0)$ ($\gamma_0 \in \Gamma_0$) poles in $\mathfrak{C}$ only at places over $z=\infty$? Further, do $\beta_{\gamma'}(\gamma_0)$ generate the whole group of units $\mathcal{E}$ in the ring of meromorphic functions on $\mathfrak{C}$ whose poles lie only at places over $z=\infty$? In the next section we shall give a simplest case where this is true.

§ 2. Left-invariant random walk on a free group (*nearest neighbour case*).

Now we consider the special case $m=1$ *and* $\Gamma_0=\Gamma$. Then P is represented as follows:

$$(Pu)(\gamma)=\sum_{i=1}^{g}\{p_i^{(+)}u(\gamma a_i)+p_i^{(-)}u(\gamma a_i^{-1})\} \tag{2.1}$$

where $\sum_{i=1}^{g}(p_i^{(+)}+p_i^{(-)})=1$ and $p_i^{(+)}, p_i^{(-)} \geqq 0$.

In this case, $G(\gamma, \gamma'|z)$ being left-invariant with respect to Γ:

$$G(\gamma_0\gamma, \gamma_0\gamma'|z)=G(\gamma, \gamma'|z), \qquad \forall \gamma_0 \in \Gamma \tag{2.2}$$

we have only to compute the function $G(\gamma, e|z)$. Let $\mathfrak{C}$ be the algebraic curve consisting of the points (z, W), satisfying

$$z+(g-1)W=\sqrt{W^2+4p_1^{(+)}p_1^{(-)}}+\cdots+\sqrt{W^2+4p_g^{(+)}p_g^{(-)}}\,. \tag{2.3}$$

The function $W(z)$ can be determined in a unique way such that

$$W(z)=z-\frac{2\sum_{j=1}^{g}p_j^{(+)}p_j^{(-)}}{z}+\cdots \tag{2.4}$$

at the infinity. Then

THEOREM 2. *The Green function* $G(\gamma, e|z)$ *can be explicitly determined as follows:*

(2.5)

i) $G(e, e|z)=\dfrac{1}{W(z)}$

ii) $\alpha_{i,\gamma'}^{\pm}(\gamma|z)$ *are independent of* γ *and* γ', *and given by the formulae*

$$\begin{cases} \alpha_{i,\gamma'}^{+}(\gamma|z)=\alpha_i^{(+)}(z)=\dfrac{-W+\sqrt{W^2+4p_i^{(+)}p_i^{(-)}}}{2p_i^{(+)}} \\[2ex] \alpha_{i,\gamma'}^{-}(\gamma|z)=\alpha_i^{(-)}(z)=\dfrac{-W+\sqrt{W^2+4p_i^{(+)}p_i^{(-)}}}{2p_i^{(-)}}\,. \end{cases} \tag{2.6}$$

Consequently, for a reduced expression $\gamma=\alpha_{i_1}^{\varepsilon_1}\cdots\alpha_{i_m}^{\varepsilon_m}$, *we have*

$$G(\gamma, e|z)=G(e, e|z)\cdot\alpha_{i_1}^{(\varepsilon_1)}(z)\cdots\alpha_{i_m}^{(\varepsilon_m)}(z). \tag{2.7}$$

Remark that for $|z|\gg 1$, the inequality $|W|\gg 1$ holds, and therefore $|\alpha_i^{\mp}(z)|<1$. This implies,

$$\lim_{L(\gamma)\to\infty} G(\gamma, e|z)=0 \qquad \text{for} \quad |z|\gg 1. \tag{2.8}$$

PROOF. Under the condition of Theorem 2, (1.11) has a very simple form:

$$\begin{cases} \alpha_i^{+}(z)=\dfrac{p_i^{(-)}}{z-\sum\limits_{j} p_j^{(+)}\alpha_j^{(+)}-\sum\limits_{j\neq i} p_j^{(-)}\alpha_j^{(-)}} \\ \alpha_i^{-}(z)=\dfrac{p_i^{(+)}}{z-\sum\limits_{j\neq i} p_j^{(+)}\alpha_j^{(+)}-\sum\limits_{j} p_j^{(-)}\alpha_j^{(-)}} \end{cases} \quad 1\leqq i\leqq g, \quad \text{where} \tag{2.9}$$

$$z-W=\sum_{j=1}^{g} p_j^{(+)}\alpha_j^{(+)}+\sum_{j=1}^{g} p_j^{(-)}\alpha_j^{(-)}. \tag{2.10}$$

(2.5) and (2.6) follow from (2.9) and (2.10).

It seems interesting to remark that the transformation T defined by the right hand side of (2.9) is *not projective but Cremona transformation on a space of points* $(\alpha_1^{+},\cdots,\alpha_g^{+},\alpha_1^{-},\cdots,\alpha_g^{-})$[2] (For the definition see [11].). $(\alpha_i^{(+)},\alpha_i^{(-)})$ $1\leqq i\leqq g$ becomes a fixed point of T, so that it can be regarded as a limiting point of T^n for $n\to\pm\infty$. In a sense, this is a generalization of *periodic continued fractions* or more generally, *periodic Jacobi-Perron algorithms* in ordinary linear difference systems, where shift operators given by the corresponding Riccati systems were projective (See [2], [17], [19], [21].).

REMARK 2. The group of units $\mathcal{E}$ attached to the curve $\mathfrak{C}$ is isomorphic to $\boldsymbol{Z}^{g(g+1)/2}$ and generated by the units $\alpha_i^{+}(z)$, $p_i^{+}\alpha_i^{+}-p_j^{+}\alpha_j^{+}$, $1\leqq i\neq j\leqq g$. This cannot be realized by any Jacobi-Perron algorithm except for the case $g=1$. This fact is an essentially different situation from one dimensional cases. It seems to be interesting to understand what algorithm is going on in our case.

§ 3. Symmetric random walk on a free group and the spectrum.

We assume further that

2) More exactly this is called "standard Cremona transformation" by I.V. Dolgachev, See [28]

$$\text{(3.1)} \qquad \begin{cases} \text{i)} & p_j = p_j^{(+)} = p_j^{(-)} \geqq 0 \\ \text{ii)} & 2\sum_{j=1}^{g} p_j = 1 . \end{cases}$$

Then the matrix P defines *a symmetric random walk on Γ* and becomes itself a self-adjoint operator on $l^2(\Gamma)$. We are interested in the spectral property for P. The equation (2.3) becomes now

$$\text{(3.2)} \qquad z+(g-1)W=\sqrt{W^2+4p_1^2}+\cdots+\sqrt{W^2+4p_g^2} .$$

The following five Lemmas can be proved in an elementary way.

LEMMA 3.1. *The functions $z=\Phi(W)$,*

$$\text{(3.3)} \qquad \Phi(W)=-(g-1)W\pm\sqrt{W^2+4p_1^2}\pm\cdots\pm\sqrt{W^2+4p_g^2}$$

have the only two critical points $\pm W_0$ $(1\geqq W_0\geqq 0)$ on $\boldsymbol{R}$.

We denote by $\pm z_0$ the corresponding value of z, namely

$$\text{(3.4)} \qquad \begin{cases} z_0=-(g-1)W_0+\sqrt{W_0^2+4p_1^2}+\cdots+\sqrt{W_0^2+4p_g^2} \\ 0=-(g-1)+\dfrac{W_0}{\sqrt{W_0^2+4p_1^2}}+\cdots+\dfrac{W_0}{\sqrt{W_0^2+4p_g^2}} . \end{cases}$$

LEMMA 3.2. *We have $0<z_0<1$ for $g>1$ and $z_0=1$ for $g=1$.*

We now assume that

$$\text{(3.5)} \qquad p_1>p_2>\cdots>p_g .$$

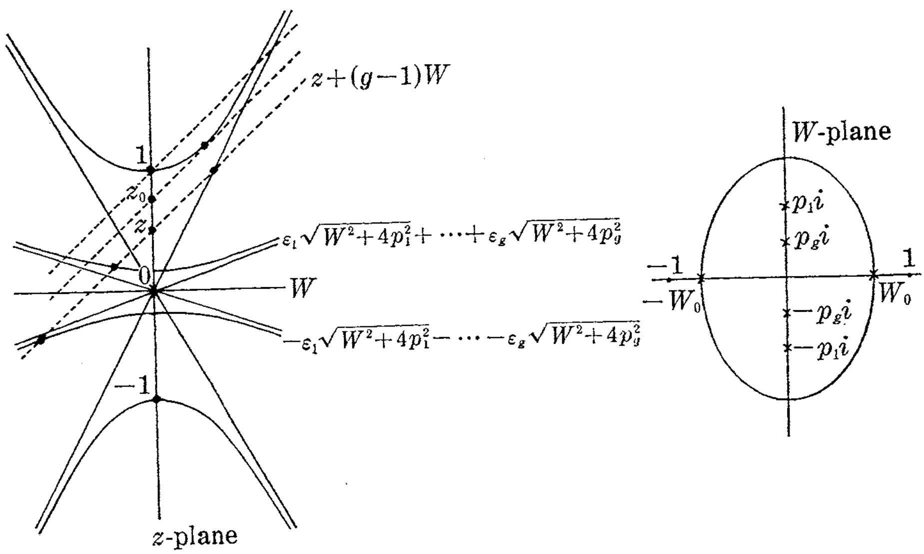
Consider the equations $(3.3)_\varepsilon$

$$(3.3)_\varepsilon : \qquad z+(g-1)W=\varepsilon_1\sqrt{W^2+4p_1^2}+\cdots+\varepsilon_g\sqrt{W^2+4p_g^2}$$

for $\varepsilon_j=\pm 1$. Then figuring out the graphs of $(3.3)_\varepsilon$, we see

LEMMA 3.3. i) *For $z\in\boldsymbol{R}$, $(3.3)_\varepsilon$ has one real root with respect to W except for $\varepsilon_1=\cdots=\varepsilon_g=\pm 1$.*

ii) *For $z>z_0$, $(3.3)_{(1,1,\dots,1)}$ has 2 different real roots. For $z<-z_0$, $(3.3)_{(-1,\dots,-1)}$ has 2 different real roots.*



As a result, we have

LEMMA 3.4. *The equation* (3.2) *with respect to* W *has* 2^g *real different solutions for* $|z|>z_0$. *The* $W(z)$ *defined by* (2.3) *is characterized as the unique maximal solution for* $z>z_0$. $(3.3)_\varepsilon$ *have* (2^g-2) *real different solutions for* $z\in(-z_0, z_0)$ *and* 2 *different imaginary roots. In particular,* $dW/dz\neq\infty$ *for* $z\in\boldsymbol{R}$ *except for* $z=\pm z_0$.

LEMMA 3.5. *The function* $W(z)$ *defined on the domain* $\varDelta=\boldsymbol{C}-[-z_0, z_0]$ *is univalent, and its image by* $W(z)$ *is disjoint from the union of intervals* $[-W_0, W_0]\cup[-p_1i, p_1i]$. *Remark that* $W(\infty)=\infty$.

PROOF. First we remark that the image $W(\varDelta\cap(\operatorname{Im} z>0))$ lies in the upper half-plane because every branch of $\Phi(W)$ is real.

Suppose that $W_1=it$ $t>0$ for $p_{j-1}\leqq t\leqq p_j$. Then $\operatorname{Im}\Phi(W_1)<0$:

$$\Phi(W_1)=-(g-1)it\pm\sqrt{p_1^2-t^2}\pm\cdots\pm\sqrt{p_j^2-t^2} \pm i\sqrt{t^2-p_{j+1}^2}\pm\cdots\pm i\sqrt{t^2-p_g^2}\,. \tag{3.6}$$

In other words the interval $[0, ip_1]$ is disjoint from $W((\operatorname{Im} z\geqq 0)\cap\varDelta)$. Similarly $[-ip_1, 0]\cap W((\operatorname{Im} z\geqq 0)\cap\varDelta)=\phi$. On the other hand the boundary correspondence is bijective because $dz/dW\neq 0$ there. This implies that $W(z)$ maps conformally from $\varDelta$ onto $W(\varDelta)$. Lemma 3.5 has been proved.

REMARK 3. It seems probable that the complement of the image $W(\Delta)$ is convex.

From Lemma 3.5, we have

PROPOSITION 2. i) *For* $-z_0 \leqq \lambda \leqq z_0$, $W(\lambda \pm i0)$ *exists and we have*

$$\begin{cases} W(\lambda+i0) \neq W(\lambda-i0) \\ W(\lambda+i0)\, W(\lambda-i0) \neq 0 \end{cases} \tag{3.7}$$

for $-z_0<\lambda<z_0$. $\pm z_0$ *are the branch points of* $W(z)$ *of order* 2.

ii) $\alpha_j^{\pm}(z)$ *are all holomorphic in* Δ. *Actually in* Δ *we have the inequality*

$$|\alpha_j^{\pm}(z)|<1 . \tag{3.8}$$

This corresponds to the following well-known properties of Green kernels:

THEOREM 3. i) $G(\gamma, \gamma'|z)$ *are holomorphic in* Δ. *We have*

$$\sum_{\gamma\in\Gamma} |G(\gamma, \gamma'|z)|^2 < \infty \tag{3.9}$$

for a fixed $\gamma' \in \Gamma$ *and*

$$G(\gamma, \gamma'|z) = G(\gamma', \gamma|z) . \tag{3.10}$$

ii) *The spectral kernel*

$$\Theta(\gamma, \gamma'|\lambda) = \frac{-1}{2i}(G(\gamma, \gamma'|\lambda+i0) - G(\gamma, \gamma'|\lambda-i0)), \qquad \lambda \in \boldsymbol{R}, \tag{3.11}$$

are different from zero if and only if $-z_0<\lambda<z_0$. $\Theta(\gamma, \gamma'|\lambda)$ *is continuous on* $[-z_0, z_0]$. *Therefore the operator* P *on* $l^2(\Gamma)$ *has an absolute continuous spectrum on the whole interval* $[-z_0, z_0]$.

In the special case where p_j are all equal to $1/2g$, this theorem is classical and has been proved by H. Kesten (See [18].) and recently by A. Figà-Talamanca and M. A. Picardello using spherical functions on free groups[3] (See [6], [7] and [22].).

§ 4. Poisson kernel and eigenfunction expansion.

As has been developed by many authors, the concept of "*wave packets*" or "*asymptotic waves*" such as plane waves, distorted plane waves, horispherical waves plays an important part for eigenfunction expansions of

3) This was communicated to the author by Mr. Y. Watatani. See also [29] and [30].

linear differential or difference operators[4]. For the Schrödinger operator $-\Delta+V$ this has been established by T. Ikebe by means of *Lippmann-Schwinger integral equations* as early as 1959 (See [12], [16] and [23].). In representation theory of semi-simple Lie groups, this has been known as *"Helgason-Okamoto conjecture"* and established by M. Kashiwara, K. Okamoto and their coworkers in full generality (See [10], [15].).

Here we want to show that a similar result holds for the transition probability matrix P. When $z=1$, from the equation (1.10) we can define Green functions for *harmonic functions* whose Martin kernels have been studied by many authors, for example see [3], [4], [5], [14], [26]. H. Furstenberg has made an interesting observation about certain equivalences between random walks on non-compact symmetric spaces and on discontinuous isometry groups acting on them (See [8], [13].). Following this principle, we shall define Poisson kernels and Poisson boundaries for arbitrary spectrum λ, $-z_0\leqq\lambda\leqq z_0$.

Let Ξ be the boundary of Γ, namely the compact totally diconnected Γ-space consisting of all infinite sequences of reduced expressions in Γ:

$$\Xi=\{\xi=a_{i_1}^{\varepsilon_1}a_{i_2}^{\varepsilon_2}a_{i_3}^{\varepsilon_3}\cdots a_{i_n}^{\varepsilon_n}\cdots\} \tag{4.1}$$

where $i_\mu\neq i_{\mu+1}$ or $i_\mu=i_{\mu+1}$, $\varepsilon_\mu\varepsilon_{\mu+1}>0$. We say that the n-th initial part of ξ, $a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n}$ *divides* ξ (or is *smaller than* ξ) and write $a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n}<\xi$.

Let $\xi\in\Xi$ and $\gamma\in\Gamma$ have reduced expressions $a_{i_1}^{\varepsilon_1}a_{i_2}^{\varepsilon_2}\cdots a_{i_n}^{\varepsilon_n}\cdots$ and $a_{j_1}^{\kappa_1}a_{j_2}^{\kappa_2}\cdots a_{j_m}^{\kappa_m}$ respectively. We denote by γ' the greatest element in Γ such that $\gamma'<\gamma$ and $\gamma'<\xi$. Then γ' has the reduced expression

$$\gamma'=a_{i_1}^{\varepsilon_1}\cdots a_{i_r}^{\varepsilon_r} \qquad (r\leqq m) \tag{4.2}$$

where $i_1=j_1,\cdots,i_r=j_r$ and $\varepsilon_1=\kappa_1,\cdots,\varepsilon_r=\kappa_r$ but that $i_{r+1}\neq j_{r+1}$ or $i_{r+1}=j_{r+1}$, $\varepsilon_{r+1}\kappa_{r+1}<0$. Under these circumstances,

PROPOSITION 3. *Let $\{\gamma'_\nu\}_{1\leqq\nu<\infty}$ be a sequence of reduced expressions which tend to a boundary point $\xi\in\Xi$. Then for $z\in\Delta$,*

$$\lim_{\gamma'_\nu\to\xi}\frac{G(\gamma,\gamma'_\nu|z)}{G(e,\gamma'_\nu|z)}=K(\gamma,\xi|z) \tag{4.3}$$

exists and defined as follows:

$$K(\gamma,\xi|z)=\frac{\alpha_{j_{r+1}}(z)\cdots\alpha_{j_m}(z)}{\alpha_{i_1}(z)\cdots\alpha_{i_r}(z)}\,. \tag{4.4}$$

$K(\gamma,\xi|z)$ satisfies the equation (1.10), *for a fixed $\xi\in\Xi$ and plays the role*

4) The author is grateful to Prof. A. Inoue for this suggestion.

of "wave packets" coming from the direction ξ at the infinity. This is the Poisson kernel for P. If $z=1$, then $K(\gamma, \xi|1)$ is nothing else than the Martin kernel for P (*See* [3], [4], [24], [26].). *Remark that*

$$\overline{K(\gamma, \xi|\lambda+i0)}=K(\gamma, \xi|\lambda-i0) \tag{4.5}$$

for $\lambda\in[-z_0, z_0]$. This is generally different from $K(\gamma, \xi|\lambda+i0)$.

Let $\Xi(\gamma)$ be the set of all $\xi\in\Xi$ such that $\gamma<\xi$. Then the family of all $\Xi(\gamma)$, $\gamma\in\Gamma$ gives a fundamental system of open neighbourhoods in Ξ. Now we define the measure $\mu(d\xi|\lambda)$ as follows:

DEFINITION.

$$\int_{\Xi}\mu(d\xi|\lambda)=\frac{1}{2i}\left(\frac{1}{W(\lambda-i0)}-\frac{1}{W(\lambda+i0)}\right) \tag{4.6}$$

$$\int_{\Xi(a_{j_1}^{\varepsilon_1}\cdots a_{j_m}^{\varepsilon_m})}\mu(d\xi|\lambda) = \frac{1}{2i}\frac{|\alpha_{j_1}(\lambda+i0)|^2\cdots|\alpha_{j_{m-1}}(\lambda+i0)|^2 p_{j_m}(\alpha_{j_m}(\lambda-i0)-\alpha_{j_m}(\lambda+i0))}{|W(\lambda+i0)|^2} \tag{4.7}$$

for $m\geqq 1$. Remark that the right hand sides are all positive because $\operatorname{Im} W(\lambda+i0)>0$, $\operatorname{Im}\alpha_j(\lambda+i0)<0$. Further

LEMMA 4.1. *We have the identity*:

$$2\sum_{j=1}^{g}\frac{|\alpha_j(\lambda+i0)|^2}{1+|\alpha_j(\lambda+i0)|^2}=1\,. \tag{4.8}$$

PROOF. (1.10) gives

$$\begin{aligned}\lambda&=2\sum_{j\neq k}p_j\alpha_j(\lambda+i0)+p_k\alpha_k(\lambda+i0)+p_k\frac{1}{\alpha_k(\lambda+i0)}\\ &=2\sum_{j\neq k}p_j\alpha_j(\lambda-i0)+p_k\alpha_k(\lambda-i0)+\frac{p_k}{\alpha_k(\lambda-i0)}\,.\end{aligned} \tag{4.9}$$

We put $\alpha_j(\lambda+i0)-\alpha_j(\lambda-i0)=A_j$. Then (4.9) implies

$$\begin{aligned}2\sum_{1}^{g}p_jA_j-p_kA_k&=p_k\left(\frac{1}{\alpha_k(\lambda-i0)}-\frac{1}{\alpha_k(\lambda+i0)}\right)\\ &=\frac{p_kA_k}{|\alpha_k(\lambda+i0)|^2}\,.\end{aligned} \tag{4.10}$$

Consequently

$$\frac{|\alpha_j(\lambda+i0)|^2}{1+|\alpha_j(\lambda+i0)|^2}=\frac{p_kA_k}{2\sum_{1}^{g}p_jA_j}\,, \tag{4.11}$$

which gives Lemma 4.1. (4.8) guarantees that $\mu(d\xi|\lambda)$ *has the countable additive property*, so that $\mu(d\xi|\lambda)$ gives a Borel measure on Ξ.

PROPOSITION 4. *$K(\gamma, \xi|z)$, $z\in\Delta$ and $\mu(d\xi|\lambda)$, $\lambda\in[-z_0, z_0]$ have the following quasi-invariant property with respect to Γ.*

$$(4.12)\quad \text{(i)}\quad K(\sigma_j^{\mp1}\gamma, \sigma_j^{\mp1}\xi|z)=\begin{cases}\alpha_j(z)K(\gamma,\xi|z) & \text{if } \sigma_j^{\mp1}<\xi\\ \dfrac{1}{\alpha_j(z)}K(\gamma,\xi|z) & \text{if } \sigma_j^{\mp1}\not<\xi.\end{cases}$$

$$(4.13)\quad \text{(ii)}\quad \mu(d(\sigma_j^{\mp1}\xi)|\lambda)=\begin{cases}\dfrac{1}{|\alpha_j(\lambda+i0)|^2}\mu(d\xi|\lambda) & \text{if } \sigma_j^{\pm1}<\xi\\ |\alpha_j(\lambda+i0)|^2\mu(d\xi|\lambda) & \text{if } \sigma_j^{\mp1}\not<\xi.\end{cases}$$

Consequently the kernel $|K(\gamma, \xi|\lambda+i0)|^2\mu(d\xi|\lambda)$ is invariant with respect to Γ:

$$(4.14)\quad K(\gamma_0\gamma, \gamma_0\xi|\lambda+i0)K(\gamma_0\gamma', \gamma_0\xi|\lambda-i0)\mu(d(\gamma_0\xi)|\lambda)$$
$$=K(\gamma, \xi|\lambda+i0)K(\gamma', \xi|\lambda-i0)\mu(d\xi|\lambda)$$

for an arbitrary $\gamma_0\in\Gamma$.

It is instructive to give $\mu(d\xi|\lambda)$ in case of $p_j=1/2g$. In this case $\alpha_j(z)$ are all equal:

$$(4.15)\quad \alpha(z)=\alpha_j(z)=\frac{gz-\sqrt{g^2z^2-(2g-1)}}{2g-1}\qquad 1\leqq j\leqq g$$

$$(4.16)\quad W(z)=\frac{(g-1)z+\sqrt{g^2z^2-(2g-1)}}{2g-1}$$

and

$$(4.17)\quad |\alpha(\lambda+i0)|^2=\frac{1}{2g-1}$$

$$(4.18)\quad \int_{\Xi}=\frac{\sqrt{(2g-1)-g^2\lambda^2}}{2g-1}$$

$$(4.19)\quad \int_{\Xi(a_{i_1}^{\varepsilon_1}\cdots a_{i_n}^{\varepsilon_n})}\mu(d\xi|\lambda)=\frac{\sqrt{2g-1-g^2\lambda^2}}{2g-1}\,\frac{1}{2g(2g-1)^{n-1}}$$

for $n\geqq1$. These relations are well-known except for the factor $\dfrac{\sqrt{2g-1-g^2\lambda^2}}{2g-1}$ (See [5], [18].).

The crucial fact is the following:

PROPOSITION 5. *We have the Poisson formula:*

$$(4.20)\quad \Theta(\gamma, e|\lambda)=\int_{\Xi}K(\gamma, \xi|\lambda+i0)\mu(d\xi|\lambda)$$

for $-z_0 \leqq \lambda \leqq z_0$.

This is proved by standard techniques in potential theory and spectral theory (See [1], [14] and [19].). In fact it can be proved by the following two Lemmas which are well-known.

LEMMA 4.2. *For two functions* $u(\gamma)$ *and* $v(\gamma)$ *satisfying* (1.10) *in a finite domain* $\mathfrak{D}$, *with smooth boundary* $\mathfrak{B}$, *we have the equalities for the Dirichlet form*:

$$(4.21) \qquad D(u, v) = \sum_{\gamma, \gamma a_i \in \mathfrak{D} \cup \mathfrak{B}} (u(\gamma) - u(\gamma a_i))(v(\gamma) - v(\gamma a_i)) p_i + (-1+z) \sum_{\gamma \in \mathfrak{D}} u(\gamma) v(\gamma)$$

$$= \sum_{\gamma \in \mathfrak{B}} u(\gamma) \frac{\partial v(\gamma)}{\partial \nu_\gamma} = \sum_{\gamma \in \mathfrak{B}} \frac{\partial u(\gamma)}{\partial \nu_\gamma} v(\gamma),$$

where $\mathfrak{B}$ *denotes the boundary of* $\mathfrak{D}$ *and*

$$(4.22) \qquad \frac{\partial u(\gamma)}{\partial \nu_\gamma} = (u(\gamma) - u(\gamma a_j^{\mp 1})) p_j$$

such that $\gamma a_j^{\mp 1} \in \mathfrak{D}$ *and* $\gamma \in \mathfrak{B}$. *Since* $\mathfrak{B}$ *is smooth*, $\gamma a_j^{\mp 1}$ *is uniquely determined by* γ.

COROLLARY. *Under the same condition as above,*

$$(4.23) \qquad \sum_{\gamma \in \mathfrak{B}} u(\gamma) \frac{\partial v(\gamma)}{\partial \nu_\gamma} = \sum_{\gamma \in \mathfrak{B}} v(\gamma) \frac{\partial u(\gamma)}{\partial \nu_\gamma}.$$

LEMMA 4.3. *Let* $G_{\mathfrak{D}}(\gamma, \gamma' | z)$ *be the Green function on* $\mathfrak{D}$. *Then* $u(\gamma)$ *satisfying* (1.10), *we have*

$$(4.24) \qquad u(\gamma) = -\sum_{\gamma \in \mathfrak{B}} u(\gamma') \frac{\partial G(\gamma, \gamma' | z)}{\partial \nu_{\gamma'}}$$

provided that z *is not any eigenvalue of the Dirichlet problem on* $\mathfrak{D}$.

PROOF OF THE PROPOSITION. Consider the Green function $G_{\mathfrak{D}_{l'}}(\gamma, \gamma' | z)$ on the domain $\mathfrak{D}_{l'}$ consisting of elements $\gamma \in \Gamma$ such that $L(\gamma) < l'$ with the boundary $\mathfrak{B}_{l'}$. We can apply the formula (4.24) to $\Theta(\gamma, e | \lambda + i\delta)$, $\delta > 0$ (Remark that $\Theta(\gamma, e | z)$ can be analytically continued near $(-z_0, z_0)$.).

$$(4.25) \qquad \Theta(\gamma, e | \lambda + i\delta) = -\sum_{\gamma' \in \mathfrak{B}_{l'}} \frac{\partial G_{\mathfrak{D}_{l'}}(\gamma, \gamma' | \lambda + i\delta)}{\partial \nu_{\gamma'}} \Theta(\gamma', e | \lambda + i\delta)$$

$$= -\sum_{\gamma' \in \mathfrak{B}_{l'}} \frac{\dfrac{\partial G_{\mathfrak{D}_{l'}}(\gamma, \gamma' | \lambda + i\delta)}{\partial \nu_{\gamma'}}}{\dfrac{\partial G_{\mathfrak{D}_{l'}}(e, \gamma' | \lambda + i\delta)}{\partial \nu_{\gamma'}}} \frac{\partial G_{\mathfrak{D}_{l'}}(e, \gamma' | \lambda + i\delta)}{\partial \nu_{\gamma'}}$$

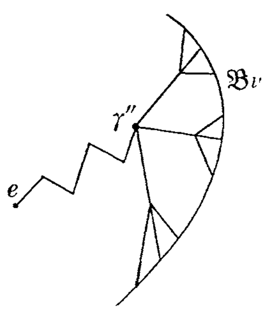
$$\times \Theta(\gamma', e | \lambda + i\delta).$$

if $L(\gamma)<l$ and $l<l'$. For $l'\to\infty$, the kernel

$$\frac{\partial G_{\mathfrak{D}_{l'}}(\gamma,\gamma'|\lambda+i\delta)}{\partial\nu_{\gamma'}}\Big/\frac{\partial G_{\mathfrak{D}_{l'}}(e,\gamma'|\lambda+i\delta)}{\partial\nu_{\gamma'}} \tag{4.26}$$

tends uniformly to the Poisson kernel $K(\gamma,\xi|\lambda+i\delta)$ for $\gamma\in\mathfrak{D}_l\subset\mathfrak{D}_{l'}$ provided that γ' tends to certain $\xi\in\Xi$. $K(\gamma,\xi|\lambda+i0)$ depends only on the first l letters of ξ. On the other hand, for a fixed $\gamma''\in\mathfrak{B}_l$, owing to (4.23),

$$\begin{aligned}
(4.27)\qquad & -\sum_{\mathfrak{S}_{\gamma''}\cap\mathfrak{B}_{l'}}\frac{\partial G_{\mathfrak{D}_{l'}}(e,\gamma'|\lambda+i\delta)}{\partial\nu_{\gamma'}}\Theta(\gamma',e|\lambda+i\delta)\\
&=-\sum_{\gamma''<\gamma''a_j^{\pm1}}p_j[(G_{\mathfrak{D}_{l'}}(e,\gamma''a_j^{\pm1}|\lambda+i\delta)-G_{\mathfrak{D}_{l'}}(e,\gamma''|\lambda+i\delta))\cdot\Theta(\gamma'',e|\lambda+i\delta)\\
&\quad-(\Theta(\gamma''a_j^{\pm1},e|\lambda+i\delta)-\Theta(\gamma'',e|\lambda+i\delta))\cdot G_{\mathfrak{D}_{l'}}(e,\gamma''|\lambda+i\delta)]\\
&=-\sum_{\gamma''<\gamma''a_j^{\pm1}}p_j[G_{\mathfrak{D}_{l'}}(e,\gamma''a_j^{\pm1}|\lambda+i\delta)\Theta(\gamma'',e|\lambda+i\delta)\\
&\quad-G_{\mathfrak{D}_{l'}}(e,\gamma''|\lambda+i\delta)\Theta(\gamma''a_j^{\pm1},e|\lambda+i\delta)].
\end{aligned}$$



For $l\to\infty$, this tends to

$$-\sum_{\gamma''<\gamma''a_j^{\pm1}}p_j[G(e,\gamma''a_j^{\pm1}|\lambda+i\delta)\Theta(\gamma'',e|\lambda+i\delta)-G(e,\gamma''|\lambda+i\delta)\Theta(\gamma''a_j^{\pm1},e|\lambda+i\delta)]. \tag{4.28}$$

When δ tends to 0, this becomes

$$-\sum_{\gamma''<\gamma''a_j^{\pm1}}p_j[G(e,\gamma''a_j^{\pm1}|\lambda+i0)\Theta(\gamma'',e|\lambda)-G(e,\gamma''|\lambda+i0)\Theta(\gamma''a_j^{\pm1},e|\lambda)] \tag{4.29}$$

which is equal to

$$-\sum_{\gamma''<\gamma''a_j^{\pm1}}p_j\Big[\frac{\alpha_{j_1}(\lambda+i0)\cdots\alpha_{j_l}(\lambda+i0)\alpha_j(\lambda+i0)}{W(\lambda+i0)} \tag{4.30}$$

$$\times\frac{1}{2i}\left(\frac{\alpha_{j_1}(\lambda-i0)\cdots\alpha_{j_l}(\lambda-i0)}{W(\lambda-i0)}-\frac{\alpha_{j_1}(\lambda+i0)\cdots\alpha_{j_l}(\lambda+i0)}{W(\lambda+i0)}\right)$$

$$-\frac{1}{2i}\left(\frac{\alpha_{j_1}(\lambda-i0)\cdots\alpha_{j_l}(\lambda-i0)\alpha_j(\lambda-i0)}{W(\lambda-i0)}\right.$$

$$\left.-\frac{\alpha_{j_1}(\lambda+i0)\cdots\alpha_{j_l}(\lambda+i0)\alpha_j(\lambda+i0)}{W(\lambda+i0)}\right)\frac{\alpha_{j_1}(\lambda+i0)\cdots\alpha_{j_l}(\lambda+i0)}{W(\lambda+i0)}\Bigg]$$

$$=-\sum_{\gamma''<\gamma''\alpha_j^{\pm1}}\frac{p_j}{2i}\left[\frac{|\alpha_{j_1}(\lambda+i0)|^2\cdots|\alpha_{j_l}(\lambda+i0)|^2(\alpha_j(\lambda+i0)-\alpha_j(\lambda-i0))}{|W(\lambda+i0)|^2}\right].$$

Therefore for $L(\gamma)<l$,

$$\Theta(\gamma, e|\lambda)=\sum_{\gamma''=\alpha_{j_1}^{\varepsilon_1}\cdots\alpha_{j_l}^{\varepsilon_l}\in\mathfrak{B}_l}K(\gamma, \alpha_{j_1}^{\varepsilon_1}\cdots\alpha_{j_l}^{\varepsilon_l}\cdots|\lambda+i0) \tag{4.31}$$

$$\cdot\sum_{\gamma''<\gamma''\alpha_j^{\pm1}}\frac{p_j}{2i}\left[\frac{|\alpha_{j_1}(\lambda+i0)|^2\cdots|\alpha_{j_l}(\lambda+i0)|^2(-\alpha_j(\lambda+i0)+\alpha_j(\lambda-i0))}{|W(\lambda+i0)|^2}\right]$$

which is exactly equal to

$$\int_{\Xi}K(\gamma, \xi|\lambda+i0)\mu(d\xi|\lambda). \tag{4.32}$$

The Proposition has thus been proved.

Proposition 4 and 5 imply immediately:

THEOREM 4. *We have the canonical decomposition*

$$\Theta(\gamma, \gamma'|\lambda)=\int_{\Xi}K(\gamma, \xi|\lambda+i0)K(\gamma', \xi|\lambda-i0)\mu(d\xi|\lambda). \tag{4.33}$$

In fact $K(e, \xi|\lambda\pm i0)=1$ and $\Theta(\gamma, \gamma'|\lambda)=\Theta(\gamma'^{-1}\gamma, e|\lambda)$.

For general theory of this kind of statements see [16], [20], [23].

Now we are in a position to give the eigenfunction expansion for the operator P as follows:

Let $u(\gamma)$ and $v(\gamma)$ be two arbitrary elements in $l^2(\Gamma)$. Firstly we define the generalized Fourier transform of $u(\gamma)$ and $v(\gamma)$:

$$\tilde{u}(\xi|\lambda)=\sum_{\gamma\in\Gamma}K(\gamma, \xi|\lambda+i0)u(\gamma) \tag{4.34}$$

$$\tilde{v}(\xi|\lambda)=\sum_{\gamma\in\Gamma}K(\gamma, \xi|\lambda+i0)v(\gamma). \tag{4.35}$$

Then we have

$$(u, v)=\sum_{\gamma\in\Gamma}u(\gamma)\overline{v(\gamma)}=\int_{-z_0}^{z_0}d\lambda\int_{\Xi}\tilde{u}(\xi|\lambda)\overline{\tilde{v}(\xi|\lambda)}\mu(d\xi|\lambda) \tag{4.36}$$

and

$$(Pu, v)=\int_{-z_0}^{z_0} \lambda d\lambda \int_{\Xi} \tilde{u}(\xi|\lambda)\overline{\tilde{v}(\xi|\lambda)}\mu(d\xi|\lambda). \tag{4.37}$$

Finally,

$$Pu(\gamma)=\text{l.i.m.}\int_{-z_0}^{z_0} \lambda d\lambda \int_{\Xi} \tilde{u}(\xi|\lambda)K(\gamma, \xi|\lambda-i0)\mu(d\xi|\lambda). \tag{4.38}$$

This gives a discrete analogue of the eigen-function expansion for the operator $(1-x^2-y^2)^2(\partial^2/\partial x^2+\partial^2/\partial y^2)$ in the unit disc $1>x^2+y^2$. See [9].

§ 5. Remarks and questions.

Green functions of the operator (2.1) can be defined for an arbitrary group with finite generators and relators.

1) The algebraicity of Green functions holds for groups which are not necessarily free, for example, for finite free products of cyclic groups or finite groups. It seems interesting to characterize groups whose Green functions are algebraic or rational:

Is a group with finite generators and relators commensurable with certain free group, if the Green functions are all algebraic?

Is it a finite group if the Green functions are all rational or more generally meromorphic in $\boldsymbol{C}$?

2) The following is in some sense an inverse scattering problem: Does the Green function $G(e, e|z)$ uniquely determine the group itself except for obvious isomorphisms?

3) For free abelian groups the Green function can easily computed by means's of Fourier transforms. The resulting functions are known to be Lauricella hypergeometric functions. It seems interesting to compute exact expressions of the Green functions in case of other suitable groups, for example nilpotent groups, Braid groups, etc.

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Added in proof.

Prof. Y. Kato and more recently Dr. T. Steger, Univ. of Washington, St. Louis pointed out to the author that the author's original proof of Theorem 1 was false. Here a corrected proof is presented. Further, Dr. T. Steger says that he has proved "almost irreducibility" of the unitary representations of Γ on the eigen-spaces (4.34) of self-adjoint operators P (See his forthcoming thesis.).

Profs. W. Woess and P. Gerl have obtained the same formula as (2.3) and computed the spectral radius $\pm z_0$ of P. They use for it the methods of system analysis of Markov processes which can be found in the book "Dynamic Probabilistic System" Vol. 1, 1971, by R. A. Howard.

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