J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
31 (1984), 373-390

Propagation of singularity and existence of real analytic solutions of locally hyperbolic equations

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(Communicated by H. Komatsu)

Introduction

Let $P=P(D)$ be a differential operator in $\boldsymbol{R}^n$ with constant coefficients, $P=P(\zeta)$ for $\zeta\in\boldsymbol{C}^n$ the characteristic polynomial, Ω an open set of $\boldsymbol{R}^n$ and denote by $A(\Omega)$ the space of real analytic functions on Ω. If P_m is the principal part of P, let $V=V(P_m)=\{\zeta\in\boldsymbol{C}^n : P_m(\zeta)=0\}$ denote the asymptotic cone of the characteristic variety $\{\zeta\in\boldsymbol{C}^n : P(\zeta)=0\}$. Decompose the germ of V at ξ, $\xi\in V\cap S^{n-1}$, $(S^{n-1}=\{\xi\in\boldsymbol{R}^n : |\xi|=1\})$, into irreducible components $V^{\xi,i}$, $1\leqq i\leqq s_\xi$, and denote by $m^{\xi,i}=m(V^{\xi,i})$ the multiplicity (or degree) of each component at ξ.

MAIN THEOREM. *Suppose that any irreducible component $V^{\xi,i}$ of V is locally hyperbolic with respect to $\pm v^{\xi,i}\in S^{n-1}$ and denote by $\pm\Gamma_\xi^{i*}=\Gamma^*(V^{\xi,i},\pm v^{\xi,i})$ the corresponding pair of local propagation cones. For an open convex set $\Omega\subset\boldsymbol{R}^n$ consider the conditions*

(1) $P(D)A(\Omega)=A(\Omega)$,

(2) $\forall\xi\in V\cap S^{n-1}$, $\forall i\leqq s_\xi$, $\forall x\in\partial\Omega$, *either* $x+\Gamma_\xi^{i*}\cap\Omega=\emptyset$ *or* $x-\Gamma_\xi^{i*}\cap\Omega=\emptyset$.

Then (2)$\Rightarrow$(1); *and conversely* (1)$\Rightarrow$(2) *if we assume in addition* $m^{\xi,i}\leqq 2$ *for every* ξ *and* i.

The source of this result is the paper [8], while the main tool of the proof given at the end of Sections 2 and 3 is found in [6]. In the latter Hörmander proves that (1) is equivalent to a kind of "Phragmén-Lindelöf principle" on the asymptotic cone V; in the former we show how the principle is affected by the perturbation which transforms the germs $V^{\xi,i}$ into the tangent cones V_ξ^i.

Notice now that the theorem permits us to exhibit a class of operators P for which (1) is fulfilled by $\Omega=\boldsymbol{R}^n$ but it is never fulfilled by $\Omega\neq\boldsymbol{R}^n$. In fact (2) is clearly void for any V when $\Omega=\boldsymbol{R}^n$. On the other hand when some component $V^{\xi,i}$ of V does not have a hyperplane as tangent cone, (and so it does not have a pair of half rays as local propagation cones), then no bounded open set with C^1 boundary satisfies (2). Furthermore, when the convex hull of $\bigcup_{\xi,i} \pm \Gamma_\xi^{i*}$ is the whole $\boldsymbol{R}^n$ for any choice of the signs $\pm$, then no open set $\Omega \neq \boldsymbol{R}^n$ satisfies (2).

Finally, we shall briefly recall the literature concerning our result. In [7] Kawai gives a sufficient condition for (1) on the class of all open bounded sets Ω. It looks like (2) but requires in addition a uniformity for ξ varying in a pair of closed subsets of $V \cap S^{n-1}$. Because of this lack of sharpness one could not treat in general the extremal situation in which (2) is satisfied but $x \pm \Gamma_\xi^{i*} \cap \bar{\Omega} = \{x\}$, $x \in \partial\Omega$, is not.

In [6] Hörmander gives necessary conditions for analytic solvability of real indefinite irreducible degenerate quadratic forms. One can see that those conditions, when applied to the defining forms of the tangent cones V_ξ^i, are all contained in (2) as one would expect in view of [8]. However they do not exhaust (2); in fact the cones V_ξ^i might decompose into pairs of hyperbolic planes, (which makes (2) trivially fulfilled for $V = V_\xi^i$), whereas the germs $V^{\xi,i}$ are assumed to be irreducible (and so (2) is possibly non-void).

I wish also to remember here my father Piero Zampieri who was the strongest support and the deepest reason of my mathematical effort.

0. Preliminaries. The Phragmén-Lindelöf principle

We state here a local version of the Phragmén-Lindelöf principle already presented in [8] and deduced from that of Hörmander. Let F be an analytic function on a neighbourhood of $\xi \in \boldsymbol{R}^n$, and $V = V(F)$ be the analytic set $F(\zeta) = 0$, $\zeta \in \boldsymbol{C}^n$. We will be concerned with the class of functions $\varphi : V \to \boldsymbol{R} \cup \{-\infty\}$, to be called weakly plurisubharmonic on V, that are upper semicontinuous everywhere in V and plurisubharmonic outside the singular set of V; and also we will work with support functions H_M of compact sets $M \Subset \boldsymbol{R}^n$.

We say that *an open convex set $\Omega \subset \boldsymbol{R}^n$ admits the Phragmén-Lindelöf principle on V at ξ if for every compact set $K \Subset \Omega$, there are another compact set $K' \Subset \Omega$ and positive constants δ, r_0, and ϱ, ($\varrho < 1$), such that $\forall r \leqq r_0$ we have*

$$\varphi(\zeta) \leqq H_{K'}(\operatorname{Im} \zeta) \quad \forall \zeta \in V \cap B(\xi, \varrho r) \tag{0.1}$$

whenever φ is a weakly plurisubharmonic function on $V \cap B(\xi, r)$ which verifies

$$\varphi(\zeta) \leqq H_K(\operatorname{Im} \zeta) + \delta r \quad \forall \zeta \in V \cap B(\xi, r); \quad \varphi(\zeta) \leqq 0 \quad \forall \zeta \in V \cap B(\xi, r) \cap \boldsymbol{R}^n, \tag{0.2}$$

(where we put $B(\xi, r) = \{\zeta \in \boldsymbol{C}^n; |\zeta - \xi| < r\}$).

Take now a polynomial P with principal part P_m, take $\xi \in V \cap S^{n-1}$, $V = V(P_m)$, and decompose the germ of V at ξ into irreducible components $V^{\xi,i}$, $1 \leqq i \leqq s_\xi$. First one recognizes that for an open convex set $\Omega \subset \boldsymbol{R}^n$ the Phragmén-Lindelöf principle on each $V^{\xi,i}$ at ξ is equivalent to the principle on V at ξ (see Proposition 6.2 of [6]). Second, one sees that the latter is equivalent, for ξ varying in $V \cap S^{n-1}$, to the principle on the whole of V in the original form given in [6] (where only plurisubharmonic functions on $\boldsymbol{C}^n$ are considered). Since the last is equivalent to $P(D)A(\Omega) = A(\Omega)$, ([6]), one concludes

Ω admits the Phragmén-Lindelöf principle on each irreducible germ of $V = V(P_m)$ at any $\xi \in V \cap S^{n-1}$ if and only if $P(D)A(\Omega) = A(\Omega)$.

1. Local properties of analytic sets

Let F be analytic on a neighbourhood of $\xi \in \boldsymbol{C}^n$. Let us define the localization F_ξ of F at ξ as the first non-vanishing term of the expansion of F at ξ into a series of homogeneous polynomials. If we denote by $m = m_\xi(F)$ the multiplicity (or degree) of F at ξ, (that is, the degree of F_ξ), we can calculate F_ξ as

$$F_\xi(\zeta) = \lim_{t \to 0} t^{-m} F(\xi + t\zeta)\,, \qquad \zeta \in \boldsymbol{C}^n.$$

The cone $V_\xi = V(F_\xi)$, associated with F_ξ, represents the tangent cone to the analytic set $V(F)$ associated with F. F and V will often be identified in the sequel with their germs at ξ and all local properties at ξ will be preferably considered as properties of the germs F and V. Let $\xi = 0$, $v = (0, \cdots, 1)$, $m = m_0(F)$, assume $F_0(v) \neq 0$, and denote by (ζ', ζ_n) the variable in $\boldsymbol{C}^{n-1} \times \boldsymbol{C}$. In this situation F can be represented, apart from an analytic factor non-vanishing at 0, as a Weierstrass polynomial of degree m in ζ_n. The relationship which links the roots $\zeta_n = \mu_j(\zeta')$, $1 \leqq j \leqq m$, of this polynomial, (these are the small roots of $F(\zeta', \zeta_n) = 0$ for ζ_n), to the roots $\zeta_n = \lambda_j(\zeta')$, $1 \leqq j \leqq m$, of the localized equation $F_0(\zeta', \zeta_n) = 0$ is classical (see [1]). We recall it in

LEMMA 1.1. *$\forall \varepsilon\, \exists r = r_\varepsilon$ s.t. $\forall |\zeta'| < r$ we have, with a suitable labelling which depends on ζ', $|\mu_j(\zeta') - \lambda_j(\zeta')| < \varepsilon |\zeta'|$, $1 \leqq j \leqq m$.*

(Analogous relations hold in a neighbourhood of ∞, between the roots of a polynomial and those of its principal part (which is its localization at ∞, say).)

Let now ξ vary within a complex neighbourhood of 0 on the analytic subset V^m of V where the multiplicity of F is equal to $m = m_0(F)$. This makes F_ξ fairly close to F_0; moreover we can see that the approximation of V by means of V_ξ is stable for ξ varying in V^m. In fact, in a neighbourhood of 0, let us denote by $(\zeta', \mu_{\xi j}(\zeta'))$, $j=1, \cdots, m$, the fiber of $F(\xi+(\zeta', \zeta_n))=0$ over ζ', and by $(\zeta', \lambda_{\xi j}(\zeta'))$, $j=1, \cdots, m$, that of $F_\xi(\zeta', \zeta_n)=0$. We then deduce from a slight modification of the proof of Lemma 1.1

LEMMA 1.2. *For ξ varying in V^m and for suitable labelling we have*

$$\mu_{\xi j}(\zeta')=\lambda_{\xi j}(\zeta')+o(|\zeta'|), \qquad |\zeta'|+|\xi'|\longrightarrow 0.$$

DEFINITION 1.1. The analytic function F is said to be locally hyperbolic at 0 if it can be normalized in some direction, e.g. the ζ_n-direction, so that:

$$F(\zeta', \zeta_n)=0, \quad \zeta \textit{ small}, \quad \zeta'\in \boldsymbol{R}^{n-1} \Longrightarrow \zeta_n \in \boldsymbol{R}. \tag{1.1}$$

(Local hyperbolicity, and all other local properties of F invariant under multiplication by analytic functions non-vanishing at 0, will be also considered as properties of (the germ) V.)

We list here some properties descending from (1.1) given in [3]. First, $F_0(v)\neq 0$, $v=(0, \cdots, 1)$; in addition since $t^{-m}F(t\zeta)$ tends to $F_0(\zeta)$ for $t\to 0$, and $m=m_0(F)$, then F_0 is hyperbolic to $\pm v$. Then it is usual to denote by $\Gamma(V_0, \pm v)$ the components of the real complements of the hypersurface $V_0\cap \boldsymbol{R}^n$ which contain $\pm v$; they are convex sets by classical results. We will put $\Gamma(V, \pm v)=\Gamma(V_0, \pm v)$ and denote also by $\pm\Gamma$ for short; regarding them, F is locally hyperbolic to any $w\in\pm\Gamma$ (i.e. (1.1) holds if, by change of coordinates, the ζ_n-direction coincides with the w-direction). An important fact concerning the real singularities of V comes from the simple remark that a Puiseux series which is real for real argument is a power series. If one then fixes small ξ', η' in $\boldsymbol{R}^{n-1}$, one recognizes that the small roots of $F(\xi'+s\eta', \zeta_n)=0$, $(s\in \boldsymbol{C},\ |s|<1)$, for ζ_n are analytic functions of s. Therefore any 2-dimensional plane through $\pm v$ cuts the surface in a curve with simple branches (possibly with multiple crossings).

We will discuss now, in a way different from [3], what happens for $\operatorname{Im}\zeta_n$ when $F(\zeta', \zeta_n)=0$ and $\zeta'\notin \boldsymbol{R}^{n-1}$. First observe that if $F_0(v)\neq 0$ then, by the argument preceding Lemma 1.1, there is a polycylinder $U=\{|\zeta'|<r\}\times\{|\zeta_n|<r_1\}$ such that

$$\Sigma=V\cap U \xrightarrow{\ \pi\ } S_r=\{|\zeta'|<r\}$$

is a ramified covering of the disc S_r with finite fibers; besides there is $k>0$ such that $|\zeta_n|<k|\zeta'|$ if $(\zeta', \zeta_n)\in\Sigma$.

Consider then the function $\varphi(\zeta')=\sup_{\zeta\in\pi^{-1}(\zeta')}|\operatorname{Im}\zeta_n|$, $\zeta'\in S_r$. It is plurisubharmonic since it is upper semicontinuous everywhere in S_r and plurisubharmonic outside the ramification set of $\Sigma \xrightarrow{\pi} S_r$; the conclusion is then given by the Riemann extension theorem for plurisubharmonic functions (see [6]).

If then (1.1) is fulfilled we have the estimate $\varphi(\zeta') \leqq 0$ if $\zeta' \in S_r \cap \boldsymbol{R}^{n-1}$; in addition to $\varphi(\zeta') \leqq k|\zeta'|$ if $\zeta' \in S_r$. So by the classical Phragmén-Lindelöf principle on $\boldsymbol{C}^{n-1}$ (see subsequent Theorem 2.1) it follows $\varphi(\zeta') \leqq k'|\mathrm{Im}\,\zeta'|$ if $\zeta' \in S_{r'}$ for some $k' > k$ and $r' < r$. We have therefore established the following

LEMMA 1.3. *If F is locally hyperbolic to the ζ_n-direction, then the small roots $\zeta_n = \mu_j(\zeta')$ of $F(\zeta', \zeta_n) = 0$ verify $|\mathrm{Im}\,\mu_j(\zeta')| = 0(|\mathrm{Im}\,\zeta'|)$, $|\zeta'| \to 0$.*

As a result of Lemma 1.1 we can characterize the tangent cone V_0 as the set of the points proportional to limits of sequences $\dfrac{\zeta^\nu}{|\zeta^\nu|}$ with $\zeta^\nu \to 0$, $\zeta^\nu \in V$. With analogous terminology the set of all points proportional to limits of sequences $\dfrac{\zeta^\nu}{|\zeta^\nu|}$ with $\zeta^\nu \to 0$, $\zeta^\nu \in V_{\boldsymbol{R}} = V \cap \boldsymbol{R}^n$ will be called the tangent cone to the real part of V. Obviously the latter is generally a proper subset of $(V_0)_{\boldsymbol{R}} = V_0 \cap \boldsymbol{R}$. However, if (1.1) holds, one immediately deduces from Lemma 1.1

LEMMA 1.4. *If V is locally hyperbolic then $(V_0)_R$ is the tangent cone to V_R.*

Assume that F is irreducible and satisfies (1.1); recall $m = m_0(F)$. Let $D(\zeta')$ be the discriminant of $F(\zeta', \zeta_n)$ thought as a polynomial in ζ_n with parameter ζ'. Then $D(\zeta') = \prod_{i<j} (\mu_i(\zeta') - \mu_j(\zeta'))^2$ and so $\deg D = m(m-1)$. The real points ζ' over which D vanishes are the points where the equation $F(\zeta', \zeta_n) = 0$ for ζ_n has a multiple real root; that is they are the projections on the ζ'-plane of the real common zeros of F and $\dfrac{\partial F}{\partial \zeta_n}$. We want to prove that in this situation all other derivatives of order 1, $\dfrac{\partial F}{\partial \zeta_i}$ for $i \neq n$, consequently vanish. In fact denoting by $S = S(F)$ the (germ of) analytic set of all singular points of $V = V(F)$ and by W (the germ of) $\{D(\zeta') = 0,\ \zeta' \in \boldsymbol{C}^{n-1}\}$, we have

LEMMA 1.5. *Let V $(= V(F))$ be as above. Then W_R coincides with the projection of S_R on the ζ'-plane.*

PROOF. Let $D(\xi') = 0$, $\xi' \in \boldsymbol{R}^{n-1}$; there are then two coincident real zeros $\xi_n = \mu_1(\xi') = \mu_2(\xi')$ of $F(\xi', \zeta_n) = 0$. For fixed i $(1 \leqq i < n)$, if we let the variable ζ_i vary within a neighbourhood of ξ_i, keeping $\zeta_k = \xi_k$ for $k \neq i$,

$k<n$, then the zeros of $F(\zeta',\zeta_n)=0$ can be labelled as analytic functions $\zeta_n=\mu_j(\zeta')$ of ζ_i, $j=1,\cdots,m$. And hence in this situation we can factorize F into the product of the analytic functions $\zeta_n-\mu_j(\zeta')$. Thus we have, for $\zeta_k=\xi_k$, $\forall k\neq i$,

$$\left[\frac{\partial}{\partial\zeta_i}\prod_{j=1}^{2}(\zeta_n-\mu_j(\zeta'))\right]_{\zeta_i=\xi_i}=-(\xi_n-\mu_2(\xi'))\frac{\partial\mu_1}{\partial\zeta_i}(\xi')-$$

$$-(\xi_n-\mu_1(\xi'))\frac{\partial\mu_2}{\partial\zeta_i}(\xi')=0\,,$$

and finally

$$\left[\frac{\partial}{\partial\zeta_i}\prod_{j=1}^{m}(\zeta_n-\mu_j(\zeta'))\right]_{\zeta=\xi}=0\,.$$

In view of the previous lemma the real codimension of W_R in $\boldsymbol{R}^{n-1}$ is equal to the codimension of S_R in V_R. Concerning the former we give now an estimate which is related to a more general result of [2].

LEMMA 1.6. *Let V be a locally hyperbolic irreducible germ. Then the real codimension of W_R in $\boldsymbol{R}^{n-1}$ is at least* 2.

PROOF. The general proof is a consequence of Lemma 1.5 above and of Proposition 27 of [2]. The particular case in which $m(V)\leqq 2$ goes as follows. Suppose V $(=V(F))$, locally hyperbolic to the ζ_n-direction. If $\mu_j(\zeta')$, $j=1,2$, are the small roots of $F(\zeta',\zeta_n)=0$, then by definition $D(\zeta')=(\mu_1(\zeta')-\mu_2(\zeta'))^2$. From the local hyperbolicity of F we infer that D is real positive semidefinite. If the germ W_R, $(=\{D(\zeta')=0,\ \zeta'\in\boldsymbol{R}^{n-1}\})$, had real codimension 1 then D should be either irreducible indefinite or the product of two locally hyperbolic germs with multiplicity 1. The first is impossible, while the second can hold only if D is the square of an analytic germ. So $\mu_1-\mu_2$ is analytic and hence both μ_j, $j=1,2$, are such. Therefore $F(\zeta)$ decomposes into the product of the analytic germs $\zeta_n-\mu_j(\zeta')$, $j=1,2$, which violates the assumption of irreducibility.

2. Existence theorems

Let $V=V(F)$ be (a germ of) an analytic set at 0. Suppose V locally hyperbolic to the (co)vectors $\pm v$ and put $\pm\Gamma=\Gamma(V,\pm v)$; finally denote by $\pm\Gamma^*$ the dual cones defined as $\{x\in\boldsymbol{R}^n : \langle x,\xi\rangle\geqq 0,\ \forall\xi\in\pm\Gamma\}$. We then have

LEMMA 2.1. *Every open half space Ω (with homogeneous boundary) whose closure does not intersect either of the cones $\pm\Gamma^*$, except at* 0, *admits the Phragmén-Lindelöf principle on V.*

PROOF. If w is the normal to Ω, then $\langle x, w\rangle > 0$ $\forall x \in +\Gamma^*$, (or $-\Gamma^*$), by which $w \in +\Gamma$, (or $-\Gamma$); thus V is forced to be locally hyperbolic to $\pm w$ because of the already quoted result of [3]. Assume, by change of the coordinates, $w=(0, \cdots, 1)$ and put $m_0(V)=m_0(F)$ if F is the defining function of V. Suppose that for small r there are exactly $m=m_0(V)$ points of V over any ζ' with $|\zeta'|<r$, and denote by $\pi: V \to \boldsymbol{C}^{n-1}$ the projection along the ζ_n-direction.

Take a polycylinder K in the form $\{x' \in \boldsymbol{R}^{n-1}, |x'| \leqq R_1\} \times \{x_n \in \boldsymbol{R}, |x_n| \leqq R_2\}$; let δ be an arbitrary positive number. Then consider a weakly plurisubharmonic function on V verifying the estimations

$$\text{(2.1)} \qquad \begin{aligned} &\varphi(\zeta) \leqq H_K(\operatorname{Im} \zeta)+\delta r, && \zeta \in V,\ |\zeta'|<r,\ (r \text{ small}); \\ &\varphi(\zeta) \leqq 0, && \zeta \in V \cap \boldsymbol{R}^n,\ |\zeta'|<r. \end{aligned}$$

Let us define now a function $\psi(\zeta')=\sup_{\zeta \in \pi^{-1}(\zeta')} \varphi(\zeta', \zeta_n)$, $|\zeta'|<r$, which is plurisubharmonic because of the same argument as in Lemma 1.3. It verifies

$$\psi(\zeta') \leqq R_1|\operatorname{Im} \zeta'|+\sup_{\zeta \in \pi^{-1}(\zeta')} R_2|\operatorname{Im} \zeta_n|+\delta r \leqq (R_1+R_2 k)|\operatorname{Im} \zeta'|+\delta r \quad \forall |\zeta'|<r,$$

if r and k are so chosen that $|\operatorname{Im} \zeta_n|<k|\operatorname{Im} \zeta'|$ whenever $\zeta \in V$, $|\zeta'|<r$ (see Lemma 1.3). It also verifies, as consequence of the local hyperbolicity of V, $\psi(\zeta') \leqq 0$, $\zeta' \in \boldsymbol{R}^{n-1}$, $|\zeta'|<r$.

Thus by the classical Phragmén-Lindelöf principle on $\boldsymbol{C}^{n-1}$ (see subsequent Theorem 2.1), $\psi(\zeta') \leqq C_{K,\delta,k,n}|\operatorname{Im} \zeta'|$, $\forall |\zeta'|<r'$ (where $\frac{r}{r'}=2(n-1)^{1/2}$). We then conclude

$$\text{(2.2)} \qquad \varphi(\zeta') \leqq H_{K'}(\operatorname{Im} \zeta), \quad \zeta \in V,\ |\zeta'|<r',$$

where K' is the ball of radius $C_{K,\delta,k,n}$ lying in the homogeneous hyperplane orthogonal to w. Observe now that a solidal translation of the sets K, K' does not affect the Phragmén-Lindelöf principle. To achieve the theorem it is then enough to notice that every compact convex subset of Ω is contained in some polycylinder with center at a point of Ω.

REMARK. The proof of the lemma shows that if $w \in \Gamma$, $(w=(0, \cdots, 1))$, then $\forall K \Subset \boldsymbol{R}^n$ and $\forall \delta>0$ we can find a compact set K' orthogonal to w so that the implication (2.1)$\Rightarrow$(2.2) is fulfilled for suitably small r. On the other hand, when $w \in \bar{\Gamma}$, $w \notin \Gamma$, (and so $w \in V_0$), then we know from [8] that if (2.1)$\Rightarrow$(2.2) is fulfilled, then the range of the projection $K' \ni x \mapsto x_n$ cannot generally shrink into a point and must even contain the range of $K \ni x \to x_n$. In such case we will show that given $K=\{|x'| \leqq R_1\} \times \{|x_n| \leqq R_2\}$ and letting $R_1' \to \infty$, $\delta \to 0$, $r \to 0$, $r'/r \to 0$, we can find $R_2' \to R_2$ so that (2.1)$\Rightarrow$

(2.2) is fulfilled for $K'=\{|x'|\leqq R_1'\}\times\{|x_n|\leqq R_2'\}$.

THEOREM 2.1. *Let V be a locally hyperbolic germ and Ω be an open half space (with a homogeneous boundary) which does not intersect either of the cones $\pm\Gamma^*$; then Ω admits the Phragmén-Lindelöf principle on V.*

PROOF. If w is the normal to Ω then $\langle w, x\rangle\geqq 0$ $\forall x\in\pm\Gamma^*$ and so $w\in\pm\bar{\Gamma}$. Let us approximate w by means of a sequence $v\to w$, belonging to $\pm\Gamma$ and running along a circumference arc which meets $\pm\partial\Gamma$ only at the extremal point w; this is possible in view of the convexity of the cones $\pm\Gamma$. As a result of the already quoted theorem in [3], V is locally hyperbolic with respect to v. Let ζ_{n-1}, ζ_n be the variables in the 2-dimensional plane of w and v, and choose the coordinate system in such plane so that v is parallel to the ζ_n axis. Denote by ζ'' the variables in the orthogonal complement. We will show that for every ε and for every sufficiently small (positive) angle ϕ between v and w, there are positive constants r_ϕ and C_ϕ such that the following properties hold for the defining function F of V

(i) $F(\zeta'',\zeta_{n-1},\zeta_n)=0$, $(|\zeta''|+|\zeta_{n-1}|<r_\phi)$, has exactly $m=m_0(F)$ roots ζ_n with $|\zeta_n|<((1+\varepsilon)\operatorname{cotg}\phi+C_\phi)r_\phi$; moreover for such roots we have the stronger estimate $|\zeta_n|<(1+\varepsilon)\operatorname{cotg}\phi|\zeta_{n-1}|+C_\phi|\zeta''|$.

(ii) $F(\zeta'',\zeta_{n-1},\zeta_n)=0$, $|\zeta''|+|\zeta_{n-1}|<r_\phi$, $(\zeta'',\zeta_{n-1})\in\boldsymbol{R}^{n-1}$, $|\zeta_n|<((1+\varepsilon)\operatorname{cotg}\phi+C_\phi)r_\phi$ implies $\zeta_n\in\boldsymbol{R}$.

(iii) $F(\zeta'',\zeta_{n-1},\zeta_n)=0$, $|\zeta''|+|\zeta_{n-1}|<r_\phi$, $|\zeta_n|<((1+\varepsilon)\operatorname{cotg}\phi+C_\phi)r_\phi$ implies $|\operatorname{Im}\zeta_n|<(1+\varepsilon)\operatorname{cotg}\phi|\operatorname{Im}\zeta_{n-1}|+C_\phi|\operatorname{Im}\zeta''|$.

First we prove the result globally on V_0 and then we transfer it to V on a neighbourhood of 0 by the aid of Lemma 1.1. Let s be the multiplicity of w as a zero of F_0 that is the multiplicity of the ray through w in the pencil defined by the equation $F_0(0,\zeta_{n-1},\zeta_n)=0$. There is then a constant k which does not depend on ϕ such that $m-s$ zeros $\{\zeta_n=\lambda_h(0,\zeta_{n-1})\}_h$ of $F_0(0,\zeta_{n-1},\zeta_n)=0$ verify $|\lambda_h(0,\zeta_{n-1})|<k|\zeta_{n-1}|$ while the s remaining $\{\zeta_n=\lambda_j(0,\zeta_{n-1})\}_j$ are equal to $-\operatorname{cotg}\phi\zeta_{n-1}$.

By observing that $F_0(\zeta'',\zeta_{n-1},\zeta_n)$, thought of as a polynomial in the only variables (ζ_{n-1},ζ_n) with ζ'' as parameters, has principal part $F_0(0,\zeta_{n-1},\zeta_n)$, we infer from Lemma 1.1, which still holds for localizations at ∞ of polynomials, that there is a constant c_ϕ such that for $\left|\frac{\zeta''}{\zeta_{n-1}}\right|<c_\phi$, $m-s$ zeros $\{\zeta_n=\lambda_h(\zeta'',\zeta_{n-1})\}_h$ of $F_0(\zeta'',\zeta_{n-1},\zeta_n)=0$ are forced to verify $|\zeta_n|<k|\zeta_{n-1}|$ and the other s zeros $\{\zeta_n=\lambda_j(\zeta'',\zeta_{n-1})\}_j$ to verify $|\zeta_n|<(\operatorname{cotg}\phi+k)|\zeta_{n-1}|$. Naturally, to separate the former sets, ϕ has been chosen small enough. On the other hand when $\left|\frac{\zeta''}{\zeta_{n-1}}\right| \geqq c_\phi$ there is C_ϕ such that for every zero we have $|\lambda(\zeta'', \zeta_{n-1})| < C_\phi |\zeta''|$ just because in our coordinate frame the ζ_n-axis is not characteristic for F_0. Thus for generic (ζ'', ζ_{n-1}) at least $m-s$ zeros $\{\lambda_h\}_h$ verify $|\lambda_h(\zeta'', \zeta_{n-1})| < k|\zeta_{n-1}| + C_\phi|\zeta''|$ and if a zero λ_j does not verify it, then $\left|\frac{\zeta''}{\zeta_{n-1}}\right| < c_\phi$ and λ_j is forced to be close to some $\lambda_j(0, \zeta_{n-1}) = -\operatorname{cotg} \phi \zeta_{n-1}$ and therefore to verify $|\lambda_j(\zeta'', \zeta_{n-1})| < (\operatorname{cotg} \phi + k)|\zeta_{n-1}|$. Thus putting $\operatorname{cotg} \phi + k = (1+\varepsilon)\operatorname{cotg} \phi$, where obviously $\varepsilon \to 0$ for $\phi \to 0$, we conclude that all zeros λ_i, $i=1, \cdots, m$, verify at least the estimate

$$|\lambda_i(\zeta'', \zeta_{n-1})| < (1+\varepsilon)\operatorname{cotg} \phi |\zeta_{n-1}| + C_\phi |\zeta''| . \tag{2.3}$$

By the aid of Lemma 1.1 we immediately obtain (i) from (2.3). From (i) and from Theorem 15.10 of [3] we also deduce (ii). In order to prove (iii), consider now the function $\varphi(\zeta') = \sup_{i=1,\cdots,m} |\operatorname{Im} \lambda_i(\zeta')|$, $\zeta' \in \boldsymbol{C}^{n-1}$, which is clearly plurisubharmonic on the whole of $\boldsymbol{C}^{n-1}$. Since $|\operatorname{Im} \lambda_i(\zeta'', \zeta_{n-1})| < \operatorname{cotg} \phi |\operatorname{Im} \zeta_{n-1}| + k|\zeta_{n-1}|$ when $\left|\frac{\zeta''}{\zeta_{n-1}}\right| < c_\phi$, then $\varphi(\zeta') < \operatorname{cotg} \phi |\operatorname{Im} \zeta_{n-1}| + k|\zeta_{n-1}| + C_\phi |\zeta''|$ for generic $\frac{\zeta''}{\zeta_{n-1}}$. Besides we have $\varphi(\zeta') \leqq 0$, $\zeta' \in \boldsymbol{R}^{n-1}$, due to the hyperbolicity of F_0. Fix now $\zeta'' \in \boldsymbol{R}^{n-2}$ and define on the half plane $\operatorname{Im} \zeta_{n-1} > 0$ the function $\psi(\zeta_{n-1}) = \varphi(\zeta'', \zeta_{n-1}) - \operatorname{cotg} \phi \operatorname{Im} \zeta_{n-1}$, which is subharmonic there. For every r it verifies $\psi(\zeta_{n-1}) \leqq kr + C_\phi |\zeta''|$ if $|\zeta_{n-1}| < r$; $\psi(\zeta_{n-1}) \leqq 0$ if $|\zeta_{n-1}| < r$, $\zeta_{n-1} \in \boldsymbol{R}$. Thus for every ε we have near the boundary of the semidisc $D^+ = \{|\zeta_{n-1}| \leqq r, \operatorname{Im} \zeta_{n-1} \geqq 0\} : \psi(\zeta_{n-1}) \leqq \frac{2}{\pi}(kr + C_\phi |\zeta''|) \arg \frac{r-\zeta_{n-1}}{r+\zeta_{n-1}} + \varepsilon$; so from the maximum principle for subharmonic functions and letting $\varepsilon \to 0$ we have in the interior of D^+ : $\psi(\zeta_{n-1}) \leqq \frac{2}{\pi}(kr + C_\phi |\zeta''|) \operatorname{arctg} \frac{2\frac{\operatorname{Im} \zeta_{n-1}}{r}}{1-\frac{|\zeta_{n-1}|^2}{r^2}}$. By developing the previous argument also in the half disc $D^- = \{|\zeta_{n-1}| \leqq r, \operatorname{Im} \zeta_{n-1} \leqq 0\}$, and by letting $r \to \infty$ we conclude: $\psi(\zeta_{n-1}) \leqq \frac{4}{\pi} k |\operatorname{Im} \zeta_{n-1}|$, $\zeta_{n-1} \in \boldsymbol{C}$ from which: $\varphi(\zeta') \leqq \left(\operatorname{cotg} \phi + \frac{4}{\pi} k\right) |\operatorname{Im} \zeta_{n-1}|$ if $\zeta'' \in \boldsymbol{R}^{n-2}$, $\zeta_{n-1} \in \boldsymbol{C}$. Fix now $(\zeta_2, \cdots, \zeta_{n-2}) \in \boldsymbol{R}^{n-3}$, $\zeta_{n-1} \in \boldsymbol{C}$, and consider the function in the only variable ζ_1 : $\psi(\zeta_1) = \varphi(\zeta_1, \cdots, \zeta_{n-1}) - \left(\operatorname{cotg} \phi + \frac{4}{\pi} k\right) |\operatorname{Im} \zeta_{n-1}|$; it verifies the estimates $\psi(\zeta_1) \leqq C_\phi |\zeta_1| + C_\phi |(\zeta_2, \cdots, \zeta_{n-2})| + k|\zeta_{n-1}|$; $\psi(\zeta_1) \leqq 0$ if

$\zeta_1 \in \boldsymbol{R}$. Thus as already seen, we obtain $\varphi(\zeta') \leqq \left(\operatorname{cotg}\phi + \frac{4}{\pi}k\right)|\operatorname{Im}\zeta_{n-1}| + \frac{4}{\pi}C_\phi|\operatorname{Im}\zeta_1|$ if $(\zeta_1, \zeta_{n-1}) \in \boldsymbol{C}^2$, $(\zeta_2, \cdots, \zeta_{n-2}) \in \boldsymbol{R}^{n-3}$. Reiterating this argument we conclude:

$$\text{(2.4)} \qquad \sup_i |\operatorname{Im}\lambda_i(\zeta')| \leqq \left(\operatorname{cotg}\phi + \frac{4}{\pi}k\right)|\operatorname{Im}\zeta_{n-1}| + \frac{4}{\pi}C_\phi(n-2)^{1/3}|\operatorname{Im}\zeta''|$$
$$= (1+\varepsilon)\operatorname{cotg}\phi|\operatorname{Im}\zeta_{n-1}| + C'_\phi|\operatorname{Im}\zeta''|, \qquad \zeta' \in \boldsymbol{C}^{n-1}.$$

To obtain (iii) we only need now to replace the roots $\zeta_n = \lambda_i(\zeta')$ of $F_0(\zeta', \zeta_n) = 0$ by the roots $\zeta_n = \mu_i(\zeta')$ of $F(\zeta', \zeta_n) = 0$. Regarding it we know from Lemma 1.1 that for some r_ϕ and with a suitable labelling we have

$$\text{(2.5)} \qquad |\mu_i(\zeta') - \lambda_i(\zeta')| \leqq k(|\zeta''| + |\zeta_{n-1}|), \qquad |\zeta''| + |\zeta_{n-1}| < r_\phi .$$

Combining (2.4) and (2.5) we then obtain

$$\sup_i |\operatorname{Im}\mu_i(\zeta')| \leqq (1+\varepsilon)\operatorname{cotg}\phi|\operatorname{Im}\zeta_{n-1}| + C'_\phi|\operatorname{Im}\zeta''| + k(|\zeta''| + |\zeta_{n-1}|),$$
$$|\zeta''| + |\zeta_{n-1}| < r_\phi ;$$
$$\sup_i |\operatorname{Im}\mu_i(\zeta')| \leqq 0, \ |\zeta''| + |\zeta_{n-1}| < r_\phi, \ \zeta' \in \boldsymbol{R}^{n-1}.$$

Reasoning on the function $\sup_i |\operatorname{Im}\mu_i(\zeta')|$, $\zeta' \in \boldsymbol{C}^{n-1}$ as in the proof of (2.4), we finally obtain (iii).

Let us consider now the initial coordinate system in which the normal w to Ω is parallel to the ζ_n axis, and let φ be weakly plurisubharmonic on $V = V(F)$ verifying:

$$\varphi(\zeta) \leqq H_K(\operatorname{Im}\zeta) + \delta r \quad \text{if } \zeta \in V \cap S_r ; \ \varphi(\zeta) \leqq 0 \quad \text{if } \zeta \in V_R \cap S_r, \ (S_r = \{|\zeta| < r\}),$$

where K, translation of a generic compact convex set of Ω, can be taken as a polycylinder in the form $\{|x''| \leqq R_1\} \times \{|x_{n-1}| \leqq R_2\} \times \{|x_n| \leqq R_3\}$ and δ is taken to be a positive number smaller than the distance of K from the boundary of Ω, whose precise determination depending on K is given in the following. After rotation which superposes the ζ_n-axis to v approaching w, set

$$\psi(\zeta') = \sup\{\varphi(\eta) : \ \eta \in V, \ \eta' = \zeta', \ |\eta_n| < ((1+\varepsilon)\operatorname{cotg}\phi + C_\phi)r'\},$$
$$|\zeta''| + |\zeta_{n-1}| < r', \quad \left(r' = \frac{r}{(1 + ((1+\varepsilon)\operatorname{cotg}\phi + C_\phi)^2)^{1/2}} \leqq r_\phi\right).$$

Because of (i), (ii), (iii) we have:

$$\phi(\zeta')\leqq(R_1+(R_2\sin\phi+R_3)C_\phi)|\mathrm{Im}\,\zeta''| \tag{2.6}$$
$$+(R_2+R_3\sin\phi+(R_2\sin\phi+R_3)(1+\varepsilon)\mathrm{cotg}\,\phi)|\mathrm{Im}\,\zeta_{n-1}|$$
$$+\delta(1+(1+\varepsilon)\mathrm{cotg}\,\phi+C_\phi)r',\quad |\zeta''|+|\zeta_{n-1}|<r'$$

in addition to $\phi(\zeta')\leqq 0$ if $|\zeta''|+|\zeta_{n-1}|<r'$, $\zeta'\in\boldsymbol{R}^{n-1}$. The argument used in proving (2.4) allows us to claim that (2.6) also holds with the third addendum replaced by $\delta(1+(1+\varepsilon)\mathrm{cotg}\,\phi+C_\phi)\frac{8}{\pi}((n-2)^{1/2}+1)(n-2)^{1/2}(|\mathrm{Im}\,\zeta''|+|\mathrm{Im}\,\zeta_{n-1}|)$ and with the first multiplied by the factor $(n-2)^{1/2}$ provided that $|\zeta''|+|\zeta_{n-1}|<\frac{r'}{2((n-2)^{1/2}+1)}$. Set $r''=\frac{r'}{2^{3/2}((n-2)^{1/2}+1)}$, $\delta''=\delta\frac{8}{\pi}((n-2)^{1/2}+1)\times(n-2)^{1/2}$, we then have in the initial coordinate system:

$$\varphi(\zeta)\leqq(R_1(n-2)^{1/2}+(R_2\sin\phi+R_3)C_\phi(n-2)^{1/2}+\delta''(1+(1+\varepsilon)\mathrm{cotg}\,\phi+C_\phi))|\mathrm{Im}\,\zeta''|$$
$$+(R_2+R_3\sin\phi+(R_2\sin\phi+R_3)(1+\varepsilon)\mathrm{cotg}\,\phi+\delta''(1+(1+\varepsilon)\mathrm{cotg}\,\phi+C_\phi))$$
$$\times|\mathrm{Im}\,\zeta_{n-1}|$$
$$+\sin\phi(R_2+R_3\sin\phi+(R_2\sin\phi+R_3)(1+\varepsilon)\mathrm{cotg}\,\phi+\delta''(1+(1+\varepsilon)\mathrm{cotg}\,\phi+C_\phi))$$
$$\times|\mathrm{Im}\,\zeta_n|$$
$$=A|\mathrm{Im}\,\zeta''|+B|\mathrm{Im}\,\zeta_{n-1}|+\sin\phi B|\mathrm{Im}\,\zeta_n|=H_{K'}(\mathrm{Im}\,\zeta)\qquad\forall\zeta\in V\cap S_{r''},$$

where K' is the polycylinder $\{|x''|\leqq A\}\times\{|x_{n-1}|\leqq B\}\times\{|x_n|\leqq\sin\phi B\}$. Observe then that $\sin\phi B$ can be made as close as we like to R_3 (whereas A and B diverge), provided that we correspondingly shrink ϕ and δ; in consequence, supposing x so chosen that $x+K\Subset\Omega$, $x+K'$ becomes a compact (convex) subset of Ω.

Now we want to free ourselves from the assumption that Ω is a half space by giving a lemma which translates analytic solvability of differential equations on open convex sets into solvability on tangent half spaces (and on complements of them).

Let Ω be an open convex set and P a differential operator with constant coefficients. In [2] it has been proven that $P(D)A(\Omega)=A(\Omega)$ implies $P(D)A(\Sigma)=A(\Sigma)$ for every half space Σ tangent to Ω, whereas the opposite implication has been proven to hold when Ω is bounded. We use now a device to omit the hypothesis of Ω to be bounded.

LEMMA 2.2. *Assume that for every tangent half space Σ to Ω both the following conditions are fulfilled $P(D)A(\Sigma)=A(\Sigma)$, $P(D)A(\boldsymbol{R}^n\setminus\bar{\Sigma})=A(\boldsymbol{R}^n\setminus\bar{\Sigma})$. Then $P(D)A(\Omega)=A(\Omega)$.*

PROOF. Let W be the real linear space engendered by all normals to Ω and H be the orthogonal complement. Thus Ω is the product of its projection on W times H. Therefore for every $K \Subset \Omega$ there are normals w_j and numbers n_j, $j=1, \cdots, d$ ($d=\dim W$), (or equivalently tangent half spaces Σ_j and points y_j), so that the following properties hold

(i) $K \Subset \bigcap_j \{\langle x, w_j \rangle > n_j\} = \bigcap_j (y_j + (\boldsymbol{R}^n \setminus \bar{\Sigma}_j))$,

(ii) the projection of $\bigcap_j \{\langle x, w_j \rangle > n_j\} \cap \Omega$ on W is a bounded set. Without loss of generality we can suppose $0 \in K$ and $K \Subset (1-\varepsilon) \bigcap_j ((y_j + (\boldsymbol{R}^n \setminus \bar{\Sigma}_j)) \cap \Omega)$. There are then finitely many tangent half spaces Σ_i, $i=1, \cdots, d'$, such that the polyhedron $\Pi = \bigcap_j (y_j + (\boldsymbol{R}^n - \bar{\Sigma}_j)) \cap \bigcap_i \Sigma_i$ is contained in $(1+\varepsilon)\Omega$. Notice that Π is the intersection of open sets on which P is analytically solvable. Since every analytic function on a finite intersection of open sets can be split into the sum of analytic functions on the single sets, we infer $P(D)A(\Pi) = A(\Pi)$ and so $P(D)A((1-\varepsilon)\Pi) = A((1-\varepsilon)\Pi)$. Observing that $K \Subset (1-\varepsilon)\Pi$ we deduce that there are $K' \Subset (1-\varepsilon)\Pi$, δ and r such that the implication (0.2)$\Rightarrow$(0.1) is fulfilled for $V = V(P_m)$, $\xi \in V \cap S^{n-1}$. Observing that $K' \Subset (1-\varepsilon)\Pi \subset \Omega$, we conclude that Ω admits the Phragmén-Lindelöf principle on V at any $\xi \in V \cap S^{n-1}$ and so $P(D)A(\Omega) = A(\Omega)$.

PROOF OF IMPLICATION (2)$\Rightarrow$(1) OF MAIN THEOREM. If (2) holds for Ω then it holds for all tangent half spaces Σ and also for all $\boldsymbol{R}^n \setminus \bar{\Sigma}$. Thus all Σ and $\boldsymbol{R}^n \setminus \bar{\Sigma}$ admit the Phragmén-Lindelöf principle on each $V^{\xi, i}$, $\xi \in V \cap S^{n-1}$, $1 \leqq i \leqq s_\xi$, due to Theorem 2.1. By Preliminaries one concludes $P(D)A(\Sigma) = A(\Sigma)$ and $P(D)A(\boldsymbol{R}^n \setminus \bar{\Sigma}) = A(\boldsymbol{R}^n \setminus \bar{\Sigma})$ for any Σ, and finally by Lemma 2.2 one obtains (1).

3. Geometric consequences of existence of real analytic solutions

Let $V = V(F)$ be an irreducible germ at 0 with multiplicity $\leqq 2$. Suppose $V(F)$ locally hyperbolic to $\pm v = (0, \cdots, \pm 1)$; then F_0 is a hyperbolic quadratic form that will be supposed, for the moment, in the canonical form

$$F_0(\zeta) = \zeta_n^2 - \sum_{i=k}^{n-1} \zeta_i^2, \qquad \zeta \in \boldsymbol{C}^n, \quad 1 \leqq k \leqq n. \tag{3.1}$$

(It is clear that by real linear change of coordinates and by suitable choice of $\pm v \in \pm \Gamma$, F_0 can be arranged in form (3.1) and $\pm v$ can be $(0, \cdots, \pm 1)$.) The cone $\Gamma = \Gamma(V_0, v)$, defined as the component of v in $\boldsymbol{R}^n \setminus (V_0)_{\boldsymbol{R}}$, is a convex wedge whose edge is the space $\boldsymbol{R}^{k-1}$ of the first $k-1$ variables and consequently its dual cone $\Gamma^* = \Gamma^*(V_0, v)$ is a subset of the space $\boldsymbol{R}^{n-k+1}$ of the last $n-k+1$. Denote by $\mathring{\tilde{\Gamma}}^*$ the interior of Γ^* as subset of $\boldsymbol{R}^{n-k+1}$. Take $w\notin\bar{\Gamma}\cup-\bar{\Gamma}$, $|w|=1$, denote by $\tilde{w}$ the projection on $\boldsymbol{R}^{n-k+1}$ and set $c=c(w)=|\tilde{w}|\left(1-2\left\langle v,\frac{\tilde{w}}{|\tilde{w}|}\right\rangle^2\right)$. For w we have the following properties.

(a) $c(w)>0$ due to the fact that $(\bar{\Gamma}\cup-\bar{\Gamma})\setminus\{0\}$ coincides with

$$\left\{x\in\dot{\boldsymbol{R}}^n\,;\ c\left(\frac{x}{|x|}\right)\leqq 0\right\}.$$

(b) The hyperplane $\langle x,w\rangle=0$ has a non-empty intersection with $\mathring{\tilde{\Gamma}}^*$; then take s in $\mathring{\tilde{\Gamma}}^*\cap\{\langle x,w\rangle=0\}$, $|s|=1$, which lies on the 2-dimensional plane spanned by v, $\tilde{w}$.

LEMMA 3.1. *Let $K=\{ts:t\in\boldsymbol{R},\,|t|\leqq 1\}$, and suppose that for some K', δ, ϱ, r_0 and $\forall r\leqq r_0$, the implication* (0.2)⇒(0.1) *is fulfilled on the class of all weakly plurisubharmonic functions on V. Then it follows*

$$c(w)|t|\leqq H_{K'}(tw)\,,\qquad t\in\boldsymbol{R}\,. \tag{3.2}$$

PROOF. We split up the proof in two steps.

(1) First assume $s=v$ and $w=\tilde{w}$; in such case we will prove (3.2) with $c(w)=1$. First note that we can assume $k\leqq n-1$ in (3.1) since otherwise $\bar{\Gamma}\cup-\bar{\Gamma}=\boldsymbol{R}^n$ and so the hypotheses of the lemma are void. We distinguish now the case $k<n-1$ from the case $k=n-1$; F_0 will be irreducible or not according to the listed cases.

(1.a) $k<n-1$. We can assume that F is a Weierstrass polynomial, of degree 2, with respect to ζ_n; we denote by $\mu_j(\zeta')$, $j=1,2$, the roots of this polynomial and by $\lambda_j(\zeta')$, $j=1,2$, those of $F_0(\zeta',\zeta_n)$ for ζ_n. The proof consists of the study of the function $\varphi(\zeta)=\left|\mathrm{Im}\left(\sum_{i=k}^{n-1}\zeta_i^2\right)^{1/2}\right|$, $\zeta\in\boldsymbol{C}^n$, which is clearly plurisubharmonic. Because of (3.1) we have $\varphi(\zeta)=|\mathrm{Im}\,\lambda_j(\zeta')|$, $j=1,2$, and, because of Lemma 1.1, $|\mathrm{Im}\,\mu_j(\zeta')|=|\mathrm{Im}\,\lambda_j(\zeta')|+o(|\zeta'|)$, $|\zeta'|\to 0$, $j=1,2$; it then follows $\varphi(\zeta)\leqq|\mathrm{Im}\,\zeta_n|+o(|\zeta|)$, $\zeta\in V$. So for small $r=r_\delta$ we have $\varphi(\zeta)\leqq|\mathrm{Im}\,\zeta_n|+\frac{\delta}{2}r$, $\zeta\in V\cap S_r$, $(S_r=\{|\zeta|<r\})$. From the continuity of φ we also deduce for $\xi\in\boldsymbol{R}^n$, $|\xi|<\varepsilon=\varepsilon_{\delta r}:\varphi(\zeta+\xi)\leqq|\mathrm{Im}\,\zeta_n|+\delta r$ $\forall\zeta\in V\cap S_r$, in addition to: $\varphi(\zeta+\xi)\leqq 0$ $\forall\zeta\in V_{\boldsymbol{R}}\cap S_r$. Observing that $|\mathrm{Im}\,\zeta_n|=|\mathrm{Im}\langle\zeta,v\rangle|=H_K(\mathrm{Im}\,\zeta)$, we deduce from our hypothesis the central inequality

$$\varphi(\zeta+\xi)\leqq H_{K'}(\mathrm{Im}\,\zeta)\,,\qquad \zeta\in V\cap S_{r'}\,,\quad r'=\varrho r. \tag{3.3}$$

Let $\eta\in\boldsymbol{R}^{n-1}$, $\langle\eta,w\rangle\neq 0$. Take $\theta\in\boldsymbol{R}^{n-1}$, $\sum_{i=k}^{n-1}\theta_i^2=\sum_{i=k}^{n-1}\eta_i^2$, $\sum_{i=k}^{n-1}\theta_i\eta_i=0$. So putting

$\zeta'=\theta+i\eta$, we have $\sum_{i=k}^{n-1}\zeta_i'^2=0$ and therefore, for $\zeta_t'=t\zeta'$, $\mu_j(\zeta_t')=o(t)$, $t\to 0$, $j=1, 2$. Now let us calculate both sides of (3.2) with $\zeta=(\zeta_t', \mu_j(\zeta_t'))$, ($j=1, 2$, $t\in\boldsymbol{R}^+$ small), and with $\xi=\varepsilon w$. First observe that for such ζ, ξ, we have $\varphi(\zeta+\xi)=\left|\operatorname{Im}\left(\sum_{i=k}^{n-1}(t\theta_i+it\eta_i+\varepsilon w_i)^2\right)^{1/2}\right|=|\operatorname{Im}(\varepsilon^2+i2\varepsilon t\langle\eta, w\rangle+0(t))^{1/2}|$, where $0(t)$ is real valued. Thus we conclude

$$\varphi(\zeta+\xi)=\varepsilon\sin\frac{1}{2}\operatorname{arctg}\frac{2t|\langle\eta, w\rangle|}{\varepsilon}+o(t)=t|\langle\eta, w\rangle|+o(t), \tag{3.4}$$

$$\zeta=(\zeta_t', \mu_j(\zeta_t')) \quad (j=1, 2), \quad \xi=\varepsilon w.$$

Next notice that we have

$$H_{K'}(\operatorname{Im}\zeta)=tH_{K'}(\eta, 0)+o(t), \quad \zeta=(\zeta_t', \mu_j(\zeta_t')), \quad j=1, 2, \tag{3.5}$$

due to the equalities $\mu_j(\zeta_t')=o(t)$, ($j=1, 2$), and $\operatorname{Im}\zeta_t'=t\eta$. Taking into account (3.4) and (3.5) we then deduce from (3.3)

$$t|\langle\eta, w\rangle|+o(t)\leqq tH_{K'}(\eta, 0)+o(t), \quad \eta\in\boldsymbol{R}^{n-1}, \quad \langle\eta, w\rangle\neq 0,$$

which gives, if we devide both sides by t, let $t\to 0$, and put $(\eta, 0)=\xi$,

$$|\langle\xi, w\rangle|\leqq H_{K'}(\xi), \quad \xi\in\boldsymbol{R}^n, \quad \langle\xi, v\rangle=0, \quad \langle\xi, w\rangle\neq 0. \tag{3.6}$$

Notice now that the condition $\langle\xi, w\rangle\neq 0$ can be removed merely by the continuity of $\xi\mapsto H_{K'}(\xi)$; at last putting $\xi=tw$, $t\in\boldsymbol{R}$, in (3.6), we obtain (3.2) with $c(w)=1$. (Untill now we partially followed Lemma 6.8 of [6].)

(1.b) $k=n-1$. Now we will deal with the function $\varphi(\zeta)=|\operatorname{Im}\zeta_{n-1}|$, $\zeta\in\boldsymbol{C}^n$. Since we have, in the present case, $|\operatorname{Im}\mu_j(\zeta')|=|\operatorname{Im}\zeta_{n-1}|+o(|\zeta'|)$, $j=1, 2$, we also have $\varphi(\zeta)\leqq|\operatorname{Im}\zeta_n|+o(|\zeta'|)$, $\zeta\in V$, in addition to $\varphi(\zeta)\leqq 0$, $\zeta\in V_{\boldsymbol{R}}$. Remembering that $|\operatorname{Im}\zeta_n|=H_K(\operatorname{Im}\zeta)$, we conclude that for small r' the following inequality holds

$$\varphi(\zeta)\leqq H_{K'}(\operatorname{Im}\zeta), \quad \zeta\in V\cap S_{r'}. \tag{3.7}$$

Consider now the discriminant $D(\zeta')$ of $F(\zeta', \zeta_n)$ as polynomial in ζ_n. Obviously its localization at 0 is equal to ζ_{n-1}^2; thus taking $\eta\in\boldsymbol{R}^{n-1}$, $\langle\eta, w\rangle\neq 0$, and remembering that w is parallel to the ζ_{n-1}-axis, one recognizes that $D(\zeta'+\tau\eta)$, $\zeta'\in\boldsymbol{C}^{n-1}$, $\tau\in\boldsymbol{C}$, is a Weierstrass polynomial of degree 2 with respect to τ, (apart from an analytic factor non-vanishing at 0). Taking into account the irreducibility of F we can find, in view of Lemma 1.6, a sequence $\theta^\nu\to 0$, $\theta^\nu\in\boldsymbol{R}^{n-1}$, such that the equation $D(\theta^\nu+\tau\eta)=0$ has two non-real, and thus complex conjugated, roots $\tau=\tau^\nu$; (we omit to distinguish them). Put $\zeta'^\nu=\theta^\nu+\tau^\nu\eta$; to achieve the proof we only need a good estimation for $H_{K'}(\mathrm{Im}\,\zeta)$, $\zeta=(\zeta'^{\nu}, \mu_j(\zeta'^{\nu}))$, $j=1,2$. So consider the function $\psi(\zeta')=\left|\mathrm{Im}\dfrac{\mu_1(\zeta')+\mu_2(\zeta')}{2}\right|$, $\zeta'\in \boldsymbol{C}^{n-1}$, which is (defined and) plurisubharmonic in a neighbourhood of 0. Since ψ is non-negative and verifies $\psi(\zeta')=o(|\zeta'|)$, $\psi(\zeta')\leqq 0$ if $\zeta'\in \boldsymbol{R}^{n-1}$, then by the classical Phragmén-Lindelöf principle on $\boldsymbol{C}^{n-1}$, (see Theorem 2.1), it also verifies $\psi(\zeta')=o(|\mathrm{Im}\,\zeta'|)$. Since by assumption $D(\zeta'^{\nu})=0$, and therefore $\mathrm{Im}\,\mu_j(\zeta'^{\nu})=\mathrm{Im}\dfrac{\mu_1(\zeta'^{\nu})+\mu_2(\zeta'^{\nu})}{2}$, $j=1,2$, we then conclude $\mathrm{Im}\,\mu_j(\zeta'^{\nu})=o(\mathrm{Im}\,\zeta'^{\nu})=o(|\mathrm{Im}\,\tau^{\nu}|)$, (remembering for the last equality that $|\mathrm{Im}\,\zeta'^{\nu}|=|\mathrm{Im}\,\tau^{\nu}||\eta|$). So putting $t^{\nu}=|\mathrm{Im}\,\tau^{\nu}|$, we have

$$H_{K'}(\mathrm{Im}\,\zeta)=t^{\nu}H_{K'}(\pm\eta, 0)+o(t^{\nu}),\quad \zeta=(\zeta'^{\nu}, \mu_j(\zeta'^{\nu})),\quad j=1,2, \tag{3.8}$$

where the sign $\pm$ corresponds to the choice of complex conjugated zeros $\tau=\tau^{\nu}$ of $D(\theta^{\nu}+\tau\eta)=0$. Observing that $\varphi(\zeta)=t^{\nu}|\eta_{n-1}|+o(t^{\nu})=t^{\nu}|\langle\eta, w\rangle|+o(t^{\nu})$, $\zeta=(\zeta'^{\nu}, \mu_j(\zeta'^{\nu}))$, $j=1,2$, recalling (3.8), and reasoning on (3.7) as in the point (1.a), we conclude

$$|\langle\xi, w\rangle|\leqq H_{K'}(\xi),\qquad \xi\in \boldsymbol{R}^n,\quad \langle\xi, v\rangle=0, \tag{3.9}$$

which gives (3.2) for $\xi=tw$.

(2) Now we prove the lemma in full generality. In the plane of $\tilde{w}$ and s, let $\underset{\sim}{w}$, $\underset{\sim}{s}$ be an orthonormal system such that $\langle \underset{\sim}{w}, v\rangle=-\left\langle\dfrac{\tilde{w}}{|\tilde{w}|}, v\right\rangle$, $\langle \underset{\sim}{s}, v\rangle=\langle s, v\rangle$. Note that $\underset{\sim}{s}\in\Gamma$ and therefore V is still (normalized and) locally hyperbolic to $\underset{\sim}{s}$ because of the already quoted result of [3]. Choose the coordinates so that $\underset{\sim}{w}=(0,\cdots,1,0)$, $\underset{\sim}{s}=(0,\cdots,1)$. This choice assures that averaging the two points of $\{F_0(\zeta', \zeta_n)=0\}$ over $\zeta'\in \boldsymbol{C}^{n-1}$, one obtains a point orthogonal to s. In fact one recognizes immediately that

$$\left\{\left(\zeta', \frac{1}{2}(\lambda_1(\zeta')+\lambda_2(\zeta'))\right),\ \zeta'\in \boldsymbol{C}^{n-1}\right\}$$

is the complex hyperplane orthogonal to s. We need also to notice that $L(\zeta')=\dfrac{1}{2}(\lambda_1(\zeta')+\lambda_2(\zeta'))$, $\zeta'\in \boldsymbol{C}^{n-1}$ is a linear form with real coefficients which does not depend on the variables of the orthogonal complement of $\underset{\sim}{w}$; so for some $a\in \boldsymbol{R}$ it coincides with $a\langle\zeta', \underset{\sim}{w}\rangle$, $\zeta'\in \boldsymbol{C}^{n-1}$. Finally we have $\prod_{j=1,2}\lambda_j(\zeta')=-\langle\zeta', \underset{\sim}{w}\rangle^2-Q(\zeta')$, where Q is a positive semidefinite quadratic form which verifies $Q(\zeta'+t\underset{\sim}{w})=Q(\zeta')$ $\forall\zeta', t$. Let us denote by $H(\zeta')$, $\zeta'\in \boldsymbol{C}^{n-1}$ the previous product. Basic in the sequel is the plurisubharmonic function $\varphi(\zeta)=\left|\mathrm{Im}\dfrac{1}{2}(\lambda_1(\zeta')-\lambda_2(\zeta'))\right|$, $\zeta\in \boldsymbol{C}^n$. First we need to express φ in other terms. Let $D_0(\zeta')$ be the discriminant of $F_0(\zeta', \zeta_n)$ for ζ_n; (it coincides with the localization at 0 of the discriminant $D(\zeta')$ of $F(\zeta', \zeta_n)$). Then we have $\varphi(\zeta)=\left|\mathrm{Im}\frac{1}{2}D_0(\zeta')^{1/2}\right|$, and moreover, remembering that $\langle(\zeta', L(\zeta')), s\rangle=0$ $\forall\zeta'\in \boldsymbol{C}^{n-1}$, $|\langle s, \underset{\sim}{s}\rangle|\varphi(\zeta)=|\langle s, \underset{\sim}{s}\rangle|\left|\mathrm{Im}\frac{1}{2}D_0(\zeta')^{1/2}\right|=\left|\left\langle \mathrm{Im}(\zeta', L(\zeta')\pm\frac{1}{2}D_0(\zeta')^{1/2}), s\right\rangle\right|$ $=|\langle \mathrm{Im}\,\zeta, s\rangle|$, $\zeta\in V_0$. Recalling that $H_K(\mathrm{Im}\,\zeta)=|\langle \mathrm{Im}\,\zeta, s\rangle|$, we then have $|\langle s, \underset{\sim}{s}\rangle|\varphi(\zeta)\leqq H_K(\mathrm{Im}\,\zeta)+o(|\zeta|)$, $\zeta\in V$; $\varphi(\zeta)\leqq 0$, $\zeta\in V_R$. We then conclude as in point (1) that for some r' and ε the following holds

$$|\langle s, \underset{\sim}{s}\rangle|\varphi(\zeta+\xi)\leqq H_{K'}(\mathrm{Im}\,\zeta), \quad \zeta\in V\cap S_{r'}, \quad \xi\in \boldsymbol{R}^n, \quad |\xi|<\varepsilon. \tag{3.10}$$

Take $\eta\in \boldsymbol{R}^{n-1}$, $\langle\eta, \underset{\sim}{w}\rangle\neq 0$. Reasoning as in (1) we can find a sequence $\theta^\nu\to 0$, $\theta^\nu\in \boldsymbol{R}^{n-1}$, such that the roots $\tau=\tau^\nu$ of the equation $D(\theta^\nu+\tau\eta)=0$ are non-real. Putting $\zeta'^\nu=\theta^\nu+\tau^\nu\eta$ we are going to calculate both sides of (3.10) on the points of $F(\zeta', \zeta_n)=0$ over $\zeta'=\zeta'^\nu$.

First observe that we have $\frac{1}{4}D_0(\zeta')=L(\zeta')^2-H(\zeta')=(a^2+1)\langle\zeta', \underset{\sim}{w}\rangle^2$ $+Q(\zeta')$. It follows $\left|\mathrm{Im}\frac{1}{2}D_0(\zeta'^\nu+\varepsilon \underset{\sim}{w})^{1/2}\right|=(a^2+1)^{1/2}|\langle \mathrm{Im}\,\zeta'^\nu, \underset{\sim}{w}\rangle|+o(|\mathrm{Im}\,\zeta'^\nu|)$ $=(a^2+1)^{1/2}|\langle\eta, \underset{\sim}{w}\rangle||\mathrm{Im}\,\tau^\nu|+o(|\mathrm{Im}\,\tau^\nu|)$, (where it is essential to remember that $Q(\zeta'^\nu+\varepsilon \underset{\sim}{w})=Q(\zeta'^\nu)$). So putting $t^\nu=|\mathrm{Im}\,\tau^\nu|$, we conclude

$$|\langle s, \underset{\sim}{s}\rangle|\varphi(\zeta+\xi)=(a^2+1)^{1/2}|\langle\eta, \underset{\sim}{w}\rangle|t^\nu+o(t^\nu), \quad \zeta'=\zeta'^\nu, \quad \xi=\varepsilon \underset{\sim}{w}. \tag{3.11}$$

On the other hand reasoning as in (1) and noting that $D(\zeta'^\nu)=0$ we have $\mathrm{Im}\,\mu_j(\zeta'^\nu)=\mathrm{Im}\frac{1}{2}(\mu_1(\zeta'^\nu)+\mu_2(\zeta'^\nu))=a\langle \mathrm{Im}\,\zeta'^\nu, \underset{\sim}{w}\rangle+o(t^\nu)$, $j=1, 2$, and we finally conclude

$$H_{K'}(\mathrm{Im}\,\zeta)=t^\nu H_{K'}(\pm(\eta, a\langle\eta, \underset{\sim}{w}\rangle))+o(t^\nu), \quad \zeta=(\zeta'^\nu, \mu_j(\zeta'^\nu)), \quad j=1, 2, \tag{3.12}$$

with ambiguity in the sign according to the ambiguity in the choice of the roots of $D(\theta^\nu+\tau\eta)=0$ for τ. Taking into account (3.12) and (3.11) we obtain from (3.10), after division by t^ν and passage to the limit over $t^\nu\to 0$,

$$|\langle s, \underset{\sim}{s}\rangle|(a^2+1)^{1/2}|\langle\eta, \underset{\sim}{w}\rangle|\leqq H_{K'}(\pm(\eta, a\langle\eta, \underset{\sim}{w}\rangle)), \quad \eta\in \boldsymbol{R}^{n-1}. \tag{3.13}$$

To conclude we need to describe the entries of (3.13). For η varying in $\boldsymbol{R}^{n-1}$ the points $\xi=(\eta, a\langle\eta, \underset{\sim}{w}\rangle)$ on the right side of (3.13) describe the whole hyperplane orthogonal to s. The entry on the left side is, apart from the factor $|\langle s, \underset{\sim}{s}\rangle|$, the length of the projection of $\xi=(\eta, a\langle\eta, \underset{\sim}{w}\rangle)$ on the plane of $\underset{\sim}{w}$, $\underset{\sim}{s}$. At last $\tilde{w}$ is parallel to such projection and therefore $(a^2+1)^{1/2}|\langle\eta, \underset{\sim}{w}\rangle|$ $=\left|\left\langle\xi, \frac{\tilde{w}}{|\tilde{w}|}\right\rangle\right|$. This allows us to rewrite (3.14) as follows

$$(3.14) \qquad |\langle s, \underset{=}{s}\rangle| \left|\left\langle \xi, \frac{\tilde{w}}{|\tilde{w}|}\right\rangle\right| \leqq H_{K'}(\xi), \quad \xi \in \boldsymbol{R}^n, \quad \langle \xi, s\rangle = 0.$$

If we put $\xi = tw$ in (3.14) and observe that $|\langle s, \underset{=}{s}\rangle| \left|\left\langle w, \frac{\tilde{w}}{|\tilde{w}|}\right\rangle\right| = \left(1-2\left\langle v, \frac{\tilde{w}}{|\tilde{w}|}\right\rangle^2\right)|\tilde{w}| = c(w)$, we finally obtain (3.2).

REMARK. We infer from (3.2) that the projection of K' on the ray $\pm w$ covers the segment tw, $|t| \leqq c(w)$; on the other hand K was assumed to coincide with the segment ts, $|t| \leqq 1$, (naturally both K and K' are supposed convex here). So the number $c(w)$ expresses a lower estimation for the ratio between the length of the projection of K' and the length of K. Regarding it, notice that when w lies in the linear space spanned by Γ^* and $\langle w, v\rangle = 0$, then we have $c(w) = 1$ which maximizes the function $c(w)$. If we now let $w \to \pm\partial\bar{\Gamma}$, $w \notin \bar{\Gamma} \cup -\bar{\Gamma}$, then either $|\tilde{w}| \to 0$ or $1-2\left\langle \frac{\tilde{w}}{|\tilde{w}|}, v\right\rangle^2 \to 0$, and in both cases $c(w) \downarrow 0$. This permits us to connect with Theorem 2.1 where for $w \in \pm\partial\bar{\Gamma}$ we allowed the ratio to take arbitrarily small, null excepted, values. At last the situation $w \in \pm\Gamma$, which gives $c(w) < 0$, was handled in Lemma 2.1 where the ratio was possibly null. Collecting these facts we are going to show that an open half space Ω with normal w does admit the Phragmén-Lindelöf principle or does not according to $c(w) \leqq 0$ or $c(w) > 0$. We point out again here that in the critical situation $c(w) = 0$, which makes Ω adherent to both sets which admit the principle and which do not, the principle is conserved according to its character of "closed" property already shown in [8].

If $V = V(F)$ is locally hyperbolic to $\pm v$, recall here the cones $\pm\Gamma^* = \Gamma^*(V, \pm v)$ defined in Section 1; with regard to them we have

THEOREM 3.1. *Let V be irreducible, locally hyperbolic and with multiplicity $\leqq 2$. If Ω is an open convex set which admits the Phragmén-Lindelöf principle on V, then the following condition is fulfilled*

$$(3.15) \qquad \text{for any } x \in \partial\Omega, \text{ either } x + \Gamma^* \cap \Omega = \emptyset, \text{ or } x - \Gamma^* \cap \Omega = \emptyset.$$

PROOF. First observe that the statement is clearly invariant under real linear change of coordinates. So we will assume (3.1) fulfilled. Second, note that if Ω admits the Phragmén-Lindelöf principle, then all tangent half spaces admit it, due to slight modification of n. 16 of [2]. On the other hand condition (3.15) for Ω is obtained by adding the correspondent conditions for the tangent half spaces. So we let Ω be a half space from the beginning. Let $0 \in \partial\Omega$ and denote by w the normal issuing from Ω. From $\Gamma^* \cap \Omega \neq \emptyset$, $-\Gamma^* \cap \Omega \neq \emptyset$, we deduce $w \notin \bar{\Gamma} \cup -\bar{\Gamma}$ and so $c(w) > 0$. If now K is an $(n-1)$-dimensional sphere orthogonal to w, with radius 1 and with center at $x\in\Omega$, then by hypothesis, Ω must contain a set K' which satisfies, in view of (3.2),

$$c(w)\leqq H_{-x+K'}(w)\,. \tag{3.16}$$

If we put $x=-\varepsilon w$ in (3.16) and observe that $H_{K'}(w)<0$, (for $K'\Subset\Omega$), we conclude $c(w)\leqq\varepsilon$ which is absurd for small ε.

PROOF OF IMPLICATION (1)⇒(2) OF MAIN THEOREM. Immediate consequence of Preliminaries and of Theorem 3.1.

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(Received November 5, 1983)

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