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S^1-*actions on twisted* CP^3

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Abstract. When a twisted CP^3 X supports a smooth S^1 (or T^2) action, the normal representations at the fixed point set and the first Pontrjagin class of X are investigated. In the case of a T^2-action, we can determine them almost completely. Here we say that X is a twisted CP^3 if X is a closed smooth manifold such that $H^*(X; Z)$ is isomorphic to $H^*(CP^3; Z)$ additively and that the cup product x^3 of a generator x of $H^2(X; Z)$ is non-zero.

§ 0. Introduction

In [2], [13] a smooth S^1-action on a homotopy CP^3 (or a Z-cohomology CP^3) X is studied in connection with Petrie's conjecture (see [8]). They completely decide the normal S^1-representation at each connected component of the S^1-fixed point set $F(S^1, X)$ and prove that the total Pontrjagin class of X (essentially the first Pontrjagin class $p_1(X)$) is of the same form as the standard CP^3. In this paper we extend the object to a twisted CP^3 and consider a similar problem to the above. The circumstances then become considerably complicated. Here we say that X is a twisted CP^3 if X is a closed smooth manifold such that $H^*(X; Z)$ is isomorphic to $H^*(CP^3; Z)$ additively and that the cup product x^3 of a generator x of $H^2(X; Z)$ is non-zero. When x^3 is k-*times* a generator of $H^6(X; Z)$ (we may assume k is a positive integer), we call it a k-*twisted* CP^3.

Our approach to the problem is similar to [2], [13]. Namely we distinguish cases by the number of connected components of $F(S^1, X)$ and by their dimensions. This is based on the Bredon-Su fixed point theorem ([1], [9]), which asserts that each component F_i of $F(S^1, X)$ is a Q-cohomology CP^{n_i} and $\sum(n_i+1)=4$ as X is, in particular, a Q-cohomology CP^3. Therefore there are the following four types (every type actually occurs as is seen in § 2):

Type I. $F(S^1, X)=F_1\cup F_2$ and $\{n_1, n_2\}=\{2, 0\}$,
Type II. $F(S^1, X)=F_1\cup F_2$ and $n_1=n_2=1$,

Type III. $F(S^1, X)=F_0\cup F_1\cup F_2$ and $\{n_0, n_1, n_2\}=\{1, 0, 0\}$,

Type IV. $F(S^1, X)=\bigcup_{i=1}^{4} F_i$ and $n_i=0$ for all i, i.e., the isolated case.

In each type we study the normal representations, $p_1(X)$ and the second Stiefel Whitney class $w_2(X)$ of X. Remark that the other Stiefel Whitney classes are trivial by Wu's formula and that if a twist k of X is odd, then $w_2(X)$ is also trivial as X is a Z_2-cohomology CP^3, see [7].

In Types I, II we can determine them almost completely (Theorems 4.2, 6.2). In Type III some cases remain undecided. The isolated case (Type IV) is far from complete. However if we assume the given S^1-action extends to a smooth T^2-action essentially, then they are completely decided (Theorems 8.2, 8.3).

This paper is organized as follows. In § 1 we review the geometric weights introduced by W. Y. Hsiang. Some examples of smooth S^1-actions on twisted CP^3 are constructed in § 2. In § 3 we introduce the equivariant Gysin homomorphism which is our main tool. S^1-actions of Types I, II, III are investigated from § 4 to § 7. We consider smooth T^2-actions in § 8 and almost complex T^2-actions in § 9.

In this paper we restrict our concern to a twisted CP^3 for brevity's sake although some results hold in a more general situation.

In concluding this introduction, I would like to express my hearty thanks to Prof. A. Hattori for useful suggestions stimulating my concern to this subject.

§ 1. Equivariant cohomology and geometric weights

Let G be a compact Lie group and $EG\to BG$ a universal principal G-bundle. For a G-space M we denote by M_G the orbit space of $EG\times M$ by the diagonal G-action. Then the notation $H_G^*(M)$ is used instead of $H^*(M_G)$ and is called the equivariant cohomology. Notice that there is a natural fibration $M\xrightarrow{\sigma} M_G\xrightarrow{\rho} BG$. We always regard $H_G^*(M)$ as an $H^*(BG)$-module via $\rho^*: H^*(BG)\to H_G^*(X)$.

Now let $G=S^1$ and $M=X$ a k-twisted CP^3. Since the cohomology groups of X and BS^1 with Z-coefficient consist of only even degree elements, the Serre spectral sequence of the above fibration collapses. This implies that $\sigma^*: H_{S^1}^*(X; Z)\to H^*(X; Z)$ is surjective. Hence, there exists a lifting of a generator x of $H^2(X; Z)$ into $H_{S^1}^2(X; Z)$, which is not unique. We fix it and denote by ξ.

Consider the restriction of ξ to a point p_i of a component F_i of

$F(S^1, X)$. Since $(p_i)_{S^1}=p_i\times BS^1$, it is naturally regarded as an element of $H^2(BS^1; Z)$. Taking a generator α of $H^2(BS^1; Z)$, it is expressed as $a_i\alpha$ with some integer a_i. Clearly a_i is independent of a choice of p_i. It is known in [4] that $\{a_i\}$ are mutually distinct, and they are called the geometric weights (at $\{F_i\}$). We remark that $\{a_i\}$ transform into $\{a_i+a\}$ with some integer a independent of i if the lifting ξ is changed into another one. Hence the differences a_i-a_j are independent of a choice of a lifting.

These geometric weights have the following geometrical meaning. Let η be the complex line bundle over X with the first Chern class $c_1(X)=x$. Since $H^1(X; Z)=0$, the given S^1-action on X lifts to an action on η as bundle isomorphisms, consequently η becomes an S^1-bundle. It is not difficult to see that taking a suitable lifting of the S^1-action, ξ coincides with the first Chern class of the complex line bundle η_{S^1} over X_{S^1}. Therefore a_i is equal to the rotation number of the complex 1-dimensional S^1-module $\eta|p_i$, that is, $\eta|p_i=t^{a_i}$ where t is a complex 1-dimensional S^1-module corresponding to α, i.e., $c_1(t_{S^1})=\alpha$.

§ 2. Examples

We shall construct some examples of smooth S^1-actions on twisted CP^3 and observe the normal representation at each S^1-fixed point set. These examples promote a better understanding for smooth S^1-actions on twisted CP^3 and are very useful to the investigation made from § 4 to § 9.

Our construction is elementary. The plan is as follows. First, prepare two copies $(S^3\times D^4)_i$ $(i=1,2)$ of $S^3\times D^4$ and select a suitable diffeomorphism $\phi_k: (S^3\times S^3)_1\to(S^3\times S^3)_2$ so that the attached manifold $\Sigma_k=(S^3\times D^4)_1\cup_{\phi_k}(S^3\times D^4)_2$ has k-torsion in $H^4(\Sigma_k; Z)$. Next find smooth T^2-actions on two copies such that ϕ_k becomes T^2-equivariant and that a certain circle subgroup of T^2 acts freely on Σ_k. The orbit space Σ_k/S^1 by the free S^1-action is exactly a k-twisted CP^3 with a smooth S^1-action.

Regard S^3 and D^3 as the unit sphere and disk of the quaternion field H respectively. Let m, n be non-zero integers and let $u+vj$ be a quaternion with the unit length where $u, v\in C$ and $j^2=-1$. We define a smooth map $\bar{\phi}_{m,n}: S^3\to S^3$ of degree mn by

$$\bar{\phi}_{m,n}(u+vj)=(u^m+v^nj)/\|u^m+v^nj\|$$

where $\| \ \|$ stands for the length of a quaternion. Then an attaching diffeomorphism $\psi_{m,n}: S^3\times S^3 \to S^3\times S^3$ is defined by

$$\psi_{m,n}(x, y)=(x\bar{\psi}_{m,n}(y), y)$$

for $(x, y)\in S^3\times S^3\subset H\times H$. The 7-manifold $\Sigma_{m,n}$ glued by $\psi_{m,n}$ has the following cohomology groups which are easily deduced from the Mayer-Vietoris exact sequence:

$$H^q(\Sigma_{m,n}; Z)=\begin{cases} Z & \text{if} \quad q=0, 7, \\ Z_{|mn|} & \text{if} \quad q=4, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Now we shall consider actions. As usual we regard S^1 as the group of the complex numbers with the unit length. Given integers p, q, r, s, we define a smooth S^1-action $\phi^i_{p,q,r,s}$ on $(S^3\times D^4)_i$ by

$$\phi^i_{p,q,r,s}(g, (u+vj, w+zj))=(g^pu+g^qvj, g^rw+g^szj)$$

where $g\in S^1$ and $(u+vj, w+zj)\in(S^3\times D^4)_i$. Suppose that S^1-actions $\phi^1_{p,q,r,s}$ and $\phi^2_{\bar{p},\bar{q},\bar{r},\bar{s}}$ are given on $(S^3\times D^4)_1$ and $(S^3\times D^4)_2$ respectively. As is easily checked, the compatible condition so that these actions define an action on $\Sigma_{m,n}$, i.e., $\psi_{m,n}$ is S^1-equivariant, is given as follows:

$$\begin{cases} \bar{p}=p+mr=q-ns, \\ \bar{q}=p+ns=q-mr, \\ \bar{r}=r, \\ \bar{s}=s. \end{cases} \tag{2.2}$$

We shall select a suitable free S^1-action. Note that there are two kinds of free S^1-actions such that their orbit spaces are spin (i.e., $w_2=0$) or not. First we treat the spin case. We define a free S^1-action ϕ_s by

$$\begin{cases} \phi_s=\phi^1_{1,1,0,0} & \text{on} \quad (S^3\times D^4)_1, \\ \phi_s=\phi^2_{1,1,0,0} & \text{on} \quad (S^3\times D^4)_2. \end{cases}$$

This clearly satisfies the compatible condition (2.2).

LEMMA 2.1. *The orbit space* $X_{m,n}=\Sigma_{m,n}/\phi_s$ *by the free action* ϕ_s *is an* $|mn|$*-twisted* CP^3 *and spin.*

PROOF. Using the Gysin exact sequence of the S^1-bundle $\Sigma_{m,n}\to X_{m,n}$

together with (2.1), the ring structure of $H^*(X_{m,n}; Z)$ is easily decided and the first assertion is proved.

The proof of second assertion is as follows. Notice that the orbit space $(S^3\times D^4)_i/\phi_s$ is naturally identified with $S^2\times D^4$. To distinguish these orbit spaces we number $S^2\times D^4$ as $(S^2\times D^4)_i=(S^3\times D^4)_i/\phi_s$. We can then regard $X_{m,n}$ as the space $(S^2\times D^4)_1 \underset{\tilde{\phi}_{m,n}}{\cup} (S^2\times D^4)_2$ glued by the map $\tilde{\phi}_{m,n}$ on $S^2\times S^3$ induced from $\phi_{m,n}$.

Let τ_i be the inclusion: $(S^2\times D^4)_i \to X_{m,n}$. Since it is an open embedding, we have

$$\tau_i^* w_2(X_{m,n}) = w_2(S^2\times D^4) = 0.$$

On the other hand the injectivity of τ_i^* in degree 2 easily follows from the Mayer-Vietoris exact sequence and the definition of $\tilde{\phi}_{m,n}$. Thus $w_2(X_{m,n})=0$, which is the desired result. Q.E.D.

Let S^1 act on $(S^3\times D^4)_1$ via $\phi^1_{p,q,r,s}$ and on $(S^3\times D^4)_2$ via $\phi^2_{p+mr,q-mr,r,s}$. Since they commute with the free action ϕ_s, they define a smooth S^1-action ϕ on $X_{m,n}$. For almost all integers p, q, r, s it has isolated fixed points. In fact, in a generic case, $F(\phi, X_{m,n})$ consists of four points given by the orbit images of points $(1, 0)_1$, $(j, 0)_1$, $(1, 0)_2$, $(j, 0)_2$ in $(S^3\times D^4)_1$ or $(S^3\times D^4)_2$, which will be denoted by p_0, p_1, p_2, p_3 in that order.

Consider the complex line bundle over $X_{m,n}$ associated with the S^1-bundle $\Sigma_{m,n} \to X_{m,n}$. The action ϕ has a natural lifting induced from the action $\phi^1_{p,q,r,s} \cup \phi^2_{p+mr,q-mr,r,s}$ on $\Sigma_{m,n}$. It is easily checked that in a generic case the geometric weights $\{a_i\}$ at points $\{p_i\}$ concerning this lifting are given by

$$a_0=p,\ a_1=q,\ a_2=p+mr,\ a_3=q-mr.$$

Furthermore the normal (or tangential) representations at $\{p_i\}$ are also easily seen. We denote the circumstances by the following figure:

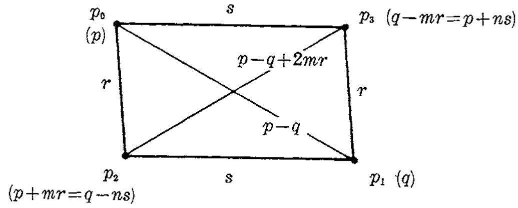


Figure 2.1

The meaning of the figure is as follows. Four vertices stands for the fixed points $\{p_i\}$, and integers in bracket assigned to $\{p_i\}$ show the geometric weights at them. A line connecting two vertices and the integer assigned to it, say r, mean the existence of $Z_{|r|}$-fixed point set component connecting two points corresponding to the two vertices. If the action ϕ is effective and $|r|>1$, then the fixed point set component of $Z_{|r|}$ is a 2-dimensional sphere. Therefore one can read the normal representations at $\{p_i\}$ from the figure.

Now let us vary integers p, q, r, s to obtain S^1-actions with non-isolated fixed point sets. For example, if we take $s=0$ (or $r=0$), then we obtain a smooth S^1-action whose fixed point set consists of two components with a same dimension, written F_1, F_2 (they are actually 2-dimensional spheres). By (2.2) $|p-q|=|p-q+2mr|=|mr|$ in this case. Hence, setting $p=0$ and changing the action into the effective one, the resulting one is represented by the following figure:

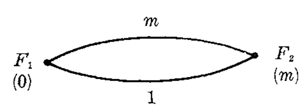


Figure 2.2

where the meaning of the figure is similar to Figure 2.1.

If we take $p=q=0$ and mutually coprime non-zero integers s, r, then we obtain a smooth effective S^1-action whose fixed point set consists of three components: two of them are p_2, p_3 and the remaining one is a 2-dimensional sphere, written F_0. Hence it is represented by the following figure:

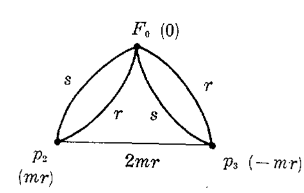


Figure 2.3

Here observe that if $|r|>1$ (resp. $|s|>1$), then $F(Z_{|r|}, X_{m,n})$ (resp. $F(Z_{|s|}, X_{m,n})$) is connected, 4-dimensional and contains all of F_0, p_2, p_3.

If we take $p-q+2mr=0$, then the resulting S^1-action is essentially the same as above.

Now we shall consider the non-spin case. We need to find a "twisted" free S^1-action in place of ϕ_s so that the orbit space is non-spin. We define such a free S^1-action ϕ_t by

$$\phi_t=\phi^1_{1,-1,0,-1} \quad \text{on} \quad (S^3\times D^4)_1,$$
$$\phi_t=\phi^2_{1,-1,0,-1} \quad \text{on} \quad (S^3\times D^4)_2.$$

Remark that n must equal 2 so that ϕ_t satisfies the compatible condition (2.2). By this reason examples of S^1-actions decrease in the non-spin case. The author does not know whether this phenomenon is essential or not.

We will denote the orbit space $\Sigma_{m,2}/\phi_t$ by $\bar{X}_{2m}$. Let S^1 act on $\bar{X}_{2m}$ similarly to the spin case with integers p, q, r, s. By a similar observation a generic S^1-action is represented by the following figure:

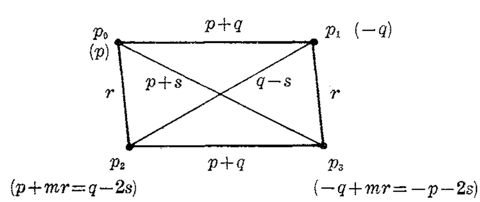


Figure 2.4

Therefore if we take $r=0$ (and $p=0$) and make the action effective, then we obtain a smooth S^1-action whose fixed point set consists of two components with a same dimension 2, written F_1, F_2, and it has the following figure:

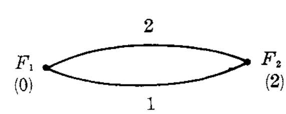


Figure 2.5

If we take $p+q=0$, in particular $p=q=0$, then the resulting S^1-action has two fixed point set components as above, written F'_1, F'_2, and its figure is as follows:

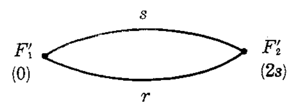
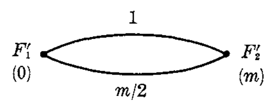


Figure 2.6

Making the action effective, this figure is divided into the following two types (note that $|mr|=|2s|$ by (2.2)):

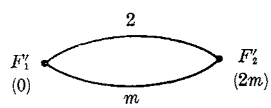
Type 1



if m is even.

Figure 2.7

Type 2



if m is odd,

Figure 2.8

where s corresponds to 1 or 2.

If we take $p+s=0$, in particular $p=s=0$, and make the action effective, then the resulting S^1-action has three fixed point set components and its figure becomes

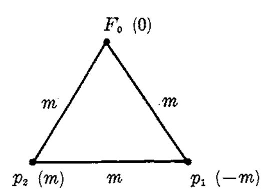


Figure 2.9

Finally it is not difficult to show that $p_1(X_{m,n})=4x^2$ and $p_1(\bar{X}_{2m})=x^2$ where x is a generator of $H^2(X_{m,n}; Z)$, $H^2(\bar{X}_{2m}; Z)$. To this end, for instance, it is convenient to choose an above S^1-action on $X_{m,n}$ (or $\bar{X}_{2m}$) with an isolated fixed point set and calculate the equivariant first Pontrjagin class by considering the restriction to the fixed point set. Restricting it to the ordinary cohomology via σ^*, we obtain the desired result, see [5] for the details.

Summarizing the facts about our examples, we have

THEOREM 2.2 (spin case). *For any positive integer k, there exists a spin k-twisted CP^3 X_k which admits many non-trivial smooth S^1-actions (extending essentially to smooth T^2-actions) and $p_1(X_k)=4x^2$ where x is a generator of $H^2(X_k; Z)$. In fact there exist smooth S^1-actions of the following types: let t be the standard complex 1-dimensional S^1-module, a_i the geometric weight at the fixed point set component F_i and ν_i the normal S^1-representation at F_i,*

Type II_0 (*cf. Figure* 2.2)

- (0) $F(S^1, X_k)$ *is of Type* II (*see Introduction*),
- (i) $\nu_1=\nu_2=t^m+t$,
- (ii) $|a_1-a_2|=|m|$,
- (iii) $k=|mn|$,

Type III_0 (*cf. Figure* 2.3)

- (0) $F(S^1, X_k)$ *is of Type* III,
- (i) $\nu_0=t^r+t^s$, $\nu_1=\nu_2=t^r+t^s+t^{2mr}$,
- (ii) $a_0=0$, $a_1=-a_2=mr(=-ns)$
- (iii) $k=|mn|$,

Type IV_0 (*cf. Figure* 2.1)

- (0) $F(S^1, X_k)$ *is of Type* IV,
- (i) $\nu_0=\nu_1=t^{mr+ns}+t^r+t^s$,
  $\nu_2=\nu_3=t^{mr-ns}+t^r+t^s$,
- (ii) $a_0=0$, $a_1=mr+ns$, $a_2=mr$, $a_3=ns$,
- (iii) $k=|mn|$,

where m, n, r, s are non-zero integers.

THEOREM 2.3 (non-spin case). *For any even positive integer k, there exists a non-spin k-twisted CP^3 $\bar{X}_k$ which admits many non-trivial smooth S^1-actions (extending essentially to smooth T^2-actions) and $p_1(\bar{X}_k)=x^2$ where x is a generator of $H^2(\bar{X}_k; Z)$. In fact there exist smooth S^1-actions of the following types: let t, a_i, ν_i be the same as Theorem 2.2,*

Type II_1 (*cf. Figure* 2.7)

- (0) $F(S^1, \bar{X}_k)$ *is of Type* II,
- (i) $\nu_1=\nu_2=t^{k/4}+t$ $(k/2: even)$,
- (ii) $|a_1-a_2|=k/2$,

Type II_2 (*cf. Figure* 2.8)

- (0) $F(S^1, \bar{X}_k)$ *is of Type* II,

(i) $\nu_1=\nu_2=t^{k/2}+t^2$ ($k/2$: *odd*),

(ii) $|a_1-a_2|=k$,

Type II_3 (*cf. Figure* 2.5)

(0) $F(S^1, \bar{X}_k)$ *is of Type* II,

(i) $\nu_1=\nu_2=t+t^2$,

(ii) $|a_1-a_2|=2$,

Type III_1 (*cf. Figure* 2.9)

(0) $F(S^1, \bar{X}_k)$ *is of Type* III,

(i) $\nu_0=t^{k/2}+t,\ \nu_1=\nu_2=2t^{k/2}+t$,

(ii) $a_0=0,\ a_2=-a_1=k/2$,

Type IV_1 (*cf. Figure* 2.4)

(0) $F(S^1, \bar{X}_k)$ *is of Type* IV,

(i) $\nu_0=\nu_3=t^{mr+2s}+t^r+t^s$,

$\nu_1=\nu_2=t^{mr+2s}+t^r+t^{mr+s}$,

(ii) $a_0=0,\ a_1=-(mr+2s),\ a_2=mr,\ a_3=-2s$,

(iii) $k=|2m|$,

where m, r, s *are non-zero integers.*

REMARKS. (1) Type II_2 and Type II_3 have a non-empty intersection, namely the cases where $k=2$ in Types II_2, II_3 have the same data to each other.

(2) The normal bundles of $F(Z_2, \bar{X}_k)$ in Types II_2, II_3 are non-trivial, which is verified by computing the Euler classes, see Lemma 5.5 of § 5.

§ 3. Equivariant Gysin homomorphisms

In this section we prepare some lemmas for the investigation into Types I, II, III and introduce the equivariant Gysin homomorphism and prove the important identities using the homomorphism. From now on until § 7 ends X will denote a k-twisted CP^3 with a non-trivial smooth S^1-action unless otherwise stated.

LEMMA 3.1. *Let* p^r *be a prime power and* Z_{p^r} *the cyclic subgroup of* S^1 *with order* p^r. *Then*

$$H^{\mathrm{odd}}(F(Z_{p^r}, X); Z_p)=0 \quad \textit{and} \quad \mathrm{rk}\, H^{\mathrm{even}}(F(Z_{p^r}, X); Z_p)=4.$$

Consequently, $H^*(F(S^1, X); Z)$ *is torsion free and consists of only even degree elements.*

PROOF. Let s be a positive integer greater than r and Z_{p^s} the cyclic subgroup of S^1 with order p^s. Note that the induced actions of Z_{p^r}, Z_{p^s} on $H^*(X; Z)$ are trivial because they are subgroups of S^1. Thus, applying Theorem 2.2 in p. 376 of [1] repeatedly, we get

$$\text{rk}\, H^{\text{odd}}(F(Z_{p^s}, X); Z_p) = \text{rk}\, H^{\text{odd}}(F(Z_{p^r}, X); Z_p) = \text{rk}\, H^{\text{odd}}(X; Z_p) = 0,$$

(3.1)

$$\text{rk}\, H^{\text{even}}(F(Z_{p^s}, X); Z_p) \leq \text{rk}\, H^{\text{even}}(F(Z_{p^r}, X); Z_p) \leq \text{rk}\, H^{\text{even}}(X; Z_p) = 4.$$

For a sufficient large s we have $F(S^1, X) = F(Z_{p^s}, X)$. Hence (3.1) shows

$$\text{rk}\, H^{\text{odd}}(F(S^1, X); Z_p) = 0,\ \text{rk}\, H^{\text{even}}(F(S^1, X); Z_p) \leq 4.$$

Since these expressions hold for any prime number p, it follows from the universal coefficient theorem that $H^*(F(S^1, X); Z)$ has no torsion and we have

$$\text{rk}\, H^*(F(S^1, X); Q) = \text{rk}\, H^*(F(S^1, X); Z) = \text{rk}\, H^*(F(Z_{p^r}, X); Z_p) \leq 4. \tag{3.2}$$

On the other hand we know that each component F_i of $F(S^1, X)$ is a Q-cohomology CP^{n_i} and $\sum(n_i+1) = 4$. From this we deduce

$$\text{rk}\, H^*(F(S^1, X); Q) = \sum \text{rk}\, H^*(F_i; Q) = \sum(n_i+1) = 4. \tag{3.3}$$

Combining (3.2) with (3.3) we conclude

$$\text{rk}\, H^*(F(Z_{p^r}, X); Z_p) = 4.$$

This completes the proof. Q.E.D.

We shall recall Su's inequality which plays an important role in our study.

LEMMA 3.2 (Su [10]). *Let Y be a fixed point set component in X of a certain subgroup of S^1. If $F(S^1, Y)$ is non-empty, then*

$$\dim Y \leq 2(\chi(Y) - 1)$$

where $\chi(Y)$ stands for the Euler number of Y.

REMARK. If Y is a Q-cohomology complex projective space, then the equality holds. However it does not hold in general as is seen in the examples of § 2.

COROLLARY 3.3. *Let Y be the same as in Lemma 3.2. Then $F(S^1, X)$ consists of at least two components.*

PROOF. This follows from Lemma 3.2 and the fact $\chi(Y)=\chi(F(S^1, Y))$ $=\sum_{F_i\subset Y}(n_i+1)$. Q.E.D.

Now we shall give some remarks about the structure of $H^*_{S^1}(X)$ as an $H^*(BS^1)$-module or as an $H^*(BS^1)$-algebra. Recall that the Serre spectral sequence of the fibration $X\xrightarrow{\sigma}X_{S^1}\xrightarrow{\rho}BS^1$ collapses. This implies that $H^*_{S^1}(X;Z)$ is isomorphic to $H^*(BS^1;Z)\otimes H^*(X;Z)$ as an $H^*(BS^1;Z)$-module. In particular $H^*_{S^1}(X;Z)$ is torsion free and consists of only even degree elements because $H^*(BS^1;Z)$ is isomorphic to the polynomial ring $Z[\alpha]$ as is well known.

Let x be a generator of $H^2(X;Z)$ and ξ its lifting as before. By the Poincaré duality $x^2/k, x^3/k$ represent generators of $H^4(X;Z), H^6(X;Z)$ respectively. There exist their liftings into $H^*_{S^1}(X;Z)$ by the surjectivity of σ^*. We choose them arbitrary and fix them, written z_1, z_2. As a direct consequence of the theorem of Leray-Hirsh (see Theorem 1.4 in p. 372 of [1] for example), we have

LEMMA 3.4. *The sets* $\{\alpha,\xi\}$, $\{\alpha^2,\alpha\xi,z_1\}$ *and* $\{\alpha^3,\alpha^2\xi,\alpha z_1,z_2\}$ *form additive basis over* Z *of* $H^2_{S^1}(X;Z)$, $H^4_{S^1}(X;Z)$ *and* $H^6_{S^1}(X;Z)$ *respectively.*

As is seen above the $H^*(BS^1)$-module structure of $H^*_{S^1}(X)$ have no information to distinguish actions. On the contrary, the $H^*(BS^1)$-algebra structure of $H^*_{S^1}(X)$ reflect behaviors of actions to some extent. For simplicity we consider cohomology groups with Q-coefficient.

LEMMA 3.5 (W. Y. Hsiang [4]). *The* $H^*(BS^1:Q)$*-algebra structure of* $H^*_{S^1}(X;Q)$ *is given by*

$$H^*_{S^1}(X;Q)=H^*(BS^1;Q)[\xi]/\Pi(\xi-a_i\alpha)^{n_i+1}.$$

In particular $H^*_{S^1}(X;Q)$ *is a free* $H^*(BS^1;Q)$*-module over* $\{1,\xi,\xi^2,\xi^3\}$.

Now we shall introduce the equivariant Gysin homomorphism. For the details we refer the reader to [5]. Generally, given a smooth G-map $f: M\to N$ between closed oriented G-manifolds M, N, the equivariant Gysin homomorphism $f_!: H^*_G(M;Z)\to H^*_G(N;Z)$ is defined. In this paper we always consider the situation that the map f is inclusion and embedding. In this case $f_!$ increases degrees by the codimension of M in N. Furthermore, as an important fact, the element $f^*f_!(1)$ of $H^*_G(M;Z)$ coincides with the equivariant Euler class $e^G(N(M,N))$ of the normal G-

vector bundle $N(M, N)$ of M in N if we give a suitable orientation to it. Here $e^G(N(M, N))$ is defined to be the ordinary Euler class of the vector bundle $N(M, N)_G \to M_G$.

We return to the previous situation. Let Y be such a space of Lemma 3.2 satisfying the equality $\dim Y = 2(\chi(Y)-1)$ and let $j: Y \to X$ be the inclusion map. The evaluation of $(j^*x)^l$ by the fundamental cycle of Y is an integer, where $l = \dim Y/2$. We will denote its absolute value by $D(Y)$. When Y is a point, $D(Y)$ is defined to be 1.

LEMMA 3.6 ([6]). *Let Y, j be as above. Then*

$$kj_!(1) = D(Y) \prod_{F_s \cap Y = \varnothing} (\xi - a_s\alpha)^{n_s+1} \qquad \text{(up to sign)}. \tag{3.4}$$

In particular, if $Y = F_i$ and the normal S^1-representation at F_i in X is given by $\sum t^{m_{ij}}$, then

$$k \prod_j |m_{ij}| = D(F_i) \prod_{s \neq i} |a_i - a_s|^{n_s+1}. \tag{3.5}$$

PROOF. Regard $j_!(1)$ as an element of $H^*_{S^1}(X; Q)$. By a property of the equivariant Gysin homomorphism the restriction of $j_!(1)$ to F_s vanishes for F_s with $F_s \cap Y = \varnothing$. This and Lemma 3.5 imply that $j_!(1)$ has a factor $\prod(\xi - a_s\alpha)^{n_s+1}$. However, since their cohomological degrees coincide with each other by the assumption $\dim Y = 2(\chi(Y)-1)$, we can conclude

$$j_!(1) = c \prod (\xi - a_s\alpha)^{n_s+1}$$

with some rational number c. Consider the restriction of this equation to the ordinary cohomology. The equivariant Gysin homomorphism then reduces to the ordinary Gysin homomorphism and $\xi - a_s\alpha$ turns into x for every s. Therefore it follows from the definition of the ordinary Gysin homomorphism that $|c| = D(Y)/k$. This proves (3.4).

The identity (3.5) is obtained by restricting (3.4) to a point of F_i.

Q.E.D.

§ 4. The case of Type I

From this section to § 7 we study smooth S^1-actions of Type I, II, III. We begin with the study of Type I. The notations in previous sections are used freely.

LEMMA 4.1. *An effective smooth S^1-action of Type* I *is semi-free.*

PROOF. First observe that ν_i is of the form $(n_j+1)t$ (or $(n_j+1)t^{-1}$) where $\{i, j\}=\{1, 2\}$. In fact if it has a factor t^m with $|m|\neq 1$, then a component Y of $F(Z_{|m|}, X)$ contains both F_1 and F_2 by Corollary 3.3. Hence, by a dimensional argument, Y must coincide with the total space X which contradicts the effectiveness of the action.

Next, in order to complete the proof we need to show $F(Z_q, X)=F(S^1, Z)$ for any positive integer q. As is easily seen we have only to show it for any prime number p. Suppose $F(Z_p, X)\neq F(S^1, X)$. The above observation means that the assumption ensures the existence of a component V of $F(Z_p, X)$ except F_1, F_2. Note that $\operatorname{rk} H^{\text{even}}(V; Z_p)$ is positive. Therefore $\operatorname{rk} H^{\text{even}}(F(Z_p, X); Z_p)$ becomes greater than 4, which contradicts Lemma 3.1. Q.E.D.

We may assume $n_1=2$ and $n_2=0$. Using (3.5) for each F_i, we get

$$\begin{aligned} k&=D(F_1)|a_1-a_2| &&\text{at } F_1,\\ k&=|a_1-a_2|^3 &&\text{at } F_2. \end{aligned} \tag{4.1}$$

Here we remark that $D(F_1)/k$ is an integer. In fact, for the inclusion map of F_1 in X, the rational number c in the proof of Lemma 3.6 is taken within Z by Lemma 3.4 as the codimension of F_1 in X is two. This fact and (4.1) show $k=1$, i.e., X is a Z-cohomology CP^3. In this case $p_1(X)$ is already calculated in [2], [13]. Thus we have proved

THEOREM 4.2. *Let X be a k-twisted CP^3 with an effective smooth S^1-action of Type* I. *Then $k=1, p_1(X)=4x^2$ and the given action is semi-free where x is a generator of $H^2(X; Z)$.*

REMARK. If X is simply connected moreover, then it is diffeomorphic to the standard CP^3 by [11].

§ 5. The case of Type II (determination of ν_i)

Suppose the given S^1-action on X is of Type II and let ν_1 be given by t^b+t^c where b, c are mutually coprime integers by the effectiveness of the action. Without loss of generality we may assume $b\geq c$. Furthermore we may assume them to be positive because the complex structure of ν_i induced from the S^1-action has ambiguity of the conjugations of irreducible factors.

Let Y_b, Y_c be the connected components of $F(Z_b, X)$, $F(Z_c, X)$ containing F_1 respectively (in fact it can be verified by a similar argument to the proof of Lemma 4.1 that $F(Z_b, X)$, $F(Z_c, X)$ are both connected). They also contain F_2 by Corollary 3.3. This implies $\nu_1=\nu_2$.

LEMMA 5.1. *Both b and c divide the difference a_1-a_2. Hence the multiplication bc also divides a_1-a_2 as $(b, c)=1$.*

PROOF. This follows from Proposition 2.4 of [6] since Y_b, Y_c connect F_1 with F_2. Q.E.D.

Let f_b (resp. f_c) be the inclusion map from Y_b (resp. Y_c) to X. We understand that $b\geq 2$ (resp. $c\geq 2$) is assumed tacitly when we use the notations Y_b, f_b (resp. Y_c, f_c). Note that $f_{b!}(1), f_{c!}(1)$ belong to $H^2_{S^1}(X; Z)$ as the codimensions of Y_b, Y_c are two. By Lemma 3.4 they are expressed as

$$f_{b!}(1)=K_b\xi+L_b\alpha,$$
$$f_{c!}(1)=K_c\xi+L_c\alpha,$$

with integers K_b, L_b, K_c, L_c.

LEMMA 5.2. *With the above situation and notations*

$$f_{b!}(1)=\pm c\alpha \quad or \quad \pm c((\xi-a_1\alpha)+(\xi-a_2\alpha))/(a_1-a_2),$$
$$f_{c!}(1)=\pm b\alpha \quad or \quad \pm b((\xi-a_1\alpha)+(\xi-a_2\alpha))/(a_1-a_2).$$

PROOF. Recall that $f_b^*f_{b!}(1)$ represents the equivariant Euler class $e^{S^1}(N(Y_b, X))$. Hence the restrictions of $f_{b!}(1)$ to points of F_1, F_2 turn into $c\alpha$ up to sign, namely we get

$$|K_ba_1+L_b|=c \qquad \text{at} \quad F_1,$$
$$|K_ba_2+L_b|=c \qquad \text{at} \quad F_2.$$

Solving these equations, $f_{b!}(1)$ has four possibilities described above. The parallel argument works for $f_{c!}(1)$. Q.E.D.

LEMMA 5.3. *If $K_b\neq 0$ (hence $b\geq 2$), then $b=2, c=1$ and $|a_1-a_2|=2$.*

PROOF. By Lemma 5.2 and the assumption $K_b\neq 0$, we have $K_b=\pm 2c/(a_1-a_2)$. This and integrality of K_b mean that a_1-a_2 divides $2c$. On the other hand since c divides a_1-a_2 by Lemma 5.1, it follows that

$$|a_1-a_2|=c \quad \text{or} \quad 2c.$$

From this and the facts that $b\geq c, b|a_1-a_2$ and $(b, c)=1$, we can deduce the desired result by an easy observation. Q.E.D.

LEMMA 5.4. *If* $K_c \neq 0$ *(hence* $c \geq 2$*), then* $K_b = 0$, $c = 2$ *and* $|a_1 - a_2| = 2b$ *where* b *is an odd number greater than* 2.

PROOF. The same argument as Lemma 5.3 yields

$$|a_1 - a_2| = b \quad \text{or} \quad 2b.$$

In case $|a_1 - a_2| = b$, from the facts that c divides $a_1 - a_2$ and $(b, c) = 1$ it follows that $c = 1$, which contradicts $c \geq 2$. In case $|a_1 - a_2| = 2b$, $c = 2$ is concluded by the same reason as above. Now suppose $K_b \neq 0$, then $c = 1$ by Lemma 5.3 which contradicts the result $c = 2$ obtained above. Q.E.D.

LEMMA 5.5. *It only happens that* $c = 1$ *or* 2. *If* $c = 2$, *then* $|a_1 - a_2| = 2b$, $f_{b!}(1) = \pm 2\alpha$ *and* $f_{c!}(1) = \pm((\xi - a_1\alpha) + (\xi - a_2\alpha))/2$.

PROOF. Suppose $c \geq 3$. Then $K_b = K_c = 0$ by Lemmas 5.3, 5.4, which means $f_{b!}(1) = \pm c\alpha$ and $f_{c!}(1) = \pm b\alpha$ by Lemma 5.2. Hence we have

$$\begin{aligned} e^{S^1}(N(F_i, X)) &= e^{S^1}(N(F_i, Y_b)) e^{S^1}(N(F_i, Y_c)) \\ &= j_i^* f_{c!}(1)\, j_i^* f_{b!}(1) \\ &= bc\alpha^2 \end{aligned} \tag{5.1}$$

up to sign, where j_i denotes the inclusion map from F_i to X.

On the other hand, applying (3.4) to the map j_i and expressing $j_i^*(x)$ as $d_i x_i$ where x_i is a generator of $H^2(F_i; Z)$ and d_i is an integer, we have

$$e^{S^1}(N(F_i, X)) = j_i^* j_{i!}(1) = |d_i|/k(d_i x_i + (a_i - a_j)\alpha)^2$$

where $\{i, j\} = \{1, 2\}$ (note that $D(F_i) = |d_i|$ as $n_i = 1$). Since $d_i \neq 0$ by the Bredon-Su's fixed point theorem and $a_i \neq a_j$ as was indicated before, the coefficient of αx_i does not vanish, which contradicts (5.1). Thus $c = 1$ or 2.

When $c = 2$, $K_b = 0$ by Lemma 5.3. Since the above argument implies that the case $K_b = K_c = 0$ does not occur, $K_c \neq 0$ is concluded. Hence the lemma follows from Lemmas 5.2, 5.4. Q.E.D.

LEMMA 5.6. (1) $|d_1| = |d_2|$,
(2) $kbc = d(a_1 - a_2)^2$ *where* $d = |d_i|$.

PROOF. From (3.5) in Lemma 3.6 we deduce two identities:

$$kbc = |d_1(a_1 - a_2)^2| \quad \text{at} \quad F_1,$$

$$kbc=|d_2(a_2-a_1)^2| \quad \text{at} \quad F_2.$$

These show the lemma. Q.E.D.

LEMMA 5.7. $|a_1-a_2|=bc$ *or* $2bc$.

PROOF. Since both $j_{1!}(1)$ and $j_{2!}(1)$ belong to the integral cohomology $H^4_{S^1}(X; Z)$, the difference (or the sum)

$$\begin{aligned} j_{2!}(1)-j_{1!}(1) &= d/k\{(\xi-a_2\alpha)^2-(\xi-a_1\alpha)^2\} \\ &= d/k\{2(a_1-a_2)\xi+(a_2^2-a_1^2)\alpha^2\} \end{aligned}$$

also belong to $H^4_{S^1}(X; Z)$ where (3.4) is used. This fact and Lemma 3.4 imply

$$k|2d(a_1-a_2). \tag{5.2}$$

Here, since $k=d(a_1-a_2)^2/bc$ by Lemma 5.6, (5.2) turns into an expression $a_1-a_2|2bc$. On the other hand we know $bc|a_1-a_2$ by Lemma 5.1. Thus the lemma is proved. Q.E.D.

Summarizing Lemmas 5.3, 5.5, 5.6, 5.7, normal representations at $F(S^1, X)$ are divided into the following three types:

Type II_0' (cf. Type II_0 in § 2)
- (i) $\nu_1=\nu_2=t^b+t$,
- (ii) $|a_1-a_2|=b$,
- (iii) $k=bd$,
- (iv) $K_b=0$, i.e., the normal bundle of $F(Z_b, X)$ in X is trivial,

Type II_1' (cf. Type II_1 in § 2)
- (i) $\nu_1=\nu_2=t^b+t$,
- (ii) $|a_1-a_2|=2b$,
- (iii) $k=4bd$ (in fact, d is odd, see Remark 5.8),
- (iv) $K_b=0$,

Type II_2' (cf. Types $\mathrm{II}_2, \mathrm{II}_3$ in § 2)
- (i) $\nu_1=\nu_2=t^b+t^2$,
- (ii) $|a_1-a_2|=2b$,
- (iii) $k=2bd$,
- (iv) $K_b=0$, $K_2\neq 0$.

Comparing these with the examples of § 2, we lack examples with $d\neq 1$ in Type II_1' for any b and in Type II_2' for any $b\neq 1$.

REMARK 5.8. In the case of Type II_1' we can deduce a further restriction such that d is odd. The proof of the case $b=1$ is as follows. In a general case we need a slightly modified argument. First recall a short exact sequence (see [1]):

$$0 \to H^2_{S^1}(X; Z) \to H^2_{S^1}(F_1 \cup F_2; Z) \to H^3_{S^1}(X, F_1 \cup F_2; Z) \to 0.$$

Since the action is semi-free in the present case, the above exact sequence holds for any Z_p-coefficient. It follows that $H^3_{S^1}(X, F_1 \cup F_2; Z)$ is torsion free. On the other hand the restriction of $\xi - a_2\alpha$ is given by $\pm dx_1 + (a_1 - a_2)\alpha \pm dx_2$. These show $\xi - a_2\alpha$ is divided by $(d, a_1 - a_2)$. Hence our assertion follows from Lemma 3.4 and $|a_1 - a_2| = 2$.

§ 6. The case of Type II (calculations of $p_1(X), w_2(X)$)

We calculate $p_1(X)$ and $w_2(X)$ under the same situation as § 5. Our aim is the following.

PROPOSITION 6.1. *If a k-twisted* CP^3 X *supports a smooth* S^1*-action of Type* $\mathrm{II}'_0, \mathrm{II}'_1$ *or* II'_2, *then we have*

(1) $w_2(X) = 0$, $p_1(X) = 4x^2$ *in case of Type* II'_0,

(2) $w_2(X) \neq 0$, $p_1(X) = x^2$ *in case of Type* II'_1 *or* II'_2.

PROOF. We distinguish three types, i.e., Types II'_0, II'_1 and II'_2, but the technique used to the computation is similar to each other. Therefore we give the proof only in Type II'_0.

First we deal with $p_1(X)$. As a matter of fact we calculate the equivariant first Pontrjagin class $p_1^{S^1}(X)$ using the method developed in [5]. Here the definition of equivariant Pontrjagin classes is similar to that of the equivariant Euler class. But, for the tangent G-bundle TM over a G-manifold M, we use the notation $p_i^G(M)$ instead of $p_i^G(TM)$.

Now observe that $N(F_i, X)$ decomposes into the Whitney sum of complex line bundles $N_i(b) \oplus N_i(1)$ such that for $g \in S^1 \subset C$, $u \in N_i(b)$ and $v \in N_i(1)$, we have

$$g_* u = g^b u, \quad g_* v = gv.$$

We remark that $c_1(N_i(1)) = e(N_i(1)) = 0$ as $K_b = 0$. Hence, setting $c(N_i(b)) = 1 + \zeta_i x_i$ $(\zeta_i \in Z)$, we get

$$\begin{aligned} c^{S^1}(N(F_i, X)) &= c^{S^1}(N_i(1))c^{S^1}(N_i(b)) \\ &= (1+\alpha)(1+\zeta_i x_i + b\alpha) \end{aligned} \tag{6.1}$$

by an easy observation (see Lemma 1.6 of [5]), in particular we have

(6.2) $$e^{S^1}(N(F_i, X))=\alpha(\zeta_i x_i+b\alpha).$$

On the other hand, since $e^{S^1}(N(F_i, X))=j_i^* j_{i!}(1)$ and $j_{i!}(1)$ is already known by (3.4), we combine this with (6.2) to obtain

$$\alpha(\zeta_i x_i+b\alpha)=d_i/k(d_i x_i+(a_i-a_j)\alpha)^2$$

up to sign where $\{i, j\}=\{1, 2\}$.

Comparing the coefficients of α^2 or α in both sides, we get

(6.3) $$\zeta_i=2bd_i/(a_i-a_j).$$

Therefore, if we note $p^{S^1}(F_i)=p(F_i)=1$ as F_i is a 2-dimensional sphere, then we have

$$\begin{aligned} j_i^* p^{S^1}(X) &= p^{S^1}(N(F_i, X))p^{S^1}(F_i) \\ &= (1+\alpha^2)(1+(\zeta_i x_i+b\alpha)^2) \\ &= j_i^*\{(1+\alpha^2)(1+((\xi-a_i\alpha)+(\xi-a_j\alpha))^2)\}, \end{aligned}$$

where (6.3) and $b=|a_1-a_2|$ are used.

Here recall the injectivity of $(j_1^*, j_2^*) : H_{S^1}^*(X; Z)\to H_{S^1}^*(F_1\cup F_2; Z)$. In fact this fact follows from Theorem 1.5 in p. 374 of [1] as X is totally nonhomologous to zero in X_{S^1}, i.e., $\sigma^* : H_{S^1}^*(X; Q)\to H^*(X; Q)$ is surjective as was indicated in § 1. Thus we conclude

$$p^{S^1}(X)=(1+\alpha^2)(1+((\xi-a_1\alpha)+(\xi-a_2\alpha))^2).$$

Restricting this to the ordinary cohomology, it reduces to

$$p(X)=1+4x^2.$$

Next we deal with $w_2(X)$. Similarly to the above we make use of the equivariant characteristic classes. We divide the proof into two cases by a value of b. In the following, for brevity, the restricted element of α to $H^2(BZ_2; Z_2)$ will be denoted by the same notation α.

Case 1. The case where b is odd. This means $F(Z_2, X)=F_1\cup F_2$. As $H^{\mathrm{odd}}(X; Z_2)=0$, it is easily checked that X is totally nonhomologous to zero in X_{Z_2} (mod 2). It follows from Theorem 1.5 in p. 374 of [1] that

(6.4) $$(j_1^*, j_2^*) : H_{Z_2}^*(X; Z_2) \to H_{Z_2}^*(F_1\cup F_2; Z_2)$$

is injective.

Therefore, let us observe the restriction $j_i^* w^{Z_2}(X)=w^{Z_2}(N(F_i, X))w^{Z_2}(F_i)$. As is well known Stiefel Whitney classes are exactly the mod 2 reduction of the Chern classes, see [7]. Since ζ_i is even by (6.3) and the fact $b=|a_1-a_2|$, (6.1) gives

$$w^{Z_2}(N(F_i, X))=1+\alpha^2.$$

Furthermore we know $w^{Z_2}(F_i)=w(F_i)=1$ as F_i is a 2-dimensional sphere. Thus it is concluded by the injectivity of (6.4) that

$$w^{Z_2}(X)=1+\alpha^2.$$

Restricting this to the ordinary cohomology, $w_2(X)=0$ is obtained.

Case 2. The case where b is even. This means $F(Z_2, X)=Y_b$. Since $H^{\text{odd}}(Y_b; Z_2)=0$ by Lemma 3.1, it follows by a similar reason to Case 1 that

$$f_b^* : H^*_{Z_2}(X; Z_2) \longrightarrow H^*_{Z_2}(Y_b; Z_2) \tag{6.5}$$

is injective. Hence, let us observe the restriction

$$f_b^* w^{Z_2}(X)=w^{Z_2}(N(Y_b, X))w^{Z_2}(Y_b)$$

as before. Since $f_{b!}(1)=\pm\alpha$ by Lemma 5.2 in the case of Type II_0', we have $w^{Z_2}(N(Y_b, X))=1+\alpha$. Therefore it suffices to prove $w^{Z_2}(Y_b)(=w(Y_b))=1$. However it is easily proved by repeating a similar argument to Case 1 for Y_b with the effective S^1-action induced from the restricted S^1-action on Y_b. Q.E.D.

Combining Proposition 6.1 with the results of § 5 we have

THEOREM 6.2. *If a smooth effective S^1-action on X a twisted CP^3 is of Type* II, *then it is of Type* II_0', II_1' *or* II_2' *in* § 5 *and*

$$p_1(X)=4x^2,\ w_2(X)=0 \quad \textit{in case of Type } \text{II}_0',$$
$$p_1(X)=x^2,\ w_2(X)\neq 0 \quad \textit{in case of Type } \text{II}_1' \textit{ or } \text{II}_2'$$

where x is a generator of $H^2(X; Z)$.

§ 7. The case of Type III

In this section we treat Type III. We may assume $n_0=1$, $n_1=n_2=0$ and

$$\nu_0=t^b+t^c \tag{7.1}$$

where b, c are mutually coprime positive integers. As before we will denote the components of $F(Z_b, X)$, $F(Z_c, X)$ containing F_0 by Y_b, Y_c respectively in case $b \geq 2$, $c \geq 2$. They contain F_1 or F_2 (or both of them) by Corollary 3.3.

LEMMA 7.1. *Let Y be Y_b or Y_c. Then the following two statements are equivalent:*

(1) *Y contains only one of F_1, F_2,*

(2) $\dim Y = 2(\chi(Y) - 1) = 4$.

PROOF. First note $\dim Y = 4$. If (1) is satisfied, then $F(S^1, Y)$ consists of two components with mutually distinct dimensions. Then a similar argument to Proposition 3.2 of [5] yields the first equality of (2). This proves that (1) implies (2).

If Y contains both of F_1 and F_2, then

$$2(\chi(Y) - 1) = 2(\chi(F(S^1, Y)) - 1) = 6 > 4 = \dim Y.$$

This proves that (2) implies (1). Q.E.D.

THEOREM 7.2. *Let X be a k-twisted CP^3 with a smooth effective S^1-action of Type* III. *If there exists a fixed point set component Y of a certain subgroup of S^1 satisfying the equation* $\dim Y = 2(\chi(Y) - 1) = 4$, *then* $k = 1$ *and* $p_1(X) = 4x^2$.

PROOF. Notice that such Y must be either Y_b or Y_c. In fact, Y contains F_0; otherwise we have

$$2(\chi(Y) - 1) = 2(\chi(F(S^1, Y)) - 1) \leq 2$$

which contradicts the assumption. We may assume that $Y = Y_b$ and that $Y_b \ni F_1$ and $Y_b \ni F_2$ by Lemma 7.1. The normal representation at F_1 in Y_b is of the form $2t^b$ by Corollary 3.3. Therefore we have an expression

$$\nu_1 = 2t^b + t^u \tag{7.2}$$

with a positive integer u prime to b as the action is effective.

Suppose Y_c contains F_1 (in case $c \geq 2$). Then c must divide b (and u) as $\dim Y_c = 4$, which contradicts $(b, c) = 1$. Thus Y_c contains F_2 and hence we have an expression

$$\nu_2 = 2t^c + t^v \tag{7.3}$$

with a positive integer v prime to c. Then, considering the Z_u, Z_v-fixed point set components containing F_1, F_2 together with Corollary 3.3, we can see

$$u=v. \tag{7.4}$$

Now consider the equivariant Gysin homomorphism of the inclusion $f_b: Y_b \to X$. Since $f_{b!}(1) \in H^2_{S^1}(X; Z)$ and its restriction to F_2 vanishes as $F_2 \cap Y_b = \emptyset$, it is expressed by Lemma 3.4 as

$$f_{b!}(1)=h(\xi-a_2\alpha)$$

with an integer h. Restricting this to F_0, F_1, it follows from (7.1) and (7.2) that

$$c=|h(a_0-a_2)| \quad \text{at} \quad F_0, \tag{7.5}$$

$$u=|h(a_1-a_2)| \quad \text{at} \quad F_1. \tag{7.6}$$

On the other hand u divides a_1-a_2 since there exists a component of $F(Z_u, X)$ connecting F_1 with F_2. It follows from (7.6) that

$$|h|=1, \ u=|a_1-a_2|. \tag{7.7}$$

This and (7.5) show

$$c=|a_0-a_1|. \tag{7.8}$$

Thus, from the formula (3.5) at F_2 together with (7.3), (7.4), (7.7) and (7.8), $k=1$ is deduced by an easy calculation. The calculation of $p_1(X)$ is then done in [2], [13]. Q.E.D.

Hereafter we will assume that such Y satisfying the hypothesis of Theorem 7.2 does not exist, namely Y_b, Y_c contain both F_1 and F_2 (in case that $b \geq 2$, $c \geq 2$).

LEMMA 7.3. (i) $\nu_1=\nu_2$, (ii) $a_1-a_0=a_0-a_2$.

PROOF. (i) is easily verified using Corollary 3.3. The proof of (ii) is as follows. The formula (3.5) at F_1, F_2 together with (i) gives two equations, from which $|a_1-a_0|=|a_2-a_0|$ is deduced. Hence (ii) follows as $\{a_i\}$ are mutually distinct. Q.E.D.

Let us put

$$\nu_1=\nu_2=t^p+t^q+t^r \tag{7.9}$$

where p, q, r are positive integers such that $(p, q, r)=1$. Since Y_b contains

F_1, F_2 and $\dim Y_b=4$, two of positive integers p, q, r are divided by b. The same fact holds for Y_c. Hence we may assume

$$b|p, q \quad (b\nmid r \text{ if } b\geq 2),$$
$$c|p, r \quad (c\nmid q \text{ if } c\geq 2),$$

in particular $bc|p$ as $(b, c)=1$.

For simplicity we normalize $a_0=0$ by taking a suitable lifting ξ and put $a_1=-a_2=a$ where we may assume that a is positive by changing indicies if necessary.

LEMMA 7.4. *If $b\geq 2$, then $f_{b!}(1)=\pm c\alpha$ and $r=c$. Similarly, if $c\geq 2$, then $f_{c!}(1)=\pm b\alpha$ and $q=b$.*

PROOF. By Lemma 3.4 we have an expression $f_{b!}(1)=A\xi+B\alpha$ with integers A, B. Restricting this to each F_i together with (7.1) and (7.9), we have

$$(7.10) \qquad \begin{aligned} &|B|=c && \text{at } F_0, \\ &|Aa+B|=r && \text{at } F_1, \\ &|A(-a)+B|=r && \text{at } F_2. \end{aligned}$$

Solutions of the latter two equations are given by

$$\begin{aligned} A&=0, \\ |B|&=r, \end{aligned} \quad \text{or} \quad \begin{aligned} A&=r/a, \\ B&=0. \end{aligned}$$

The latter solution contradicts (7.10), so we obtain the desired result for f_b. The argument for f_c is parallel to that of f_b. Q.E.D.

Let $j_0: F_0 \to X$ be the inclusion. Then $j_0^*(x)$ is expressed as d_0x_0 with a generator x_0 of $H^2(F_0; Z)$ and a non-zero integer d_0. We may assume that d_0 is positive by taking $-x_0$ as a generator instead of x_0 if necessary.

LEMMA 7.5. *If $b\geq 2$ and $c\geq 2$, then $q=b, r=c$ and $p=2a/d_0$.*

PROOF. The first two identities are trivial consequences of Lemma 7.4. The formula (3.5) at F_0, F_1 together with these results gives

$$\begin{aligned} kbc&=d_0a^2 && \text{at } F_0, \\ kbcp&=2a^3 && \text{at } F_1. \end{aligned}$$

These show the last identity. Q.E.D.

LEMMA 7.6. *If* $b\geq 2$ *and* $c\geq 2$, *then* $p_1(X)=4x^2/d_0^2$.

PROOF. As usual we shall compute $p^{S^1}(X)$. By Lemmas 7.4, 7.5 we have

$$j_0^* p^{S^1}(X)=(1+b^2\alpha^2)(1+c^2\alpha^2)$$
$$j_i^* p^{S^1}(X)=(1+b^2\alpha^2)(1+c^2\alpha^2)(1+4a^2\alpha^2/d_0^2)$$

for $i=1,2$. Hence we can conclude

$$p^{S^1}(X)=(1+b^2\alpha^2)(1+c^2\alpha^2)(1+4(\xi-a_0\alpha)^2/d_0^2)$$

from the injectivity of the restriction map: $H_{S^1}^*(X;Z)\to H_{S^1}^*(F_0\cup F_1\cup F_2;Z)$. Restricting this to the ordinary cohomology, the desired result is obtained. Q.E.D.

REMARK. It is natural to expect $d_0=1$ or 2. In fact it can be verified under the condition that $H^*(Y_b;Z)$, $H^*(Y_c;Z)$ are both torsion free. Furthermore it is not difficult to see that $w_2(X)=0$ if $d_0=1$ and $w_2(X)\neq 0$ if $d_0=2$.

§ 8. Smooth T^2-actions

We have studied smooth S^1-actions in previous sections, but it seems difficult to determine normal representations completely. On the other hand notice that every twisted CP^3 constructed in § 2 supports essential smooth T^2-actions. So, in this section, we consider the case that T^2 acts on a k-twisted CP^3 X effectively. In this setting we can determine normal representations almost completely.

For a while we shall review weights of a T^2-action. By the same reason as the case of an S^1-action there exists a lifting of x into $H_{T^2}^2(X;Z)$. Its restriction to a point of a component F_i of $F(T^2,X)$ is regarded as an element of $H^2(BT^2;Z)$. We call it the weight at F_i and write it by w_i. It is known in [4] that $\{w_i\}$ are mutually distinct. The weights $\{w_i\}$ depend on a choice of a lifting but the differences $\{w_i-w_j\}$ are independent of i. In fact they have an important geometric meaning as follows.

We first notice that an element of $H^2(BT^2;Z)$ is regarded as a homomorphism: $T^2\to S^1$ through the natural isomorphisms $H^2(BT^2;Z)\cong H^1(T^2;Z)\cong\mathrm{Hom}(T^2,S^1)$. By this correspondence the kernel of w_i-w_j determines a circle subgroup of T^2, which is often written $w_i=w_j$. Then, it is known in [4] that the fixed point set component of $w_i=w_j$ containing F_i contains F_s if and only if $w_i=w_s$ coincides with $w_i=w_j$, that is, w_i, w_j, w_s lie on a same line in $H^2(BT^2; R)$. In particular it contains F_j. From this fact we see that $\{w_i\}$ do not lie on a same line if the given T^2-action is essential. Therefore $F(T^2, X)$ consists of at least three components.

Suppose that $F(T^2, X)$ consists of three components, say F_1, F_2, F_3. Then one of them is a 2-dimensional sphere by Bredon-Su's fixed point theorem, say F_1. By the fact stated above the action of $w_1=w_2$ (or $w_1=w_3$) is of Type I and hence $k=1$, $p_1(X)=4x^2$ by Theorem 4.2.

Therefore we assume from now on that $F(T^2, X)$ consists of four components F_1, F_2, F_3, F_4, i.e., is isolated. If three of $\{w_i\}$ lie on a same line, then $k=1$ and $p_1(X)=4x^2$ by the same reason as above. Hence we may assume that any three of $\{w_i\}$ do not lie on a same line, that is, $\{w_i\}$ form vertices of a quadrilateral in $H^2(BT^2; R)$.

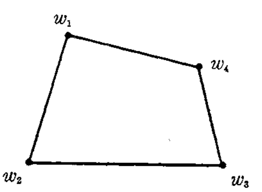


Figure 8.1

LEMMA 8.1. *Unless $\{w_i\}$ form vertices of a parallelogram in $H^2(BT^2; R)$, then $k=1$ and $p_1(X)=4x^2$.*

PROOF. By the assumption there exists such a side of the quadrilateral formed by $\{w_i\}$ that the distances between the side and two vertices not connected with it are mutually distinct. Let such a side connect w_3 with w_4 for example and consider the restricted action of $w_3=w_4$, written T. Then $F(T, X)$ consists of three components F_0, F_1, F_2 where F_0 is a 2-dimensional sphere containing F_3, F_4. Furthermore, if we represent the weights of the T-action as $\{a_i\}$, then $|a_0-a_1|\neq|a_0-a_2|$. This fact and (ii) of Lemma 7.3 ensure the existence of a fixed point set component satisfying the hypothesis of Theorem 7.2. This proves the lemma. Q.E.D.

If $\{w_i\}$ form vertices of a parallelogram, then the restricted action of $w_i=w_j$ for some distinct i, j is of Type II. Thus, by Theorem 6.2, we have

THEOREM 8.2. *If a twisted CP^3 X supports an essential smooth T^2-action, then*

$$p_1(X)=4x^2 \qquad \textit{if } X \textit{ is spin,}$$
$$p_1(X)=x^2 \qquad \textit{if } X \textit{ is non-spin,}$$

where x is a generator of $H^2(X; Z)$.

Now we investigate the type of the normal representations at $F(T^2, X)$. First recall the local geometric weight system at F_i, written $\Omega_i(X)$, which is introduced by W. Y. Hsiang and is defined by

$$\Omega_i(X)=\{\pm(w_i-w_j) \ \ (j\neq i)\}$$

in the present isolated case. To compare the type of the normal representation at F_i with $\Omega_i(X)$ we need to translate the data of the normal representations into the words of the cohomology group $H^*(BT^2; Q)$. The process is as follows.

As is well known ν_i splits into a sum of complex 1-dimensional irreducible representations: $\nu_i=\sum_s \rho_{is}$ where ρ_{is} is a complex 1-dimensional irreducible representation. The first Chern class of a complex line bundle $ET^2 \times_{\rho_{is}} C \to BT^2$ is an element of $H^2(BT^2; Z)$ which is denoted by $\tilde{\rho}_{is}$. Then the representation weight system $RW_i(X)$ at F_i is defined by

$$RW_i(X)=\{\pm\tilde{\rho}_{is}\}.$$

W. Y. Hsiang pointed out in [4] that $\Omega_i(X)$ and $RW_i(X)$ coincides with each other if we ignore the length of the weight vectors, i.e.,

$$RW_i(X)=\{\pm(w_i-w_s)/h_{i,s}\}$$

for some positive numbers $h_{i,s}$ where $h_{i,s}=h_{s,i}$. As a matter of fact $h_{i,s}$ are all integers by Proposition 2.4 of [6] and the product $\prod_s h_{i,s}$ are equal to k by (3.5). Hence, unless $\{w_i\}$ form vertices of a parallelogram, then all $h_{i,s}$ equal one as $k=1$, which is called of the linear type in [3], [5], [6].

Suppose $\{w_i\}$ form vertices of a parallelogram shown in Figure 8.2. Consider the restricted action of $w_1=w_3$ (or $w_2=w_4$), which is of Type III and denote by F_0 the fixed point set component of $w_1=w_3$ containing

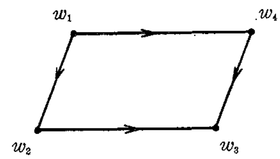


Figure 8.2

F_1, F_3. It follows from Lemma 7.6 and Theorem 8.2 that $d_0=1$ if X is spin and $d_0=2$ is X is non-spin. This and (3.5) imply that $h_{i,i+2}=1$ in the spin case and $h_{i,i+2}=2$ in the non-spin case where indices are taken modulo 4 (although $b\geq 2$ and $c\geq 2$ are assumed in Lemma 7.6, they can be removed in the case of T^2-action).

Next consider the restricted action of $w_1=w_2$, which is of Type II. By the results of § 5, it is of Type $\text{II}_1', \text{II}_2', \text{II}_3'$. This and (3.5) imply $h_{1,4}=h_{2,3}$. Similarly we get $h_{1,2}=h_{4,3}$ by making use of $w_1=w_4$.

Thus, setting $w=w_1-w_4$ and $w'=w_1-w_2$, we have

$$RW_i(X)=\{\pm(w_i-w_{i+2})/d_0, w/\gamma, w'/\delta\} \tag{8.1}$$

where indices are taken modulo 4 as before, $d_0=1$ or 2 according as X is spin or non-spin and γ, δ are positive integers such that the multiplication $\gamma\delta$ equals k/d_0. Summing up these facts, we have

THEOREM 8.3. *Let T^2 act on a k-twisted CP^3 X smoothly and essentially. Then it is of the linear type (and $k=1$) or of the type given by* (8.1).

REMARK. It is checked that, in the spin case, every k-twisted CP^3 constructed in § 2 supports a smooth T^2-action with the type (8.1) for such any γ, δ.

§ 9. Almost complex T^2-actions

Our aim in this section is the following.

THEOREM 9.1. *If a k-twisted CP^3 X admits an effective T^2-action preserving an almost complex structure of X, then*

$$k=1 \textit{ and } p_1(X)=4x^2 \quad \textit{if } X \textit{ is spin,}$$
$$k=2 \textit{ and } p_1(X)=x^2 \quad \textit{if } X \textit{ is non-spin,}$$

where x is a generator of $H^2(X; Z)$.

Combining this with the classification theorem of [14], we have

COROLLARY 9.2. *A simply connected k-twisted CP^3 with an effective almost complex T^2-action is diffeomorphic to the standard CP^3 or the complex quadric Q_3.*

REMARKS. (1) It is known in [11] that every k-twisted CP^3 admits infinitely many almost complex structures which are classified by their first Chern classes.

(2) Concerning the type of the normal representations at $F(T^2, X)$, we show in the proof of Theorem 9.1 that it is of the linear type if $k=1$ and is of the type (8.1) with $\gamma=\delta=1$ and $d_0=2$ if $k=2$. They are realizable by actions preserving the standard complex structure of CP^3 or Q_3.

(3) The total Chern classes $c(X)$ of the almost complex structure preserved by the given T^2-action are of the same form as CP^3 or Q_3, i.e.,

$$c(X)=(1+x)^4 \quad \text{if} \quad k=1,$$
$$c(X)=(1+x)^5(1+2x)^{-1} \quad \text{if} \quad k=2.$$

For the calculation we need to decide the normal representations with the complex structure compatible with the given almost complex structure of X. The outline of the determination is as follows. By the above remark (2) we have only to decide the signs of the exponents of the normal representations. When the geometric weights $\{w_i\}$ take a general position, the signs are decided by comparing the normal representations at p_i, p_j of circle subgroups $w_i=w_j$. Then the following two types remain undecided by this argument:

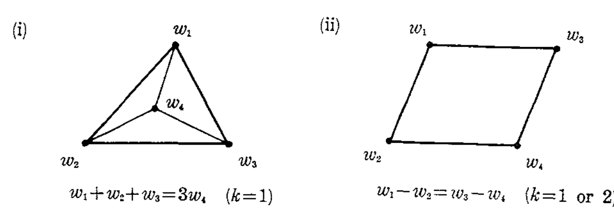


Figure 9.1

where indicies are suitably chosen. In these cases, we use the Lefschetz formula of the equivariant Todd genus with an appropriate bundle coefficient for the restricted action of the circle subgroup $w_1=w_4$. By the Atiyah-Singer Index Theorem the value must belong to $R(S^1)$, from which we can deduce that the signs are the same as those of the corresponding standard action on CP^3 or Q_3 with the same geometric weights as (i) or (ii) of Figure 9.1.

Before beginning the proof of Theorem 9.1 we prepare a lemma which asserts that fixed point sets of any subgroup are well-regulated in the case of almost complex actions.

LEMMA 9.3. *If $F(T^2, X)$ is isolated, then there does not exist such a fixed point set component Y of a certain subgroup of T^2 that* $\dim Y=4$, *$F(T^2, Y)=F(T^2, X)$ and $e^{T^2}(N(Y, X))=\tilde{w}$ for some $\tilde{w}\in H^2(BT^2; Z)$.*

PROOF. This is essentially proved in Proposition 4.2 of [6], but we give the proof here for the convenience of the readers.

Take a general circle subgroup S^1 of T^2 such that $F(S^1, X)$ is isolated. Then we have

Assertion. $c_3^{S^1}(X)=e^{S^1}(X)=1/k\sum_i\prod_{s\neq i}(\xi-a_i\alpha)$.

PROOF. The first sign of equality is a well known fact. The second sign of equality follows from the injectivity of the restriction $H^*_{S^1}(X; Z)\to H^*_{S^1}(F(S^1, X))$ and the following identities

$$j_i^*j_{i!}(1)=j_i^*\left(1/k\prod_{s\neq i}(\xi-a_s\alpha)\right)\quad\text{by (3.4)},$$
$$j_i^*j_{i!}(1)=e^{T^2}(\nu_i)=j_i^*e^{T^2}(X).$$

This proves Assertion.

Suppose that there exists such Y of Lemma 9.3. It follows from the condition $e^{T^2}(N(Y, X))=\tilde{w}$ that

$$c^{S^1}(N(Y, X))=1+w$$

for some $w\in H^2(BS^1; Z)$. On the other hand the product formula of characteristic classes yields

$$j^*c^{S^1}(X)=c^{S^1}(Y)c^{S^1}(N(Y, X))$$

where $j: Y\to X$ is the inclusion. These two identities yield

$$c_3^{S^1}(Y)=j^*(c_3^{S^1}(X)-wc_2^{S^1}(X)+w^2c_1^{S^1}(X)-w^3). \tag{9.1}$$

The left hand side of this identity vanishes as $\dim Y=4<6$. However the right side does not vanish at all which we shall observe in the following.

By Lemma 3.5 $c_q^{S^1}(X)$ is expressed uniquely as a polynomial of ξ over $H^*(BS^1;Q)$ whose polynomial degree is not greater than q. Furthermore the coefficient of ξ^3 in $c_3^{S^1}(X)$ is non-zero Assertion. Thus the sum in the bracket of the right side of (9.1) does not vanish in $H^*_{S^1}(X;Q)$ because, by Lemma 3.5, a non-zero polynomial of ξ over $H^*(BS^1;Q)$ must contain a multiple of ξ with at least degree four to vanish. Setting $S=H^*(BS^1;Q)-\{0\}$, the natural map to the localized ring: $H^*_{S^1}(X;Q)\to S^{-1}H^*_{S^1}(X;Q)$ is injective as is easily seen and $j^*:S^{-1}H^*_{S^1}(X;Q)\to S^{-1}H^*_{S^1}(Y;Q)$ is an isomorphism by the assumption $F(S^1,X)=F(S^1,Y)$ and the localization theorem (see [4]). These show that the right side of (9.1) does not vanish, which is a contradiction. Q.E.D.

PROOF OF THEOREM 9.1. We have only to treat the case where $F(T^2,X)$ is isolated and their weights form vertices of a parallelogram; otherwise $k=1$ is concluded in § 8. Let $\{w_i\}$ be arranged in the manner of Figure 8.2 and $RW_i(X)$ be given by (8.1).

We first treat the spin case. Consider the restricted action of $w_1=w_2$, which is of Type II. Since X is spin, it is of Type II_0' of § 5. Hence $\gamma=1$ follows from Lemma 9.3 and $d_0=1$. Similarly, making use of $w_1=w_4$, $\delta=1$ is concluded. Thus, counting the twist k using (3.5) at F_i, $k=1$ is deduced.

Next we treat the non-spin case. By a similar argument to the above, we can see that γ,δ are 1 or 2. Furthermore, considering the restricted action of $w_1=w_3$, it follows from Lemma 9.3 that $\gamma=\delta$. Again, consider the restricted action of $w_1=w_2$ which is of Type II. If $\gamma=\delta=2$, then the fixed point sets of $w_1=w_2$ have defect two by (3.5), which contradicts Remark 5.8. Hence $\gamma=\delta=1$ is established. Thus, counting the twist k using (3.5) at F_i, $k=2$ is deduced. Q.E.D.

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