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On 3-blocks with an elementary abelian defect group of order 9

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0. Introduction

Let G be a finite group and p be a fixed prime number. Let B be a p-block of G with a defect group D. It is a major problem in modular representation theory to determine the structure of B when the defect group D is given. Here by the structure of B, we mean certain numerical invariants associated with B such as $k(B)$, the number of the irreducible ordinary characters in B, $l(B)$, the number of the irreducible Brauer characters in B, the decomposition numbers of B and the Cartan matrix of B. In case D is cyclic or a 2-group of special type, the above problem is answered successfully. For more detailed arguments, we refer the reader to Feit [10] p. 465.

In this paper, we deal with the case $p=3$ and D is elementary abelian of order 9. Let $p=3$ and D be elementary abelian of order 9. Let b be a 3-block of $\mathrm{C}_G(D)$ with $b^G=B$ and $T(b)$ be the group of all x in $\mathrm{N}_G(D)$ with $b^x=b$. In order to determine the numbers $k(B)$ and $l(B)$, we have to distinguish nine cases according to the action of $T(b)$ on D. In each case the numbers $k(B)$ and $l(B)$ are conjectured as in the following table and the last column of this table shows the main results in this paper.

In Table 1, $e(B)$ denotes the order of the quotient group $T(b)/\mathrm{C}_G(D)$. The symbols ?, $\triangle$ and $\bigcirc$ have the following meanings. ? means that it is proved only when the structure of G is restricted such as G is 3-solvable. $\triangle$ means that it is proved when the structure of G is arbitrary but B is assumed to be principal, while $\bigcirc$ means that it is proved for any G and any B.

The lines of the proofs are as follows. First we determine the difference $k(B)-l(B)$ by using a method of Brauer [6]. Next we investigate the generalized decomposition numbers associated with B and get the

Table 1

$e(B)$	structure of $T(b)/C(D)$		$k(B)$	$l(B)$	result
1	1		9	1	○
2	fixed point free action		6	2	○
	not fixed point free action		9	2	○
4	cyclic		6	4	○
	four-group	subcase (a)	9	4	○
		subcase (b)	6	1	○
8	cyclic		9	8	?
	dihedral	subcase (a)	9	5	○
		subcase (b)	6	2	○
	quaternion		6	5	△
16	semi-dihedral		9	7	△ [1]

number $k(B)$. In case $e(B)=8$ and $T(b)/C_G(D)$ is cyclic or quaternion, however, the above methods are not sufficient to determine $k(B)$ and $l(B)$ uniquely. When $T(b)/C_G(D)$ is quaternion of order 8 and B is principal, we construct a covering group of G to get the result by using an idea of Higman [12]. But when $T(b)/C_G(D)$ is cyclic of order 8, this method does not work even in case B is principal.

In the section 1, we will prepare some lemmas which are needed later.

In the section 2, we will prove the results listed in the table 1. In the end of this section, we will prove a general result which has a corollary that the values of $k(B)$ and $l(B)$ in the table 1 are true in all cases if G is 3-solvable.

In the section 3, we will consider the decomposition numbers and the Cartan matrix with respect to a suitable basic set, assuming that B is principal.

In the section 4, using the results of the sections 2 and 3, we will investigate the structure of finite groups with an elementary abelian Sylow 3-subgroup of order 9. In particular we will give a characteriza-

1) After this paper was written, A. Watanabe proved that in case $e(B)=16$ the values of $k(B)$ and $l(B)$ in the table 1 are true for any B.

tion of the simple groups A_6 and A_7 in terms of their Sylow 3-normalizers.

In the section 5, we will state without proofs some partial results for general abelian defect groups.

1. Preliminaries

Let B be a p-block of a finite group G with a defect group D. A p-block b of $DC_G(D)$ with $b^G=B$ is called a *root* of B in $DC_G(D)$. From now on we fix B, D and b and always use them in the above meaning. We define the *inertial group* $T(b)$ and the *inertial index* $e(B)$ as follows;

$$T(b)=\{x\in \mathrm{N}_G(D)\mid b^x=b\},$$
$$e(B)=|T(b):DC_G(D)|.$$

It is well-known that $e(B)$ is prime to p.

A pair $s=(\pi, b_1)$ is called a *subsection*, if π is a p-element of G and b_1 is a p-block of $\mathrm{C}_G(\pi)$. When $b_1^G=B, s$ is called a subsection associated with B. G acts by conjugation on the set of all subsections associated with B (i.e. $s^g=(\pi^g, b_1^g)$ for $g\in G$).

The following lemmas will be needed in the next section.

LEMMA (1A) (Brauer [6] (4G), (6C)). *Assume that D is abelian. Let $\{x_i\mid i=1,\cdots,n\}$ be a set of representatives for the $T(b)$-conjugacy classes of D. Then $\{(x_i, b_i)\mid i=1,\cdots,n\}$ is a set of representatives for the classes of conjugate subsections associated with B, where $b_i=b^{\mathrm{C}(x_i)}$.*

We denote by $\mathrm{Irr}(B)$, $\mathrm{IBr}(B)$ the set of all irreducible ordinary or Brauer characters in B, respectively. And set $k(B)=|\mathrm{Irr}(B)|$, $l(B)=|\mathrm{IBr}(B)|$.

LEMMA (1B) (Brauer [6] (6D)). *Let $\{(x_i, b_i)\mid i=1,\cdots,n\}$ be a set of representatives for the classes of conjugate subsections associated with B. Then we have*

$$k(B)=\sum_{i=1}^{n} l(b_i).$$

LEMMA (1C) (Brauer [3] (4C)). *Let (π, b_1) be a subsection associated with B. Suppose that the defect of b_1 is equal to that of B. Then for any $\chi\in\mathrm{Irr}(B)$ there exists $\varphi\in\mathrm{IBr}(b_1)$ such that $d^\pi(\chi,\varphi)\neq 0$. Here $d^\pi(\chi,\varphi)$ is* *the generalized decomposition number of* χ *with respect to* φ.

LEMMA (1D) (Brauer-Feit [7], Landrock [16] Corollary 1.6). *Suppose that* D *is elementary abelian of order* 9. *Then we have* $k(B)=3$, 6 *or* 9.

LEMMA (1E) (Brandt [1] p. 513). *If* $k(B)=3$ *and* $l(B)=1$, *then* D *is of order* 3.

According to Brauer, we consider a *column* $\boldsymbol{a}$ of complex numbers whose length is $k(B)$ and which is indexed by $\chi\in\mathrm{Irr}(B)$. Thus, for each $\chi\in\mathrm{Irr}(B)$, we have a coefficient $\boldsymbol{a}_\chi$ of $\boldsymbol{a}$. The column is an *integral column*, if all $\boldsymbol{a}_\chi$ are integers. The *inner product* $(\boldsymbol{a}, \boldsymbol{b})$ of two columns is defined as follows;

$$(\boldsymbol{a}, \boldsymbol{b})=\sum_{\chi\in\mathrm{Irr}(B)}\boldsymbol{a}_\chi\overline{\boldsymbol{b}}_\chi,$$

where - denotes the complex conjugation.

Typical example of a column is a column $\boldsymbol{d}^\pi(\varphi)$ of generalized decomposition numbers for φ whose χ-th coefficient is $\boldsymbol{d}^\pi(\chi, \varphi)$, where φ is an irreducible Brauer character of $\mathrm{C}_G(\pi)$.

These columns play an essential role in the next section.

2. Determination of $k(B)$ and $l(B)$

In this section, we assume that $p=3$ and D is elementary abelian of order 9. We will try to determine the numbers $k(B)$ and $l(B)$ in this case.

First we distinguish nine cases according to the action of $T(b)$ on D. Since $T(b)/\mathrm{C}_G(D)$ is isomorphic to a $3'$-subgroup of $\mathrm{Aut}(D)\cong GL(2, 3)$, we have

$T(b)/\mathrm{C}_G(D)\cong 1$, Z_2, Z_4, Z_8 (cyclic of order 2, 4 and 8),
E_4 (four group), D_8 (dihedral of order 8),
Q_8 (quaternion of order 8) or SD_{16} (semi-dihedral of order 16).

In each case except for the Z_2-case, the action of $T(b)$ on D is unique. If $T(b)/\mathrm{C}_G(D)\cong Z_2$, there are two non-equivalent actions on D; one is fixed point free action and the other is not.

DEFINITION. If $T(b)/C_G(D) \cong X$, we say that B is *of type* X.

PROPOSITION (2A). *If* $e(B)=1$, *then we have* $k(B)=9$ *and* $l(B)=1$.

PROOF. This is a special case of Brauer [5] (6G).

Let x and y be generators of D.

PROPOSITION (2B). *Suppose that* $e(B)=2$ *and* $T(b)$ *acts fixed point freely on* D. *Then we have* $k(B)=6$ *and* $l(B)=2$.

PROOF. In this case, we have the following $T(b)$-orbits on D;

$$\{1\}, \{x, x^{-1}\}, \{y, y^{-1}\}, \{xy, x^{-1}y^{-1}\}, \{xy^{-1}, x^{-1}y\}.$$

By (1A), a set of representatives for the classes of conjugate subsections associated with B is as follows;

$$\{(1, B), (x, b_x), (y, b_y), (xy, b_{xy}), (xy^{-1}, b_{xy^{-1}})\},$$

where $b_x = b^{C(x)}$ and so on. Using (1B), we get

$$k(B)=l(B)+l(b_x)+l(b_y)+l(b_{xy})+l(b_{xy^{-1}}).$$

Since $T(b) \cap C_G(x) = C_G(D)$, $e(b_x)=1$. So, $l(b_x)=1$ by (2A). Then we have

$$k(B)=l(B)+4.$$

Thus, $k(B)=6$ or 9 by (1D).

Let d^x be the column of generalized decomposition numbers for φ^x, where $\{\varphi^x\}=\mathrm{IBr}(b_x)$. As (x, b_x) is conjugate to (x^{-1}, b_x) in $T(b)$, d^x is integral. Let d^y be the corresponding column of generalized decomposition numbers for b_y. It follows from [10] p. 173 Lemma 6.2 that

$$(d^x, d^x)=(d^y, d^y)=9 \tag{1}$$

$$(d^x, d^y)=0 \tag{2}$$

Suppose that $k(B)=9$, then by (1) and (1C) we get

$$d^x = {}^t(\pm 1, \cdots, \pm 1),$$
$$d^y = {}^t(\pm 1, \cdots, \pm 1).$$

But this contradicts (2). Hence $k(B)=6$ and $l(B)=2$.

PROPOSITION (2C). *Suppose that* $e(B)=2$ *and* $T(b)$ *fixes a non-trivial* *element of D. Then we have $k(B)=9$ and $l(B)=2$.*

PROOF. We may assume that the $T(b)$-orbits are the following;

$$\{1\}, \{x\}, \{x^{-1}\}, \{y, y^{-1}\}, \{xy, xy^{-1}\}, \{x^{-1}y, x^{-1}y^{-1}\}.$$

By (1A) and (1B), we have

$$k(B)=l(B)+2l(b_x)+l(b_y)+l(b_{xy})+l(b_{x^{-1}y}).$$

Note that $b_{x^{-1}}=b_x$ by definition. As before we see that $l(b_y)=l(b_{xy})=l(b_{x^{-1}y})=1$.

Next we will show that $l(b_x)=2$. Note that $e(b_x)=2$. It is well-known that b_x corresponds to the unique block $\tilde{b}_x$ of $C_G(x)/\langle x\rangle$ with respect to the relation $\mathrm{Irr}(\tilde{b}_x)\subseteq\mathrm{Irr}(b_x)$ ([10] p. 204 Lemma 4.5). $D/\langle x\rangle$ is a defect group of $\tilde{b}_x$, since D is a defect group of b_x. By Olsson [18] Theorem 1.5, $e(\tilde{b}_x)=e(b_x)=2$. Using Dade's theorem ([10] p. 275), we get

$$l(b_x)=l(\tilde{b}_x)=2.$$

Then, $k(B)=l(B)+7$. By (1D), $k(B)=9$ and $l(B)=2$.

PROPOSITION (2D). *If B is of type Z_4, then $k(B)=6$ and $l(B)=4$.*

PROOF. We may assume that the $T(b)$-orbits of D are the following;

$$\{1\}, \{x, x^{-1}, y, y^{-1}\}, \{xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}\}.$$

By (1A) and (1B), we have

$$k(B)=l(B)+l(b_x)+l(b_{xy}).$$

As before we see that $l(b_x)=l(b_{xy})=1$. So, $k(B)=l(B)+2$. By (1D) and (1E), $k(B)=6$ or 9. If $k(B)=9$, then we have a contradiction using the same arguments as in the proof of (2B). Hence $k(B)=6$ and $l(B)=4$.

When B is of type E_4, we have to divide into three subcases to determine the values $k(B)$ and $l(B)$. Suppose that B is of type E_4. We may assume that the $T(b)$-orbits of D are as follows;

$$\{1\}, \{x, x^{-1}\}, \{y, y^{-1}\}, \{xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}\}.$$

As usual we set $b_x=b^{C(x)}$ and $b_y=b^{C(y)}$. Since $e(b_x)=2$, it follows that $l(b_x)=2$ by (2C). Let $C_G^*(x)$ be the extended centralizer of x i.e. $C_G^*(x)=\{g\in G\,|\,x^g=x \text{ or } x^{-1}\}$. For any $g\in T(b)\cap C_G^*(x)$,

$$(b_x)^g = (b^{C(x)})^g = (b^g)^{C(x)^g} = b^{C(x)} = b_x.$$

So $C_G^*(x) = C(x)(T(b) \cap C_G^*(x))$ fixes b_x. Therefore $C_G^*(x)$ acts on $\mathrm{IBr}(b_x) = \{\varphi_1^x, \varphi_2^x\}$ by conjugation.

Now we can divide into three subcases.

Subcase (a): $C_G^*(x)$ fixes φ_1^x and φ_2^x, and the same occurs for y.

Subcase (b): $C_G^*(x)$ interchanges φ_1^x and φ_2^x, and the same occur for y.

Subcase (c): Otherwise.

Then we have the following;

PROPOSITION (2E).

(i) *In the subcase* (a), *we have* $k(B)=9$ *and* $l(B)=4$.
(ii) *In the subcase* (b), *we have* $k(B)=6$ *and* $l(B)=1$.
(iii) *The subcase* (c) *does not occur.*

PROOF. We have

$$\begin{aligned} k(B) &= l(B) + l(b_x) + l(b_y) + l(b_{xy}) \\ &= l(B) + 2 + 2 + 1. \end{aligned}$$

Then $k(B)=6$ or 9.

As in the proof of (2C), we let $\tilde{b}_x$ be the block of $C_G(x)/\langle x\rangle$ which corresponds to b_x. Again by Dade's Theorem, the Cartan matrix of $\tilde{b}_x$ is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Then the Cartan matrix of b_x is $\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}$. Let $\boldsymbol{d}_1^x, \boldsymbol{d}_2^x$ be the column of generalized decomposition numbers for φ_1^x or φ_2^x, respectively. It follows from [10] p. 173 Lemma 6.2 that

$$\begin{aligned} &(\boldsymbol{d}_1^x, \boldsymbol{d}_1^x) = (\boldsymbol{d}_2^x, \boldsymbol{d}_2^x) = 6 \\ &(\boldsymbol{d}_1^x, \boldsymbol{d}_2^x) = 3. \end{aligned} \qquad (*)$$

Assume that the subcase (a) occurs. Then $\boldsymbol{d}_1^x$ and $\boldsymbol{d}_2^x$ are integral. We can easily show that

$$\begin{aligned} &\boldsymbol{d}_1^x = {}^t(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6, 0, 0, 0), \\ &\boldsymbol{d}_2^x = {}^t(0, 0, 0, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7, \varepsilon_8, \varepsilon_9) \qquad (\varepsilon_i = \pm 1) \end{aligned}$$

is the essentially unique solution for $(*)$. Hence (i) holds.

Next assume that the subcase (b) occurs. Then $\overline{\boldsymbol{d}_1^x} = \boldsymbol{d}_2^x$. Set $\boldsymbol{d}_1^x = \boldsymbol{a} + \boldsymbol{b}\omega$ where ω is a cubic root of unity and $\boldsymbol{a}, \boldsymbol{b}$ are integral columns. From

(*) we get

$$(\boldsymbol{a}, \boldsymbol{a})=5,\ (\boldsymbol{a}, \boldsymbol{b})=1,\ (\boldsymbol{b}, \boldsymbol{b})=2. \qquad (**)$$

In this case,

$$\boldsymbol{a}={}^t(0, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6),$$
$$\boldsymbol{b}={}^t(\varepsilon_1, \varepsilon_2, 0, 0, 0, 0) \qquad (\varepsilon_i=\pm 1)$$

is the essentially unique solution for (**). Hence (ii) holds.

Next suppose that the subcase (c) occurs, then we may assume that $C_G^*(x)$ fixes φ_1^x and φ_2^x, and $C_G^*(y)$ interchanges φ_1^y and φ_2^y. Considering $\boldsymbol{d}_1^x$ and $\boldsymbol{d}_2^x$, we get $k(B)=9$ by the same argument as in the subcase (a). On the other hand, we apply the same argument as in the subcase (b) for b_y and get $k(B)=6$. This is a contradiction, so (iii) holds.

REMARKS.

(1) If B is principal, then the subcase (a) occurs.

(2) The subcase (b) actually occurs. We have the following example.

We let Q_8 act on D in such a way that $Z(Q_8)$ acts trivially and $Q_8/Z(Q_8)$ acts faithfully. Let $G=DQ_8$ be the semidirect product. Then the non-principal 3-block of G is an example of the subcase (b).

When B is of type D_8, the situation is very similar to the E_4-case. Suppose that B is of type D_8. We may assume that the $T(b)$-orbits of D are the following;

$$\{1\}, \{x, x^{-1}, y, y^{-1}\}, \{xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}\}.$$

As before we set $b_x=b^{C(x)}$ and $b_{xy}=b^{C(xy)}$. Since $l(b_x)=l(b_{xy})=2$, we can divide into three subcases (a), (b) and (c) in the similar way. We have the following;

PROPOSITION (2F).

(i) *In the subcase* (a), *we have* $k(B)=9$ *and* $l(B)=5$.

(ii) *In the subcase* (b), *we have* $k(B)=6$ *and* $l(B)=2$.

(iii) *The subcase* (c) *does not occur.*

PROOF. Omitted.

The remarks after (2E) hold also in this case if we replace Q_8 by SD_{16}.

PROPOSITION (2G). *Suppose that B is of type SD_{16}. If B is principal, then $k(B)=9$ and $l(B)=7$.*

PROOF. The $T(b)$-orbits of D are $\{1\}$ and $D^{\#}$. By (1A) and (1B),

$$\begin{aligned} k(B) &= l(B)+l(b_x) \\ &= l(B)+2. \end{aligned}$$

Since b_x is the principal block of $C_G(x)$, the remaining proof is quite similar to that of (2E) (i). So we stop the proof here.

THEOREM (2H). *Suppose that B is of type Q_8. If B is principal, then $k(B)=6$ and $l(B)=5$.*

PROOF. By (1A), $\{(1, B), (x, b_x)\}$ is a set of representatives for the conjugate subsections associated with B, where b_x is the principal block of $C_G(x)$. $l(b_x)=1$ because $e(b_x)=1$. So,

$$\begin{aligned} k(B) &= l(B)+l(b_x) \\ &= l(B)+1. \end{aligned}$$

Let $\boldsymbol{d}^x$ be the column of generalized decomposition numbers for φ^x, where $\{\varphi^x\}=\mathrm{IBr}(b_x)$. Since (x, b_x) is cojugate to (x^{-1}, b_x) in $T(b)$, $\boldsymbol{d}^x$ is integral. On the other hand, we have $(\boldsymbol{d}^x, \boldsymbol{d}^x)=9$. Hence,

$$\boldsymbol{d}^x={}^t(1, \varepsilon_1, \varepsilon_2, \cdots, \varepsilon_8),\ {}^t(1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, 2\varepsilon_5) \quad \text{or} \quad {}^t(1, 2\varepsilon_1, 2\varepsilon_2) \qquad (\varepsilon_i=\pm 1).$$

Suppose that $\boldsymbol{d}^x={}^t(1, 2\varepsilon_1, 2\varepsilon_2)$. Let $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \chi_2\}$, where χ_0 denotes the principal character of G. It follows from the orthogonality relations that

$$1+2\varepsilon_1\chi_1(1)+2\varepsilon_2\chi_2(1)=0.$$

This is a contradiction.

Next suppose that $\boldsymbol{d}^x={}^t(1, \varepsilon_1, \cdots, \varepsilon_8)$. We seek a contradiction. We have $k(B)=9$ and $l(B)=8$. Set $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}$. It follows from the orthogonality relations that

$$1+\sum_{i=1}^{8}\varepsilon_i\chi_i=0 \qquad \text{on } p\text{-regular elements of } G.$$

The decomposition matrix $\boldsymbol{D}$ and the Cartan matrix $\boldsymbol{C}$ with respect to the basic set $\{\chi_0, \varepsilon_1\chi_1, \cdots, \varepsilon_7\chi_7\}$ are the following;

$$D=\begin{pmatrix} 1 & & & 0 \\ & \varepsilon_1 & & \\ & & \ddots & \\ 0 & & & \varepsilon_7 \\ -\varepsilon_8 & \cdots & & -\varepsilon_8 \end{pmatrix}, \qquad C=\begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Now we consider a 3-fold covering group of G and derive a contradiction from the structure of the principal 3-block of the covering group. Here a group H is called a covering group of G if there exists a subgroup Z of H such that $Z\subseteq H'\cap Z(H)$ and $H/Z\cong G$. When Z is of order 3, H is called a 3-fold covering group. First of all, we show the existence of a 3-fold covering group of G.

(1) $N=N_G(D)$ has a 3-fold covering group.

Set $L=$a 3-complement of N and $K=O_{3'}(N)$. Note that $K=L\cap C_G(D)$, $L/K\cong Q_8$. There exist $x, y\in D$ and $\sigma, \tau\in L$ with

$$D=\langle x, y\rangle,\ L=\langle K, \sigma, \tau\rangle,\ x^\sigma=y^{-1},\ y^\sigma=x,\ x^\tau=xy^{-1},\ y^\tau=x^{-1}y^{-1}.$$

Set $P=\langle x_1, y_1 \mid x_1^3=y_1^3=[x_1, y_1]^3=[x_1, y_1, x_1]=[x_1, y_1, y_1]=1\rangle$. We define the action of L on P by the following;

$$x_1^\sigma=y_1^{-1},\ y_1^\sigma=x_1,\ x_1^\tau=x_1y_1^{-1},\ y_1^\tau=x_1^{-1}y_1^{-1} \text{ and } K \text{ is trivial on } P.$$

It is easily checked that this action is well-defined and that $[Z(P), L]=1$. Let $M=PL$ be the semidirect product. Because $M/Z(P)\cong N$, M is the desired covering group.

(2) Since D is abelian, it follows from a theorem of Swan [21] that the 3-part of the Schur multiplier of G is isomorphic to that of N.

(3) By (1) and (2), G has a 3-fold covering group H.

There exists $z\in H'\cap Z(H)$ of order 3 with $H/\langle z\rangle\cong G$. Let Q be a Sylow 3-subgroup of H. Q does not split over $\langle z\rangle$. As $N_H(Q)/\langle z\rangle\cong N$ and $N_H(Q)/QC_H(Q)\cong Q_8$, Q is isomorphic to the above P.

Now we consider the principal 3-block $\hat{B}$ of H. A set of representatives for the conjugate subsections associated with $\hat{B}$ is;

$$\{(1, \hat{B}), (z, \hat{B}), (z^{-1}, \hat{B}), (w, \hat{b}_w)\},$$

where $w\in Q-Z(Q)$ and $\hat{b}_w$ is the principal block of $C_H(w)$. Note that the above representatives are essentially the ones for conjugacy classes of 3-elements in H. Since $C_H(w)$ is 3-nilpotent, $l(\hat{b}_w)=1$. So,

$$k(\hat{B})=3l(\hat{B})+l(\hat{b}_w)=3l(\hat{B})+1=3l(B)+1=25.$$

We can view that $\mathrm{Irr}(B)\subseteqq\mathrm{Irr}(\hat{B})$. $\mathrm{Irr}(\hat{B})-\mathrm{Irr}(B)$ divide into eight pairs $\{\eta_i, \eta_i'\}$ of 3-conjugate characters $(i=1, \cdots, 8)$:

$$\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}\cup\{\eta_1, \cdots, \eta_8\}\cup\{\eta_1', \cdots, \eta_8'\}.$$

The Cartan matrix $\hat{C}$ of $\hat{B}$ with respect to the basic set $\{\chi_0, \varepsilon_1\chi_1, \cdots, \varepsilon_7\chi_7\}$ is 3 times of C;

$$\hat{C}=\begin{pmatrix} 6 & 3 & \cdots & 3 \\ 3 & 6 & & \vdots \\ \vdots & & \ddots & \vdots \\ 3 & \cdots & \cdots & 6 \end{pmatrix}.$$

The decomposition matrix $\hat{D}$ of $\hat{B}$ has the following form;

$$\hat{D}=\begin{array}{c} \chi_0 \\ \vdots \\ \vdots \\ \chi_8 \\ \eta_1 \\ \vdots \\ \eta_8 \\ \eta_1' \\ \vdots \\ \eta_8' \end{array}\left(\begin{array}{c} \begin{matrix} 1 & & & 0 \\ & \varepsilon_1 & & \\ & & \ddots & \\ & 0 & & \varepsilon_7 \\ -\varepsilon_8 & \cdots & \cdots & -\varepsilon_8 \end{matrix} \\ \hline \hat{D}_1 \\ \hline \hat{D}_1' \end{array}\right).$$

Because η_i and η_i' are 3-conjugate, we have $\hat{D}_1=\hat{D}_1'$. Set $\hat{D}_1=(\boldsymbol{a}_1, \cdots, \boldsymbol{a}_8)$. ${}^t\hat{D}\hat{D}=\hat{C}$ implies that

$$(\boldsymbol{a}_i, \boldsymbol{a}_i)=2,\ (\boldsymbol{a}_i, \boldsymbol{a}_j)=1 \quad \text{for } i\neq j \text{ and } i, j=1, \cdots, 8.$$

Hence we may assume that $\hat{D}_1$ has the form;

$$\hat{D}_1=\begin{pmatrix} \delta_1 & \delta_1 & \cdots & \delta_1 \\ \delta_2 & & & \\ & \delta_3 & & 0 \\ & & \ddots & \\ & 0 & & \delta_9 \end{pmatrix} \qquad (\delta_i=\pm 1).$$

On the other hand $\hat{D}_1$ is a $(8, 8)$-type matrix. This is a contradiction. This completes the proof.

If B is principal and of type Z_8, then we have $k(B)=l(B)+1$ and $k(B)=9$ or 6 as in the first part of the proof of (2H). It seems that $k(B)=9$. In fact we can prove it in case G is 3-solvable or $D \lhd G$ (see (2J)). But I have not proved it in general.[2)]

Next we will show that in case G is 3-solvable or $D \lhd G$, the values of $k(B)$ and $l(B)$ in the table 1 are true. At first we will prove a general result.

THEOREM (2I). *Let B be a p-block of G with an abelian defect group D and b be a root of B in $\mathrm{C}_G(D)$. Suppose that G is p-solvable or $D \lhd G$. Then we have an inequality*

$$l(B) \leqq k(T(b)/\mathrm{C}_G(D)),$$

where the right hand of the inequality means the number of conjugacy classes of $T(b)/\mathrm{C}_G(D)$. Furthermore, if one of the following conditions holds, then the equality holds in the above inequality;
(i) *B is principal,*
(ii) *the Schur multiplier of $T(b)/\mathrm{C}_G(D)$ is trivial.*

Note that in the above theorem p is any prime number not necessarily 3 and D is any abelian p-group. We introduce some notations and conventions. We denote by $\mathrm{Irr}(G)$, $\mathrm{IBr}(G)$ the set of all irreducible ordinary or Brauer characters of G, respectively. If $N \lhd G$, then we view that $\mathrm{Irr}(G/N) \subseteqq \mathrm{Irr}(G)$. If $\theta \in \mathrm{Irr}(N)$, we set

$$\mathrm{Irr}(G \mid \theta)=\{\chi \in \mathrm{Irr}(G) \mid (\chi, \theta^G)_G \neq 0\}.$$

PROOF OF (2I). By Okuyama [17] Theorem 4.1 and Knorr [15], we may assume that $D \lhd G$. Using Reynolds [19] Theorem 1, we may assume that $T(b)=G$. By Theorem 6 in the same paper, we may assume that D is a Sylow p-subgroup of G and $\mathrm{C}_G(D)=D \times Z$, where $Z \subseteqq Z(G)$.

For each $\lambda_i \in \mathrm{Irr}(Z)$, there is a p-block b_i of $\mathrm{C}_G(D)$ with

$$\mathrm{Irr}(b_i)=\{\phi \times \lambda_i \mid \phi \in \mathrm{Irr}(D)\}.$$

This correspondence defines a bijection from $\mathrm{Irr}(Z)$ onto the set of all

2) If we are allowed to use the classification theorem of finite simple groups, we can show that $k(B)=9$ holds in case B is principal and of type Z_8.

p-blocks of $C_G(D)$. So $b_1, \cdots, b_n$ are all the p-blocks of $C_G(D)$, where $n=|\mathrm{Irr}(Z)|$. Set $B_i=b_i^G$ $(i=1, \cdots, n)$. As $T(b_i)=G$ for any i, $i\neq j$ implies $B_i\neq B_j$. Then $B_1, \cdots, B_n$ are all the p-blocks of G.

Since G/D is a p'-group, we have

$$\mathrm{IBr}(G)=\mathrm{Irr}(G/D)=\bigcup_{\lambda_i\in \mathrm{Irr}(Z)}\mathrm{Irr}(G\,|\,1_D\times\lambda_i),$$

where 1_D denotes the principal character of D. By Brauer [2] (4F), $\mathrm{Irr}(G\,|\,1_D\times\lambda_i)\subseteqq\mathrm{Irr}(B_i)$. Therefore $\mathrm{IBr}(B_i)=\mathrm{Irr}(G\,|\,1_D\times\lambda_i)$.

Using [14] p. 196 (11.10), we get

$$|\mathrm{Irr}(G\,|\,1_D\times\lambda_i)|\leqq k(G/C_G(D)). \qquad (*)$$

Hence $l(B_i)\leqq k(G/C_G(D))$.

If B is principal, then we may assume that $O_{p'}(G)=1$. So $Z=1$ and B is the only p-block of G. Hence,

$$l(B)=k(G/D)=k(G/C_G(D)).$$

If the Schur multiplier of $G/C_G(D)$ is trivial, then the equality holds in $(*)$ by [14] p. 195 (11.9). This completes the proof.

COROLLARY (2J). *If G is 3-solvable or $D\lhd G$, then the values of $k(B)$ and $l(B)$ in the table 1 are true in all cases.*

PROOF. We may assume that B is of type Z_8, Q_8 or SD_{16}. By [13] p. 643 Satz 25.3, the Schur multipliers of Z_8 and Q_8 are trivial.

We will show that the Schur multiplier of SD_{16} is also trivial. Suppose that it is non-trivial. Then we have a covering group T with $T/\langle z\rangle\cong SD_{16}$ and $z\in T'\cap Z(T)$ where z is an involution of T. By the structure of SD_{16}, $Z(T)=\langle z\rangle$ or $|Z(T)|=4$. If $Z(T)=\langle z\rangle$, T is a 2-group of maximal class. So $T\cong D_{32}, Q_{32}$ or SD_{32}, by [13] p. 339 Satz 11.9. Hence $T/\langle z\rangle\cong D_{16}$, a contradiction. If $|Z(T)|=4$, then $Z(T)\subseteqq T'$ since $Z(T)/\langle z\rangle\subseteqq (T/\langle z\rangle)'$. We have $T/Z(T)\cong D_8$. But this is a contradiction because the Schur multiplier of D_8 is Z_2 ([13] p. 646 Satz 25.6). Therefore the Schur multiplier of SD_{16} is trivial.

It follows from the equality condition (ii) of (2I) that $l(B)=8, 5$ or 7 if B is of type Z_8, Q_8 or SD_{16}, respectively. Since the difference $k(B)-l(B)$ is easily determined in each case, the proof is complete.

REMARK. Theorem (2I) holds also for any (not necessarily abelian) defect group D if we replace $l(B)$, $C_G(D)$ by $l_0(B)$, $DC_G(D)$ respectively, where $l_0(B)$ denotes the number of irreducible Brauer characters of height 0 in B. The proof is similar to that of Theorem (2I).

3. Decomposition numbers and Cartan matrix

In this section, we suppose that B is the principal 3-block of G with an elementary abelian defect group D of order 9. We consider the decomposition numbers and the Cartan matrix of B with respect to a suitable basic set. Since the case $e(B)=1$ is trivial, we treat the remaining eight cases in the following propositions. Let χ_0 be the principal character of G and G_0 be the set of all 3-regular elements of G.

PROPOSITION (3A). *Suppose that $e(B)=2$ and $T(b)$ acts fixed point freely on D. Let $\mathrm{Irr}(B)=\{\chi_0, \cdots, \chi_5\}$, then we have*

$$1-\varepsilon_1\chi_1+\varepsilon_2\chi_2=0,\ \chi_2=\chi_3=\cdots=\chi_5 \quad on \quad G_0,$$

where $\varepsilon_i=\pm 1$. The decomposition matrix and the Cartan matrix of B with respect to the basic set $\{\chi_0, \varepsilon_2\chi_2\}$ are the following;

$$D=\begin{pmatrix} 1 & 0 \\ \varepsilon_1 & \varepsilon_1 \\ 0 & \varepsilon_2 \\ 0 & \varepsilon_2 \\ 0 & \varepsilon_2 \\ 0 & \varepsilon_2 \end{pmatrix}, \qquad C=\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}.$$

If we have $k(B)-l(B)$ relations for characters on G_0 and choose a suitable basic set, then the decomposition matrix and the Cartan matrix with respect to this basic set is easily written down as is seen in (3A). Therefore, in the following propositions we state only relations for characters on G_0 without showing the decomposition matrix and the Cartan matrix.

PROPOSITION (3B). *Suppose that $e(B)=2$ and $T(b)$ fixes a non-trivial element of D. Let $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}$. Then we have*

$$1+\chi_1=\chi_2,\ \chi_3=\chi_4=1,\ \chi_5=\chi_6=\chi_1,\ \chi_7=\chi_8=\chi_2 \quad on\ G_0.$$

We can choose $\{\chi_0, \chi_1\}$ as a basic set.

PROPOSITION (3C). *Suppose that B is of type Z_4. Let* $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_5\}$. *Then we have*

$$1+\varepsilon_1\chi_1+\varepsilon_2\chi_2+\varepsilon_3\chi_3=\varepsilon_4\chi_4,\ \chi_4=\chi_5 \quad on\ G_0,$$

where $\varepsilon_i=\pm 1$. We can choose $\{\chi_0, \varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3\}$ as a basic set.

PROPOSITION (3D). *Suppose that B is of type E_4. Let* $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}$. *Then we have*

$$1+\varepsilon_1\chi_1+\varepsilon_2\chi_2=\varepsilon_3\chi_3+\varepsilon_4\chi_4+\varepsilon_5\chi_5=\varepsilon_6\chi_6+\varepsilon_7\chi_7+\varepsilon_8\chi_8=0,$$
$$1+\varepsilon_3\chi_3+\varepsilon_6\chi_6=\varepsilon_1\chi_1+\varepsilon_4\chi_4+\varepsilon_7\chi_7=\varepsilon_2\chi_2+\varepsilon_5\chi_5+\varepsilon_8\chi_8$$

on G_0, where $\varepsilon_i=\pm 1$. We can choose $\{\chi_0, \varepsilon_1\chi_1, \varepsilon_3\chi_3, \varepsilon_4\chi_4\}$ as a basic set.

PROPOSITION (3E). *Suppose that B is of type D_8. Let* $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}$. *Then we have*

$$1+\varepsilon_1\chi_1+\varepsilon_2\chi_2=\varepsilon_3\chi_3+\varepsilon_4\chi_4+\varepsilon_5\chi_5=\varepsilon_6\chi_6+\varepsilon_7\chi_7+\varepsilon_8\chi_8,$$
$$1+\varepsilon_3\chi_3+\varepsilon_6\chi_6=\varepsilon_1\chi_1+\varepsilon_4\chi_4+\varepsilon_7\chi_7=\varepsilon_2\chi_2+\varepsilon_5\chi_5+\varepsilon_8\chi_8$$

on G_0, where $\varepsilon_i=\pm 1$. We can choose $\{\chi_0, \varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3, \varepsilon_4\chi_4\}$ as a basic set.

PROPOSITION (3F). *Suppose that B is of type Q_8. Let* $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_5\}$. *Then we have*

$$1+\sum_{i=1}^{4}\varepsilon_i\chi_i+2\varepsilon_5\chi_5=0 \quad on\ G_0,$$

where $\varepsilon_i=\pm 1$. We can choose $\{\chi_0, \varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3, \varepsilon_5\chi_5\}$ as a basic set.

PROPOSITION (3G). *Suppose that B is of type Z_8. Assume further that $k(B)=9$. Let* $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}$. *Then we have*

$$1+\sum_{i=1}^{8}\varepsilon_i\chi_i=0 \quad on\ G_0,$$

where $\varepsilon_i=\pm 1$. We can choose $\{\chi_0, \varepsilon_1\chi_1, \cdots, \varepsilon_7\chi_7\}$ as a basic set.

PROPOSITION (3H). *Suppose that B is of type SD_{16}. Let* $\mathrm{Irr}(B)=\{\chi_0, \chi_1, \cdots, \chi_8\}$. *Then we have*

$$1+\varepsilon_1\chi_1+\varepsilon_2\chi_2=\varepsilon_3\chi_3+\varepsilon_4\chi_4+\varepsilon_5\chi_5=\varepsilon_6\chi_6+\varepsilon_7\chi_7+\varepsilon_8\chi_8$$

on G_0, where $\varepsilon_i=\pm 1$. We can choose $\{\chi_0, \varepsilon_1\chi_1, \varepsilon_2\chi_2, \varepsilon_3\chi_3, \varepsilon_4\chi_4, \varepsilon_6\chi_6, \varepsilon_7\chi_7\}$ as a basic set.

The proofs of the above propositions are very similar, so we prove (3C) only, which will be used in the next section.

PROOF OF (3C). We use the same notation as in the proof of (2D). By (2D), $k(B)=6$ and $l(B)=4$. B has two proper subsections (x, b_x) and (xy, b_{xy}) with $l(b_x)=l(b_{xy})=1$. Let $\boldsymbol{d}^x, \boldsymbol{d}^{xy}$ be the column of generalized decomposition numbers with respect to b_x or b_{xy}, respectively. Then $\boldsymbol{d}^x$ and $\boldsymbol{d}^{xy}$ are both integral. By [10] p. 173, Lemma 6.2,

$$(\boldsymbol{d}^x, \boldsymbol{d}^x)=(\boldsymbol{d}^{xy}, \boldsymbol{d}^{xy})=9,$$
$$(\boldsymbol{d}^x, \boldsymbol{d}^{xy})=0.$$

Since $\chi(x)\equiv\chi(1)\equiv\chi(xy) \pmod 3$ for any $\chi\in \mathrm{Irr}(B)$, we have $\boldsymbol{d}^x\equiv\boldsymbol{d}^{xy}$ (mod 3). Then we get the essentially unique solution;

$$\boldsymbol{d}^x={}^t(1, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, -2\varepsilon_5),$$
$$\boldsymbol{d}^{xy}={}^t(1, \varepsilon_1, \varepsilon_2, \varepsilon_3, -2\varepsilon_4, \varepsilon_5) \qquad (\varepsilon_i=\pm 1).$$

For $g\in G$, we let $\boldsymbol{\chi}_B(g)$ be the column whose χ-th coefficient is $\chi(g)$. As $(\boldsymbol{\chi}_B(1), \boldsymbol{d}^x-\boldsymbol{d}^{xy})=0$, $\varepsilon_4=\varepsilon_5$. For any $s\in G_0$, we get

$$(\boldsymbol{\chi}_B(s), \boldsymbol{d}^x)=(\boldsymbol{\chi}_B(s), \boldsymbol{d}^{xy})=0.$$

So we obtain

$$1+\varepsilon_1\chi_1(s)+\varepsilon_2\chi_2(s)+\varepsilon_3\chi_3(s)=\varepsilon_4\chi_4(s),$$
$$\chi_4(s)=\chi_5(s).$$

The proof is complete.

4. Applications

In this section, we assume that G has an elementary abelian Sylow 3-subgroup P of order 9. G is called *of type* X if $\mathrm{N}_G(P)/\mathrm{C}_G(P)\cong X$. Let B_0 be the principal 3-block of G. So P is a defect group of B_0 and we can apply the results of the sections 2 and 3 for B_0. We will investigate the structure of G by using informations on the structure of B_0. Note that G is of type X if and only if B_0 is of type X.

If G is of type 1, then G is 3-nilpotent by Burnside's Theorem.

If G is of type Z_2 or E_4, we have the following results.

THEOREM (4A) (Smith-Tyrer [20]). *If G is of type Z_2, then $\mathrm{O}^3(G)<G$ or G is 3-solvable of 3-length* 1.

THEOREM (4B) (Higman [12] Theorem 3.1). *Let G be of type E_4. If $\mathrm{C}_G(P)=P$, then G is not simple.*

The following corollary will be needed in the proof of (4I).

COROLLARY (4C). *Let G be of type E_4 and suppose that $\mathrm{C}_G(P)=P$. If G is a direct product of isomorphic simple groups, then $G\cong A_5\times A_5$ or $PSL(2, 7)\times PSL(2, 7)$.*

PROOF. By assumption, $G=S_1\times \cdots \times S_r$ and $S_i\cong S$ is simple. Because $|P|=9$, it follows from (4B) that $r=2$. There exists $x\in G$ of order 3 with $P\cap S_1=\langle x\rangle$. Clearly $\mathrm{C}_G(x)=\mathrm{C}_{S_1}(x)\times S_2$. Since $\mathrm{C}_G(P)=P$, $\mathrm{C}_{S_1}(x)=\langle x\rangle$. By Feit-Thompson [11], $S_1\cong A_5$ or $PSL(2, 7)$. This completes the proof.

In case G is of type Z_4, we will prove the following;

THEOREM (4D). *Let G be of type Z_4. Suppose that $\mathrm{C}_G(P)=P$ and $\mathrm{O}_{3'}(G)=1$. Then one of the following holds;*

(i) $G\cong A_6$ *or* A_7,

(ii) $G\triangleright P$.

First we will prepare some lemmas.

LEMMA (4E). *Assume the hypothesis of* (4D). *Then G is simple or $G\triangleright P$.*

PROOF. We omit the proof of (4E) since it is similar to that of (4H).

Next lemma is a key result for the proof of (4D).

LEMMA (4F). *Assume the hypothesis of* (4D). *If G is simple, then G has an ordinary irreducible character of degree* 5 *or* 10.

PROOF. G has two conjugacy classes of 3-elements. Let x and y be representatives for these classes such that $\langle x, y\rangle=P$. Let t be an involution of $\mathrm{N}_G(P)$. By the assumption, the principal 3-block B_0 of G has the following form (see (2D), (3C));

	1	x	y	t
χ_0	1	1	1	1
χ_1	d_1	ε_1	ε_1	a_1
χ_2	d_2	ε_2	ε_2	a_2
χ_3	d_3	ε_3	ε_3	a_3
χ_4	d	$-\varepsilon$	2ε	a
χ_5	d	2ε	$-\varepsilon$	a

$(\varepsilon, \varepsilon_i=\pm 1)$.

Suppose $x^g y^h=t$ for some $g, h \in G$. It follows from [13] p. 142 Aufgabe 75 that $\langle x^g, y^h\rangle \cong A_4$, so x^g is conjugate to y^h or $(y^h)^{-1}$. Therefore x is conjugate to y, a contradiction. Hence there exists no pair (u, v) such that u is conjugate to x, v is conjugate to y and $uv=t$. By [14] p. 45 (3.9), we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x)\chi(y)\chi(t)}{\chi(1)}=0.$$

If $\chi(x)\neq 0$ and $\chi(y)\neq 0$, then χ is in a 3-block of full defect. So $\chi \in B_0$ since $C_G(P)=P$. Therefore we get

$$1+\sum_{i=1}^{3}\frac{a_i}{d_i}-4\frac{a}{d}=0. \qquad (*)$$

The character table of $N=\mathrm{N}_G(P)$ is as follows;

	1	x	y	t
λ_1	1	1	1	1
λ_2	1	1	1	1
λ_3	1	1	1	-1
λ_4	1	1	1	-1
θ_1	4	1	-2	0
θ_2	4	-2	1	0

Here we omit the two columns corresponding to elements of order 4, which are not needed.

Now we consider the decomposition of $\chi|_N$ into the sum of irreducible characters of N for $\chi \in \mathrm{Irr}(B_0)$. If $\varepsilon_i=1$, then we have

$$\chi_i|_N=\lambda_j+\sum_{k=1}^{n}(\theta_1+\theta_2+\lambda_{i_k}).$$

Therefore $\chi_i(1)=9n+1$, $|\chi_i(t)|\leqq n+1$. So,

$$\left|\frac{a_i}{d_i}\right|\leqq\frac{n+1}{9n+1}\qquad n=1,2,\cdots.$$

Similarly we get that

$$\text{if } \varepsilon_i=-1, \text{ then } \left|\frac{a_i}{d_i}\right|\leqq\frac{n}{9n+8}\qquad n=0,1,\cdots;$$

$$\text{if } \varepsilon=1, \text{ then } \left|\frac{a}{d}\right|\leqq\frac{n+1}{9n+5}\qquad n=0,1,\cdots;$$

$$\text{if } \varepsilon=-1, \text{ then } \left|\frac{a}{d}\right|\leqq\frac{n}{9n+4}\qquad n=0,1,\cdots.$$

Hence if $d_i>10$, then $\left|\frac{a_i}{d_i}\right|\leqq\frac{3}{19}$ $(\varepsilon_i=1)$ or $<\frac{1}{9}$ $(\varepsilon_i=-1)$ and if $d>5$, then $\left|\frac{a}{d}\right|\leqq\frac{2}{14}$ $(\varepsilon=1)$ or $<\frac{1}{9}$ $(\varepsilon=-1)$.

Case 1. $\varepsilon=1$.

Note that $\varepsilon_i=-1$ for some i in this case. Assume that $d_i>10$ for all i and $d>5$. By $(*)$ and the above inequalities, we have

$$\begin{aligned}1&=4\cdot\frac{a}{d}-\sum_{i=1}^{3}\frac{a_i}{d_i}\leqq4\left|\frac{a}{d}\right|+\sum_{i=1}^{3}\left|\frac{a_i}{d_i}\right|\\&<4\cdot\frac{2}{14}+\frac{1}{9}+\frac{3}{19}+\frac{3}{19}=0.9983\cdots.\end{aligned}$$

This is a contradiction.

Case 2. $\varepsilon=-1$.

Suppose that $d_i>10$ for all i. As in Case 1, we have

$$\begin{aligned}1&=4\cdot\frac{a}{d}-\sum_{i=1}^{3}\frac{a_i}{d_i}\leqq4\left|\frac{a}{d}\right|+\sum_{i=1}^{3}\left|\frac{a_i}{d_i}\right|\\&<4\cdot\frac{1}{9}+\frac{3}{19}+\frac{3}{19}+\frac{3}{19}=0.9181\cdots.\end{aligned}$$

This is a contradiction.

Therefore $d_i=10$ for some i or $d=5$ in any case. This completes the proof.

REMARKS. If we investigate the equality $(*)$ more carefully, we can obtain the following results;

(i) $\varepsilon=-1$ does not occur.

(ii) If we assume that $d>5$, then the values of the character table of B_0 at $1, x, y$ and t are uniquely determined as follows;

	1	x	y	t
χ_0	1	1	1	1
χ_1	10	1	1	-2
χ_2	10	1	1	-2
χ_3	35	-1	-1	-1
χ_4	14	-1	2	2
χ_5	14	2	-1	2

LEMMA (4G). *Let θ be a faithful Z-valued character of G of degree n. Set $\{\theta(g) \mid g\in G,\ g\neq 1\}=\{a_1, \cdots, a_l\}$. Then $|G|$ divides $\prod_{i=1}^{l}(n-a_i)$.*

PROOF. Put $\hat{\theta}=\prod_{i=1}^{l}(\theta-a_i\chi_0)$. It is well-known that $\hat{\theta}$ is a generalized character. So, we have

$$\hat{\theta}(1)/|G|=(\hat{\theta}, \chi_0)_G\in Z.$$

Hence $|G|$ divides $\hat{\theta}(1)=\prod_{i=1}^{l}(n-a_i)$, the desired result.

Now we are ready to prove (4D).

PROOF OF (4D). At first we introduce some notations and convention. We denote by $B_0(p)$ the principal p-block of G and identify $B_0(p)$ and $\mathrm{Irr}(B_0(p))$ in this proof. Let S_p be a Sylow p-subgroup of G for $p\neq 3$. We use the notations in the proof of (4F). We use the results on the structure of p-blocks of defect one freely.

By (4E), we may assume that G is simple.

Case 1. $d=5$.

Since every algebraically conjugate of χ_4 lies in $B_0(3)$, it follows from the shape of $B_0(3)$-table in the proof of (4F) that χ_4 is Z-valued. Then we have that the primes dividing $|G|$ are 2, 3 and 5 and that $5^2\nmid|G|$. So,

$$|G|=2^a3^25.$$

Here we may quote a theorem of Brauer [4] to get the conclusion $G\cong A_6$, but we will prove it without using such classification theorems. By (4G), $|G|\,|\,9!$, therefore $3\leqq a\leqq 7$. Since $d_i\,|\,2^75$, $d_i\equiv\varepsilon_i \pmod 9$ and $1+\sum_{i=1}^{3}\varepsilon_i d_i+5=0$, we have that d_i's are 8, 8 and 10. If $\chi_i(1)=8$, then $\chi_i(t)=0$ by considering $\chi_i|_N$. Then the values of $B_0(3)$-table are uniquely determined as the following:

	1	x	y	t
χ_0	1	1	1	1
χ_1	8	-1	-1	0
χ_2	8	-1	-1	0
χ_3	10	1	1	-2
χ_4	5	-1	2	1
χ_5	5	2	-1	1

By Brauer [4] Proposition 1, $C_G(S_5)=S_5$. We see that $B_0(3)\cap B_0(5)\subseteqq\{\chi_0, \chi_1, \chi_2\}$. Using Brauer-Tuan [8] Lemma 3 (block separation), we have

$$\sum_{\chi\in B_0(3)\cap B_0(5)}\chi(1)\chi(x)\equiv 0 \pmod 5.$$

So,

$$B_0(3)\cap B_0(5)=\{\chi_0, \chi_1, \chi_2\}.$$

Because $\chi_1, \chi_2\in B_0(5)$ are exceptional, it follows that $|N_G(S_5):C_G(S_5)|=2$. The Brauer tree for $B_0(5)$ is

$$\underset{1}{\circ}\overset{1}{\text{——}}\underset{8}{\circ}\overset{7}{\text{——}}\underset{7}{\circ} \quad \text{or} \quad \underset{1}{\circ}\overset{1}{\text{——}}\underset{9}{\circ}\overset{8}{\text{——}}\underset{8}{\circ}\ .$$

As $7\nmid |G|$, the former does not occur. Then

$$B_0(5)=\{\chi_0, \chi_1, \chi_2, \phi\},$$

where ϕ is of degree 9. By [8] Lemma 2, $\chi_1, \chi_2\notin B_0(2)$. Therefore $B_0(2)\cap B_0(5)\subseteqq\{\chi_0, \phi\}$. By block separation, we have

$$B_0(2)\cap B_0(5)=\{\chi_0, \phi\} \qquad \text{and} \quad 8\equiv 0 \pmod{2^a}.$$

So $a=3$ and $|G|=2^3 3^2 5$. Hence $G\cong A_6$.

Case 2. $d>5$.

By the remarks after (4F), we have the following $B_0(3)$-table;

	1	x	y	t
χ_0	1	1	1	1
χ_1	10	1	1	-2
χ_2	10	1	1	-2
χ_3	35	-1	-1	-1
χ_4	14	-1	2	2
χ_5	14	2	-1	2

If p is a prime divisor of $|G|$, then $p\leqq 13$ and $11^2\nmid |G|$, $13^2\nmid |G|$, because χ_4 is Z-valued. Suppose that $13\,|\,|G|$. By Feit [9] Theorem 1, $C_G(S_{13})=S_{13}$. So, if $\chi\in \mathrm{Irr}(G)-B_0(13)$, then $13\,|\,\chi(1)$. Therefore $\chi_1, \chi_3\in B_0(13)$, a contradiction. Hence $13\nmid |G|$. Similarly $11\nmid |G|$. Therefore

$$|G|=2^a3^25^b7^c.$$

By (4G) we have $|G|\,|\,27!$, so $b\leqq 6$ and $c\leqq 3$. Because χ_4 is Z-valued, G has no element of order 7^2. If S_7 is extraspecial of order 7^3 or elementary abelian of order 7^3, then S_7 does not have a faithful Z-valued character of degree 14. Therefore $S_7\cong Z_7\times Z_7$ or Z_7. Hence every character in $B_0(7)$ is of height 0. So, $B_0(3)\cap B_0(7)\subseteqq\{\chi_0, \chi_1, \chi_2\}$. Take $z\in S_7$, $z\neq 1$. Because $9\nmid |C_G(z)|$, $[x^g, z]\neq 1$ for all $g\in G$ or $[y^g, z]\neq 1$ for all $g\in G$. It follows from [10] p. 174 Lemma 6.4 that

$$\sum_{\chi\in B_0(3)\cap B_0(7)}\chi(x)\chi(z)=0 \quad\text{or}\quad \sum_{\chi\in B_0(3)\cap B_0(7)}\chi(y)\chi(z)=0.$$

Hence $B_0(3)\cap B_0(7)\neq\{\chi_0\}$ and so $B_0(7)$ has a character of degree 10. By [8] Lemma 1, $C_G(S_7)=S_7$. By block separation,

$$\sum_{\chi\in B_0(3)\cap B_0(7)}\chi(1)\chi(x)\equiv 0 \pmod{7^c}.$$

Therefore

$$c=1 \quad\text{and}\quad B_0(3)\cap B_0(7)=\{\chi_0, \chi_1, \chi_2\}.$$

As $\chi_1, \chi_2\in B_0(7)$ are exceptional, $|N_G(S_7):C_G(S_7)|=3$. Degree equation:

$$1+\delta_1 x_1+\delta_2 x_2-10=0$$

has the unique solution $1+15-6-10=0$, because $3|x_i$, $x_i|2^a3^25^b$ and $x_i\equiv\delta_i$ (mod 7), where $\delta_i=\pm1$. So,

$$B_0(7)=\{\chi_0, \chi_1, \chi_2, \phi_1, \phi_2\}$$

where $\phi_1(1)=6$, $\phi_2(1)=15$.

Because ϕ_1 is Z-valued, G has no subgroup of order 5^2 and so $b=1$. By (4G), $|G|\,|\,11!$ and so $3\leqq a\leqq 8$. We have

$$B_0(5)\cap B_0(7)\subseteqq\{\chi_0, \phi_1\}.$$

By block separation,

$$B_0(5)\cap B_0(7)=\{\chi_0, \phi_1\}.$$

We get $C_G(S_5)=S_5$ because of [8] Lemma 1. By block separation,

$$B_0(3)\cap B_0(5)=\{\chi_0, \chi_4, \chi_5\}.$$

So,

$$B_0(5)=\{\chi_0, \chi_4, \chi_5, \phi_1, \phi_3\},$$

where $\phi_3(1)=21$.

Then we have $|N_G(S_5):C_G(S_5)|=4$. Hence $|N_G(S_5)|=20$ and $|N_G(S_7)|=21$. By Sylow's Theorem for $p=5$ and 7, we have $a=3$. So,

$$|G|=2^33^25\,7.$$

Hence $G\cong A_7$. This completes the proof of (4D).

PROPOSITION (4H). *Let G be of type Z_8. Suppose that $C_G(P)=P$ and $O_{3'}(G)=1$. Then one of the following holds;*

(i) *G is simple,*

(ii) *$G\cong PGL(2, 9)$,*

(iii) *$G\rhd P$.*

PROOF. Let M be a minimal normal subgroup of G. Then $P\cap M\neq1$ and so $P\subseteqq M$ because G is of type Z_8. By the Frattini argument, $G=MN_G(P)$. So $|G:M|=1, 2, 4$ or 8.

If $|G:M|=1$, then $G=M$ is simple. (i) holds in this case.

If $|G:M|=2$, then M is of type Z_4. By (4D), $M\cong A_6$ or A_7. So G is a subgroup of $\mathrm{Aut}(A_6)\cong P\Gamma L(2, 9)$ or $\mathrm{Aut}(A_7)\cong S_7$. Therefore $G\cong PGL(2, 9)$, (ii) holds in this case.

If $|G:M|=4$, then M is of type Z_2. By (4A), $M'\subsetneqq M$. This is a contradiction.

If $|G:M|=8$, then M is of type 1. Since $O_{3'}(G)=1$, $M=P$ by Burnside's Theorem. (iii) holds in this case.

The proof is complete.

REMARK. It seems that the case (i) of (4H) does not occur. But I have not proved it.

The proofs of the next three propositions are very similar to that of (4H). So we omit their proofs. Note that (4C) is needed in the proof of (4I).

PROPOSITION (4I). *Let G be of type D_8. Suppose that* $C_G(P)=P$ *and* $O_{3'}(G)=1$. *Then one of the following holds*:

(i) *G is simple,*

(ii) $G\cong S_6, S_7, A_5 \wr Z_2$ *or* $PSL(2,7) \wr Z_2$,

(iii) $G \triangleright P$.

PROPOSITION (4J). *Let G be of type Q_8. Suppose that* $C_G(P)=P$ *and* $O_{3'}(G)=1$. *Then one of the following holds*:

(i) *G is simple,*

(ii) $G\cong M_{10}$ *(one point stabilizer of the Mathieu group* M_{11}*),*

(iii) $G \triangleright P$.

PROPOSITION (4K). *Let G be of type SD_{16}. Suppose that* $C_G(P)=P$ *and* $O_{3'}(G)=1$. *Then one of the following holds*:

(i) *G is simple,*

(ii) *G has a simple subgroup of index* 2,

(iii) $G\cong P\Gamma L(2,9)$,

(iv) $G \triangleright P$.

REMARKS. In (4I), (4J) and (4K), each case actually occurs. $G\cong A_8$, $PSL(3,4)$, M_{11} or $\mathrm{Aut}(M_{22})$ is an example of (i) of (4I), (i) of (4J), (i) of (4K) or (ii) of (4K), respectively.

5. Some results for abelian defect groups

In this section, we state some results on blocks with an abelian defect group without proofs. Let B be a p-block of G with an abelian defect D and b be a root of B in $C_G(D)$. The following theorem treats of the simplest case in $e(B)=2$.

THEOREM (5A). *Suppose that p is an odd prime and $e(B)=2$. If $[D, T(b)]\cong Z_p$, then we have*

$$k(B)=\frac{p+3}{2p}|D| \quad and \quad l(B)=2.$$

First the author proved this theorem under the additional hypothesis $p\neq 7$, but A. Watanabe of Kumamoto University proved the theorem in the present form. She showed $l(B)\geqq 2$ by using the theory of lower defect groups, and this result together with column calculation methods that we used in the section 2 yields the theorem.

In case $e(B)=2$ and $T(b)$ acts fixed point freely on D, we have the following;

PROPOSITION (5B). *Let $p=5$ and D be elementary abelian of order 25. Suppose that $e(B)=2$ and $T(b)$ acts fixed point freely on D. Then we have*

$$k(B)=14 \quad and \quad l(B)=2.$$

This proposition was also proved by A. Watanabe. Our first version of the proposition contained the additional hypothesis that D is a Sylow 5-subgroup of G and $x^G\cap D=\{x, x^{-1}\}$ for all $x\in D$.

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