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A 2-local geometry for the Fischer group F_{24}

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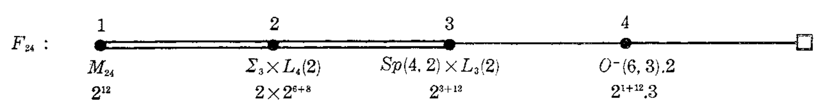
0. Introduction

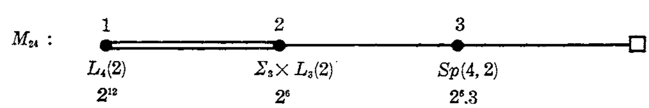
The notion of a 2-local geometry and its diagram for a sporadic simple group is introduced by Ronan-Smith [12]. A 2-local geometry is a group geometry defined by maximal 2-constrained, 2-local subgroups of the group (for the meaning of a group geometry, see the beginning of section 5). This was inspired by a building and a Dynkin diagram for a group of Lie type.

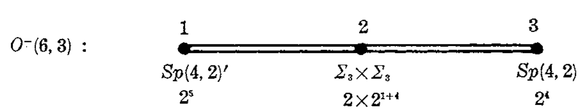
The purpose of this paper is to give a description of a 2-local geometry for the Fischer group F_{24}, which is associated with the 2-local diagram for F_{24}. This is obtained as a natural extention of the geometry for the Mathieu group M_{24}.

In Ronan-Smith [12], the following diagram is given for F_{24}.



In this diagram, we find those for M_{24} and $O^-(6,3)$ as subdiagrams ([9], [12]).





In these diagrams, each node ● denotes a maximal 2-constrained, 2-local subgroup of the group, and we write $\frac{A}{B}$ under the node to mean that the corresponding 2-local subgroup is an extension of the group B by the group A. If we remove the node above $\frac{A}{B}$, we get the diagram for the group A. (A square □ vanishes if the removed node is a neighbor to the square; otherwise it is treated as a node ●, except when we remove the first node in the diagram for F_{24}; the remainder diagram is for M_{24}.)

Now a geometry associated with the diagram for M_{24} is described by using the Steiner system $S(24, 8, 5)$, whose automorphism group is M_{24} ([12]). The set of vertices is the union of the sets of octads, trios, and sextets, which are denoted by the three nodes of the diagram (the first node for octads, the second for trios, and the third for sextets). Those are natural objects in the study of $S(24, 8, 5)$, and their stabilizers coincide with the corresponding 2-local subgroups (section 1). The subdiagram obtained by removing a node indicates the subgeometry given as the residue of a vertex belonging to the node (that is, the subgeometry consisting of vertices incident with the vertex). In the case of the first node, we get the geometry of points and lines of a projective 3-space which is indicated by ●——●——□. In the case of the third node, the Sp(4, 2)-generalized quadrangle is indicated by ●══●.

Now we will sketch roughly the geometry for F_{24}. The group F_{24} is generated by a unique class D of 3-transpositions (i.e. $x^2=1\neq x$ and $|xy|=2$ or 3, for any x,y $(x\neq y)$ in D). Let L be a maximal set of mutually commuting elements of D. Then we have $|L|=24$, and we can define the structure of the Steiner system $S(24, 8, 5)$ on L. Hence we can define octads, trios, and sextets on L, and, furthermore, some subgroups of $\langle L\rangle$ are naturally associated with them (section 4). Thus we have four classes of elementary abelian 2-subgroups of F_{24} ($\langle L\rangle$ and its subgroups defined by octads, trios, and sextets). Our geometry for F_{24} is obtained by defining the set of vertices and the incidence as the union of those classes of elementary abelian 2-subgroups and the inclusion relation, respectively. Then the stabilizer of each vertex is a maximal 2-constrained, 2-local subgroup which appears in the diagram (Theorem 4-10, 11). Moreover, as expressed by the diagram, we get the geometry for M_{24} and $O^-(6, 3)$ as residues in this geometry (Theorem 5-3).

The 2-local geometry for $O^-(6, 3)$ also appears in Kantor [9], from the point of view that $O^-(6, 3)$ is a subgroup of PSU(6, 2). But for our convenience, we will represent this geometry in a slightly different form, although Kantor's description is more natural than ours.

For structures of groups, we will use the following notation. We write $X\simeq B.A$ to mean that the group X is an extension of the group

B by the group A. For a prime number p, and integers n and m, p^n denotes an elementary abelian p-group of order p^n, and p^{n+m} denotes a special p-group P such that $P'=Z(P)=\Phi(P)\simeq p^n$, and $P/P'\simeq p^m$ (where P', $Z(P)$, and $\Phi(P)$ denote the commutator subgroup, the center, and the Frattini subgroup of P, respectively). $Sp(4,2)$, $L_3(2)$, etc, denote the groups of Lie type as usual, except for $O^-(6,3)$ which is defined in section 3. Σ_n and A_n denote the symmetric and the alternating group respectively.

In addition, for subsets X, Y of a group, we set $X^Y=\{x^y | x\in X,\ y\in Y\}$. In the case of $X=\{x\}$, we write x^Y instead of $\{x\}^Y$.

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1. The Steiner system $S(24, 8, 5)$ and the Mathieu group M_{24}

The *Steiner system* $S(24, 8, 5)$ is the pair $(\Omega, \mathscr{B})$ of a 24-element set Ω and a family $\mathscr{B}$ of 8-element subsets of Ω such that any 5-element subset of Ω is contained in just one member of $\mathscr{B}$.

Witt proved that such a system is unique up to isomorphism.

Members of $\mathscr{B}$ are called (special) *octads*. The following lemma is easily proved (see Conway [5], Curtis [6]).

LEMMA 1-1. *Let $(\Omega, \mathscr{B})$ be the Steiner system $S(24, 8, 5)$, and $\mathcal{O}$ be an octad. Then we have*

(1) *There exist 759 octads.*

(2) *For any two elements a, b of $\mathcal{O}$, there exists an octad $\mathcal{O}'$ with the property $\mathcal{O}\cap\mathcal{O}'=\{a, b\}$. Moreover there exist exactly 77 octads containing a and b.*

(3) *For any three elements of $\Omega-\mathcal{O}$, there exists an octad $\mathcal{O}'$ containing them with the property $|\mathcal{O}\cap\mathcal{O}'|=4$.*

Let $(\Omega, \mathscr{B})$ be the Steiner system $S(24, 8, 5)$. We regard the set $\mathscr{P}(\Omega)$ of all subsets of Ω as a 24-dimensional vector space over F_2 by defining the sum $X+Y$ of two subsets X, Y of Ω as their symmetric difference $(X\cup Y)-(X\cap Y)$. A subspace C of $\mathscr{P}(\Omega)$ is the space spanned by all members of $\mathscr{B}$.

THEOREM 1-2. (1) *The space* $\mathcal{C}$ *is* 12-*dimensional.*

(2) *For* $X \in \mathcal{C}$, $|X|=0, 8, 12, 16$, *or* 24. *Moreover* $|X|=8$, *if and only if* $X \in \mathcal{B}$.

PROOF. See Conway [5].

DEFINITION. (1) A *trio* is a triplet of mutually disjoint octads.

(2) A *sextet* is a system of mutually disjoint six tetrads (i.e. 4-element subsets of Ω) such that the union of any two of them is an octad.

(3) A sextet $\mathcal{S}$ is called a *refinement* of a trio $\mathcal{T}$, if each octad of $\mathcal{T}$ is the union of two tetrads of $\mathcal{S}$.

LEMMA 1-3. (1) *Each tetrad is contained in exactly one sextet. If* $\{T_1, \cdots, T_6\}$ *is a sextet, then the octads containing* T_1 *are just* $T_1 \cup T_2, \cdots, T_1 \cup T_6$.

(2) *Each trio has exactly seven refinements.*

(3) *Let* $\mathcal{T}$ *be a trio, and* $\{\mathcal{S}_1, \cdots, \mathcal{S}_7\}$ *be the set of all refinements of* $\mathcal{T}$. *If an octad* $\mathcal{O}$ *has the property that* $\mathcal{O}$ *is the union of two tetrads of* $\mathcal{S}_i$ *for each* $i \in \{1, \cdots, 7\}$, *then* $\mathcal{O}$ *is one of the three octads of* $\mathcal{T}$.

(4) *There exist* 3795 *trios.*

PROOF. See Curtis [6].

The *Mathieu group* M_{24} is defined as the full automorphism group of $(\Omega, \mathcal{B})$ (i.e. the group of all permutations on Ω that stabilize $\mathcal{B}$ globally).

THEOREM 1-4. (1) M_{24} *acts* 5-*transitively on* Ω, *and transitively on the set of octads, trios, and sextets, respectively.*

(2) *Let* N_1 *be the stabilizer of an octad, and* K_1 *be the kernel of the action of* N_1 *on the eight elements of that octad. Then* N_1/K_1 *is isomorphic to the alternating group* A_8.

(3) *Let* N_2 *be the stabilizer of a trio, and* K_2 *be the kernel of the action of* N_2 *on the seven refinements of that trio. Then* N_2/K_2 *is isomorphic to the projective linear group* $L_3(2)$.

PROOF. See Curtis [6].

REMARK. Let N_3 be the stabilizer of a sextet. The structures of the N_i $(i=1, 2, 3)$ are as follows (see Curtis [6]).

$$N_1 \simeq 2^4.\ A_8.$$
$$N_2 \simeq 2^6.\ (\Sigma_3 \times L_3(2)).$$
$$N_3 \simeq 2^6.\ 3.\ \Sigma_6.$$

2. 3-transpositions

DEFINITION. Let G be a finite group, and D be a set of involutions of G. If D satisfies the conditions

(i) $G=\langle D\rangle$, and $D^G=D$,

(ii) if $x, y\in D$, then the order of xy is 1, 2, or 3,

then D is a set of 3-*transpositions* of G.

We set $D_d=\{x\in D|dx=xd\neq 1\}$ for $d\in D$, and $\mathcal{L}=\mathcal{L}_G=\{$maximal sets of mutually commuting elements of $D\}$. Then members of $\mathcal{L}$ are given as the intersections of D with Sylow 2-subgroups of G. In particular, G acts transitively on $\mathcal{L}$. For $L\in\mathcal{L}$, we set $w(G)=|L|$, and it is called the (D-) *width* of G. Subgroups generated by elements of D are called D-*subgroups* of G.

The following three lemmas are in Fischer [7].

LEMMA 2-1. *Let D be a set of 3-transpositions of G.*

(1) *If N is a proper normal subgroup of G, then DN/N is a set of 3-transpositions of G/N.*

(2) *Let a, b, and c be distinct commuting elements of D. Then we have $C_D(ab)=C_D(a, b)$ and $C_D(abc)=C_D(a, b, c)$.*

LEMMA 2-2. *Let D be a conjugacy class of 3-transpositions of G, and $d\in D$. Then we have,*

(1) $d^{O_2(G)}=dO_2(G)\cap D=\{e\in D|C_D(d)=C_D(e)\}$.

(2) $d^{O_3(G)}=dO_3(G)\cap D=\{e\in D|D_d=D_e\}$.

(3) *Set* $V=\langle ed|C_D(d)=C_D(e),\ d, e\in D\rangle$. *Then* $V\leq O_2(G)$, *and* $O_2(G/V)\leq Z(G/V)$.

The number $|d^{O_2(G)}|$ is called the 2-*depth* of G. In the remainder of this section, we assume D and G are as in Lemma 2-2.

LEMMA 2-3. *Suppose G' is simple. Let a, b, c, and d be elements of D with the properties $a\neq b$, $c\neq d$, and $ab=cd$. Then $\{a, b\}=\{c, d\}$.*

LEMMA 2-4. *Let N be a proper normal 2-subgroup of G. Set $\bar{D}=DN/N$ and $\bar{d}$ to be the image of $d\in D$ in $\bar{D}$. Then the following conditions are equivalent for d, $e\in D$.*

(i) $C_D(d)=C_D(e)$,

(ii) $C_{\bar{D}}(\bar{d})=C_{\bar{D}}(\bar{e})$.

PROOF. Easy.

LEMMA 2-5. *Suppose $G \neq Z(G)$. Set $\bar{D}=DZ(G)/Z(G)$, and $\bar{d}$ to be the image of $d \in D$ in $\bar{D}$. Then $|D|=|\bar{D}|$. Moreover, for $d, e \in D$, d commutes with e if and only if $\bar{d}$ commutes with $\bar{e}$.*

PROOF. Suppose $de \in Z(G)$ for $d, e \in D$. Since $d=d^{de}=d^e$, d commutes with e. Hence if $d \neq e$, we have $D \leq C_D(de)=C_D(d, e)$ by Lemma 2-1(2). Since D is a conjugacy class of G, we have $D \leq C_D(D)$, a contradiction. Hence $d=e$, and so $|D|=|\bar{D}|$.

Suppose $\bar{d}$ commutes with $\bar{e}$, but that d does not commute with e. Then $(de)^3=1$ and $(de)^2 \in Z(G)$. Hence $ed=(de)^2 \in Z(G)$, and this contradicts $|D|=|\bar{D}|$. Thus the lemma is proved.

LEMMA 2-6. *Suppose $G/Z(G)$ is isomorphic to the symmetric group Σ_n. Then $G \simeq \Sigma_n$.*

PROOF. We can choose distinct elements $d_1, \cdots, d_{n-1} \in D$ such that $(d_i d_j)^3=1$ if $|i-j|=1$, and $(d_i d_j)^2=1$ if $|i-j|>1$. Then $K=\langle d_1, \cdots, d_{n-1}\rangle \simeq \Sigma_n$. By Lemma 2-5, we have $|D|=|DZ(G)/Z(G)|=n(n-1)=|K \cap D|$. Hence we have $G=K$ as required.

LEMMA 2-7. *Suppose the 2-depth of G equals 4. Let $a \in D$, and set $a^{O_2(G)}=\{a, b, c, d\}$. Then we have $abcd \in Z(G)$. Moreover for any $x \in D$, the product of the elements of $x^{O_2(G)}$ equals $abcd$.*

PROOF. If $x \in C_D(a)$, then $x \in C_D(a, b, c, d)$ by Lemma 2-2, and so $x \in C_D(abcd)$. Let $x \notin C_D(a)$. Then $x \notin C_D(a) \cup C_D(b) \cup C_D(c) \cup C_D(d)$. Since ab, ac, ad belong to $O_2(G)$ by Lemma 2-2, we have x^{ab}, x^{ac}, $x^{ad} \in x^{O_2(G)}$. If $x^{ab}=x^{ac}$, then $x^{bc}=x$, and so $x \in C_D(b, c)$, a contradiction. Thus we have $x^{O_2(G)}=\{x, x^{ab}, x^{ac}, x^{ad}\}$, since the 2-depth of G equals 4.

Now $x^{cd} \in x^{O_2(G)}$, too. If $x^{cd} \in \{x, x^{ac}, x^{ad}\}$, we have a contradiction in the similar way. Thus $x^{cd}=x^{ab}$, and so $x \in C_D(abcd)$. Thus $abcd \in Z(\langle D\rangle)=Z(G)$. Since D is a conjugacy class of G, the rest of this lemma is immediate.

3. The orthogonal group $O^-(6, 3)$

Let V be a 6-dimensional vector space over F_3, and $(\ ,\ )$ be a non-degenerate symmetric bilinear form on V with Witt index 2. Set $V_+=\{v \in V \mid (v, v)=1\}$, and E^* be the set of reflections

$$x \longmapsto x+(x, a)a \qquad (x \in V)$$

for all $a\in V_{+}$. We define $G=O^{-}(6,3)=\langle E^{*}\rangle/Z(\langle E^{*}\rangle)=\langle E\rangle$, where $E=E^{*}Z(\langle E^{*}\rangle)/Z(\langle E^{*}\rangle)$. Then E is a conjugacy class of 3-transpositions of G, with $|E|=126$ and $w(G)=6$. Moreover we have $|G:G'|=2$, and G' is simple. These facts and the following lemma are in Fischer [8].

LEMMA 3-1. (1) *Let e and f be distinct commuting elements of E. Then $|E_{e}|=45$, $|E_{e}\cap E_{f}|=12$, $\langle E_{e}\rangle/\langle e\rangle\simeq \mathrm{PSU}(4,2)$, and $\langle E_{e}\cap E_{f}\rangle\simeq W(D_{4})$, where $W(D_{4})$ is the Weyl group of type D_{4}.*

(2) *Let $L\in\mathcal{L}_{G}$. Then $N_{G}(L)/C_{G}(L)\simeq A_{6}$, and $C_{G}(L)=\langle L\rangle\simeq 2^{5}$. Moreover $N_{G}(L)$ acts 4-transitively on the six elements of L.*

LEMMA 3-2. (1) *G has a unique class of E-subgroups isomorphic to $W^{*}(D_{6})=W(D_{6})/Z(W(D_{6}))$.*

(2) *Let $L\in\mathcal{L}_{G}$, and $L=\{a_{1},a_{1}'\}\cup\{a_{2},a_{2}'\}\cup\{a_{3},a_{3}'\}$ be a partition of L into 2-element subsets of L. Then there exists just one E-subgroup W as in (1) such that $W>L$ and $a_{i}^{O_{2}(W)}=\{a_{i},a_{i}'\}$ for each $i\in\{1,2,3\}$.*

(3) *Let a and b be distinct commuting elements of E, and W be an E-subgroup as in (1) such that W contains a and b, and $a^{O_{2}(W)}=\{a,b\}$. Then exactly three members of $\mathcal{L}_{G}$ contain a and b, and they are all contained in W.*

PROOF. The existence of an E-subgroup W as in (1) is proved in [8]. Let $L=\{a_{1},a_{1}'\}\cup\{a_{2},a_{2}'\}\cup\{a_{3},a_{3}'\}\in\mathcal{L}_{G}$, and W, W_{0} be E-subgroups isomorphic to $W^{*}(D_{6})$ with the property in (2). By Lemma 3-1(1), and the structure of $W(D_{6})$, we have $W=\langle E_{a_{1}}\cap E_{a_{1}'},\ E_{a_{2}}\cap E_{a_{2}'}\rangle=W_{0}$. By Lemma 3-1(2) and the fact that G acts transitively on $\mathcal{L}_{G}$, we obtain (1) and (2). Suppose $L'\in\mathcal{L}_{G}$ contains a_{1},a_{1}'. Then $L'-\{a_{1},a_{1}'\}$ is contained in $E_{a_{1}}\cap E_{a_{1}'}$, and so we obtain (3).

PROPOSITION 3-3. *Let W be an E-subgroup isomorphic to $W^{*}(D_{6})$. And let $L\in\mathcal{L}_{G}$, and a,b be distinct commuting elements of E.*

(1) *$N_{G}(\langle a,b\rangle)\simeq(2\times 2^{1+4})\cdot(\Sigma_{3}\times\Sigma_{3})$.*

(2) *W, $N_{G}(\langle L\rangle)$, and $N_{G}(\langle a,b\rangle)$ are maximal subgroups of G.*

PROOF. These are derived from calculations similar to those in section 4. So we omit the proof.

4. The Fischer group F_{24}

Let $G=F_{24}$ (or $M(24)$ in Fischer [8]). There is a unique conjugacy class D of 3-transpositions of G. The following facts are in [8].

THEOREM 4-1. (1) $|G|=2^{22}\cdot 3^{16}\cdot 5^2\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29$, $|D|=306936$, *and* $w(G)=24$.

(2) $|G:G'|=2$, *and* G' *is simple.*

(3) *For* $L\in\mathcal{L}=\mathcal{L}_G$, $N_G(L)/C_G(L)\simeq M_{24}$, *and* $C_G(L)=\langle L\rangle\simeq 2^{12}$. *Moreover* $N_G(L)$ *acts* 5*-transitively on the elements of* L.

(4) *Let* a, b, *and* c *be distinct commuting elements of* D. *Then we have* $|D_a|=31671$, $|D_a\cap D_b|=3510$, *and* $|D_a\cap D_b\cap D_c|=693$. *Moreover* $\langle D_a\rangle/\langle a\rangle\simeq F_{23}$, $\langle D_a\cap D_b\rangle/\langle a, b\rangle\simeq F_{22}$, *and* $\langle D_a\cap D_b\cap D_c\rangle/\langle a, b, c\rangle\simeq \mathrm{PSU}(6,2)$.

The unitary group PSU(6, 2) is generated by a unique conjugacy class E of 3-transpositions, which is the class of unitary transvections. The following facts are found in [8], or derived from direct calculations for unitary transvections (See the remark after the lemma).

LEMMA 4-2. *Let* x, y *be distinct commuting elements of* E.

(1) $|E_x|=180$, *and* $\langle E_x\rangle/O_2(\langle E_x\rangle)\simeq\mathrm{PSU}(4,2)$.

(2) *The* 2*-depth of* $\langle E_x\rangle$ *equals* 4.

(3) *Set* $[x:y]=\{x\}\cup y^{O_2(\langle E_x\rangle)}$. *Then the product of the elements of* $[x:y]$ *equals* 1.

(4) *Let* $L\in\mathcal{L}_{\langle E\rangle}$, *and* $x\in E-L$. *Then we have* $|L\cap E_x|=5$. *Moreover* $L\cap E_x=[a:b]$ *for all* $a, b\in L\cap E_x$, $a\neq b$.

(5) *Set* $X=C_E([x:y])-[x:y]$. *Then we have* $\langle X\rangle/O_2(\langle X\rangle)\simeq\Sigma_3$, *and the* 2*-depth of* $\langle X\rangle$ *equals* 16. *In particular,* $|X|=48$.

REMARK. $\langle E_x\rangle$ is a maximal parabolic subgroup of $\langle E\rangle\simeq\mathrm{PSU}(6,2)$. A member of $\mathcal{L}_{\langle E\rangle}$ is the set of transvections with respect to vectors in a maximal totally isotropic plane of the related unitary space. Similarly $[x:y]$ corresponds to a totally isotropic line.

Now we return to $G=F_{24}=\langle D\rangle$. We can define (special) octads which are subsets of $L\in\mathcal{L}=\mathcal{L}_G$.

LEMMA 4-3. *Let* $L\in\mathcal{L}$, *and* $a_1,\cdots,a_r$ *be distinct elements of* L *with the property* $a_1a_2\cdots a_r=1$. *Then* $r\geq 8$.

PROOF. Clearly $r\neq 1,2$. If r is odd, we have $a_1=a_2\cdots a_r\in G'$. This contradicts Theorem 4-1(2). Lemma 2-3 implies $r\neq 4$. Suppose $r=6$. By Theorem 4-1(4), there exists an element g of $N_G(L)$ with the properties $a_i^g=a_i$ if $i\in\{1,2,3,4\}$, and $a_5^g\notin\{a_1,\cdots,a_6\}$. Hence we have $a_1\cdots a_6=1=a_1a_2a_3a_4a_5^ga_6^g$ and so $a_5a_6=a_5^ga_6^g$. This contradicts Lemma 2-3. Thus $r\geq 8$ as required.

Let $a_1, \cdots, a_5$ be distinct commuting elements of D. By Theorem 4-1(4) and Lemma 4-2(2), we can set

$$\{a_1, \cdots, a_4\} \cup a_5^{O_2(\langle D_{a_1} \cap \cdots \cap D_{a_4} \rangle)} = \{a_1, \cdots, a_8\}.$$

By Lemma 4-2(3), we have $a_4 a_5 a_6 a_7 a_8 \in \langle a_1, a_2, a_3 \rangle$. Hence $a_1 \cdots a_8 = 1$ by Lemma 4-3.

DEFINITION. Let $a_1, \cdots, a_8$ be distinct commuting elements of D. The set $\{a_1, \cdots, a_8\}$ is a (special) *octad* if $a_1 a_2 \cdots a_8 = 1$.

LEMMA 4-4. *Let $a_1, \cdots, a_5$ be distinct commuting elements of D. Then $\{a_1, \cdots, a_5\}$ is contained in just one octad.*

PROOF. We have shown $\{a_1, \cdots, a_4\} \cup a_5^{O_2(\langle D_{a_1} \cap \cdots \cap D_{a_4} \rangle)}$ is an octad. Let $\{a_1, \cdots, a_5, x, y, z\}$ and $\{a_1, \cdots, a_5, u, v, w\}$ be octads. Then we have $a_1 \cdots a_5 xyz = 1 = a_1 \cdots a_5 uvw$, and so $xyz = uvw$. Since

$$x, y, z \in C_D(xyz) = C_D(uvw) = C_D(u, v, w),$$

there exists $L \in \mathcal{L}$ containing x, y, z, u, v, w. By Lemma 4-3, we have $\{x, y, z\} = \{u, v, w\}$ as required.

Let $\mathcal{O}$ be an octad. There exists $L \in \mathcal{L}$ containing $\mathcal{O}$. Thus by Theorem 4-1(3) and Lemma 4-4, for any five elements of L, the octad containing them is a subset of L. Let $\mathcal{B}(L)$ be the set of all octads contained in L. Lemma 4-4 implies that $(L, \mathcal{B}(L))$ is the Steiner system $S(24, 8, 5)$. So we can define a trio of L, a sextet of L, and refinements of a trio of L in the same manner as in section 1. We regard the set $\mathcal{P}(L)$ as a 24-dimensional vector space over F_2 as in section 1. Moreover a subspace $\mathcal{C}(L)$ is the space spanned by all members of $\mathcal{B}(L)$. The facts $|C(L)| = 2^{12}$, and $|\langle L \rangle| = 2^{12}$ imply the following lemma.

LEMMA 4-5. *Let $L \in \mathcal{L}$, and $\{a_1, \cdots, a_r\} \leq L$. Then the following conditions are equivalent.*

(i) $a_1 \cdots a_r = 1$,
(ii) $\{a_1, \cdots, a_r\} \in \mathcal{C}(L)$.

LEMMA 4-6. *Let $L \in \mathcal{L}$, and $x \in D - L$. Then one of the following holds.*

(1) *$D_x \cap L$ is an octad (for 759×2^5 elements of $D - L$).*

(2) $|D_x \cap L|=2$ *(for* $\binom{24}{2}\times 2^{10}$ *elements of* $D-L$*).*

PROOF. Let $\mathcal{O}$ be an octad of L. By Lemma 4-2(5), $2^5(=48-(24-8))$ elements of $D-L$ commute with all elements of $\mathcal{O}$. Let x be such an element. By Lemma 4-2(4), $L\cap D_x=\mathcal{O}$. By Lemma 1-1, there exist 759 octads in L. Thus 759×2^5 elements have the property in (1). Let a, b be distinct elements of L. If $D_y\cap L\supsetneq\{a, b\}$ for $y\in D-L$, then $D_y\cap L$ is an octad by Lemma 4-2(4). Since there exist 77 octads in L containing a, b by Lemma 1-1, the number of elements x of $D-L$ with the property $D_x\cap L=\{a, b\}$ equals $|D_a\cap D_b|-(77\times 2^5+22)=2^{10}$. Since $|D|=306936=24+759\times 2^5+\binom{24}{2}\times 2^{10}$, the proof of this lemma is complete.

Let $L\in\mathcal{L}$, and a, b, c, and d be distinct elements of L. By Aschbacher-Seitz [1], any involution in $\langle L\rangle$ is fused to a, ab, abc, or $abcd$ in $N_G(L)$, and, furthermore, they are representatives for the conjugacy classes of involutions in G.

The involution $abcd$ is related to a sextet of L. Set $T_1=\{a, b, c, d\}$, and let $\{T_1, \cdots, T_6\}$ be a sextet of L. Since $T_1\cup T_i(i\neq 1)$ is an octad, the product of the elements of T_i equals $abcd$. Let $x, y, z, u\in L$, and suppose $xyzu=abcd$. Then either $\{x, y, z, u\}=\{a, b, c, d\}$, or $\{a, b, c, d, x, y, z, u\}$ is an octad. By Lemma 1-3(1), we conclude $\{x, y, z, u\}=T_i$ for some $i\in\{1, \cdots, 6\}$. Hence the product of four elements of L is in one-to-one correspondence to a sextet of L. The product of the elements of T_i is called the involution defined by the sextet $\{T_1, \cdots, T_6\}$.

Now we define four classes $\mathcal{P}_i$ $(i=1, 2, 3, 4)$ of elementary abelian 2-subgroups of G as follows.

$\mathcal{P}_1=\{\langle L\rangle | L\in\mathcal{L}\}$,
$\mathcal{P}_2=\{\langle\mathcal{O}\rangle | \mathcal{O}$ is an octad$\}$,
$\mathcal{P}_3=\{\langle\mathcal{O}_1\rangle\cap\langle\mathcal{O}_2\rangle\cap\langle\mathcal{O}_3\rangle | \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$ is a trio of some $L\in\mathcal{L}\}$,
$\mathcal{P}_4=\{\langle T_1\rangle\cap\cdots\cap\langle T_6\rangle | \{T_1, \cdots, T_6\}$ is a sextet of some $L\in\mathcal{L}\}$.

LEMMA 4-7. *Let* $V_i\in\mathcal{P}_i(i=1, 2, 3, 4)$. *Then we have,*

(1) $|V_1|=2^{12}$, $|V_2|=2^7$, $|V_3|=2^3$, *and* $|V_4|=2$.

(2) $V_3^\#$ $(=V_3-\{1\})$ *consists of the seven involutions defined by the refinements of the trio defining* V_3.

(3) $V_4^\#$ *consists of the involution defined by the sextet defining* V_4.

PROOF. $|V_1|$ is in Theorem 4-1. Set $V_2=\langle\mathcal{O}\rangle$ where $\mathcal{O}$ is an octad.

For a subset $\{a_1, \cdots, a_r\}$ of $\mathcal{O}$, the product $a_1 \cdots a_r$ equals 1 if and only if $\{a_1, \cdots, a_r\} = \mathcal{O}$ by Theorem 1-2(2) and Lemma 4-5. Hence $|V_2| = 2^7$.

Set $V_3 = \langle \mathcal{O}_1 \rangle \cap \langle \mathcal{O}_2 \rangle \cap \langle \mathcal{O}_3 \rangle$ where $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$ is a trio of $L \in \mathcal{L}$. Fix $a \in \mathcal{O}_3$. By Lemma 1-1(2), for any element b of $\mathcal{O}_3 - \{a\}$, there exist $x_1, \cdots, x_6 \in \mathcal{O}_1 \cup \mathcal{O}_2$ such that $\{x_1, \cdots, x_6, a, b\}$ is an octad. Thus we have $b = x_1 \cdots x_6 a \in \langle \mathcal{O}_1, \mathcal{O}_2, a \rangle$, and so $\langle \mathcal{O}_1, \mathcal{O}_2, a \rangle = \langle L \rangle$. Hence $|\langle \mathcal{O}_1, \mathcal{O}_2 \rangle| \geq 2^{11}$, and $|V_3| \leq |\langle \mathcal{O}_1 \rangle \cap \langle \mathcal{O}_2 \rangle| \leq (2^7)^2/2^{11} = 2^3$. On the other hand, the seven involutions defined by the refinements of the trio $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\}$ are contained in V_3. Hence $|V_3| = 2^3$, and (3) is obtained.

Set $V_4 = \langle T_1 \rangle \cap \cdots \cap \langle T_6 \rangle$ where $\{T_1, \cdots, T_6\}$ is a sextet of $L \in \mathcal{L}$. Since $|\langle T_1, T_2 \rangle| = 2^7$ and $|\langle T_1 \rangle| = 2^4$, we have $|\langle T_1 \rangle \cap \langle T_2 \rangle| = (2^4)^2/2^7 = 2$. Hence $|V_4| = 2$, and (3) is obtained.

LEMMA 4-8. *Let $V \in \mathcal{P}_4$, and set $X = C_D(V)$, and $\bar{X} = XO_2(\langle X \rangle)/O_2(\langle X \rangle)$. Then we have*

(1) $\langle \bar{X} \rangle / Z(\langle \bar{X} \rangle) \simeq O^-(6, 3)$.

(2) *The 2-depth of $\langle X \rangle$ equals 4.*

(3) *Let $a, b, c,$ and d be distinct commuting elements of X. Then $V = \langle abcd \rangle$ if and only if $a^{O_2(\langle X \rangle)} = \{a, b, c, d\}$.*

PROOF. See Aschbacher-Seitz [1].

LEMMA 4-9. *Let $V \in \mathcal{P}_3$, and set $Y = C_D(V)$, and $\bar{Y} = YO_2(\langle Y \rangle)/O_2(\langle Y \rangle)$. For $y \in Y$, we set $\mathcal{O}(y) = y^{O_2(\langle Y \rangle)}$. Then we have*

(1) $\langle \bar{Y} \rangle \simeq \Sigma_6$.

(2) *For $y \in Y$, $\mathcal{O}(y)$ is an octad.*

(3) *Let $L \in \mathcal{L}_{\langle Y \rangle}$. There exist $a, b, c \in L$ such that $L = \mathcal{O}(a) \cup \mathcal{O}(b) \cup \mathcal{O}(c)$. In particular, $\{\mathcal{O}(a), \mathcal{O}(b), \mathcal{O}(c)\}$ is a trio of L. Moreover $V = \langle \mathcal{O}(a) \rangle \cap \langle \mathcal{O}(b) \rangle \cap \langle \mathcal{O}(c) \rangle$.*

PROOF. Set $V = \langle \mathcal{O}_1 \rangle \cap \langle \mathcal{O}_2 \rangle \cap \langle \mathcal{O}_3 \rangle$, $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 = L \in \mathcal{L}$, and $\mathcal{O}_1 = \{a_1, \cdots, a_8\}$. We may assume $V = \langle a_1a_2a_3a_4, a_1a_2a_5a_6, a_1a_3a_5a_7 \rangle$ (see Curtis [6]). Let $x \in Y$, and $x \in D_{a_1}$. Since $x = x^{a_1a_2a_3a_4} = x^{a_2a_3a_4}$, we have $x \in C_D(a_2, a_3, a_4)$. By the same arguments, we have $x \in C_D(a_1, \cdots, a_8)$. Thus $Y \cap C_D(a_1) = Y \cap C_D(a_1, \cdots, a_8) = C_D(a_1, \cdots, a_8)$.

Set $Y_0 = Y \cap C_D(a_1) - \{a_1, \cdots, a_8\}$. By Lemma 4-2(5), $\langle Y_0 \rangle / O_2(\langle Y_0 \rangle) \simeq \Sigma_3$, and $|b^{O_2(\langle Y_0 \rangle)}| = 16$ for $b \in Y_0$. In particular, Y_0 is a conjugacy class of $\langle Y_0 \rangle$.

We prove that Y is a conjugacy class of $\langle Y \rangle$. It is sufficient to show that each a_i is conjugate to an element of $\mathcal{O}_2$ in $\langle Y \rangle$. Let $b \in \mathcal{O}_3$, and set $Y_1 = Y \cap C_D(b) - \mathcal{O}_3$. Then Y_1 is a conjugacy class of $\langle Y_1 \rangle$. Since $Y_1 > \mathcal{O}_1, \mathcal{O}_2$, we have the required statement.

Hence by Lemma 2-2, we have $\mathcal{O}(a_1) \geq \mathcal{O}_1$. If $b \in \mathcal{O}(a_1) - \mathcal{O}_1$, then $b \in Y \cap C_D(a_1) = Y \cap C_D(b)$, and so $b \in Z(\langle Y_0 \rangle)$. This contradicts the fact that Y_0 is a cojugacy class of $\langle Y_0 \rangle$. Hence $\mathcal{O}(a_1) = a_1^{O_2(\langle Y \rangle)} = \mathcal{O}_1$. In particular, the 2-depth of $\langle Y \rangle$ is 8, and so the 2-depth of $\langle Y_0 \rangle O_2(\langle Y \rangle)/O_2(\langle Y \rangle)$ is 2. Thus we have $\langle Y_0 \rangle O_2(\langle Y \rangle)/O_2(\langle Y \rangle) \simeq \Sigma_4$, and so $\langle Y \rangle O_2(\langle Y \rangle)/O_2(\langle Y \rangle) \simeq \Sigma_6$ by Fischer's main theorem in [7].

Now $\mathcal{O}(a_1)$ is the octad $\mathcal{O}_1$. Since Y is a conjugacy class of $\langle Y \rangle$, $\mathcal{O}(y)$ is an octad for any $y \in Y$. Furthermore $L = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3 \in \mathcal{L}_{\langle Y \rangle}$. Since any member of $\mathcal{L}_{\langle Y \rangle}$ is conjugate to L in $\langle Y \rangle$, we obtain (3).

THEOREM 4-10. *Let $V_i \in \mathcal{P}_i$ $(i=1, 2, 3, 4)$. Then we have*

(1) $N_G(V_1) \simeq 2^{12}. M_{24}$.

(2) $N_G(V_2) \simeq (2 \times 2^{6+8}). (\Sigma_3 \times A_8)$.

(3) $N_G(V_3) \simeq 2^{3+12}. (\Sigma_6 \times L_3(2))$.

(4) $N_G(V_4) \simeq 2^{1+12}. 3.0^-(6, 3). 2$.

PROOF. Statement (1) is in [8], and (4) is in [1].

(2) Set $V_2 = \langle \mathcal{O} \rangle$ where $\mathcal{O} = \{a_1, \cdots, a_8\}$ is an octad. Moreover we set $X = C_D(\mathcal{O})$, $X_0 = X - \mathcal{O}$, $Q = O_2(\langle X \rangle)$, and $Q_0 = O_2(\langle X_0 \rangle)$. Let $x \in X_0$. Then $\langle X_0 \rangle / Q_0 \simeq \Sigma_3$, and $|x^{Q_0}| = 16$ by Lemma 4-2(5). Thus we can choose $x = x_1$, x_2, and x_3 in X_0 such that $X_0 = x_1^{Q_0} \cup x_2^{Q_0} \cup x_3^{Q_0}$. Set $X_i = x_i^{Q_0}$, and $L_i = \mathcal{O} \cup X_i$ for $i \in \{1, 2, 3\}$. Then we have $L_i \in \mathcal{L}$ for each i.

Now $\langle X \rangle = \langle X_0, \mathcal{O} \rangle$. But for each $j \in \{2, \cdots, 8\}$, there exist $y_1, \cdots, y_6 \in X$ such that $\{a_1, a_j, y_1, \cdots, y_6\}$ is an octad by Lemma 1-1(2). Thus $a_j = a_1 y_1 \cdots y_6 \in \langle X_0, a_1 \rangle$, and so we have $\langle X \rangle = \langle X_0, a_1 \rangle$. Since $\langle X \rangle \triangleright Q_0$, and $a_1 \in Q$, we have $Q = \langle Q_0, a_1 \rangle$, and $\langle X \rangle / Q \simeq \Sigma_3$.

Set $Q_1 = \langle bb' | b, b' \in L_i - \mathcal{O}\ (i=1, 2, 3) \rangle$. By Lemma 2-2, we have $O_2(\langle X_0 \rangle / Q_1) \leq Z(\langle X_0 \rangle / Q_1)$. Since $(\langle X_0 \rangle / Q_1)/Z(\langle X_0 \rangle / Q_1) \simeq \Sigma_3$, we have $\langle X_0 \rangle / Q_1 \simeq \Sigma_3$ by Lemma 2-4. Thus $Q_0 = Q_1$. Since $G' > Q_0$, and $a_1 \notin G'$, we have $Q = \langle a_1 \rangle \times Q_0$.

Since X_0 is a conjugacy class of $\langle X_0 \rangle$, X_0 is a conjugacy class of $\langle X \rangle$. Since $|X_0| = 3 \times 16$, we have $|\langle X \rangle : C_{\langle X \rangle}(x)| = 2^4 \cdot 3$. On the other hand, $C_{\langle X \rangle}(x) \geq \langle L_1 \rangle \simeq 2^{12}$. Hence $|\langle X \rangle| \geq 2^{16} \cdot 3$.

Set $M = N_G(L_1)$. By Lemma 1-4(2), $N_M(\mathcal{O})/C_M(\mathcal{O}) \simeq A_8$. In particular $N_G(V_2)/C_G(V_2) \gtrsim A_8$. Since $C_G(V_2) \geq \langle X \rangle$, we have $|N_G(V_2)| \geq |A_8| \times 2^{16} \cdot 3 = 2^{22} \cdot 3^3 \cdot 5 \cdot 7$.

Now $|G : N_G(V_2)|$ equals the number of octads. Hence by Theorem 4-1(4) and Lemma 4-3, we have

$$|G:N_G(V_2)|=\frac{306936\times 31671\times 3510\times 693\times 4\times 45}{8\quad\times\quad 7\quad\times\ 6\ \times\ 5\ \times\ 4}=\frac{|G|}{2^{22}\cdot 3^3\cdot 5\cdot 7}.$$

Thus we have $N_G(V_2)/C_G(V_2)\simeq A_8$, $C_G(V_2)=\langle X\rangle$, and $|O_2(\langle X\rangle)|=2^{15}$.

Let K be the kernel of the action of $N_G(V_2)$ on the set $\{X_1, X_2, X_3\}$. Then $K>Q$, and $N_G(V_2)/K\simeq\Sigma_3$. Thus we have $K/Q\simeq A_8$, and so $N_G(V_2)/Q\simeq\Sigma_3\times A_8$.

We determine the structure of Q. Set $Q_\infty=\langle a_ia_j|1\leq i, j\leq 8\rangle$. Let $b, b'\in X_i$, $c, c'\in X_j (b\neq b', c\neq c')$ for some i, $j\in\{1, 2, 3\}$. If $i=j$, then bb' and cc' commute. Suppose $i\neq j$. Then each element of X_i commutes with no elements of X_j. Thus we have $X_j=\{c^{bd}|d\in X_i\}$ by the same argument as in the proof of Lemma 2-6, and the fact $X_j=c^{Q_0}$.

Hence there exists an element b'' of X_i with the property $c'=c^{bb''}$. If $b=b''$, then $cc'=1$. If $b'=b''$, then $(cc')^{bb'}=(cc^{bb'})^{bb'}=c^{bb'}c=cc'$. Suppose $b''\notin\{b\cdot b'\}$. By Lemma 1-1(2), there exist $b'''\in X_i$, and $\{i_1, i_2, i_3, i_4\}<\{1, \cdots, 8\}$ such that $\{b, b', b'', b''', a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$ is an octad. Then $bb'b''b'''=a_{i_1}a_{i_2}a_{i_3}a_{i_4}$, and

$$\begin{aligned}(bb'cc')^2&=bb'cc^{bb''}bb'cc^{bb''}=c^{bb'}c^{bb'''}cc^{bb''}\\&=(c^{b'}c^{b'''}c^{b}c^{b''})^{b}=b'b'''bb''\\&=a_{i_1}a_{i_2}a_{i_3}a_{i_4}\in Q_\infty.\end{aligned}$$

Hence Q_0/Q_∞ is an elementary abelian 2-group. Moreover it is easily seen that any 4-element subset of $\{1, \cdots, 8\}$ appears as $\{i_1, i_2, i_3, i_4\}$ in the above argument. Thus we conclude $Q_\infty=Q_0'=\Phi(Q_0)$.

Finally we show $Q_\infty=Z(Q_0)$. For each $i\in\{1, 2, 3\}$, the product of four elements of X_i equals either 1 or the product of two elements of X_i modulo Q_∞ by Lemma 1-1(2). Thus if $z\in Q_0-Q_\infty$, there exist $b, b'\in X_i$, $c, c'\in X_j$, and $d, d'\in X_k$ where $\{i, j, k\}=\{1, 2, 3\}$, such that z equals bb', $bb'cc'$, or $bb'cc'dd'$ modulo Q_∞. But since $X_i^c=X_k$, there exist $b'', b'''\in X_j$ such that $d=b''^c$, and $d'=b'''^c$. Then we have

$$\begin{aligned}dd'&=(b''b''')^c=c^{b''}c^{b'''}=(c^{b''}c^{b'''})^{b''b'''}=(b''b''')^{cb''b'''}\\&=b''b'''cb''b'''cb''b'''=b''b'''cc^{b''b'''}.\end{aligned}$$

Hence the case that z equals to $bb'cc'dd'$ modulo Q_∞ may be omitted. Now it is easily seen that there exists an element of Q_0 which does not commute with z. In fact, bb' does not commute with cc' when $c'\neq c, c^{bb'}$. Moreover if $c'=c^{bb'}$ (resp. $c'\neq c^{bb'}$), then $bb'cc'$ does not commute with $c'c''$ where $c''\neq c', c^{bb'}$ (resp. $c''=c^{bb'}$). Hence we have $Q_\infty=Z(Q_0)$.

The order of $\langle a_1, Q_\infty\rangle=\langle\mathcal{O}\rangle$ equals 2^7 by Lemma 4-7(1). Thus $|Q_\infty|=2^6$, and so $Q_0=2^{6+8}$.
(3) Set $Y=C_D(V_3)$, and $R=O_2(\langle Y\rangle)$. Let $y\in Y$. By Lemma 4-6, we have $\langle Y\rangle/R\simeq\Sigma_6$, and y^R is an octad. Since $|Y|=15\times 8$, and Y is a conjugacy class of $\langle Y\rangle$, we have $|\langle Y\rangle:C_{\langle Y\rangle}(y)|=2^3\cdot 3\cdot 5$. Moreover $\langle y^R\rangle\geq V_3$ implies $C_{\langle Y\rangle}(y)\geq\langle C_D(y^R)\rangle$. Now that we already know $|\langle C_D(y^R)\rangle|=2^{16}\cdot 3$, we have $|\langle Y\rangle|\geq 2^{19}\cdot 3^2\cdot 5=(6!)\times 2^{15}$.

Let $L\in\mathcal{L}_{\langle Y\rangle}$, and $M=N_G(L)$. By Theorem 1-4(3), $N_M(V_3)/C_M(V_3)\simeq L_3(2)$. In particular, $N_G(V_3)/C_G(V_3)\gtrsim L_3(2)$.

We calculate $|N_G(V_3)|$. By Lemma 1-3(4), each member of $\mathcal{P}_1$ contains 3795 members of $\mathcal{P}_3$ (see the proof of Theorem 5-3(1)). On the other hand, V_3 is contained in 15 members of $\mathcal{P}_1$ since $\langle Y\rangle/R\simeq\Sigma_6$. Hence $|G:N_G(V_3)|=(|G|\times 3795)/(|M_{24}|\times 2^{12}\times 15)$. Thus $|N_G(V_3)|=2^{22}\cdot 3^3\cdot 5\cdot 7=(2^{15})\cdot(6!)\cdot(168)$, and we have $N_G(V_3)/C_G(V_3)\simeq L_3(2)$, $C_G(V_3)=\langle Y\rangle$, and $|\langle Y\rangle|=2^{15}$.

Let K be the kernel of the action of $N_G(V_3)$ on YR/R. Then we have $N_G(V_3)/K\simeq\Sigma_6$ and $K/R\simeq L_3(2)$, since $L_3(2)$ is simple and $\mathrm{Aut}(\Sigma_6)/\Sigma_6$ is solvable. Hence $N_G(V_3)/R\simeq\Sigma_6\times L_3(2)$.

The structure of R is determined by an argument similar to the one used for the case of Q. So we omit the proof.

THEOREM 4-11. *Let $V_i\in\mathcal{P}_i$ $(i=1, 2, 3, 4)$. Then $N_G(V_i)$ is a maximal subgroup of G for each i.*

PROOF. It is easily seen that there are no inclusion relations among the $N_G(V_i)$ $(i=1, 2, 3, 4)$.

Set $X_i=C_D(V_i)$. Then we have $N_G(V_i)=N_G(X_i)$. Suppose a subgroup M of G contains $N_G(V_i)$ for some i, and $M\neq N_G(V_i)$. Set $X=X_i$, $V=V_i$, and $E=M\cap D$. Then we have $E\geq X$.

We first show that E is a conjugacy class of $\langle E\rangle$. Let E_0 be a cojugacy class of $\langle E\rangle$ with the property $E_0\cap X\neq\emptyset$. Then we have $E-E_0<C_D(E_0)$, since two non-commuting elements a, b of D are conjugate in $\langle a, b\rangle$. Moreover E_0 is a set of imprimitivity for the action of M on E.

Suppose $i=3, 4$. Then X is a conjugacy class of $\langle X\rangle$. Hence $E_0\geq X$. Since $\langle X\rangle\geq V$, we have $C_D(E_0)\leq C_D(X)\leq C_D(V)=X$. Hence we have $E=E_0$.

Suppose $i=1$. Then $X\in\mathcal{L}$, and so $X=C_D(X)$. Since $M>N_G(X)$ and $N_G(X)$ acts 5-transitively on X, we have either $E_0\geq X$ or $|E_0\cap X|=1$. If $E_0\geq X$, we have $E=E_0$ since $E-E_0\leq C_D(X)=X$. If $|E_0\cap X|=1$, all elements of E_0 commute with the 23 elements of $X-E_0$. By Lemma 4-6, we have $|E_0|=1$. Thus $X^{\langle E\rangle}=X$, and so $E-X<C_D(X)=X$. This contradicts the assumption $E\neq X$.

Finally suppose $i=2$. Then $X=\mathcal{O}\cup X_0$ where $\mathcal{O}$ is an octad and $X_0=C_D(\mathcal{O})-\mathcal{O}$. Then X_0 is a conjugacy class of $\langle X_0\rangle$. Considering the action of $N_G(V)=N_G(\mathcal{O})$ on $\mathcal{O}$, we have $E=E_0$ in a manner similar to the case of $i=1$. Thus we have shown that E is a conjugacy class of $\langle E\rangle$.

Now the width of $\langle E\rangle$ is 24, for so is that of $\langle X\rangle$. Let $L\in\mathcal{L}_{\langle E\rangle}$, and $x\in L$. Then $x^{O_3(\langle E\rangle)}\cap L=\{x\}$, and elements of $x^{O_3(\langle E\rangle)}$ commute with the 23 elements of $L-\{x\}$ by Lemma 2-2(2). Thus Lemma 4-6 implies $x^{O_3(\langle E\rangle)}=\{x\}$, and so $O_3(\langle E\rangle)\leq Z(\langle E\rangle)$.

Let n and m be the 2-depth of $\langle E\rangle$ and the width of $\langle E\rangle O_2(\langle E\rangle)/O_2(\langle E\rangle)$ respectively. Then n is a power of 2, and $nm=24$. Thus $(n, m)=(1, 24)$, $(2, 12)$, $(4, 6)$, or $(8, 3)$.

The condition $(n, m)=(1, 24)$ means $O_2(\langle E\rangle)\leq Z(\langle E\rangle)$. The list of groups H generated by 3-transpositions which satisfy $O_2(H)\leq Z(H)\geq O_3(H)$ is found in [7]. By that list we conclude $\langle E\rangle=G$.

Suppose $(n, m)=(2, 12)$. Let $e\in E$ and $e^{O_2(\langle E\rangle)}=\{e, d\}$. Then $\{e, d\}$ is a set of imprimitivity for the action of M on E. Let $L\in\mathcal{L}_{\langle E\rangle}$, and set $K=N_M(L)/C_M(L)$. Then K is a subgroup of $N_G(L)/C_G(L)\simeq M_{24}$. If $i=1$, we have $K\simeq M_{24}$. In the other cases, K contains an octad stabilizer if $i=2$, a trio stabilizer if $i=3$, or a sextet stabilizer if $i=4$. Thus there are no 2-element sets of imprimitivity for the action of K on L (see Curtis [6]). Hence the case $(n, m)=(2, 12)$ is impossible.

The condition $(n, m)=(4, 6)$ implies that M is contained in a conjugate of $N_G(V_4)$ by Lemma 2-7. This contradicts the assumption for M.

Finaly suppose $(n, m)=(8, 3)$. Let $L\in\mathcal{L}_{\langle E\rangle}$. Then $L=a^{O_2(\langle E\rangle)}\cup b^{O_2(\langle E\rangle)}\cup c^{O_2(\langle E\rangle)}$ for some elements a, b, and c in L. Let $x\in E-L$. Then by Lemma 4-6, we may assume x commutes with a. Then by Lemma 2-2, x commutes with the elements of $a^{O_2(\langle E\rangle)}$. Thus $a^{O_2(\langle E\rangle)}$ is an octad by Lemma 4-6. Hence $\{a^{O_2(\langle E\rangle)}, b^{O_2(\langle E\rangle)}, c^{O_2(\langle E\rangle)}\}$ is a trio of L, and $E\leq C_D(W)$ where $W=\langle a^{O_2(\langle E\rangle)}\rangle\cap\langle b^{O_2(\langle E\rangle)}\rangle\cap\langle c^{O_2(\langle E\rangle)}\rangle\in\mathcal{P}_3$. Thus we conclude $M\leq N_G(W)$, a contradiction.

Now the proof of Theorem 4-11 is complete.

REMARK. It is easily seen that the N_i $(i=1, 2, 3, 4)$ are 2-constrained.

5. 2-local geometries

In this section, we will describe some geometries for M_{24}, $O^-(6,3)$, and F_{24} associated with their 2-local diagrams given in the introduction.

A geometry which we consider here is the pair (Δ, I) of a set $\Delta=\Delta_1 \cup \cdots \cup \Delta_r$ (disjoint union) and a symmetric, reflective relation I on Δ such that for each $i \in \{1, \cdots, r\}$ and $x, y \in \Delta_i$, xIy implies $x=y$.

A trivial geometry is the geometry (Δ, I) such that if $i \neq j$ then xIy for any $x \in \Delta_i$ and $y \in \Delta_j$.

A *flag* of (Δ, I) is a subset F of Δ such that each pair of elements of F is incident.

Let G be a group, and $\{G_i | i=1, \cdots, r\}$ be a family of subgroups of G. Then the *group geometry* $\Gamma(G, \{G_i\})$ is the geometry $(\Gamma, *)$, where $\Gamma=\Gamma_1 \cup \cdots \cup \Gamma_r$, Γ_i is the coset space $G_i \backslash G$ for each $i \in \{1, \cdots, r\}$, and the incidence $*$ is defined by

$$G_i x * G_j y \Longleftrightarrow G_i x \cap G_j y \neq \emptyset.$$

(I) M_{24}

We consider the Steiner system $S(24, 8, 5)$. Let Δ_1, Δ_2, and Δ_3 be the sets of all octads, trios, and sextets respectively.

We set $\Delta=\Delta_1 \cup \Delta_2 \cup \Delta_3$. Let $X_i \in \Delta_i$ $(i=1, 2, 3)$. We define I by the following conditions.

$X_1 I X_2 \Longleftrightarrow$ the octad X_1 is one of the octads in the trio X_2,
$X_1 I X_3 \Longleftrightarrow$ the octad X_1 is the union of two tetrads in the sextet X_3,
$X_2 I X_3 \Longleftrightarrow$ the sextet X_3 is a refinement of the trio X_2.

Let $i \in \{1, 2, 3\}$, and $A_i \in \Delta_i$. Set $\{j, k\}=\{1, 2, 3\}-\{i\}$. Then we set

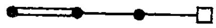
$\bar{\Delta}_i=\bar{\Delta}_i(A_i)=\{X_j \in \Delta_j,\ X_k \in \Delta_k | X_j I A_i,\ X_k I A_i\}$.

By Theorem 1-4(1), the structure of $\bar{\Delta}_i$ depends on only i. The next theorem is well-known ([12]).

THEOREM 5-1. (1) *$(\bar{\Delta}_1, I)$ is isomorphic to the geometry of points and lines of a 3-dimensional projective space over* F_2 *with the relation defined by the inclusion relation.*

(2) *$(\bar{\Delta}_2, I)$ is a trivial geometry.*

(3) *$(\bar{\Delta}_3, I)$ is isomorphic to the $Sp(4,2)$-generalized quadrangle; that is, the geometry of isotropic points and lines of a 3-dimensional projective symplectic space over* F_2 *with the relation defined by the inclusion relation.*

REMARK. Theorem 5-1 means that the geometry (Δ, I) belongs to the diagram  in the sense of Buekenhout [2]. (See [11])

(II) $O^-(6,3)$

We use the notations in section 3. So $\langle E\rangle = O^-(6,3)$. We define $\Delta_i'(i=1,2,3)$ as follows, and set $\Delta' = \Delta_1' \cup \Delta_2' \cup \Delta_3'$.

$\Delta_1' = \mathcal{L}_{\langle E\rangle}$,
$\Delta_2' = \{\{a, b\} \mid a \in E,\ b \in E_a\}$,
$\Delta_3' = \{E\text{-subgroups which are isomorphic to } W^*(D_6)\}$.

We define I' by the following conditions. Let $X_i \in \Delta_i'$ $(i=1,2,3)$, and set $X_1 = L$, $X_2 = \{a, b\}$, and $X_3 = W$.

$X_1 I' X_2 \Longleftrightarrow L > \{a, b\}$,
$X_1 I' X_3 \Longleftrightarrow W > L$,
$X_2 I' X_3 \Longleftrightarrow W > \{a, b\}$, and $a^{O_2(W)} = \{a, b\}$.

Moreover we define $\bar{\Delta}_i'$ as in (I).

Theorem 5-2. (1) *$(\bar{\Delta}_i', I')$ is isomorphic to the $Sp(4,2)$-generalized quadrangle.*

(2) *$(\bar{\Delta}_2', I')$ is a trivial geometry.*

(3) *$(\bar{\Delta}_3', I')$ is isomorphic to the $Sp(4,2)$-generalized quadrangle.*

PROOF. Since $W/O_2(W) \simeq \Sigma_6$, (3) is easy. Lemma 3-2(3) implies (2), and Lemma 3-1(2) implies (1).

REMARK. Theorem 5-2 means that the geometry (Δ', I') belongs to the diagram .

(III) F_{24}

We use the notations in section 4. So $F_{24} = \langle D\rangle$, and the $\mathcal{P}_i$ $(i = 1, 2, 3, 4)$ denote the conjugacy classes of elementary abelian 2-subgroups of F_{24} defined in section 4.

We set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$, and define the relation J by the inclusion relation, that is, for $V_i \in \mathcal{P}_i$,

$$V_i J V_j \Longleftrightarrow V_i \geq V_j \text{ or } V_i \leq V_j.$$

Moreover we define $\bar{\mathcal{P}}_i$ as before.

THEOREM 5-3. (1) *$(\bar{\mathcal{P}}_1, J)$ is isomorphic to the geometry (Δ, I) in* (I).

(2) $(\overline{\mathcal{P}}_4, J)$ *is isomorphic to the geometry* (Δ', I') *in* (II).

PROOF. Set $\overline{\mathcal{P}}_1=\overline{\mathcal{P}}_1(\langle L\rangle)$ where $L\in\mathcal{L}$. Let $V_i\in\overline{\mathcal{P}}_1\cap\mathcal{P}_i$ $(i=2,3,4)$. Then $\langle L\rangle>V_i$. Since $\langle L\rangle\cap D=L$, $\mathcal{O}=V_2\cap D$ is a unique octad such that $V_2=\langle\mathcal{O}\rangle$. Moreover we had shown in section 4 that there is just one sextet $\{T_1,\cdots,T_6\}$ of L such that $\langle T_1\rangle\cap\cdots\cap\langle T_6\rangle=V_4$. Set $V_3=\langle\mathcal{O}_1\rangle\cap\langle\mathcal{O}_2\rangle\cap\langle\mathcal{O}_3\rangle$ where $\{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3\}$ is a trio of some $L'\in\mathcal{L}$. Set $K=C_D(V_3)$. Then $K>L, L'$ and so $L, L'\in\mathcal{L}_{\langle K\rangle}$. Hence there exists an element k in K such that $L'^k=L$. Thus $V_3=V_3^k=\langle\mathcal{O}_1^k\rangle\cap\langle\mathcal{O}_2^k\rangle\cap\langle\mathcal{O}_3^k\rangle$, and $\{\mathcal{O}_1^k,\mathcal{O}_2^k,\mathcal{O}_3^k\}$ is a trio of L. Moreover by Lemma 1-3(3) and Lemma 4-7(2), $\{\mathcal{O}_1^k,\mathcal{O}_2^k,\mathcal{O}_3^k\}$ is a unique trio of L with the property $V_3=\langle\mathcal{O}_1^k\rangle\cap\langle\mathcal{O}_2^k\rangle\cap\langle\mathcal{O}_3^k\rangle$.

Hence we can identify the members of $\overline{\mathcal{P}}_1$ with those of Δ. We will show that this identification preserves the relation.

Let $V_i\in\overline{\mathcal{P}}_1\cap\mathcal{P}_i$ $(i=2,3,4)$, and $\mathcal{O}, \mathcal{T}=\{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3\}$, and $\mathcal{S}=\{T_1,\cdots,T_6\}$ be an octad, a trio, and a sextet of L such that $V_2=\langle\mathcal{O}\rangle$, $V_3=\langle\mathcal{O}_1\rangle\cap\langle\mathcal{O}_2\rangle\cap\langle\mathcal{O}_3\rangle$, and $V_4=\langle T_1\rangle\cap\cdots\cap\langle T_6\rangle$, respectively.

(i) $V_2>V_4\Longleftrightarrow\mathcal{O}I\mathcal{S}$.

Set $\mathcal{O}=\{x_1,\cdots,x_8\}$ and $T_1=\{a,b,c,d\}$. Suppose $V_2>V_4$. Then we may assume $abcd=x_1x_2x_3x_4=x_5x_6x_7x_8$. Hence $\{x_1,x_2,x_3,x_4\}$ and $\{x_5,x_6,x_7,x_8\}$ are tetrads of the sextet $\mathcal{S}$ as we have shown in section 4. The converse is easy.

(ii) $V_3>V_4\Longleftrightarrow\mathcal{T}I\mathcal{S}$.

This is easily obtained by Lemma 4-7(2).

(iii) $V_2>V_3\Longleftrightarrow\mathcal{O}I\mathcal{T}$.

Suppose $V_2>V_3$, and let $\mathcal{S}'=\{T_1',\cdots,T_6'\}$ be a sextet of L such that $\mathcal{S}'$ is a refinement of $\mathcal{T}$. By (ii), we have $V_3>\langle T_1'\rangle\cap\cdots\cap\langle T_6'\rangle$ and so $\mathcal{O}I\mathcal{S}'$ by (i). Thus we conclude $\mathcal{O}I\mathcal{T}$ by Lemma 1-3(3). The converse is easy.

Now the proof of (1) is complete.

(2) Set $\overline{\mathcal{P}}_4=\mathcal{P}_4(V_4)$, and $V_4=\langle abcd\rangle\in\mathcal{P}_4$ where $a,b,c,$ and d are distinct commuting elements of D. Moreover set $F=C_D(V_4)$, $Q=O_2(\langle F\rangle)$, $K=\langle F\rangle/Q$, and $\bar{F}=FQ/Q$. Since $K/Z(K)\simeq O^-(6,3)$ by Lemma 4-8(1), K has the same properties as stated in Lemma 3-2. In fact, by Lemma 2-5, there is a one-to-one correspondence between the set of E-subgroups of $O^-(6,3)$ isomorphic to $W^*(D_6)$ and the set of $\bar{F}$-subgroups U of K such

that $U/Z(U)\simeq W^*(D_6)$. Hence we assume $(\varDelta', I')$ is defined in K.

Let $V_i\in\overline{\mathcal{P}}_4\cap\mathcal{P}_i$ $(i=1,2,3)$. And set $V_1=\langle L\rangle$, $V_2=\langle\mathcal{O}\rangle$, and $V_3=\langle\mathcal{O}_1\rangle\cap\langle\mathcal{O}_2\rangle\cap\langle\mathcal{O}_3\rangle$, where $L\in\mathcal{L}$, $\mathcal{O}$ is an octad, and $\{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3\}$ is a trio of some member of $\mathcal{L}$. For any subset A of F, $\bar{A}$ denotes the image of A in $\bar{F}$.

$V_1>V_4$ implies that $F>L$, and so $\bar{L}\in\mathcal{L}_K$. Set $\mathcal{O}=\{x_1,\cdots,x_8\}$. Then since $V_2>V_4$, we may assume $abcd=x_1x_2x_3x_4=x_5x_6x_7x_8$. Hence by Lemma 4-8(3), $x_1{}^Q=\{x_1,x_2,x_3,x_4\}$, $x_5{}^Q=\{x_5,x_6,x_7,x_8\}$, and so $\bar{\mathcal{O}}=\{\bar{x}_1,\bar{x}_5\}$ which is a pair of mutually commuting elements of $\bar{F}$. These correspondences, $V_1\longleftrightarrow\bar{L}$ and $V_2\longleftrightarrow\bar{\mathcal{O}}$, are one-to-one, respectively. Moreover it is trivial that $V_1>V_2\Longleftrightarrow\bar{L}>\bar{\mathcal{O}}$ i.e. $\bar{L}I\bar{\mathcal{O}}$.

Set $X=C_D(V_3)$. $V_3>V_4$ implies $F>X$. By Lemma 4-9(1)(2), we have $\langle X\rangle/O_2(\langle X\rangle)\simeq\Sigma_6$, and $x^{O_2(\langle X\rangle)}$ is an octad for any $x\in X$. Set $\mathcal{O}(x)=x^{O_2(\langle X\rangle)}$. Then $\langle\mathcal{O}(x)\rangle>V_3$ by Lemma 4-9(3). In particular, $\langle\mathcal{O}(x)\rangle>V_4$, and so $\overline{\mathcal{O}(x)}=\{\bar{x},\bar{x}'\}$ for some $x'\in\mathcal{O}(x)$ as already shown. Hence the 2-depth of $\langle\bar{X}\rangle$ is 2, and so $\langle\bar{X}\rangle\simeq W^*(D_6)$.

Let $V_3'\in\overline{\mathcal{P}}_4\cap\mathcal{P}_3$, and $V_3'\neq V_3$. Then by Lemma 4-9(3), we have $C_D(V_3')\neq X$, and so $\langle\overline{C_D(V_3')}\rangle\neq\langle\bar{X}\rangle$. Hence the correspondence, $V_3\longleftrightarrow\langle\overline{C_D(V_3)}\rangle$, is one-to-one.

We will show that these correspondences preserve the relation.

(iv) $V_1>V_2\Longleftrightarrow\bar{L}I'\bar{\mathcal{O}}$.

We have already shown.

(v) $V_2>V_3\Longleftrightarrow\bar{\mathcal{O}}I'\langle\bar{X}\rangle(X=C_D(V_3))$.

Suppose $V_2>V_3$, and set $\bar{\mathcal{O}}=\{\bar{x},\bar{y}\}$. Clearly $\bar{\mathcal{O}}<\langle\bar{X}\rangle$. By (iii) in this proof, we may assume $\mathcal{O}=\mathcal{O}_1$. Thus we have $\mathcal{O}=x^{Q_2(\langle X\rangle)}$. Hence $C_X(x)=C_X(y)$ by Lemma 2-2(1), and so $C_{\bar{X}}(\bar{x})=C_{\bar{X}}(\bar{y})$ by Lemma 2-4. Thus we have $\bar{x}^{O_2(\langle\bar{X}\rangle)}=\{\bar{x},\bar{y}\}$.

Conversely suppose $\bar{x}^{O_2(\langle\bar{X}\rangle)}=\{\bar{x},\bar{y}\}$. Then we have $C_{\bar{X}}(\bar{x})=C_{\bar{X}}(\bar{y})$, and so $C_X(x)=C_X(y)$. Thus we have $x^{O_2(\langle X\rangle)}=\mathcal{O}$, and so $V_2=\langle\mathcal{O}\rangle$ by Lemma 4-9.

(vi) $V_1>V_3\Longleftrightarrow\bar{L}I'\langle\bar{X}\rangle$.

If $V_1>V_3$, then $L<C_D(V_3)=X$, and so $\bar{L}<\langle\bar{X}\rangle$. If $\bar{L}<\langle\bar{X}\rangle$, then $L<X$. Thus we have $\langle L\rangle>V_3$ by Lemma 4-9(3).

Now the proof of Theorem 5-3 is complete.

REMARK. It is easily seen from Theorems 5-1, 2, 3, that the geometry $(\mathcal{P}, J)$ belongs to the diagram .

Finally we rewrite this geometry into the form of the group geometry defined by the 2-local subgroups $N_G(V_i)$ $(V_i \in \mathcal{P}_i)$.

THEOREM 5-4. *Let* $V_i \in \mathcal{P}_i$ *and* $N_i = N_G(V_i)$ $(i=1, \cdots, 4)$ *such that* $N_1 \cap \cdots \cap N_4$ *contains a Sylow 2-subgroup of* G. *Then the group geometry* $\Gamma(G, \{N_i\})$ *is isomorphic to the geometry* $(\mathcal{P}, J)$.

We need some lemmas. The facts in the following lemma follow from the same properties of M_{24}.

LEMMA 5-5. (1) *G acts transitively on the set of all maximal flags of* $(\mathcal{P}, J)$.

(2) *The stabilizer of a maximal flag in G contains a Sylow 2-subgroup of G.*

(3) *Let S be a Sylow 2-subgroup of G. Then* $S = N_G(S)$.

LEMMA 5-6. *Let* $\{V_1, \cdots, V_4 (V_i \in \mathcal{P}_i)\}$ *be a maximal flag, and set* $N_i = N_G(V_i)$. *Then the following conditions are equivalent for any* $i, j \in \{1, \cdots, 4\}$ *and any* $g, h \in G$.

(1) $V_i^g J V_j^h$,

(2) $N_i g \cap N_j h \neq \varnothing$,

(3) $N_i^g \cap N_j^h$ *contains a Sylow 2-subgroup of G.*

PROOF. We may assume $h=1$.

$(1) \Longrightarrow (2)$; Since $V_i^g J V_j$, there exists some element x of N_j such that $V_i^g = V_i^x$ by Lemma 5-5(1). Thus $xg^{-1} \in N_i$, and so $x \in N_i g \cap N_j$.

$(2) \Longrightarrow (3)$; Let $x \in N_i g \cap N_j$. Then by Lemma 5-5(2), $N_i^g \cap N_j = N_i^x \cap N_j = (N_i \cap N_j)^x$ contains a Sylow 2-subgroup of G.

$(3) \Longrightarrow (1)$; Let S be a Sylow 2-subgroup of G contained in $N_i^g \cap N_j$, and x be an element of N_j such that $N_i \cap N_j > S^x$. Then we have $N_i^g > S, S^{xg}$, and so $S^y = S^{xg}$ for some element y of N_i^g. Thus $yg^{-1}x^{-1} \in N_G(S) = S < N_i^g$, and $N_i^g = (N_i^g)^{yg^{-1}x^{-1}} = N_i^x$. Thus we have $V_i^g = V_i^{x^{-1}} J V_j^{x^{-1}} = V_j$, as required.

Now Theorem 5-4 is derived from Lemma 5-6 immediately.

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