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On representations of $\mathfrak{p}$-adic split and non-split symplectic groups, and their character relations

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Introduction

Let F be a non-archimedean local field, $\bar{F}$ its algebraic closure, and D a division quaternion algebra over F. Then, up to local isomorphisms over F, there are two algebraic groups over F which are $\bar{F}$-isomorphic to $GSp(n)$, the symplectic group of genus n with similitudes. They are $GSp(n)$ and $GUq(n)$, the latter being the quaternionic unitary group of size n with similitudes.

Jacquet and Langlands stated in [6] that there exists a 'good' correspondence in terms of characters between the irreducible admissible representations of $GSp(1, F)\cong GL(2, F)$ and those of $GUq(1, F)\cong D^{\times}$. The main purpose of this note is to find an analogous good correspondence between the admissible representations of $GSp(2, F)$ and those of $GUq(2, F)$. This is a representation-theoretic approach to a problem raised in Y. Ihara [5]: Are there any connections between Dirichlet series attached to spherical functions of $USp(2)$ and those attached to Siegel modular forms of degree two? This problem was later posed under a more general setting as the Langlands' functoriality problem. We set

$$G=GSp(2, F)=\{g\in GL(4, F)\,;\ gJ{}^tg=n(g)J,\ n(g)\in F^{\times}\},$$

$$G^*=GUq(2, F)=\left\{g\in GL(2, D);\ g\begin{pmatrix} & 1\\ 1 & \end{pmatrix}{}^t\bar{g}=n(g)\begin{pmatrix} & 1\\ 1 & \end{pmatrix},\ n(g)\in F^{\times}\right\},$$

where $J=\begin{pmatrix} & I_2\\ -I_2 & \end{pmatrix}$ and '$-$' denotes the main involution of D.

This paper is divided into three sections. In §1, we classify the F-conjugacy classes of maximal F-tori of $GSp(2)$ and $GUq(2)$. There are five types of F-conjugacy classes in the case of $GSp(2)$: (1) the class of maximal F-split tori, (2), (3) classes which are parametrized by quadratic extensions of F, (4) classes which are parametrized by pairs of two quadratic extensions of F, (5) classes which are parametrized by pairs $\{M, E\}$, where M/E, E/F are quadratic extensions. In the case of $GUq(2)$, there are three types: (3)*, (4)*, (5)* parametrized in a similar way as (3), (4), (5) respectively. The classification of F-conjugacy classes of maximal F-tori is useful for describing the character formulae (§3). It also gives some insight for the classification of irreducible representations of G and G^*. The absolutely cuspidal representations are considered as corresponding (by the Harish-Chandra principle) to the maximal tori of types (4), (5), (4)*, (5)* which are F-compact type, and the induced representations defined in §2, to the maximal tori of types (1), (2), (3), (3)*. In §2, we define some induced representations of G and G^*, and study their properties. In §3, we give character formulae for the induced representations of G and of G^*. As for absolutely cuspidal representations, we continue to study them.

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The summary of this paper has been presented in [4].

§1. Classification of F-conjugacy classes of maximal tori of $GSp(2)$ and $GUq(2)$

Only in this §1, we assume that F has odd residue characteristic.

Let T be any F-maximal torus of $GSp(2)$. By the method used in K. Hashimoto-T. Ibukiyama [3], we can classify the G-conjugacy classes of semi-simple elements of $G=GSp(2)$. Observing this classification, we see easily that each $T(F)$, the group of F-rational points of T, contains a regular semi-simple element. Here, an element of G is said to be regular if its eigenvalues are distinct.

LEMMA 1-1. *Let g be a regular semi-simple element of G, with the characteristic polynomial $f_g(X)$ and the similitude $n(g)$. Then, either one of the following five cases occurs.*

(1) $f_g(X)=(X-a)(X-b)(X-sa^{-1})(X-sb^{-1})$,

$n(g)=s,\ a\neq b,\ a^2\neq s,\ ab\neq s,\ b^2\neq s,\ a, b, s\in F^{\times}$.

(2) $f_g(X)=(X-a)(X-sa^{-1})(X^2-\sigma X+s)$,
$n(g)=s,\ a,\ s\in F^{\times},\ \sigma\in F,\ a^2\neq s,\ X^2-\sigma X+s$ *is F-irreducible.*

(3) $f_g(X)=(X^2-\sigma X+\tau)(X^2-\sigma\tau^{-1}sX+\tau^{-1}s^2)$,
$n(g)=s,\ \sigma, \tau\in F,\ s\in F^{\times},\ \sigma\neq 0,\ \tau\neq s$ *or* $\sigma=0,\ \tau\neq\pm s$,
$X^2-\sigma X+\tau$ *is F-irreducible.*

(4) $f_g(X)=(X^2-\sigma X+s)(X^2-\tau X+s)$,
$n(g)=s,\ \sigma, \tau\in F,\ s\in F^{\times},\ \sigma\neq\tau$,
$X^2-\sigma X+s$ *and* $X^2-\tau X+s$ *are F-irreducible.*

(5) $f_g(X)=X^4-\sigma X^3+\tau X^2-s\sigma X+s^2$,
$n(g)=s,\ \sigma, \tau\in F,\ s\in F^{\times},\ f_g(X)$ *is F-irreducible.*

PROOF. This is a consequence of the reciprocal property of $f_g(X)$: $s^{-2}X^4f_g(sX^{-1})=f_g(X)$. q.e.d.

Let $M\supset E\supset F$ be a tower of two separable quadratic extensions, i_E be an F-isomorphism of E into $M_2(F)$, extended naturally to an injective F-homomorphism of $M_2(E)$ into $M_4(F)$. Then there exists an element $h\in GL(4, F)$ such that, for any $g\in GL(2, E)$,

$$J^t(h\cdot i_E(g)\cdot h^{-1})J^{-1}=h\cdot i_E\left(\begin{pmatrix} & 1\\ -1 & \end{pmatrix}^t g\begin{pmatrix} & 1\\ -1 & \end{pmatrix}^{-1}\right)\cdot h^{-1}$$
$$=\det(g)\cdot h\cdot i_E(g)^{-1}\cdot h^{-1}.$$

LEMMA 1-2. *In each cases of Lemma 1-1, g is G-conjugate to an element of the following form;*

(1) $\begin{pmatrix} a &&& \\ & b && \\ && sa^{-1} & \\ &&& sb^{-1}\end{pmatrix}$.

(2) $\begin{pmatrix} a &&& \\ & \alpha && \beta \\ && sa^{-1} & \\ & \gamma && \delta\end{pmatrix}$, $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in GL(2, F),\ \alpha+\delta=\sigma,\ \alpha\delta-\beta\gamma=s$.

(3) $\begin{pmatrix} A & \\ & s^tA^{-1}\end{pmatrix}$, $A\in GL(2, F),\ \mathrm{tr}(A)=\sigma,\ \det(A)=s$.

(4) *We denote the splitting field of $X^2-\sigma X+s$, $X^2-\tau X+s$ by E_1, E_2, respectively.*

(i) $\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}$ *if* $E_1 \neq E_2$,

(ii) $\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix}$ *or* $\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & tb_2 \\ c_1 & & d_1 & \\ & t^{-1}c_2 & & d_2 \end{pmatrix}$ *if* $E_1 = E_2$,

where $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in GL(2, F)$, $a_i d_i - b_i c_i = s$, $(i=1, 2)$, $a_1 + d_1 = \sigma$, $a_2 + d_2 = \tau$, $t \in F^\times \setminus \mathrm{N}_{E_1/F}(E_1^\times)$.

(5) *Denote the splitting fields of* $f_g(X)$ *and* $X^2 - \sigma X + (\tau - 2s)$, *by* M *and* E, *respectively.*

(i) $h \cdot i_E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot h^{-1}$, *if* $[E^\times : F^\times \mathrm{N}_{M/E}(M^\times)] = 1$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, E)$.

(ii) $h \cdot i_E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot h^{-1}$ *or* $h \cdot i_E\left(\begin{pmatrix} a & tb \\ t^{-1}c & d \end{pmatrix}\right) \cdot h^{-1}$, *if* $[E^\times : F^\times \mathrm{N}_{M/E}(M^\times)] = 2$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, E)$, $ad - bc = s$, $X^2 - \sigma X + (\tau - 2s)$ *is the minimal polynomial of* $a + d$ *over* F, *and* $t \in E^\times \setminus F^\times \mathrm{N}_{M/E}(M^\times)$.

The proof of this lemma is a slight modification of [3].

REMARK. The index $[E^\times : F^\times \mathrm{N}_{M/E}(M^\times)]$ which appears in (5) is 2 if M is a $(2, 2)$-extension of F, and 1 otherwise.

By Lemma 1-2 we obtain

PROPOSITION 1-3. *Let* $\boldsymbol{T}$ *be a maximal F-torus of* G, *then it is F-conjugate to one and only one of the following five types of tori.*

(1) $\boldsymbol{T}_1(F) = \left\{ t_1 = \begin{pmatrix} a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1} \end{pmatrix} ; a, b, s \in F^\times \right\}$: *an F-maximal split torus.*

(2) *For each separable quadratic extension* E *of* F, *we fix an F-isomorphism* i_E *of* E *into* $M_2(F)$. *We set*

$$\boldsymbol{T}_{2,E}(F) = \left\{ t_2 = \begin{pmatrix} a & & & \\ & \alpha & & \beta \\ & & sa^{-1} & \\ & \gamma & & \delta \end{pmatrix} ; \begin{array}{l} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in i_E(E^\times) \\ a,\ s \in F^\times,\ \alpha\delta - \beta\gamma = s \end{array} \right\}.$$

(3) *For each separable quadratic extension E of F, set*

$$\boldsymbol{T}_{3,E}(F)=\left\{t_3=\begin{pmatrix} A & \\ & s\,{}^tA^{-1}\end{pmatrix};\ s\in F^{\times},\ A\in i_E(E^{\times})\right\}.$$

(4) *For each pair (E_1, E_2) of separable quadratic extensions of F, we set*

(i) *if $E_1\neq E_2$*

$$\boldsymbol{T}_{4,(E_1,E_2)}(F)=\left\{t_4=\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2\end{pmatrix};\ \begin{matrix}\begin{pmatrix} a_i & b_i \\ c_i & d_i\end{pmatrix}\in i_{E_i}(E_i^{\times})\ \ (i=1,2) \\ a_1d_1-b_1c_1=a_2d_2-b_2c_2\end{matrix}\right\}.$$

(ii) *If $E_1=E_2$ and*

(a) $\mathrm{N}_{E_1/F}(E_1)\not\ni -1$

$$\boldsymbol{T}_{4,E_1}(F)=\left\{t_4=\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2\end{pmatrix};\ \begin{matrix}\begin{pmatrix} a_i & b_i \\ c_i & d_i\end{pmatrix}\in i_{E_1}(E_1^{\times})\ \ (i=1,2) \\ a_1d_1-b_1c_1=a_2d_2-b_2c_2\end{matrix}\right\}.$$

(b) $\mathrm{N}_{E_1/F}(E_1)\ni -1$

$$\boldsymbol{T}_{4,E_1,t}(F)=\left\{t_4=\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & tb_2 \\ c_1 & & d_1 & \\ & t^{-1}c_2 & & d_2\end{pmatrix};\ \begin{matrix}\begin{pmatrix} a_i & b_i \\ c_i & d_i\end{pmatrix}\in i_{E_1}(E_1^{\times})\ \ (i=1,2) \\ a_1d_1-b_1c_1=a_2d_2-b_2c_2\end{matrix}\right\},$$

where $t\in F^{\times}/\mathrm{N}_{E_1/F}(E_1^{\times})$.

(5) *For each ordered pair (M, E) such that $M\supset E\supset F$ is a tower of two separable quadratic extensions, we fix an E-isomorphism j of M into $M_2(E)$. We set*

(i) *if M is a $(2, 2)$-extension of F*

$$\boldsymbol{T}_{5,(M,E),t}(F)=\left\{t_5=h\cdot i_E\left(\begin{pmatrix} a & tb \\ t^{-1}c & d\end{pmatrix}\right)\cdot h^{-1};\ \begin{matrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in j(M^{\times})\subset GL(2,E) \\ ad-bc\in F^{\times}\end{matrix}\right\}$$

where $t\in E^{\times}/F^{\times}\mathrm{N}_{M/E}(M^{\times})$.

(ii) *If M is not a $(2, 2)$-extension of F*

$$\boldsymbol{T}_{5,(M,E)}(F)=\left\{t_5=h\cdot i_E\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right)\cdot h^{-1};\ \begin{matrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in j(M^{\times})\subset GL(2,E) \\ ad-bc\in F^{\times}\end{matrix}\right\}.$$

The above tori are not F-conjugate to each other.

PROOF. The exhaustion of F-conjugacy classes of maximal F-tori is obvious from Lemma 1-2. Let $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ be elements of $GL(2, F)$ for $i=1, 2, 3$ such that the splitting field of their characteristic polynomials is E. Then regular semi-simple elements

$$\begin{pmatrix} a_1 & & b_1 & \\ & a_2 & & b_2 \\ c_1 & & d_1 & \\ & c_2 & & d_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_1 & & b_1 & \\ & a_3 & & b_3 \\ c_1 & & d_1 & \\ & c_3 & & d_3 \end{pmatrix}$$

are G-conjugate if and only if there exists an element $g \in GL(2, F)$ such that $g^{-1} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \cdot g = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$ and $\det(g) \in \mathrm{N}_{E/F}(E^\times)$. In addition, if $k^{-1} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \cdot k = \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix}$ holds for some $k \in GL(2, F)$, then $\det(k) \in (-1)\mathrm{N}_{E/F}(E^\times)$. These prove (4)-(ii). The other part of Proposition 1-3 is obvious. q.e.d.

The classification of F-conjugacy classes of maximal F-tori of $G^* = GUq(2)$ is obtained by the same procedure as in the case of G. Let $M \supset E \supset F$ be a tower of two separable quadratic extensions, i_E^* be an F-isomorphism of E into D, extended naturally to an injective F-homomorphism of $M_2(E)$ into $M_2(D)$. Then there exists $h^* \in GL(2, D)$ such that for any $g \in GL(2, E)$, $h^* \cdot i_E^*(g) \cdot h^{*^{-1}}$ belongs to G^* if and only if $\det(g) \in F^\times$. We obtain

PROPOSITION 1-4. *Let T^* be any maximal F-torus of G^*. Then it is F-conjugate to one and only one of the following tori.*

(3)* *For each separable quadratic extension E of F, fix an F-isomorphism i_E^* of E into D. We set*

$$T_{3,E}^*(F) = \left\{ t_3^* = \begin{pmatrix} \alpha & \\ & s\bar{\alpha}^{-1} \end{pmatrix};\ s \in F^\times,\ \alpha \in i_E^*(E^\times) \subset D^\times \right\}.$$

(4)* *Here in (4)*, we use the hermitian form $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ instead of $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ to express elements of G^* easily (cf. [3] p. 578). For each pair (E_1, E_2) of separable quadratic extensions of F, set*

(i) *if $E_1 \neq E_2$*

$$T^*_{4,(E_1,E_2)}(F)=\left\{t_4^*=\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix};\ \alpha\in i^*_{E_1}(E_1^\times),\ \beta\in i^*_{E_2}(E_2^\times),\ \alpha\bar{\alpha}=\beta\bar{\beta}\right\}.$$

(ii) *If* $E_1=E_2$ *and*

(a) $\mathrm{N}_{E_1/F}(E_1)\not\ni -1$

$$T^*_{4,E_1,t}(F)=\left\{t_4^*=\begin{pmatrix}\alpha & \\ & t^{-1}\beta t\end{pmatrix};\ \alpha,\beta\in i^*_{E_1}(E_1^\times),\ \alpha\bar{\alpha}=\beta\bar{\beta}\right\},$$

where $t\in D^\times$ *is* 1 *or the one fixed element such that* $t\bar{t}\notin \mathrm{N}_{E_1/F}(E_1^\times)$.

(b) $\mathrm{N}_{E_1/F}(E_1)\ni -1$

$$T^*_{4,E_1}(F)=\left\{t_4^*=\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix};\ \alpha,\beta\in i^*_{E_1}(E_1^\times),\ \alpha\bar{\alpha}=\beta\bar{\beta}\right\}.$$

(5)* *For each ordered pair* (M, E) *such that* $M\supset E\supset F$ *is a tower of two separable quadratic extensions, fix an E-isomorphism* j^* *of* M *into* $M_2(E)$. *We set*

(i) *if* M *is a* $(2,2)$*-extension of* F

$$T^*_{5,(M,E),t}(F)=\left\{t_5^*=h^*\cdot i_E^*\left(\begin{pmatrix}\alpha & t\beta\\ t^{-1}\gamma & \delta\end{pmatrix}\right)\cdot h^{*^{-1}};\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in j^*(M^\times)\subset GL(2,E),\ \alpha\delta-\beta\gamma\in F^\times\right\}$$

where $t\in E^\times/F^\times\mathrm{N}_{M/E}(M^\times)$.

(ii) *If* M *is not a* $(2,2)$*-extension of* F

$$T^*_{5,(M,E)}(F)=\left\{t_5^*=h^*\cdot i_E^*\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)\cdot h^{*^{-1}};\ \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in j^*(M^\times)\subset GL(2,E),\ \alpha\delta-\beta\gamma\in F^\times\right\}.$$

The above tori are not F-conjugate to each other.

In view of Propositions 1–3 and 1–4, we notice the existence of a correspondence between F-conjugacy classes of maximal F-tori of G and those of G^*. More precisely, let T, T^* be maximal F-tori of type (ν), $(\nu)^*$, respectively $(\nu=3,4,5)$. Let $g\in T(F)$ and $g^*\in T^*(F)$ be regular semi-simple elements with the same characteristic polynomial over F. Then, there is a unique F-isomorphism from $T(F)$ to $T^*(F)$ which sends $g\in T(F)$ to $g^*\in T^*(F)$. Thus, we say that T and T^* are corresponding maximal F-tori. Of course, this F-isomorphism depends on the choice of g and

g^*. So the correspondence is one to one for types (3) and (3)*, (4)-(i) and (4)*-(i), (5)-(ii) and (5)*-(ii), one to two for types (4)-(ii)-(a) and (4)*-(ii)-(a), two to one for types (4)-(ii)-(b) and (4)*-(ii)-(b), and two to two for types (5)-(i) and (5)*-(i).

§ 2. Construction of induced representations

We define the following parabolic subgroups P_1, P_2, P_3 of G and P_* of G^* in order to introduce some induced representations:

$$P_1=\left\{p_1=\begin{pmatrix} a & * & & \\ 0 & b & & * \\ & & sa^{-1} & 0 \\ & O_2 & * & sb^{-1}\end{pmatrix}\in G;\ a, b, s\in F^\times\right\}$$

$$P_2=\left\{p_2=\begin{pmatrix} a_1 & * & * & * \\ 0 & a & * & b \\ 0 & 0 & sa_1^{-1} & 0 \\ 0 & c & * & d\end{pmatrix}\in G;\ ad-bc=s,\ \ s, a_1\in F^\times\right\}$$

$$P_3=\left\{p_3=\begin{pmatrix} A & * \\ O_2 & s\,{}^tA^{-1}\end{pmatrix}\in G;\ A\in GL(2, F),\ s\in F^\times\right\}$$

$$P_*=\left\{p_*=\begin{pmatrix} \alpha & * \\ 0 & s\bar{\alpha}^{-1}\end{pmatrix}\in G^*;\ \alpha\in D^\times,\ s\in F^\times\right\}.$$

These are the representatives of isomorphism classes over F of parabolic subgroups of G and G^*. We denote by M_v, N_v a Levi subgroup and the unipotent radical, respectively, of the parabolic subgroup P_v for $v=1, 2, 3, *$. Let W (V, U, etc.) be a vector space over $\boldsymbol{C}$. Then we denote by $F(G, W)$ the set of all locally constant W-valued functions on G. Let $R(GL(2, F))$ be the set of all equivalence classes of irreducible admissible representations of $GL(2, F)$, etc. We define the following representation spaces of G and G^*:

$$B_1(\mu_1, \mu_2, \eta)=\{f\in F(G, \boldsymbol{C});\ f(p_1g)=\mu_1(a)\mu_2(b)\eta(s)d_1^{-1/2}(p_1)f(g),\ p_1\in P_1,\ g\in G\}$$

$$B_2(\mu, \pi)=\{f\in F(G, V);\ f(p_2g)=\mu(a)d_2^{-1/2}(p_2)\pi(A)f(g),\ p_2\in P_2,\ g\in G\}$$

$$B_3(\pi, \eta)=\{f\in F(G, V);\ f(p_3g)=\eta(s)d_3^{-1/2}(p_3)\pi(A)f(g),\ p_3\in P_3,\ g\in G\}$$

$$B_*(r, \eta)=\{f\in F(G^*, U);\ f(p_*g)=\eta(s)d_*^{-1/2}(p_*)r(\alpha)f(g),\ p_*\in P_*,\ g\in G^*\}$$

where $p_v \in P_v$ is expressed as in the above form for $v=1, 2, 3, *$ and $\mu_1, \mu_2, \mu, \eta \in R(F^\times)$, $(\pi, V) \in R(GL(2, F))$, $(r, U) \in R(D^\times)$. Let $d_v(p_v)$ be a measure of P_v defined by $d_v(p_v)=d(p_v^{-1}n_v p_v)/dn_v$ where dn_v is a Haar measure of N_v for $v=1, 2, 3, *$. More explicitly, $d_1(p_1)=|s^3a^{-4}b^{-2}|$, $d_2(p_2)=|s^2a^{-4}|$, $d_3(p_3)=|s\det(A)^{-1}|^3$, and $d_*(p_*)=|s(\alpha\bar{\alpha})^{-1}|^3$ where $|\cdot|$ is the normalised $\mathfrak{p}$-adic absolute value of F.

The group G acts on the vector spaces B_1, B_2, and B_3 by right translations and G^* acts on B_* by right translations. We denote these representations by $\rho_1(\mu_1, \mu_2, \eta)$, $\rho_2(\mu, \pi)$, $\rho_3(\pi, \eta)$, and $\rho_*(r, \eta)$, respectively.

We shall discuss several fundamental properties of these induced representations. First, some equivalences of these representations.

LEMMA 2-1. *Let $\mu_1, \mu_2, \eta \in R(F^\times)$, then $\rho_1(\mu_1, \mu_2, \eta)$, $\rho_2(\mu_1, \rho(\mu_2\eta, \eta))$, and $\rho_3(\rho(\mu_1, \mu_2), \eta)$ are equivalent. Here $\rho(\mu_1, \mu_2)$ and $\rho(\mu_2\eta, \eta)$ are induced representations of $GL(2, F)$ defined in Jacquet-Langlands* [6].

PROOF. We define linear operators $L_{1,3}$ (resp. $L_{3,1}$) from $B_1(\mu_1, \mu_2, \eta)$ to $B_3(\rho(\mu_1, \mu_2), \eta)$ (resp. from $B_3(\rho(\mu_1, \mu_2), \eta)$ to $B_1(\mu_1, \mu_2, \eta)$) as follows:

$$(L_{1,3}f_1)(g)(A)=f_1\left(\begin{pmatrix} A & \\ & {}^tA^{-1}\end{pmatrix}g\right)|\det(A)|^{-3/2}$$
$$(L_{3,1}f_3)(g)=f_3(g)(1_2)$$

for $f_1 \in B_1(\mu_1, \mu_2, \eta)$, $f_3 \in B_3(\rho(\mu_1, \mu_2), \eta)$, $g \in G$ and $A \in GL(2, F)$. We see that $L_{1,3}$ and $L_{3,1}$ are compatible with the action of G, and that $L_{3,1}\circ L_{1,3}f_1 =f_1$, $L_{1,3}\circ L_{3,1}f_3=f_3$. Hence, $\rho_1(\mu_1, \mu_2, \eta)$ is equivalent to $\rho_3(\rho(\mu_1, \mu_2), \eta)$. In the same way, we see that $\rho_1(\mu_1, \mu_2, \eta)$ is equivalent to $\rho_2(\mu_1, \rho(\mu_2\eta, \eta))$. q.e.d.

COROLLARY 2-2. *If $\mu_1, \mu_2, \mu_1\mu_2, \mu_1\mu_2^{-1} \neq |\cdot|^{\pm1}$, then $\rho_1(\mu_1, \mu_2, \eta)$, $\rho_1(\mu_1^{-1}, \mu_2, \mu_1\eta)$, $\rho_1(\mu_1, \mu_2^{-1}, \mu_2\eta)$, $\rho_1(\mu_1^{-1}, \mu_2^{-1}, \mu_1\mu_2\eta)$, $\rho_1(\mu_2, \mu_1, \eta)$, $\rho_1(\mu_2, \mu_1^{-1}, \mu_1\eta)$, $\rho_1(\mu_2^{-1}, \mu_1, \mu_2\eta)$, and $\rho_1(\mu_2^{-1}, \mu_1^{-1}, \mu_2\mu_1\eta)$ are equivalent.*

PROOF. Obvious from Lemma 2-1 and Jacquet-Langlands [6] (Theorem 3-3). q.e.d.

We denote by $\tilde{\rho}_1(\mu_1, \mu_2, \eta)$ the contragradient representation of $\rho_1(\mu_1, \mu_2, \eta)$, etc. Then we get the following equivalences:

LEMMA 2-3.

$\tilde{\rho}_1(\mu_1, \mu_2, \eta)$ is equivalent to $\rho_1(\mu_1^{-1}, \mu_2^{-1}, \eta^{-1})$.

$\tilde{\rho}_2(\mu, \pi)$ *is equivalent to* $\rho_2(\mu^{-1}, \tilde{\pi})$.
$\tilde{\rho}_3(\pi, \eta)$ *is equivalent to* $\rho_3(\tilde{\pi}, \eta^{-1})$.
$\tilde{\rho}_*(r, \eta)$ *is equivalent to* $\rho_*(r, \eta^{-1})$.

PROOF. Applying the same argument as in [6] (p. 92-p. 94) to this case, we can easily prove this lemma. q.e.d.

As for the decomposition of these induced representations into irreducible components, we have partial results. For example, if $\rho_1(\mu_1, \mu_2, \eta)$ is irreducible, then none of $\mu_1, \mu_2, \mu_1\mu_2$, and $\mu_1\mu_2^{-1}$ can be equal to $|\cdot|^{\pm 3/2}$. If r is one dimensional representation of $D^\times$, then $\rho_*(r, \eta)$ is irreducible if and only if $r \neq |\cdot|^{\pm 3/2}$.

§ 3. Character formulae

In this section, we calculate the distributive characters of induced representations defined in § 2. Let $\mathcal{S}(G)$ be the set of all functions on G which are locally constant and compactly supported. For $f \in \mathcal{S}(G)$ and $(\Pi, W) \in R(G)$, put $\Pi(f) = \int_G f(g)\Pi(g)dg \in \mathrm{End}(W)$, where dg is a Haar measure on G. Since Π is admissible, the operator $\Pi(f)$ has finite range. Hence $\mathrm{Tr}\,\Pi(f)$ is defined. If there exists a locally integrable function $\chi_\Pi(g)$ such that $\mathrm{Tr}\,\Pi(f) = \int_G f(g)\chi_\Pi(g)dg$ holds for any $f \in \mathcal{S}(G)$, then we call this function the *character* of Π.

LEMMA 3-1 (Weyl integral formula; [2] p. 86). *Let* $\boldsymbol{G}$ *be a connected reductive linear algebraic group over* F, *and* G *be the group of all* F-*rational points of* $\boldsymbol{G}$. *We denote by* T, *the group of* F-*rational points of a maximal torus* $\boldsymbol{T}$ *of* $\boldsymbol{G}$. *Then we have*

$$\int_G f(g)dg = \sum_T |W_T|^{-1} \int_T D(t) \left\{ \int_{T\backslash G} f(g^{-1}tg)d\bar{g} \right\} dt,$$

the summation being over the complete set of representatives of F-*conjugacy classes of the maximal* F-*tori, and* W_T *is the Weyl group of* T, *and* $d\bar{g}$ *is the* G-*invariant quotient measure on* $T\backslash G$.

Let $X^4 - AX^3 + BX^2 - sAX + s^2 \in F[X]$ be the characteristic polynomial of $g \in G = GSp(2, F)$, where $s = n(g)$. Then *in our case*,

$$D(g) = |(A^2 - 4(B-2s)^2)((B+2s)^2 - 4sA^2)s^{-3}|.$$

In particular,

$$D\begin{pmatrix} a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1} \end{pmatrix} = |(a-b)^2(s-ab)^2(a-sa^{-1})^2(b-sb^{-1})^2a^{-2}b^{-2}s^{-3}|.$$

The same formula holds for G^*. Then we obtain the following character formulae.

THEOREM 3-2. *Characters of induced representations defined in* § 2 *exist. The characters of* $\rho_1(\mu_1, \mu_2, \eta)$, $\rho_2(\mu, \pi)$, $\rho_3(\pi, \eta)$, *and* $\rho_*(r, \eta)$ *are given respectively by* $\chi_1(\mu_1, \mu_2, \eta)$, $\chi_2(\mu, \pi)$, $\chi_3(\pi, \eta)$, *and* $\chi_*(r, \eta)$ *defined as follows*:

$$(1)\quad \chi_1(\mu_1, \mu_2, \eta)(g) = \begin{cases} \Big(\sum_{W_1} \mu_1(a)\mu_2(b)\eta(s)\Big) D(g)^{-1/2} \\ \qquad \text{if } g \approx \begin{pmatrix} a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1} \end{pmatrix} \in T_1^{\mathrm{reg}} \\ 0 \qquad \text{otherwise.} \end{cases}$$

Here '≈' means 'to be G-conjugate to'; T_1^{reg} *is the set of all regular elements of* T_1, *and* W_1 *is the Weyl group of* T_1. *The summation runs through all the conjugates of* $\begin{pmatrix} a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1} \end{pmatrix}$ *by the action of* W_1.

$$(2)\quad \chi_2(\mu, \pi)(g) = \begin{cases} \Big[\sum_{W_1} \chi_\pi\begin{pmatrix} b & \\ & sb^{-1} \end{pmatrix}\mu(a)d^{1/2}\begin{pmatrix} b & \\ & sb^{-1} \end{pmatrix}\Big] D(g)^{-1/2} \times 1/2 \\ \qquad \text{if } g \approx \begin{pmatrix} a & & & \\ & b & & \\ & & sa^{-1} & \\ & & & sb^{-1} \end{pmatrix} \in T_1^{\mathrm{reg}} \\ \Big[\sum_{W_{2\ E}} \chi_\pi(A)\mu(a_1)d^{1/2}(A)\Big] D(g)^{-1/2} \times 1/2 \\ \qquad \text{if } g \approx \begin{pmatrix} a_1 & & & \\ & a & & b \\ & & sa_1^{-1} & \\ & c & & d \end{pmatrix} \in T_{2,E}^{\mathrm{reg}} \\ 0 \qquad \text{otherwise} \end{cases}$$

where χ_π *is the character of* $\pi \in R(GL(2, F))$, A *is* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *and* $W_{2,E}$ *is* *the Weyl group of* $T_{2,E}$. *Finally, let* $X^2-\sigma X+\tau\in F[X]$ *be the characteristic polynomial of* A; *then* $d(A)=|(\sigma^2-4\tau)\tau^{-1}|$.

$$
(3)\quad \chi_3(\pi,\eta)(g)=\begin{cases}
\left[\sum_{W_1}\chi_\pi\begin{pmatrix}a & \\ & b\end{pmatrix}d^{1/2}\begin{pmatrix}a & \\ & b\end{pmatrix}\eta(s)\right]D(g)^{-1/2}\times 1/2 \\
\qquad \text{if } g\approx\begin{pmatrix}a &&& \\ & b && \\ && sa^{-1} & \\ &&& sb^{-1}\end{pmatrix}\in T_1^{\mathrm{reg}} \\
\left[\sum_{W_{3,E}}\chi_\pi(A)d^{1/2}(A)\eta(s)\right]D(g)^{-1/2}\times 1/2 \\
\qquad \text{if } g\approx\begin{pmatrix}A & \\ & s\,{}^tA^{-1}\end{pmatrix}\in T_{3,E}^{\mathrm{reg}} \\
0 \qquad \text{otherwise}
\end{cases}
$$

where $W_{3,E}$ *is the Weyl group of* $T_{3,E}$.

$$
(*)\quad \chi_*(r,\eta)(g)=\begin{cases}
\left[\sum_{W^*_{3,E}}\chi_r(\alpha)d_*^{1/2}(\alpha)\eta(s)\right]D_*^{-1/2}(g)\times 1/2 \\
\qquad \text{if } g\approx\begin{pmatrix}\alpha & \\ & s\bar{\alpha}^{-1}\end{pmatrix}\in T_{3,E}^{*\,\mathrm{reg}} \\
0 \qquad \text{otherwise}
\end{cases}
$$

where $W^*_{3,E}$ *is the Weyl group of* $T^*_{3,E}$, χ_r *is the character of* $r\in R(D^\times)$, *and* d_* *is defined in the same way as* d.

PROOF. We prove only in the case of (3). The other parts are proved in a same way. Set $\chi=\chi_3(\pi,\eta)$. For $f\in\mathcal{S}(G)$, we have

$$
\int_G f(g)\chi(g)dg=1/8\int_{T_1}D(t_1)\left\{\int_{T_1\backslash G}\chi(g^{-1}t_1g)f(g^{-1}t_1g)d\bar{g}\right\}dt_1
$$
$$
+1/4\sum_{E/F:\ \text{quadratic}}\int_{T_{3,E}}D(t_3)\left\{\int_{T_{3,E}\backslash G}\chi(g^{-1}t_3g)f(g^{-1}t_3g)d\bar{g}\right\}dt.
$$

Set $K=GSp(2,O_F)$, where O_F is the ring of all the integers of F. The above summation runs through all the quadratic extensions of F. Put $t_1=\begin{pmatrix}a &&& \\ & b && \\ && sa^{-1} & \\ &&& sb^{-1}\end{pmatrix}$ and $g=mnk$, where $m\in M_3$, $n\in N_3$, $k\in K$. Using the product formula for invariant measures, the first term is equal to

$$1/2\int_{T_1} D(t_1)^{1/2}\left\{\int_{(T_1\backslash M_3)\times N_3\times K} \chi_\pi\begin{pmatrix} a & \\ & b\end{pmatrix} d\begin{pmatrix} a & \\ & b\end{pmatrix}^{1/2}\right.$$
$$\left.\times\eta(s)f(k^{-1}n^{-1}m^{-1}t_1mnk)dkdnd\overline{m}\right\}dt_1,$$

where $d\overline{m}$ is the M_3-invariant quotient measure on $T_1\backslash M_3$. Put $n^{-1}(m^{-1}t_1m)n=(m^{-1}t_1m)n'$ $(n'\in N_3)$. Then $dn/dn'=D(t_1)^{-1/2}d^{1/2}\begin{pmatrix} a & \\ & b\end{pmatrix}d_3^{-1/2}(t_1)$. Thus the first term becomes

$$1/2\int_{T_1}\chi_\pi\begin{pmatrix} a & \\ & b\end{pmatrix}d_3^{-1/2}(t_1)\eta(s)d\begin{pmatrix} a & \\ & b\end{pmatrix}$$
$$\times\left\{\int_{(T_1\backslash M_3)\times N_3\times K} f(k^{-1}m^{-1}t_1mn'k)dkdn'd\overline{m}\right\}dt_1.$$

In the same way, the second term is equal to

$$1/2\sum_{E/F}\int_{T_{3,E}}\chi_\pi(A)d_3^{-1/2}(t_3)\eta(s)d(A)\left\{\int_{(T_1\backslash M_3)\times N_3\times K} f(k^{-1}m^{-1}t_3mn'k)dkdn'd\overline{m}\right\}dt_3$$

where $t_3=\begin{pmatrix} A & \\ & s\,{}^tA^{-1}\end{pmatrix}$. Using Weyl integral formula for $GL(2,F)$, we have

$$\int_G f(g)\chi(g)dg=\int_{M_3\times N_3\times K}\chi_\pi(A)\eta(s)d_3^{-1/2}(m)f(k^{-1}mn'k)dkdn'dm,$$

where $m=\begin{pmatrix} A & \\ & s\,{}^tA^{-1}\end{pmatrix}$.

Now, for $\varphi\in B_3(\pi,\eta)$,

$$\begin{aligned}[\rho(f)\varphi](k_1)&=\int_G f(g)(\rho(g)\varphi)(k_1)dg\\ &=\int_G f(g)\varphi(k_1g)dg\\ &=\int_G f(k_1^{-1}g)\varphi(g)dg\\ &=\int_{M_3\times N_3\times K} f(k_1^{-1}mnk_2)d_3^{-1/2}(m)\eta(s)\pi(A)\varphi(k_2)dmdndk_2\end{aligned}$$

where $k_1\in K$, $\rho=\rho_3(\pi,\eta)$ and $m=\begin{pmatrix} A & \\ & s\,{}^tA^{-1}\end{pmatrix}$. Put

$$\mathrm{K}(k_1,k_2)=\int_{M_3\times N_3} f(k_1^{-1}mnk_2)d_3^{-1/2}(m)\eta(s)\pi(A)dmdn.$$

Then

$$[\rho(f)\varphi](k_1)=\int_K \mathrm{K}(k_1, k_2)\varphi(k_2)dk_2.$$

Hence,

$$\begin{aligned}\mathrm{Tr}\,\rho(f)&=\int_K \mathrm{Tr}\,\mathrm{K}(k, k)dk\\&=\int_{M_3\times N_3\times K} f(k^{-1}mnk)d_3^{-1/2}(m)\eta(s)\chi_\pi(A)dmdndk.\end{aligned}$$

Hence $\chi_3(\pi, \eta)$ is the character of $\rho_3(\pi, \eta)$. q.e.d.

COROLLARY 3-3. *For $\mu_1, \mu_2, \eta, \nu_1, \nu_2, \xi\in R(F^\times)$, assume that $\rho_1(\mu_1, \mu_2, \eta)$ and $\rho_1(\nu_1, \nu_2, \xi)$ are irreducible. Then they are equivalent to each other if and only if $\{\nu_1, \nu_2\}$ is equal to $\{\mu_1, \mu_2\}$, $\{\mu_1^{-1}, \mu_2\}$, $\{\mu_1, \mu_2^{-1}\}$, or $\{\mu_1^{-1}, \mu_2^{-1}\}$, and $\nu_1\nu_2\xi^2=\mu_1\mu_2\eta^2$.*

PROOF. This follows from Theorem 3-2 and linear independence of characters of irreducible admissible representations of G which can be proved as that of $GL(2)$ [6]. q.e.d.

Now, let $\rho_*(r, \eta)$ correspond to $\rho_3(\pi(r), \eta)$, where $r\in R(D^\times)\longmapsto\pi(r)\in R(GL(2, F))$ is the correspondence defined in Jacquet and Langlands [6]. Then, we obtain the following character relation. That is our main results.

THEOREM 3-4. *Let T and T^* be corresponding F-tori of G and G^*, respectively. Then the following character relation holds independently of the choice of an F-isomorphism σ of $T^*(F)$ into $T(F)$.*

$$\chi_*(r, \eta)(g)+\chi_3(\pi(r), \eta)(\sigma(g))=0 \qquad (g\in T^*(F)^{\mathrm{reg}}).$$

Furthermore, the central character of $\rho_*(r, \eta)$ and that of $\rho_3(\pi(r), \eta)$ are the same.

PROOF. According to the character relation of the corresponding representations of $GL(2, F)$ and $D^\times$ ([6] Proposition 15-5), we can prove the above equality by comparing the character formulae (3) and $(*)$ in Theorem 3-2. q.e.d.

The correspondences of representations of G and G^* parametrized by the duals of maximal F-tori of types (4) and (5) are yet unknown. In order to find them, we are now concerned with the construction of irreducible absolutely cuspidal representations which are considered as corresponding to the maximal tori of types (4) and (5), and with the calculation of their characters. We expect that the correspondence defined above can be extended to the whole of $R(G^*)$.

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