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Local energy decay of solutions to the free Schrödinger equation in exterior domains

By Yoshio TSUTSUMI

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§ 1. Introduction and a theorem.

We shall investigate the local energy decay for solutions of the following free Schrödinger equation:

$$i\frac{\partial u}{\partial t}=\Delta u \quad \text{in} \quad (0,\infty)\times\Omega, \tag{1.1}$$

$$u(0,x)=u_0(x), \tag{1.2}$$

$$u|_{(0,\infty)\times\partial\Omega}=0, \tag{1.3}$$

where Ω is the exterior domain of a compact set in $\boldsymbol{R}^n$, $n\geqq 3$, and the boundary $\partial\Omega$ is smooth. We shall make the assumption that Ω is "non-trapping", which will be more precisely described later.

For hyperbolic equations in exterior domains the local energy decay of solutions has been extensively studied by many mathematicians (e.g. Vainberg [11], Rauch [6], Shibata [7] and Melrose [2]). For the Schrödinger equation in exterior domains, however, the local energy decay have not been studied well enough. It follows immediately from the results of Vainberg [9, 10, 11] that the local energy of solutions for Problem (1.1)-(1.3) decays like $t^{-3/2}$ as $t\to\infty$ if n ($\geqq 3$) is odd and decays like t^{-1} as $t\to\infty$ if n ($\geqq 3$) is even. In the present paper we shall evaluate the decay rate more precisely. Namely, we shall prove that if $n\geqq 3$ the decay rate as $t\to\infty$ is $O(t^{-n/2})$. Such precise information about the decay rate will play an important role in establishing the existence of global solutions of the nonlinear Schrödinger equation in exterior domains (see [8]).

We first give some notations which will be used below. Let D be an open subset in $\boldsymbol{R}^n$. We denote by $L^2(D)$ the Banach space consisting of complex-valued measurable functions on D that are square-integrable.

For a positive integer m we put

$$(1.5) \qquad H^m(D)=\left\{u\in L^2(D);\ \left(\frac{\partial}{\partial x}\right)^\alpha u\in L^2(D) \quad \text{for all} \quad |\alpha|\leqq m\right\}$$

with the norm

$$(1.6) \qquad \|u\|_{H^m(D)}=\left(\sum_{|\alpha|\leqq m}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha u\right\|^2_{L^2(D)}\right)^{1/2}.$$

Let $\overset{\circ}{H}{}^m(D)$ be the closure in $H^m(D)$ of the set of functions in $H^m(D)$ with compact support in D. Let $H^m_e(D)$ be the Banach space $\{u;\ e^{-|x|^2}u(x)\in H^m(D)\}$ with the norm

$$(1.7) \qquad \|u\|_{H^m_e(D)}=\left(\sum_{|\alpha|\leqq m}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha (e^{-|x|^2}u)\right\|^2_{L^2(D)}\right)^{1/2}.$$

Let R be a positive constant such that $\partial\Omega\subset\{x\in \boldsymbol{R}^n;\ |x|<R\}$. For $r>R$ we denote by $H^m_r(\Omega)$, $\overset{\circ}{H}{}^m_r(\Omega)$ and $L^2_r(\Omega)$ the closed subspaces of $H^m(\Omega)$, $\overset{\circ}{H}{}^m(\Omega)$ and $L^2(\Omega)$, respectively, consisting of functions that vanish for $|x|>r$. For $r>R$ we write $\Omega_r=\{x\in\Omega;\ |x|<r\}$. For two Banach spaces X and Y we denote the Banach space consisting of all bounded linear operators from X to Y and its norm by $\mathrm{Hom}\,(X, Y)$ and $\|\cdot\|_{X,Y}$, respectively. For any subset $E\subset \boldsymbol{R}^n$ we denote by $\bar{E}$ the closure of E.

Let $G=G(t, x, x_0)$ be the Green function for the following problem:

$$\left(\frac{\partial^2}{\partial t^2}-\Delta_x\right)G=0 \qquad \text{in} \quad (0,\infty)\times\Omega,$$

$$\lim_{t\to+0}\frac{\partial^j}{\partial t^j}G(t, x, x_0)=\begin{cases}0, & j=0,\\ \delta(x-x_0), & j=1,\end{cases}$$

$$G|_{(0,\infty)\times\partial\Omega}=0,$$

where x_0 is an arbitrary point in Ω and Δ_x is the Laplace operator with respect to the variable x. For any $v\in L^2(\Omega)$ we put

$$(Fv)(t, x)=\int_\Omega G(t, x, x_0)v(x_0)dx_0.$$

Now we formulate the non-trapping condition on the domain Ω.

Assumption [A]: Let a and b be arbitrary positive constants such that $a, b>R$. Then there exists a positive constant T_0 depending only on a, b, n and Ω such that

$$(Fv)(t, x) \in C^\infty([T_0, \infty) \times \overline{\Omega}_b)$$

for any $v \in L^2_a(\Omega)$.

REMARK 1.1. Assumption [A] implies that singularities of the Green function of the wave equation in the exterior domain Ω go to infinity as $t \to \infty$. Assumption [A] is satisfied, for example, if the complement of Ω is convex (see Melrose [2] and Rauch [6]). Assumption [A] is almost the same as assumptions that Vainberg and Rauch supposed in their works (see Vainberg [11, the hypothesis D′, p.11] and Rauch [6, the hypothesis (9.3), p. 476]). A condition needed in their proof is, in fact, such a condition as Assumption [A].

Our main theorem is the following:

THEOREM 1.1. *Let $n \geqq 3$ and let Assumption* [A] *be satisfied for Ω. Let $U(t)$ be the evolution operator associated with the equation* (1.1)-(1.3). *For two positive constants a and b with $a, b > R$ there exists a positive constant C such that*

$$\|U(t)\|_{L^2_a(\Omega), L^2(\Omega_b)} \leqq Ct^{-n/2}, \qquad t > 1, \tag{1.8}$$

where C depends only on n, a, b and Ω.

We may assume that $b > a > R+1$. We fix a and b as above from now on.

§ 2. Lemmas.

As is well known, we have

$$U(t) = (2\pi i)^{-1} \int_{-d+i\infty}^{-d-i\infty} e^{-\tau t}(i\tau + \Delta)^{-1} d\tau, \qquad d > 0. \tag{2.1}$$

Therefore, we have only to investigate the resolvent $(i\tau + \Delta)^{-1}$ in order to estimate $U(t)$. Such resolvents as $(k^2 + \Delta)^{-1}$ were intensively investigated by Vainberg [9, 10].

Here we shall summarize his results needed for the proof of Theorem 1.1. Let $D(P)$ be the entire complex plane if n is odd and the Riemann surface on which the function $\ln k$ is single-valued if n is even. Let D^+ be the region $\{k \in D(P);\ 0 < \arg k < \pi,\ k \neq 0\}$. Since the resolvent $(k^2 + \Delta)^{-1}$ is a Hom $(L^2(\Omega), H^2(\Omega))$-valued analytic function with respect to $k \in D^+$, we can regard $(k^2 + \Delta)^{-1}$ as a Hom $(L^2_a(\Omega), H^2(\Omega_b))$-valued analytic function with respect to $k \in D^+$. Then we have the following two lemmas (see Vainberg [9, 11]).

LEMMA 2.1. *Let $n \geqq 3$. Then the resolvent $(k^2+\Delta)^{-1}$ admits a meromorphic extension to $D(P)$ as a* Hom $(L_a^2(\Omega), H_e^2(\Omega))$*-valued function, and the set of all poles of the meromorphic extension has no limit point in $D(P)$.*

We also denote the extension by $(k^2+\Delta)^{-1}$.

LEMMA 2.2. *Let $n \geqq 3$ and let Assumption* [A] *be satisfied for the domain Ω. Then there exist positive constants α, β, C and T such that*

$$\|(k^2+\Delta)^{-1}\|_{L_a^2(\Omega), L^2(\Omega_b)} \leqq C|k|^{-1} \exp(T|\operatorname{Im} k|) \tag{2.2}$$

in the region $V=\{k \in D(P);\ |\operatorname{Im} k|<\alpha \ln |\operatorname{Re} k|-\beta\}$ if n is odd and in the region $V'=\left\{k \in D(P);\ |\operatorname{Im} k|<\alpha \ln |\operatorname{Re} k|-\beta,\ -\frac{\pi}{2}<\arg k<\frac{3}{2}\pi\right\}$ if n is even.

In order to prove Theorem 1.1 by using (2.1) we have to know the behaviour of $(k^2+\Delta)^{-1}$ near $k=0$. In addition to Lemmas 2.1 and 2.2 we need the following lemma, which makes Vainberg's results [10, Theorems 2 and 3] more precise for $(k^2+\Delta)^{-1}$.

LEMMA 2.3. *Let $n \geqq 3$. Then there exists a positive constant ε_1 such that:*

1) *If n is odd,*

$$(k^2+\Delta)^{-1}=\sum_{j=0}^{\infty} B_{2j} k^{2j}+\sum_{j=(n-3)/2}^{\infty} B_{2j+1} k^{2j+1} \tag{2.3}$$

in the region $W=\{k \in D(P);\ |k|<\varepsilon_1\}$, where the operators B_j $(j=0,1,2,\cdots)$ are bounded linear operators from $L_a^2(\Omega)$ to $H_e^2(\Omega)$ and the expansion (2.3) converges uniformly and absolutely in the operator norm;

2) *If n is even,*

$$(k^2+\Delta)^{-1}=\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} B_{mj}(k^{n-2} \ln k)^m k^{2j} \tag{2.4}$$

in the region $W'=\left\{k \in D(P);\ |k|<\varepsilon_1,\ -\frac{\pi}{2}<\arg k<\frac{3}{2}\pi\right\}$, where the operators B_{mj} $(m, j=0,1,2,\cdots)$ are bounded linear operators from $L_a^2(\Omega)$ to

$H^2_e(\Omega)$ and the expansion (2.4) converges uniformly and absolutely in the operator norm.

REMARK 2.1. As a consequence of Lemma 2.3, the meromorphic extension $(k^2+\Delta)^{-1}$ has no pole and is bounded in a neighbourhood of $k=0$.

PROOF OF LEMMA 2.3. From [9, § 3], [10, Lemma 3] and [4, Theorems 7.2 and 7.3] we know that the main problem is to prove that a right regularizer G_k is one to one at $k=0$. The right regularizer G_k is a Hom $(L^2_a(\Omega), H^2_e(\Omega))$-valued function defined by

$$G_k g = \beta_1(x) L_{k_0}(\alpha_1(x) g) + \beta_2(x) A_k(\alpha_2(x) g) \tag{2.5}$$

for all $g \in L^2_a(\Omega)$. Here L_{k_0} is the operator which maps a function $f(x) \in L^2_a(\Omega)$ into the solution $u(x) \in H^2(\Omega_a)$ of the problem

$$(\Delta + k_0^2) u = f \qquad (x \in \Omega_a), \tag{2.6}$$

$$u|_{\partial\Omega_a} = 0, \tag{2.7}$$

where k_0 is a pure imaginary number and the absolute value $|k_0|$ of k_0 is sufficiently large (see [9, § 3]). A_k is the analytic extension of $(k^2+\Delta)^{-1}$ for the case $\Omega = \mathbf{R}^n$ and A_k maps a function $f(x) \in L^2_a(\mathbf{R}^n)$ into the solution $u(x) \in H^2_e(\mathbf{R}^n)$ of the problem

$$(\Delta + k^2) u = f \quad \text{in} \quad \mathbf{R}^n, \tag{2.8}$$

(see [9, Theorem 1]). $\alpha_1(x)$ and $\alpha_2(x)$ are step functions such that

$$\alpha_1(x) = \begin{cases} 0, & \text{if } |x| > R + \dfrac{1}{2}, \\ 1, & \text{otherwise,} \end{cases}$$

and $\alpha_2(x) = 1 - \alpha_1(x)$. $\beta_1(x)$ and $\beta_2(x)$ are real-valued C^∞-functions such that

$$\beta_1(x) = \begin{cases} 1, & \text{if } |x| < R + \dfrac{2}{3}, \\ 0, & \text{if } |x| > R + 1, \end{cases}$$

$$\beta_2(x) = \begin{cases} 0, & \text{if } |x| < R, \\ 1, & \text{if } |x| > R + \dfrac{1}{3}. \end{cases}$$

Note that Vainberg defined G_k with $\alpha_1(x)$ and $\alpha_2(x)$ being C^∞-functions (see Vainberg [9, (3.32)]), but that since we consider the operator G_k as a bounded operator from $L^2_a(\Omega)$ to $H^2_e(\Omega)$ we can define $\alpha_1(x)$ and $\alpha_2(x)$ as above. Put

$$S_k = (\Delta + k^2)G_k - I, \tag{2.9}$$

where I is the identity operator. From [9, § 3] we already know that S_k is a compact operator of $L^2_a(\Omega)$ to $L^2_a(\Omega)$ for all k, $(I+S_k)$ has the meromorphic inverse and

$$(k^2+\Delta)^{-1} = G_k(I+S_k)^{-1}. \tag{2.10}$$

Furthermore, the expansions of the types (2.3) and (2.4) hold also for A_k (see, e.g., [9], [10] and [4]). Therefore, we obtain the expansions of the types (2.3) and (2.4) for G_k and S_k. Since $I+S_0=\Delta G_0$, $G_0 g \to 0$ ($|x|\to\infty$), and $u\equiv 0$ if u satisfies

$$\Delta u = 0 \quad \text{in} \quad \Omega, \tag{2.11}$$

$$u|_{\partial\Omega} = 0, \tag{2.12}$$

$$u(x) \longrightarrow 0 \quad \text{as} \quad |x|\to\infty, \tag{2.13}$$

it follows from the Fredholm theorem that the operator $(I+S_0)$ has the bounded inverse operator if and only if G_0 is one to one. We assume for the moment that $(I+S_0)^{-1}$ exists. Then we see by the Neumann series expansion that

$$(I+S_k)^{-1} = \sum_{j=0}^{\infty} (-1)^j \{(I+S_0)^{-1}(S_k - S_0)\}^j (I+S_0)^{-1} \tag{2.14}$$

near $k=0$. Combining (2.10), (2.14) and the expansions of G_k and S_k, we obtain (2.3) and (2.4). Consequently it remains only to show that G_0 is one to one.

Let g be a real-valued function in $L^2_a(\Omega)$ satisfying $G_0 g=0$. We easily see by the relation $G_0 g = 0$ and the definitions of $\beta_1(x)$ and $\beta_2(x)$ that $L_{k_0}(\alpha_1 g)=0$ for $|x|<R$ and $A_0(\alpha_2 g)=0$ for $|x|>R+1$. Hence, we have by the definitions of $\alpha_1(x)$ and $\alpha_2(x)$ that

$$g(x) = \alpha_1(x)g(x) = (\Delta + k_0^2)L_{k_0}(\alpha_1 g) = 0 \qquad \text{for} \quad |x|<R,$$
$$g(x) = \alpha_2(x)g(x) = \Delta A_0(\alpha_2 g) = 0 \qquad \text{for} \quad |x|>R+1.$$

From the definitions of L_{k_0}, A_0, $\alpha_1(x)$ and $\alpha_2(x)$ and the relation $L_{k_0}(\alpha_1 g) = -A_0(\alpha_2 g)$ in $R+\frac{1}{3}<|x|<R+\frac{2}{3}$ it follows that

$$(2.15) \qquad (\Delta+\gamma_1(x))L_{k_0}(\alpha_1 g)=0 \quad \text{a.e. in} \quad \left\{x\in \boldsymbol{R}^n;\ R+\frac{1}{3}<|x|<a\right\},$$

$$(2.16) \qquad L_{k_0}(\alpha_1 g)=0 \quad \text{at} \quad |x|=a,$$

and

$$(2.17) \qquad (\Delta+\gamma_2(x))A_0(\alpha_2 g)=0 \quad \text{a.e. in} \quad \left\{x\in \boldsymbol{R}^n;\ |x|<R+\frac{2}{3}\right\},$$

where

$$\gamma_1(x)=\begin{cases} -|k_0|^2, & \text{if} \quad R+\frac{1}{2}<|x|\leqq a, \\ 0, & \text{if} \quad R+\frac{1}{3}<|x|<R+\frac{1}{2}, \end{cases}$$

$$\gamma_2(x)=\begin{cases} 0, & \text{if} \quad |x|<R+\frac{1}{2}, \\ -|k_0|^2, & \text{if} \quad R+\frac{1}{2}<|x|<R+\frac{2}{3}. \end{cases}$$

From (2.15), (2.17) and [1, Theorem 8.24] we see that $L_{k_0}(\alpha_1 g)$ is Hölder continuous for $R+\frac{1}{3}<|x|\leqq a$ and that $A_0(\alpha_2 g)$ is Hölder continuous for $|x|<R+\frac{2}{3}$. Therefore, it follows by the relation $\beta_1 L_{k_0}(\alpha_1 g)=-\beta_2 A_0(\alpha_2 g)$ that $L_{k_0}(\alpha_1 g)$ and $A_0(\alpha_2 g)$ are Hölder continuous in $\bar{\Omega}_a$. Applying the strong maximum principle for a weak solution (see [1, Theorems 8.1 and 8.19]) to $L_{k_0}(\alpha_1 g)$ in $R+\frac{1}{3}<|x|\leqq a$ and to $A_0(\alpha_2 g)$ in $|x|<R+\frac{2}{3}$, we have

$$\max\left\{|L_{k_0}(\alpha_1 g)|;\ |x|=R+\frac{1}{3}\right\}\geqq \max\left\{|L_{k_0}(\alpha_1 g)|;\ |x|=R+\frac{2}{3}\right\},$$

$$\max\left\{|A_0(\alpha_2 g)|;\ |x|=R+\frac{2}{3}\right\}\geqq \max\left\{|A_0(\alpha_2 g)|;\ |x|=R+\frac{1}{3}\right\}.$$

Since $L_{k_0}(\alpha_1 g)=-A_0(\alpha_2 g)$ in $R+\frac{1}{3}\leqq|x|\leqq R+\frac{2}{3}$, it follows that

$\max\left\{|L_{k_0}(\alpha_1 g)|;\ |x|=R+\frac{1}{3}\right\}=\max\left\{|L_{k_0}(\alpha_1 g)|;\ |x|=R+\frac{2}{3}\right\}$. Applying the strong maximum principle to $L_{k_0}(\alpha_1 g)$ in $R+\frac{1}{3}<|x|<a$ again, we have by (2.16)

$$L_{k_0}(\alpha_1 g)=\text{constant}=0 \quad \text{in} \quad R+\frac{1}{3}\leqq|x|\leqq a. \tag{2.18}$$

Therefore, we have by the relation $G_0 g=0$ and (2.17)

$$A_0(\alpha_2 g)=0 \quad \text{in} \quad \boldsymbol{R}^n, \tag{2.19}$$

and consequently

$$L_{k_0}(\alpha_1 g)=0 \quad \text{in} \quad \Omega_a. \tag{2.20}$$

We conclude by (2.19) and (2.20) that g vanishes identically, that is, G_0 is one to one. This completes the proof of Lemma 2.3. (Q.E.D.)

REMARK 2.1. Recently Shibata [7] investigated the behaviour of the resolvent $(k^2-ik+\Delta)^{-1}$ near $k=0$, but he did not give its expansion near $k=0$.

We shall next translate the results on $(k^2+\Delta)^{-1}$ into those on $(i\tau+\Delta)^{-1}$, because $(i\tau+\Delta)^{-1}$ actually appears in the integral representation (2.1). For $0<\varepsilon_0<\frac{1}{\sqrt{2}}\varepsilon_1$ we consider the region D_k on the k-plane, which is hatched in Figure 1. From Lemmas 2.1, 2.2 and 2.3 we can choose ε_0 so

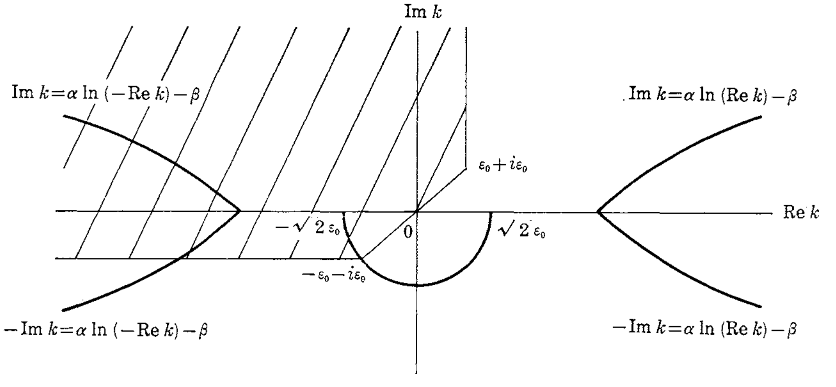

Fig. 1. The region D_k on the k-plane.

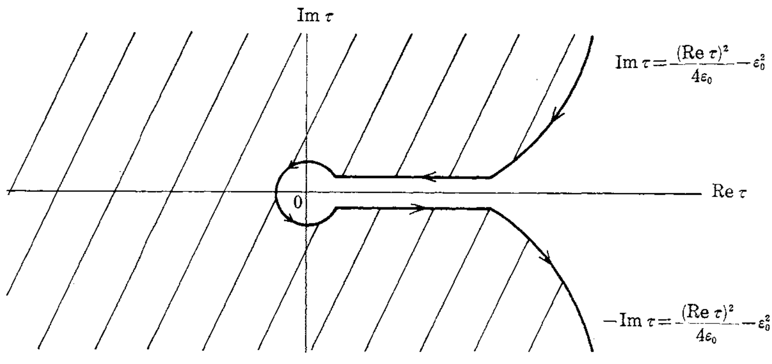

Fig. 2. The region D_τ on the τ-plane.

small that $(k^2+\Delta)^{-1}$ has no pole in the region D_k. Under the mapping $i\tau=k^2$, the region D_k is taken one to one onto the region D_τ on the τ-plane, which is hatched in Figure 2.

We intend to shift the contour of the integral in (2.1) into the half plane $\operatorname{Re}\tau>0$. But only the estimate (2.2) dose not suffice for this purpose. It is well known that the Laplace operator Δ with the domain $\mathring{H}^1(\Omega)\cap H^2(\Omega)$ is a generator of holomorphic semi-group and that there exist three positive constants ξ,η and M such that

$$\|(\lambda-\Delta)^{-1}\|_{L^2(\Omega),L^2(\Omega)}\leqq\frac{M}{|\lambda|} \tag{2.21}$$

for all $\lambda\in\left\{\lambda;\ |\arg(\lambda-\xi)|<\frac{\pi}{2}+\eta\right\}$. Combining (2.21) and (2.2), we obtain

$$\|(i\tau+\Delta)^{-1}\|_{L^2_a(\Omega),L^2(\Omega_b)}\leqq C|\tau|^{-1/2} \tag{2.22}$$

for all $\tau\in\left\{\tau\in D(P);\ |\operatorname{Im}\tau|\geqq\frac{(\operatorname{Re}\tau)^2}{4\varepsilon_0}-\varepsilon_0^2,\ |\tau|>K\right\}$, where C and K are positive constants independent of τ.

§ 3. Proof of Theorem 1.1.

Now we shall give a proof of Theorem 1.1. The line of our proof is the same as that of [11], [5] and [6].

We shall first verify that the following integral converges:

$$U(t)+iI=(2\pi i)^{-1}\int_{-d+i\infty}^{-d-i\infty} e^{-\tau t}\left\{(i\tau+\Delta)^{-1}+\frac{i}{\tau}I\right\}d\tau, \qquad d>0. \tag{3.1}$$

Since we have

$$(i\tau+\Delta)^{-1}+\frac{i}{\tau}I=\frac{i}{\tau}(i\tau+\Delta)^{-1}\Delta, \tag{3.2}$$

we obtain from (2.22)

$$\begin{aligned}\left\|(i\tau+\Delta)^{-1}+\frac{i}{\tau}I\right\|_{H_a^2(\Omega)\cap\mathring{H}^1(\Omega),L^2(\Omega_b)} &\leqq \frac{1}{|\tau|}\|(i\tau+\Delta)^{-1}\|_{L_a^2(\Omega),L^2(\Omega_b)}\|\Delta\|_{H_a^2(\Omega)\cap\mathring{H}^1(\Omega),L_a^2(\Omega)} \\ &\leqq C|\tau|^{-3/2}\end{aligned} \tag{3.3}$$

for all $\tau\in\left\{\tau\in D(P);\ |\text{Im}\,\tau|\geqq\frac{(\text{Re}\,\tau)^2}{4\varepsilon_0}-\varepsilon_0^2,\ |\tau|>K\right\}$. Therefore, the integral in (3.1) converges absolutely in Hom $(H_a^2(\Omega)\cap\mathring{H}^1(\Omega), L^2(\Omega_b))$.

By (3.3) and the Cauchy theorem we can shift the contour of the integral in (3.1) into the right half plane as in Figure 2. By Γ we denote the whole contour in Figure 2. By Γ_1^+ and Γ_1^- we denote the parabolic parts of Γ which are situated on the upper half plane and on the lower half plane, respectively. By Γ_2^+ and Γ_2^- we denote the straight line parts of Γ which are situated on the upper half plane and on the lower half plane, respectively. By Γ_3 we denote the circular part of Γ.

All constants which will appear in the course of calculations below will be simply denoted by C.

Since

$$(2\pi i)^{-1}\int_\Gamma e^{-\tau t}\left(\frac{i}{\tau}I\right)d\tau=iI, \tag{3.4}$$

we have

$$U(t)=(2\pi i)^{-1}\int_\Gamma e^{-\tau t}(i\tau+\Delta)^{-1}d\tau. \tag{3.5}$$

For the integral on Γ_3 we can shrink the circular part Γ_3 to the origin by the Cauchy theorem and Lemma 2.3.

Since $\|(i\tau+\Delta)^{-1}\|_{L_a^2(\Omega),L^2(\Omega_b)}$ is bounded on Γ_1^+ or Γ_1^- and $\text{Re}\,\tau\leqq C(1+|\text{Im}\,\tau|^{1/2})$ on Γ_1^+ or Γ_1^-, we obtain

$$\left\|(2\pi i)^{-1}\int_{\Gamma_1^+\cup\Gamma_1^-} e^{-\tau t}(i\tau+\Delta)^{-1}d\tau\right\|_{L_a^2(\Omega),L^2(\Omega_b)}\leqq Ce^{-Ct}, \qquad t\geqq 1. \tag{3.6}$$

It remains only to evaluate the integral on Γ_2^+ and Γ_2^-. If n is odd, we have by Lemma 2.3

$$(3.7) \qquad (i\tau+\Delta)^{-1}=B_1(\tau)+(i\tau)^{(n-3)/2}B_2(\tau)$$

for all $\tau\in\{\tau\in D(P);\ |\tau|<2\varepsilon_0^2\}$, where $B_1(\tau)$ and $B_2(\tau)$ are Hom $(L_a^2(\Omega), H^2(\Omega_b))$-valued holomorphic functions on $\{\tau\in D(P);\ |\tau|<2\varepsilon_0^2\}$. If n is even, we have by Lemma 2.3

$$(3.8) \qquad (i\tau+\Delta)^{-1}=B_3(\tau)+B_4\tau^{(n-2)/2}\ln\sqrt{i\tau}+\tau^{(n-2)/2}B_5(\tau)$$

for all $\tau\in\{\tau\in D(P);\ |\tau|<2\varepsilon_0^2,\ -\pi<\arg\tau<3\pi\}$, where $B_3(\tau)$ is a Hom $(L_a^2(\Omega), H^2(\Omega_b))$-valued holomorphic function on $\{\tau;\ |\tau|<2\varepsilon_0^2\}$, B_4 is a bounded operator from $L_a^2(\Omega)$ to $H^2(\Omega_b)$, and $B_5(\tau)$ is a Hom $(L_a^2(\Omega), H^2(\Omega_b))$-valued bounded continuous function on $\{\tau\in D(P);\ |\tau|<2\varepsilon_0^2,\ -\pi<\arg\tau<3\pi\}$. Therefore, the routine calculation (see, e.g., Rauch [5]) gives

$$(3.9) \qquad \left\|\int_{\Gamma_2^+\cup\Gamma_2^-} e^{-\tau t}(i\tau+\Delta)^{-1}d\tau\right\|_{L_a^2(\Omega),L^2(\Omega_b)}\leqq C\,t^{-n/2}, \qquad t\geqq 1.$$

Combining (3.6) and (3.9), we obtain (1.8). This completes the proof of Theorem 1.1.

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References

[1] Gilbarg, D. and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[2] Melrose, R. B., Singularities and energy decay in acoustical scattering, Duke Math. J. **46** (1) (1979), 43-59.

[3] Murata, M., Scattering solutions decay at least logarithmically, Proc. Japan Acad. **54**, Ser. A (1978), 42-45.

[4] Murata, M., Asymptotic expansions in time for solutions of Schrödinger-type equations, J. Funct. Anal. **49** (1982), 10-56.

[5] Rauch, J., Local decay of scattering solutions to Schrödinger's equation, Comm. Math. Phys. **61** (1978), 149-168.

[6] Rauch, J., Asymptotic behaviour of solutions to hyperbolic partial differential equations with zero speeds, Comm. Pure Appl. Math. **31** (1978), 431-480.

[7] Shibata, Y., On the global existence of classical solutions of second order fully nonlinear hyperbolic equations with first order dissipation in the exterior domain,

Tsukuba J. Math. **7** (1983), 1-68.
[8] Tsutsumi, Y., Global solutions of the nonlinear Schrödinger equation in exterior domains, Comm. Partial Differential Equations **8** (1983), 1337-1374.
[9] Vainberg, B. R., On the analytical properties of the resolvent for a certain class of operator-pencils, Math. USSR-Sb. **6** (1968), 241-273.
[10] Vainberg, B. R., On the exterior elliptic problems polynomially depending on a spectral parameters, and the asymptotic behaviour for large time of solutions of non-stationary problems, Math. USSR-Sb. **21** (1973), 221-239.
[11] Vainberg, B. R., On the short wave asymptotic behaviour of solutions of stationary problems and the asymptotic behaviour as $t\to\infty$ of solutions of non-stationary problems, Russian Math. Surveys **30** (1975), 1-58.

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Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Meguro-ku, Tokyo
153 Japan