J. Fac. Sci. Univ. Tokyo
Sect. IA, Math.
31 (1984), 109–146.

Time-decay of the high energy part of the solution for a Schrödinger equation

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(Communicated by H. Fujita)

§ 1. Introduction

In this paper, we shall consider the time-decay of the high energy part of the solution $e^{-itH}f$ of the Schrödinger equation in R^N

$$\frac{1}{i}\frac{\partial u}{\partial t}(t)+Hu(t)=0, \quad u(0)=f \qquad (f\in L^2(R^N)), \tag{1.1}$$

$$H=H_0+V(x), \qquad H_0=-\frac{1}{2}\Delta=-\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}. \tag{1.2}$$

Here $V(x)$ is a potential decomposable as $V(x)=V^L(x)+V^S(x)$ with $V^L(x)$ and $V^S(x)$ satisfying the following assumptions.

ASSUMPTION (L). $V^L(x)$ is a real-valued C^∞-function on R^N $(N\geqq 1)$ and satisfies with some $\varepsilon\in(0,1)$

$$|\partial_x^\alpha V^L(x)|\leqq C_\alpha\langle x\rangle^{-|\alpha|-\varepsilon} \tag{1.3}$$

for all multi-index α and $x\in R^N$, where $\langle x\rangle=\sqrt{1+|x|^2}$.

ASSUMPTION (S). $V^S(x)$ is a real-valued measurable function on R^N and satisfies with some $\delta\in[0,1/2)$

$$\|V^S(x)\langle x\rangle^\sigma(H_0+1)^{-\delta}\|<\infty \tag{1.4}$$

for all $\sigma\geqq 0$, where $\|\quad\|$ stands for the operator norm in $L^2(R^N)$.

$V^L(x)$ is a long-range potential, and $V^S(x)$ is a short-range potential which decreases rapidly as $|x|\to\infty$ but may have local singularities like $|x|^{-1+\varepsilon}$ $(\varepsilon>0)$ e.g. at $x=0$ when $N=3$.

Accordingly, H is a self-adjoint operator in $L^2(R^N)$ with domain $\mathcal{D}(H)=H^2(R^N)$ (Sobolev space of order two).

Let $\chi\in C^\infty(R^1)$ satisfy $\chi(\lambda)=1$ ($|\lambda|\geqq R_0+1$), $=0$ ($|\lambda|\leqq R_0$) for some R_0 large enough. The high energy part χe^{-itH} of e^{-itH} is defined by

$$\chi e^{-itH}=\chi(H)e^{-itH}=\int_{R^1}\chi(\lambda)e^{-it\lambda}dE(\lambda). \tag{1.5}$$

Here $E(\lambda)$ is the spectral resolution of the identity associated with the self-adjoint operator H in $L^2(R^N)$.

For $s\in R^1$, $L_s^2(R^N)=L_s^2$ denotes the weighted L^2 space $L^2(R^N, (1+|x|^2)^s dx)$, and $\|\ \ \|_s$ and $(\ ,\ )_s$ stand for its norm and inner product:

$$\begin{cases} (f, g)_s=\int f(x)\overline{g(x)}(1+|x|^2)^s dx, \\ \|f\|_s=(f, f)_s^{1/2}. \end{cases} \tag{1.6}$$

We also use the following notation:

$$\|T\|_{s\to r}=\|T\|_{L_s^2\to L_r^2}, \qquad s, r\in R^1 \tag{1.7}$$

for any bounded linear operator T from L_s^2 into L_r^2.

Our main result is the following theorem.

THEOREM 1.1. *Let Assumptions* (L) *and* (S) *be satisfied, and let* $s\geqq 0$ *and* $\varepsilon>0$ *be arbitrarily fixed. Then there exist a* C^∞ *function* $\chi(\lambda)=\chi_{s,\varepsilon}(\lambda)$ *on* R^1 *as above with* $R_0=R_{0,s,\varepsilon}$ *large enough and a constant* $C_{s,\varepsilon}>0$ *such that*

$$\|\chi e^{-itH}\|_{s\to -s}\leqq C_{s,\varepsilon}\langle t\rangle^{-s+\varepsilon}, \qquad t\in R^1. \tag{1.8}$$

The analogous estimates were obtained by e.g. Rauch [11], Jensen and Kato [3] and Murata [7~9] for some short-range potentials. Our result is new in the following two points: i) We include the long-range potentials and some singular potentials and ii) our estimate (1.8) is quite close to the best possible estimate $\|\chi e^{-itH}\|_{s\to -s}\leqq C_s\langle t\rangle^{-s}$, which holds for the unperturbed Hamiltonian $H=H_0$.

In proving Theorem 1.1, we shall construct the total approximate propagator $E(t)$ ($t\geqq 0$) which approximates the behavior of χe^{-itH} when $t\to\infty$ in some sense. The construction of $E(t)$ consists of two steps. We shall first construct the outgoing approximate propagator $E_+(t)$, which describes the outgoing behavior of the quantum mechanical particles. We shall next construct the incoming approximate propagator $E_-(t)$. $E(t)$ will then be defined by $E(t)=E_+(t)+E_-(t)$, roughly speaking.

The outgoing part $E_+(t)$ will be constructed in a form of the Fourier integral operator in the sense of Kitada and Kumano-go [5]. The method of construction and estimation of $E_+(t)$ is similar to that of the "approximate propagator" in Kitada and Yajima [6]. The incoming part will need some geometrical considerations regarding the location of the particles. In this respect, we shall follow Murata's idea [8] but some modifications will be necessary. In our construction of $E_-(t)$, we shall use the genuine propagator e^{-itH^L} for the long-range part $H^L=H_0+V^L$ of the Hamiltonian H. The estimation of $E_-(t)$ will be carried out by using the "ingoing approximate propagator" $\tilde{E}_-(-t)$ $(t\geqq 0)$, which will be constructed and estimated in a way quite similar to that of the outgoing part $E_+(t)$. Theorem 1.1 will then be proved by Laplace transform methods analogous to those in Vainberg [12], Rauch [11] and Murata [8].

The plan of the paper is as follows. In sections 2 and 3 we shall construct the outgoing and ingoing approximate propagators and give some basic estimates for them, leaving the proof of a key theorem to the Appendix. In section 4, we shall first construct the incoming propagator and give some crucial estimates for it by using the estimates obtained for the outgoing and ingoing propagators in sections 2 and 3. Then in the same section 4 we shall construct the total approximate propagator and state some estimates for it. In the final section 5, using those estimates, we shall prove our main theorem, Theorem 1.1. Appendix will be devoted to proving a key theorem in section 2.

We shall use the following notations and conventions throughout the paper. For multi-index $\alpha=(\alpha_1, \cdots, \alpha_N)$, $\alpha_j\in N$, $|\alpha|=\alpha_1+\cdots+\alpha_N$, $x^\alpha=x_1^{\alpha_1}\cdots x_N^{\alpha_N}$ and $\partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_N}^{\alpha_N}$. We write $\partial_{x_j}=\partial/\partial x_j$, $\partial_x=(\partial_{x_1}, \cdots, \partial_{x_N})$, $D_x=-i\partial_x$, and $D_t=-i\partial_t$. For any $x\in R^m$ $(m\geqq 1)$, we write $\langle x\rangle=\sqrt{1+|x|^2}$. $H_s^m=H_s^m(R^N)$ $(m, s\in R^1)$ is the weighted Sobolev space with the norm $\|f\|_{H_s^m}=\|\langle x\rangle^s\langle D_x\rangle^m f\|_{L^2}$. $\mathcal{S}$ denotes the Schwarz space of rapidly decreasing functions on R^N. $\mathcal{B}^\infty(\Omega)$ (Ω is a domain in R^N) is the set of all C^∞ functions on Ω whose derivatives are uniformly bounded in Ω. For $f(x)$, $g(x)\in C^\infty(R^N)$, $f\sim g$ means that $f(x)-g(x)=O(\langle x\rangle^{-L})$ as $|x|\to\infty$ for any $L\geqq 0$. When we use the definition of the form $f(x)\sim\sum_{k=1}^\infty g_k(x)$, we mean that $g_k(x)=O(\langle x\rangle^{-L_k})$, $L_1\leqq L_2\leqq\cdots\leqq L_k\leqq\cdots\to\infty$ $(k\to\infty)$ and that $f(x)$ is defined by $f(x)=\sum_{k=1}^\infty \chi(\langle x\rangle^{-1}\varepsilon_k^{-1})g_k(x)$ with $\chi\in C_0^\infty([0,\infty))$ such that $\chi(\theta)=1$ $(0\leqq\theta\leqq 1)$, $=0$ $(\theta\geqq 2)$ and with a suitable sequence $\{\varepsilon_k\}_{k=1}^\infty$ such that $\varepsilon_k\downarrow 0$ (as $k\to\infty$). For $a, b\in R^1$, $a\gg b$ means that $a>b$ and a is sufficiently away from b. For $p(x,\xi)$ and $q(\xi,y)\in\mathscr{B}^\infty(R^{2N})$, the corresponding pseudo-differential operators $p(X,D)$ and $q(D,Y)$ are defined by

$$
(1.9)\qquad \begin{cases} p(X,D)f(x)=O_s\text{-}\displaystyle\iint e^{i(x-y)\cdot\xi}p(x,\xi)f(y)dy\bar{d}\xi, \\ q(D,Y)f(x)=O_s\text{-}\displaystyle\iint e^{i(x-y)\cdot\xi}q(\xi,y)f(y)dy\bar{d}\xi, \end{cases}\qquad (f\in\mathcal{S}),
$$

where $\bar{d}\xi=(2\pi)^{-N}d\xi$ and $O_s\text{-}\iint\cdots dy\bar{d}\xi$ means the oscillatory integral (see e.g. [5]). For any Banach spaces X, Y, $B(X,Y)$ denotes the Banach space of all bounded linear operators from X into Y with operator norm $\|T\|_{X\to Y}$. We write $B(X)=B(X,X)$. When $X=L^2_s$ and $Y=L^2_r$ $(r,s\in R^1)$, we write $\|T\|_{s\to r}=\|T\|_{L^2_s\to L^2_r}$.

Acknowledgement: The author expresses his sincere appreciation to Professor M. Murata for his helpful discussions.

§ 2. Outgoing approximate propagators

In this and the next sections, we shall study the outgoing and ingoing behavior of the particles which are subject to the long-range potential $V^L(x)$. By "outgoing" or "ingoing" particle we mean the one which is pointing outward from the scatterer and leaving it to infinity as $t\to\infty$ or $t\to-\infty$. The method we adopt here is similar to that in Kitada and Yajima [6]. Namely we shall first study some properties of the classical orbits associated with the outgoing or ingoing particles. Then, utilizing those properties, we shall construct the outgoing or ingoing approximate propagators in the form of Fourier integral operators (cf. Kitada and Kumano-go [5]), whose phase function is a generating function of the classical orbits.

Let $\chi_0(x)\in C^\infty(R^N)$ and $\varphi(t)\in C^\infty(R^1)$ be real-valued C^∞ functions satisfying.

$$
(2.1)\qquad \chi_0(x)=\begin{cases} 1, & |x|\geqq 2 \\ 0, & |x|\leqq 1 \end{cases}
$$

and

$$
(2.2)\qquad \varphi(t)\begin{cases} =\log\langle t\rangle, & |t|\geqq 2, \\ \geqq 1, & 2\geqq|t|\geqq 1, \\ =1, & |t|\leqq 1. \end{cases}
$$

For $\rho \in (0, 1)$, we set

$$V_{\rho}(x) = V_{\rho}^{L}(x) = \chi_0(\rho x) V^{L}(x) \tag{2.3}$$

and

$$\begin{cases} V_{\rho}(t, x) = \chi_0\left(\dfrac{\varphi(t)}{\langle t \rangle} x\right) V_{\rho}(x), \\ H_{\rho}(t, x, \xi) = \dfrac{1}{2}|\xi|^2 + V_{\rho}(t, x), \\ H_{\rho}^{L}(x, \xi) = \dfrac{1}{2}|\xi|^2 + V_{\rho}(x). \end{cases} \tag{2.4}$$

Then the following proposition is obvious.

PROPOSITION 2.1. *Let Assumption* (L) *be satisfied, and let* $\rho \in (0, 1)$. *Then:*

i) *For* $|x| \leqq \rho^{-1}$,

$$V_{\rho}(t, x) = 0, \qquad V_{\rho}(x) = 0. \tag{2.5}$$

ii) *For* $|x| \geqq 2\langle t \rangle / \varphi(t)$,

$$V_{\rho}(t, x) = V_{\rho}(x). \tag{2.6}$$

iii) *For* $T > 0$, *let* $\rho^{-1} \geqq \sup_{0 \leqq t \leqq T} 2\langle t \rangle / \varphi(t)$. *Then for any* $0 \leqq t \leqq T$,

$$H_{\rho}(t, x, \xi) = H_{\rho}^{L}(x, \xi). \tag{2.7}$$

iv) *For any* α, *we have*

$$|\partial_x^{\alpha} V_{\rho}(t, x)| \leqq \tilde{C}_{\alpha} \rho^{\varepsilon_0} \langle t \rangle^{-|\alpha| - \varepsilon_0}, \qquad \varepsilon_0 = \varepsilon/3, \tag{2.8}$$

where the constant $\tilde{C}_{\alpha}$ *is independent of* t, x *and* ρ.

2.1. Classical orbits

The classical orbit $(q, p)(t, s) = (q, p)(t, s; y, \xi)$ is the solution $(q, p)(t)$ of the Hamilton equation

$$\begin{cases} \dfrac{dq}{dt}(t) = \nabla_{\xi} H_{\rho}(t, q(t), p(t)) = p(t), \\ \dfrac{dp}{dt}(t) = -\nabla_x H_{\rho}(t, q(t), p(t)) = -\nabla_x V_{\rho}(t, q(t)), \\ q(s) = y, \quad p(s) = \xi. \end{cases} \tag{2.9}$$

This is equivalent to the integral equation

$$(2.10)\qquad \begin{cases} q(t,s)=y+\int_s^t p(\tau,s)d\tau, \\ p(t,s)=\xi-\int_s^t \nabla_x V_\rho(\tau,q(\tau,s))d\tau. \end{cases}$$

The following two propositions are the special cases of Propositions 2.2, 2.3 and Corollary 2.4 of Kitada and Yajima [6].

PROPOSITION 2.2. *Let Assumption* (L) *be satisfied, and let* $\rho\in(0,1)$. *Then the solution* $(q,p)(t,s;y,\xi)$ *of* (2.10) *exists uniquely and is of class* C^∞ *in* $(y,\xi)\in R^{2N}$ *for each fixed* $(t,s)\in R^2$, *and the derivative* $\partial_\xi^\alpha\partial_y^\beta(q,p)\cdot(t,s;y,\xi)$ *is of class* C^1 *in* $(t,s;y,\xi)$ *for any* α *and* β. *Furthermore the following estimates hold*:

i) *There exists a constant* C_0 *independent of* ρ *such that for* $y,\xi\in R^N$ *and* $t\geqq s\geqq 0$ *or* $t\leqq s\leqq 0$:

$$(2.11)\qquad |p(s,t;y,\xi)-\xi|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0};$$

$$(2.12)\qquad \begin{cases} |\nabla_y q(s,t;y,\xi)-I|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0}, \\ |\nabla_y p(s,t;y,\xi)|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_0}; \end{cases}$$

$$(2.13)\qquad \begin{cases} |\nabla_\xi q(t,s;y,\xi)-(t-s)I|\leqq C_0\rho^{\varepsilon_0}|t-s|\langle s\rangle^{-\varepsilon_0}, \\ |\nabla_\xi p(t,s;y,\xi)-I|\leqq C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0}; \end{cases}$$

and for any α

$$(2.14)\qquad |\partial_\xi^\alpha[q(t,s;y,\xi)-y-(t-s)p(t,s;y,\xi)]|\leqq C_\alpha\rho^{\varepsilon_0}\min\{\langle t\rangle^{1-\varepsilon_0},|t-s|\langle s\rangle^{-\varepsilon_0}\}.$$

ii) *For* $|\alpha+\beta|\geqq 2$, *we have*

$$(2.15)\qquad \begin{cases} |\partial_\xi^\alpha\partial_y^\beta q(t,s;y,\xi)|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}|t-s|\langle s\rangle^{-\varepsilon_0}, \\ |\partial_\xi^\alpha\partial_y^\beta p(t,s;y,\xi)|\leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0} \end{cases}$$

where $C_{\alpha\beta}$ *is independent of* ρ,t,s,y *and* ξ *such that* $t\gtreqless s\gtreqless 0$.

PROPOSITION 2.3. *Let Assumption* (L) *be satisfied. Let* $\rho\in(0,1)$ *satisfy*

$$(2.16)\qquad C_0\rho^{\varepsilon_0}<1/2,$$

where C_0 *is the constant appeared in Proposition* 2.2-i). *Then for* $t\gtreqless s\gtreqless 0$ *there exist the inverse* C^∞ *diffeomorphisms* $x\longmapsto y^\pm(s,t;x,\xi)$ *and* $\xi\longmapsto$

$\eta^{\pm}(t, s; x, \xi)$ of the mappings $y \longmapsto x=q(s, t; y, \xi)$ and $\eta \longmapsto \xi=p(t, s; x, \eta)$, respectively. These mappings $y^{\pm}$ and $\eta^{\pm}$ are of class C^{∞} in (x, ξ) for each $t \gtreqless s \gtreqless 0$, and their derivatives $\partial_{\xi}^{\alpha}\partial_{x}^{\beta}y^{\pm}$ and $\partial_{\xi}^{\alpha}\partial_{x}^{\beta}\eta^{\pm}$ are of class C^{1} in $(t, s; x, \xi)$. Furthermore $y^{\pm}$ and $\eta^{\pm}$ satisfy the following properties:

i) $$\begin{cases} q(s, t; y^{\pm}(s, t; x, \xi), \xi)=x, \\ p(t, s; x, \eta^{\pm}(t, s; x, \xi))=\xi. \end{cases}$$

ii) $$\begin{cases} y^{\pm}(s, t; x, \xi)=q(t, s; x, \eta^{\pm}(t, s; x, \xi)), \\ \eta^{\pm}(t, s; x, \xi)=p(s, t; y^{\pm}(s, t; x, \xi), \xi). \end{cases}$$

iii) *There exists a constant $C_1>0$ independent of ρ such that the following estimates hold:*

a) $$\begin{cases} |\nabla_{x}y^{\pm}(s, t; x, \xi)-I| \leqq C_1\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0}, \\ |\nabla_{x}\eta^{\pm}(t, s; x, \xi)| \leqq C_1\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_0}. \end{cases}$$

b) *For any α*

$$\begin{cases} |\partial_{\xi}^{\alpha}[\eta^{\pm}(t, s; x, \xi)-\xi]| \leqq C_{\alpha}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0}, \\ |\partial_{\xi}^{\alpha}[y^{\pm}(s, t; x, \xi)-x-(t-s)\xi]| \leqq C_{\alpha}\rho^{\varepsilon_0}\min\{\langle t\rangle^{1-\varepsilon_0}, |t-s|\langle s\rangle^{-\varepsilon_0}\}. \end{cases} \tag{2.17}$$

iv) *For $|\alpha+\beta| \geqq 2$, we have*

$$\begin{cases} |\partial_{\beta}^{\alpha}\partial_{x}^{\beta}\eta^{\pm}(t, s; x, \xi)| \leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_0}, \\ |\partial_{\xi}^{\alpha}\partial_{x}^{\beta}y^{\pm}(s, t; x, \xi)| \leqq C_{\alpha\beta}\rho^{\varepsilon_0}\langle t-s\rangle\langle s\rangle^{-\varepsilon_0}, \end{cases} \tag{2.18}$$

where $C_{\alpha\beta}$ is independent of ρ, t, s, x and ξ.

v) *Set for $\sigma_0 \in (-1, 1)$*

$$\Gamma_{\pm}(\sigma_0) \equiv \{(x, \xi) \in R^{2N} \,|\, |x| \geqq 1,\ |\xi| \geqq 1,\ \pm x\cdot\xi \geqq \sigma_0|x||\xi|\}. \tag{2.19}$$

Then for any $\sigma_0 \in (-1, 1)$ there exist constants $c>0$ and $T=T(\sigma_0)>1$ such that

$$|y^{\pm}(s, t; x, \xi)| \geqq c(|x|+|t-s||\xi|) \geqq 2\langle t-s\rangle/\varphi(t-s) \tag{2.20}$$

for $(x, \xi) \in \Gamma_{\pm}(\sigma_0)$ and $t \gtreqless s \gtreqless 0$ satisfying $|t-s| \geqq T$.

Throughout the rest of the paper, $\rho \in (0, 1)$ will be taken so small that (2.16) is satisfied.

2.2 Phase functions

We define the phase function of the outgoing and ingoing approximate propagators as follows.

DEFINITION 2.4. Let $\rho \in (0, 1)$ satisfy (2.16). For $t \gtreqless s \gtreqless 0$, we define

$$\phi^{\pm}(t, y, \xi)=u(t, 0; y, \eta^{\pm}(t, 0; y, \xi)), \tag{2.21}$$

where

$$u(t, 0; y, \eta)=y\cdot\eta+\int_0^t \{H_\rho - x\cdot\nabla_x H_\rho\}(\tau, q(\tau, 0; y, \eta), p(\tau, 0; y, \eta))d\tau. \tag{2.22}$$

Then we have the following proposition, whose proof is similar to that of Proposition 2.6 in Kitada and Yajima [6], hence is omitted here.

PROPOSITION 2.5. *Let Assumption* (L) *be satisfied, and let* $\rho\in(0,1)$ *satisfy* (2.16). *Then for* $t\gtreqless 0$, $\phi^{\pm}(t, y, \xi)$ *satisfies*:

i) $\begin{cases} \nabla_y \phi^{\pm}(t, y, \xi)=\eta^{\pm}(t, 0; y, \xi), \\ \nabla_\xi \phi^{\pm}(t, y, \xi)=y^{\pm}(0, t; y, \xi). \end{cases}$

ii) $\begin{cases} \partial_t \phi^{\pm}(t, y, \xi)=H_\rho(t, \nabla_\xi \phi^{\pm}(t, y, \xi), \xi), \\ \phi^{\pm}(0, y, \xi)=y\cdot\xi. \end{cases}$

In particular, when $(y, \xi)\in\Gamma_{\pm}(\sigma_0)$ $(\sigma_0\in(-1, 1))$ *and* $|t|\geqq T(\sigma_0)$ *with* $T=T(\sigma_0)>1$ *large enough, one has*

$$\partial_t \phi^{\pm}(t, y, \xi)=H_\rho^L(\nabla_\xi \phi^{\pm}(t, y, \xi), \xi). \tag{2.23}$$

2.3. Outgoing approximate propagators

We now construct the outgoing approximate propagators, which describe the behavior of the outgoing particles. For our purpose of estimating χe^{-itH}, we need to use a more precise approximation than in Kitada and Yajima [6] to the outgoing behavior of the particles, and so we solve the transport equation in defining the symbol function $e_+(t, \xi, y)$ of the outgoing propagator $E_+(t)$.

Choose C^∞ functions $\gamma\in C^\infty(R^N)$ and $\psi_{\pm}\in C^\infty([-1, 1])$ such that

$$\gamma(\xi)=\begin{cases} 1, & |\xi|\geqq 2, \\ 0, & |\xi|\leqq 1 \end{cases} \tag{2.24}$$

and

$$\begin{cases} \psi_+(\sigma)=\begin{cases} 1, & 1\geqq\sigma\geqq\sigma_0, \\ 0, & \sigma_0'\geqq\sigma\geqq -1, \end{cases} \\ 0\leqq\psi_+(\sigma)\leqq 1, \\ \psi_+(\sigma)+\psi_-(\sigma)=1, \end{cases} \tag{2.25}$$

where $0>\sigma_0>\sigma_0'>-1$. Set

$$g_{\pm}(\xi, y)=\gamma(\xi)\gamma(y)\psi_{\pm}(\cos(\xi, y))\in C^\infty(R^{2N}),\ \cos(\xi, y)=\frac{\xi\cdot y}{|\xi||y|}. \tag{2.26}$$

We define the symbol $e_+(t,\xi,y)\sim\sum_{l=0}^{\infty} i^l e_+^l(t,\xi,y)$ $(t\geqq 0)$ as the solution of the transport equation

$$-\partial_t e_+^l(t,\xi,y)+\sum_{k=1}^{N}(\partial_{x_k}H_\rho)(t,y^+(0,t;y,\xi),\xi)(\partial_{\xi_k}e_+^l)(t,\xi,y)$$
$$+\frac{1}{2}\sum_{j,k=1}^{N}(\partial_{x_k}\partial_{x_j}H_\rho)(t,y^+(0,t;y,\xi),\xi)(\partial_{\xi_k}\partial_{\xi_j}\phi^+)(t,y,\xi)e_+^l(t,\xi,y)$$
$$+B_l(t,\xi,y)=0\qquad (l\geqq 0,\ B_0\equiv 0) \tag{2.27}$$

with the initial condition

$$\begin{cases} e_+^0(0,\xi,y)=g_+(\xi,y),\\ e_+^l(0,\xi,y)=0 \qquad (l\geqq 1).\end{cases} \tag{2.28}$$

Here $B_l(t,\xi,y)$ $(l\geqq 1)$ is defined inductively by

$$B_l(t,\xi,y)=\sum_{2\leqq|\alpha|\leqq l+1}\frac{1}{\alpha!}\partial_{\xi'}^{\alpha}\{(\partial_x^{\alpha}H_\rho)(t,\nabla_\xi\phi^+(t;\xi,y,\xi'),\xi)$$
$$\times e_+^{l+1-|\alpha|}(t,\xi',y)\}|_{\xi'=\xi}, \tag{2.29}$$

where

$$\nabla_\xi\phi^+(t;\xi,y,\xi')=\int_0^1\nabla_\xi\phi^+(t,y,\xi'+\theta(\xi-\xi'))d\theta. \tag{2.30}$$

Then the solutions $e_+^l(t,\xi,y)$ $(l\geqq 0)$ are given by the classical theory of first order partial differential equations as follows:

$$e_+^0(t,\xi,y)=\exp\left\{-\frac{1}{2}\sum_{l,k=1}^{N}\int_t^0(\partial_{x_k}\partial_{x_l}H_\rho)(\tau,y^+(0,\tau;y,\Xi(\tau)),\Xi(\tau))\right.$$
$$\left.\times(\partial_{\xi_k}\partial_{\xi_l}\phi^+)(\tau,y,\Xi(\tau))d\tau\right\}g_+(\eta^+(t,0;y,\xi),y) \tag{2.31}$$

and

$$e_+^l(t,\xi,y)=\exp\left\{-\frac{1}{2}\sum_{j,k=1}^{N}\int_t^0(\partial_{x_k}\partial_{x_j}H_\rho)(\tau,y^+(0,\tau;y,\Xi(\tau)),\Xi(\tau))\right.$$
$$\left.\times(\partial_{\xi_k}\partial_{\xi_j}\phi^+)(\tau,y,\Xi(\tau))d\tau\right\}$$
$$\times\int_t^0\exp\left\{-\frac{1}{2}\sum_{j,k=1}^{N}\int_t^\theta(\partial_{x_k}\partial_{x_j}H_\rho)(\tau,y^+(0,\tau;\ y,\Xi(\tau)),\Xi(\tau))\right.$$
$$\left.\times(\partial_{\xi_k}\partial_{\xi_j}\phi^+)(\tau,y,\Xi(\tau))d\tau\right\}B_l(\theta,\Xi(\theta),y)d\theta,\qquad (l\geqq 1) \tag{2.32}$$

where

$$\Xi(\tau)=p(\tau,0;y,\eta^+(t,0;y,\xi)). \tag{2.33}$$

REMARK. 1° From (2.28), (2.31) and the relations

$$e_+^0(\theta, \Xi(\theta), y)=\exp\Big\{-\frac{1}{2}\sum_{l,k=1}^{N}\int_\theta^0 (\partial_{x_k}\partial_{x_l}H_\rho)(\tau, y^+(0, \tau; y, \Xi(\tau)), \Xi(\tau)) \times(\partial_{\xi_k}\partial_{\xi_l}\phi^+)(\tau, y, \Xi(\tau))d\tau\Big\}g_+(\eta^+(t, 0; y, \xi), y), \quad \text{etc.,} \tag{2.34}$$

we can easily see that for some $\delta\in(0, \sigma_0'+1)$

$$B_l(t, \xi, y)=0, \quad e_+^l(t, \xi, y)=0 \qquad \text{for} \quad \cos(\xi, y)\leqq\sigma_0'-\delta \tag{2.35}$$

and that

$$\langle y\rangle^{l/2}\langle\xi\rangle^{l/2}\langle\xi\rangle^{|\alpha|}\partial_\xi^\alpha e_+^l(t, \xi, y)\in\mathcal{B}^\infty([0, \infty)\times R_\xi^N\times R_y^N). \tag{2.36}$$

2° (2.36) justifies the asymptotic definition: $e_+(t, \xi, y)\sim\sum_{l=0}^{\infty} i^l e_+^l(t, \xi, y)$.

Now we define $E_+(t)$ as follows.

DEFINITION 2.7. For a sufficiently large $R>1$, we define for $f\in\mathcal{S}$ and $t\geqq0$

$$E_+(t)f(x)=\chi_0\Big(\frac{Rx}{\langle t\rangle}\Big)O_s\text{-}\iint e^{i(x\cdot\xi-\phi^+(t,y,\xi))}e_+(t, \xi, y)f(y)dy đ\xi. \tag{2.37}$$

Furthermore we set

$$P_+f(x)=E_+(0)f(x)=\chi_0(Rx)O_s\text{-}\iint e^{i(x-y)\cdot\xi}g_+(\xi, y)f(y)dy đ\xi. \tag{2.38}$$

Then we have the following key estimate.

THEOREM 2.8. *Let Assumption* (L) *be satisfied, and set* $K_+^L(t)=(D_t+H_\rho^L)E_+(t)$, $H_\rho^L=H_0+V_\rho^L$. *Then:*

i) *For any* $s\geqq0$ *we have with some* $C_s>0$

$$\|E_+(t)\|_{0\to-s}\leqq C_s\langle t\rangle^{-s}. \tag{2.39}$$

ii) *We can write* $K_+^L(t)$ *as*

$$K_+^L(t)f(x)=O_s\text{-}\iint e^{i(x\cdot\xi-\phi^+(t,y,\xi))}k_+^L(t, x, \xi, y)f(y)dy đ\xi, \tag{2.40}$$

where $k_+^L(t, x, \xi, y)$ *can be written as*

$$k_+^L(t, x, \xi, y)=\chi_0\Big(\frac{Rx}{\langle t\rangle}\Big)k_+^1(t, \xi, y)+k_+^2(t, \xi, y) \tag{2.41}$$

and $k_+^j(t,\xi,y)$ *satisfies*

$$|\partial_t^a\partial_\xi^\alpha\partial_y^\beta k_+^j(t,\xi,y)|\leqq C_{\alpha\beta,n}\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n}, \quad |a|\leqq 1, \quad j=1,2 \tag{2.42}$$

for any $\alpha,\beta,n\geqq 0, t\geqq 0, \xi, y\in R^N$ *provided that* $\rho\in(0,1)$ *is taken sufficiently small. In particular, we have*

$$\|K_+^L(t)\|_{0\to s}\leqq C_{s,n}\langle t\rangle^{-n} \tag{2.43}$$

for any $s\geqq 0$ *and* $n\geqq 0$.

The proof of this theorem is given in the appendix.

The following proposition will be used in section 4 when dealing with the short-range part $(V^L(x)-V_\rho^L(x))+V^S(x)$.

PROPOSITION 2.9. *Let Assumptions* (L) *and* (S) *be satisfied, and let* $\delta\in[0,1/2)$ *be the constant appeared in Assumption* (S). *Then*:

i) *For* $t>0$ *and any* $\sigma\geqq 0$

$$\|\langle x\rangle^{-\sigma}(H_0+1)^\delta E_+(t)\|_{0\to 0}\leqq C_\sigma t^{-2\delta}\langle t\rangle^{2\delta-\sigma}. \tag{2.44}$$

ii) *Set* $K_+(t)=(D_t+H)E_+(t)=(D_t+H_\rho^L+V^L-V_\rho^L+V^S)E_+(t)$. *Then we have*

$$\|K_+(t)\|_{0\to s}\leqq C_{s,n}t^{-2\delta}\langle t\rangle^{-n}, \qquad t>0 \tag{2.45}$$

for any $s\geqq 0$ *and* $n\geqq 0$.

iii) *For* $\theta\geqq 0$ *satisfying* $0\leqq\theta+\delta<1/2$ *and for any* $s\geqq 0$ *and* $n\geqq 0$

$$\limsup_{h\downarrow 0} h^{-\theta}\|K_+(t+h)-K_+(t)\|_{0\to s}\leqq C_{n,s,\theta}t^{-2(\delta+\theta)}\langle t\rangle^{-n}, \quad t>0. \tag{2.46}$$

PROOF. The proof of i) is similar to that of Proposition 4.5 of [6], hence is omitted here. Then ii) follows from i), Assumption (S) and Theorem 2.8-ii). iii) is proved in a way similar to that in the proof of i) by using Proposition 2.5-ii) and Theorem 2.8-(2.42) with $|a|=1$. □

REMARK. We can give another proof of the main theorems of Kitada and Yajima [6] if we use our Theorem 2.8 instead of Theorem 4.4 of [6]. In this new proof we need not use the parameter $r\geqq 1$, but we use the compactness of the relevant operators. The details will be discussed elsewhere.

§ 3. Ingoing approximate propagators

In this section, we shall study the operator $F(t)=(I-P_{+})\ \varphi(H_{\rho}^{L})e^{-itH_{\rho}^{L}}$ $(t\geqq 0)$ with $\varphi\in C^{\infty}(R^{1})$ satisfying $\varphi(\lambda)=1$ (for large $|\lambda|$), $=0$ (for small $|\lambda|$), and construct the ingoing propagator $\tilde{E}_{-}(-t)$ $(t\geqq 0)$ in such a way that $\tilde{E}_{-}(-t)^{*}$ approximates $F(t)$ in some sense (see Theorem 3.2 below). This property of $\tilde{E}_{-}(-t)$ will be used in the next section in estimating the incoming propagators. Because of the existence of the factor $I-P_{+}$ in $F(t)$, the ingoing propagator $\tilde{E}_{-}(-t)$ $(t\geqq 0)$ can be constructed in a way similar to that for $E_{+}(t)$ $(t\geqq 0)$ in the previous section, hence it has the properties appropriate for its name. In this section, we sometimes write $H_{\rho}^{L}=H_{0}+V_{\rho}^{L}$ only as H, omitting the super- and sub- scripts L and ρ. We begin with studying the properties of the operator $\varphi(H)=\varphi(H_{\rho}^{L})$.

Let $\chi_{\infty}, \chi_{1}\in C^{\infty}(R^{N})$ satisfy

$$(3.1)\qquad \begin{cases} 0\leqq\chi_{\infty}(\xi),\ \chi_{1}(\xi),\ \chi_{2}(\xi)\leqq 1; \\ \chi_{1}(\xi)=\begin{cases} 0, & |\xi|\geqq 3, \\ 1, & |\xi|\leqq 2; \end{cases} \end{cases}$$

and

$$(3.2)\qquad \chi_{1}(\xi)+\chi_{\infty}(\xi)=1.$$

Choose $\varphi_{1}(\lambda)\in C_{0}^{\infty}(R^{1})$ such that

$$(3.3)\qquad \begin{cases} 0\leqq\varphi_{1}(\lambda)\leqq 1, \\ \varphi_{1}(\lambda)=\begin{cases} 0, & |\lambda|\geqq 6, \\ 1, & |\lambda|\leqq 5 \end{cases} \end{cases}$$

and set

$$(3.4)\qquad \varphi(\lambda)=1-\varphi_{1}(\lambda).$$

Moreover set

$$(3.5)\qquad \phi(t, y, \xi)=\begin{cases} \phi^{+}(t, y, \xi), & t\geqq 0, \\ \phi^{-}(t, y, \xi), & t\leqq 0, \end{cases}$$

and let $u(t, \xi, y)\sim\sum_{l=1}^{\infty}i^{l}u^{l}(t, \xi, y)$ $(t\in R^{1})$ be defined by the solution $u^{l}(t, \xi, y)$ of the transport equation (2.27) (for $t\leqq 0$, replace the superscript "+" by "−") with the initial condition

$$(3.6)\qquad u^{0}(0, \xi, y)=1,\quad u^{l}(0, \xi, y)=0\quad (l\geqq 1).$$

Then denoting the inverse Fourier transform of $\varphi(\lambda)$ by $\tilde{\varphi}(t)=\delta(t)-\tilde{\varphi}_1(t)$, we define for $f\in\mathcal{S}$

$$\Gamma_j f(x)=\int_{-\infty}^{\infty}dt\tilde{\varphi}(t)\iint e^{i(x\cdot\xi-\phi(t,y,\xi))}u(t,\xi,y)\chi_j(\xi)f(y)dy đ\xi, \tag{3.7}$$

where $j=1$ or ∞.

PROPOSITION 3.1. *Let Assumption* (L) *be satisfied, and let* $\rho\in(0,1)$ *be sufficiently small. Then for any integer* $m\geqq 0$, *we have*

$$\|\varphi(H)-\Gamma_\infty^*\|_{H_0^0\to H_m^m}<\infty. \tag{3.8}$$

Moreover, Γ_∞ *is represented as a pseudo-differential operator with a* $\mathscr{B}^\infty(R^{2N})$ *symbol.*

PROOF. We decompose $\varphi(H)$ as

$$\varphi(H)=\chi_\infty(D)\varphi(H)+\chi_1(D)\varphi(H).$$

Denoting the inner integral in the r.h.s. of (3.7) by $F_j(t)f(x)$, we have

$$\begin{aligned}\varphi(H)\chi_\infty(D)-\Gamma_\infty&=(\chi_\infty(D)-\varphi_1(H)\chi_\infty(D))-\left(F_\infty(0)-\int_{-\infty}^{\infty}\tilde{\varphi}_1(t)F_\infty(t)dt\right)\\&=\int_{-\infty}^{\infty}\tilde{\varphi}_1(t)F_\infty(t)dt-\varphi_1(H)\chi_\infty(D)\\&=\int_{-\infty}^{\infty}\tilde{\varphi}_1(t)(F_\infty(t)-e^{-itH}\chi_\infty(D))dt\\&=i\int_{-\infty}^{\infty}\tilde{\varphi}_1(t)e^{-itH}\int_0^t e^{i\tau H}(D_\tau+H)F_\infty(\tau)d\tau dt.\end{aligned} \tag{3.9}$$

In a way similar to the proof of Theorem 2.8 (cf. Appendix), we can show that the symbol function $f_\infty(\tau,\xi,y)$ of $(D_\tau+H)F_\infty(\tau)$ satisfies

$$|\partial_\xi^\alpha\partial_y^\beta f_\infty(\tau,\xi,y)|\leqq C_{\alpha\beta,l}\langle y\rangle^{-l}\langle\xi\tau\rangle^l$$

for any $l\geqq 0$. Since $\tilde{\varphi}_1(t)\in\mathcal{S}$, it follows from this and (3.9) that

$$\|\varphi(H)\chi_\infty(D)-\Gamma_\infty\|_{H_{-m}^m\to H_0^0}<\infty,\qquad m\geqq 0. \tag{3.10}$$

On the other hand, we can write

$$\int_{-\infty}^{\infty}\tilde{\varphi}_1(t)F_\infty(t)f\,dt=\iint e^{i(x-y)\cdot\xi}\left[\int_{-\infty}^{\infty}e^{i(y\cdot\xi-\phi(t,y,\xi))}u(t,\xi,y)\tilde{\varphi}_1(t)dt\right]\chi_\infty(\xi)f(y)dy đ\xi.$$

Applying the integration by parts to the integral in [ ] above by using the differential operator

$$L=\frac{1-i\partial_t(y\cdot\xi-\phi(t,y,\xi))\cdot\partial_t}{\langle\partial_t(y\cdot\xi-\phi(t,y,\xi))\rangle^2},$$

we have

$$\left|\partial_\xi^\alpha\partial_y^\beta\left[\int_{-\infty}^{\infty}e^{i(y\cdot\xi-\phi(t,y,\xi))}({}^tL)^l(u(t,\xi,y)\tilde{\varphi}_1(t))dt\right]\right|\leqq C_{\alpha\beta,l}\langle\xi\rangle^{-2l+|\alpha|},$$

if ρ in section 2 is taken small enough. Thus we have

$$\left\|\int_{-\infty}^{\infty}\tilde{\varphi}_1(t)F_\infty(t)dt\right\|_{H_0^{-2m}\to H_0^0}<\infty,\qquad m\geqq 0.$$

Combining this with the obvious estimate $\|\varphi_1(H)\chi_\infty(D)\|_{H_0^{-2m}\to H_0^0}<\infty$, we get from (3.9)

$$\|\varphi(H)\chi_\infty(D)-\Gamma_\infty\|_{H_0^{-2m}\to H_0^0}<\infty,\qquad m\geqq 0. \tag{3.11}$$

An interpolation between (3.10) and (3.11) yields

$$\|\varphi(H)\chi_\infty(D)-\Gamma_\infty\|_{H_{-m}^{-m}\to H_0^0}<\infty,\qquad m\geqq 0. \tag{3.12}$$

Then we have now only to show

$$\|\varphi(H)\chi_1(D)-\Gamma_1\|_{H_{-m}^{-m}\to H_0^0}<\infty \tag{3.13}$$

and

$$\|\Gamma_1\|_{H_{-m}^{-m}\to H_0^0}<\infty. \tag{3.14}$$

We first prove (3.13). Choosing C^∞ functions $\tilde{\chi}_0(\tau)$, $\tilde{\chi}_\infty(\tau)\in C^\infty(R^1)$ such that

$$\begin{cases} 0\leqq\tilde{\chi}_0(\tau),\ \tilde{\chi}_\infty(\tau)\leqq 1,\\ \tilde{\chi}_0(\tau)+\tilde{\chi}_\infty(\tau)=1,\\ \tilde{\chi}_0(\tau)=\begin{cases}0, & |\tau|\geqq C_2,\\ 1, & |\tau|\leqq C_1\end{cases} \quad (C_2>C_1>6),\end{cases} \tag{3.15}$$

we define $T_l(\tau,t)$ by

$$T_l(\tau,t)f(x)=O_s\text{-}\iint e^{i(x\cdot\xi-\phi(\tau,y,\xi))}u(\tau,\xi,y)\tilde{\chi}_l\left(\frac{\langle y\rangle}{\langle t\rangle}\right)\chi_1(\xi)f(y)dy\bar{d}\xi \tag{3.16}$$

for $l=0,\infty$. Then we have

$$(3.17)\qquad (\varphi(H)\chi_1(D)-\Gamma_1)f(x)=-\int_{-\infty}^{\infty}dt\tilde{\varphi}_1(t)e^{-itH}$$
$$\times\int_0^t e^{i\tau H}(D_\tau+H)(T_0(\tau,t)f+T_\infty(\tau,t)f)(x)d\tau.$$

We can write for some function $k_l(\tau,\xi,y)$

$$(3.18)\qquad (D_\tau+H)T_l(\tau,t)f(x)=O_s\text{-}\iint e^{i(x\cdot\xi-\phi(\tau,y,\xi))}k_l(\tau,\xi,y)$$
$$\times\tilde{\chi}_l\left(\frac{\langle y\rangle}{\langle t\rangle}\right)f(y)dyd\!\!\bar{}\,\xi,\qquad l=0,\infty.$$

For $l=\infty$, since $\langle y\rangle\geqq C_1\langle t\rangle\geqq C_1\langle\tau\rangle$ and $|\xi|\leqq 3$ when $\tilde{\chi}_\infty\left(\frac{\langle y\rangle}{\langle t\rangle}\right)\neq 0$ and $\chi_1(\xi)\neq 0$, we can prove in a way quite similar to that in the proof of Theorem 2.8 (cf. Appendix) that

$$(3.19)\qquad |\partial_\xi^\alpha\partial_y^\beta k_\infty(\tau,\xi,y)|\leqq C_{\alpha\beta,m}\langle|y|+\langle\tau\rangle\langle\xi\rangle\rangle^{-m}$$

for any $m\geqq 0$. Thus we have

$$(3.20)\qquad \|(D_\tau+H)T_\infty(\tau,t)\|_{H^{-m}_{-m}\to H^0_0}\leqq C_m,\qquad m\geqq 0.$$

For $l=0$, because of the existence of the factor $\chi_1(\xi)$ in (3.16), the symbol $k_0(\tau,\xi,y)$ of $(D_\tau+H)T_0(\tau,t)$ belongs to $\mathcal{B}^\infty(R^1_\tau\times R^N_\xi\times R^N_y)$ and decays rapidly with respect to ξ as $|\xi|\to\infty$. Since $\langle t\rangle^{-1}\leqq C_2\langle y\rangle^{-1}$ when $\tilde{\chi}_0\left(\frac{\langle y\rangle}{\langle t\rangle}\right)\neq 0$, and since $\tilde{\varphi}_1(t)\in\mathcal{S}$, we thus have

$$(3.21)\qquad \tilde{\varphi}_1(t)\|(D_\tau+H)T_0(\tau,t)\|_{H^{-m}_{-m}\to H^0_0}\leqq C_m$$

for some $C_m>0$. From (3.20)-(3.21) and (3.17) we obtain (3.13).

We next prove (3.14). We can write

$$(3.22)\qquad \Gamma_1f(x)=\iint e^{i(x-y)\cdot\xi}\int_{-\infty}^{\infty}\varphi(\lambda)\left[\int_{-\infty}^{\infty}e^{i(y\cdot\xi-\phi(t,y,\xi)+t\lambda)}u(t,\xi,y)dt\right]d\lambda$$
$$\times\chi_1(\xi)f(y)dyd\!\!\bar{}\,\xi.$$

On the support of $\varphi(\lambda)\chi_1(\xi)$, the differential operator

$$(3.23)\qquad L=\frac{\partial_t}{\lambda-\partial_t\phi(t,y,\xi)}$$

is well-defined if $\rho\in(0,1)$ is sufficiently small (see Proposition 2.5-ii)).

Thus the formula in the parenthesis [ ] of (3.22) is equal to

$$\int_{-\infty}^{\infty} e^{i(y\cdot\xi-\phi(t,y,\xi)+t\lambda)}({}^tL)^l(u(t,\xi,y))dt. \tag{3.24}$$

From the expression of $u(t,\xi,y)$ (cf. (2.31)-(2.32)), we have

$$\begin{aligned}|({}^tL)^l u(t,\xi,y)| &\leqq C_l\langle t\rangle^{-3l/4}\langle y+t\xi\rangle^{-l/4}\langle\lambda\rangle^{-l} \\ &\leqq C_l\langle t\rangle^{-l/2}\langle y\rangle^{-l/4}\langle\lambda\rangle^{-l}\end{aligned} \tag{3.25}$$

on $\operatorname{supp}\varphi(\lambda)\chi_1(\xi)$ for any $l\geqq 0$. Thus we obtain (3.14). □

Set

$$g_-^\infty(\xi,y)\sim\sum_\alpha\frac{1}{\alpha!}\partial_\xi^\alpha D_y^\alpha g_-(\xi,y), \tag{3.26}$$

and

$$p(\xi,y)=\int_{-\infty}^{\infty}\varphi(\lambda)\int_{-\infty}^{\infty}e^{i(y\cdot\xi-\phi(t,y,\xi)+t\lambda)}u(t,\xi,y)dtd\lambda. \tag{3.27}$$

Define $\tilde{e}_-(t,\xi,y)\sim\sum_{l=0}^{\infty}i^l\tilde{e}_-^l(t,\xi,y)$ $(t\leqq 0)$ as the solution of the transport equation (2.27) with "+" replaced by "−" and with the initial condition

$$\begin{aligned}\tilde{e}_-^0(0,\xi,y)&=\chi_\infty(\xi)\sum_{|\alpha|<\infty}\frac{1}{\alpha!}\partial_z^\alpha[p(\xi,z)\cdot D_\eta^\alpha\{(1-\gamma(\eta))+\gamma(\eta)(1-\chi_0(Ry)) \\ &\quad+\gamma(\eta)(1-\gamma(z))\chi_0(Ry)+g_-^\infty(\eta,z)\chi_0(Ry)\}]\Big|_{\substack{z=y\\ \eta=\xi}}, \\ \tilde{e}_-^l(0,\xi,y)&=0 \qquad (l\geqq 1).\end{aligned} \tag{3.28}$$

We then define the ingoing approximate propagator $\tilde{E}_-(t)$ for $t\leqq 0$ by

$$\tilde{E}_-(t)f(x)=O_s\text{-}\iint e^{i(x\cdot\xi-\phi^-(t,y,\xi))}\tilde{e}_-(t,\xi,y)f(y)dy\,d\!\!\,\bar{}\,\xi. \tag{3.29}$$

Then we have the following estimate, which will be used in section 4.

THEOREM 3.2. *Let Assumption* (L) *be satisfied, and set* $\tilde{K}_-(t)=(D_t+H_\rho^L)\tilde{E}_-(t)$ *for* $t\leqq 0$. *Then we can write* $\tilde{K}_-(t)$ *as*

$$\tilde{K}_-(t)f(x)=O_s\text{-}\iint e^{i(x\cdot\xi-\phi^-(t,y,\xi))}\tilde{k}_-(t,\xi,y)f(y)dy\,d\!\!\,\bar{}\,\xi, \tag{3.30}$$

and the symbol $\tilde{k}_-(t,\xi,y)$ *satisfies*

$$|\partial_\xi^\alpha\partial_y^\beta\tilde{k}_-(t,\xi,y)|\leqq C_{\alpha\beta,n}\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n} \tag{3.31}$$

for any $\alpha, \beta, n \geqq 0, t \leqq 0, \xi, y \in R^N$ *provided that* $\rho \in (0, 1)$ *is taken sufficiently small. In particular, we have*

$$\sup_{t \geqq 0} \| (I - P_+) \Gamma_\infty^* U(t) - \tilde{E}_-(-t)^* \|_{H_0^0 \to H_m^m} < \infty, \qquad m \geqq 0, \tag{3.32}$$

where $U(t) = e^{-itH} = e^{-itH_\rho^L}$, *hence by Proposition* 3.1

$$\sup_{t \geqq 0} \| (I - P_+) \varphi(H) U(t) - \tilde{E}_-(-t)^* \|_{H_0^0 \to H_m^m} < \infty, \qquad m \geqq 0. \tag{3.33}$$

Proof. The estimate (3.31) is proved in a way quite similar to that of Theorem 2.8 (cf. Appendix). We prove (3.32). We have for $t \geqq 0$

$$\begin{aligned} & \| (I - P_+) \Gamma_\infty^* U(t) - \tilde{E}_-(-t)^* \|_{H_0^0 \to H_m^m} \\ & \leqq \| (I - P_+) \Gamma_\infty^* - \tilde{E}_-(0)^* \|_{H_0^0 \to H_m^m} + \| \tilde{E}_-(0)^* U(t) - \tilde{E}_-(-t)^* \|_{H_0^0 \to H_m^m}. \end{aligned} \tag{3.34}$$

By (3.28) and the definition of Γ_∞, the first term on the r. h. s. of (3.34) is finite for any $m \geqq 0$. The second term is estimated by using (3.31) as follows:

$$\begin{aligned} & = \| \tilde{E}_-(0) - U(t) \tilde{E}_-(-t) \|_{H_{-m}^{-m} \to H_0^0} \\ & \leqq \int_0^t \| D_\tau (U(\tau) \tilde{E}_-(-\tau)) \|_{H_{-m}^{-m} \to H_0^0} d\tau \\ & = \int_0^t \| \tilde{K}_-(-\tau) \|_{H_{-m}^{-m} \to H_0^0} d\tau \leqq C_m, \end{aligned} \tag{3.35}$$

where C_m is independent of $t \geqq 0$. □

§ 4. Total approximate propagators

The main purpose of this section is to construct the total approximate propagator $E(t)$ $(t \geqq 0)$ such that $E(0) = I$, and prove the weighted L^2-estimate for $E(t)$ and $K(t) \equiv (D_t + H) E(t)$, $H = H_0 + V$, $V = V^L + V^S$, which will be used in the next section in proving Theorem 1.1. In doing so, we shall first construct the incoming approximate propagator $E_-(t)$, which describes the incoming behavior of the particles. By "incoming" particle we mean the particle that is pointing inward to the scatterer and coming into it. Therefore it must become an outgoing or bounded particle when a certain finite time passed. Since we shall only consider the particle with sufficiently high total energy, it all becomes the outgoing one after a finite time. When constructing the incoming propagator, we must therefore take into account the location of each particle (which we label by $k\geqq 0$), and in accordance with it we determine the "incoming time" $(\tilde{\tau}_k)$ only after which the particle behaves like an outgoing one. We denote the incoming propagator by $E_{-,k}(t)$ $(t\geqq 0)$ corresponding to the location k. Then the total approximate propagator $E(t)$ is defined by $E(t)=E_+(t)+\sum_{k\geqq 0}E_{-,k}(t)+E_\infty(t)$ with some smoothing operator $E_\infty(t)$.

4.1. Incoming approximate propagators

Let $\delta\in(0,1)$ and let Z_δ be a finite set included in the sphere $S^{N-1}=\{x\in R^N\,|\,|x|=1\}$ such that

$$\bigcup_{\beta\in Z_\delta} B_\beta(\delta/2)\supset S^{N-1}, \tag{4.1}$$

where $B_\beta(\delta)$ denotes the open ball with center β and radius δ. We set for $\delta\in(0,1)$

$$J_\delta=\{\alpha\in R^N\,|\,\alpha=(1+\delta)^j\beta \text{ for some integer } j\geqq 0 \text{ and some } \beta\in Z_\delta\}. \tag{4.2}$$

Then

$$\bigcup_{\alpha\in J_\delta} B_\alpha(\delta|\alpha|)\supset\{x\in R^N\,|\,|x|\geqq 1\}. \tag{4.3}$$

Let $f\in C_0^\infty(R^N)$ satisfy

$$f(x)=\begin{cases}1, & |x|\leqq 1\\ 0, & |x|\geqq 2,\end{cases}\qquad 0\leqq f(x)\leqq 1 \tag{4.4}$$

and set for $\alpha\in J_\delta$

$$\tilde{f}_\alpha(x)=f\Big(\frac{x-\alpha}{\delta\langle\alpha\rangle}\Big). \tag{4.5}$$

Then

$$\begin{cases}\operatorname{supp}\tilde{f}_\alpha\subset\{x\in R^N\,|\,|x-\alpha|\leqq 2\delta\langle\alpha\rangle\},\\ \tilde{f}_\alpha(x)=1 \qquad \text{for}\quad |x-\alpha|\leqq\delta\langle\alpha\rangle,\\ |\partial_x^\gamma\tilde{f}_\alpha(x)|\leqq C_\gamma\langle x\rangle^{-|\gamma|}.\end{cases} \tag{4.6}$$

Thus for a sufficiently small $\delta\in(0,1)$

$$f_\alpha(x)=\tilde{f}_\alpha(x)\Big/\Big(\sum_{\beta\in J_\delta}\tilde{f}_\beta(x)\Big)\qquad(\alpha\in J_\delta,\ |x|\geqq 1) \tag{4.7}$$

is well-defined and there is a constant $R_1(>1)$ such that

$$\left\{\begin{array}{ll}\sum_{\alpha\in J_\delta} f_\alpha(x)=1, & |x|\geqq R_1,\\ |\partial_x^\gamma f_\alpha(x)|\leqq C_\gamma\langle x\rangle^{-|\gamma|}, & \alpha\in J_\delta,\ |x|\geqq 1,\end{array}\right. \tag{4.8}$$

where C_γ is independent of $\alpha\in J_\delta$ and $|x|\geqq 1$.

In the following, we shall take and fix σ_0 (in the previous sections) and σ_1 as

$$-1<\sigma_0<\sigma_1<0. \tag{4.9}$$

Set $\tau_0=0$ and

$$\tau_k=(1+\delta)^{k-1} \quad \text{for} \quad k\geqq 1, \tag{4.10}$$

and for any $\tau\geqq 0$

$$\Gamma_-(\sigma_1,\tau)=\{(x,\xi)\in\Gamma_-(\sigma_1)\,|\cos(x+\tau\xi,\xi)\leqq\sigma_1\}. \tag{4.11}$$

Let $I_0=\phi$ ($=$the empty set) and let for $k\geqq 1$

$$I_k=\Big\{(\alpha,\beta)\in J_\delta^2\,\Big|\,(\alpha,\beta)\notin\bigcup_{j=0}^{k-1}I_j,\ [(\operatorname{supp}f_\alpha)\times\{\beta\}]\cap[\Gamma_-(\sigma_1,\tau_{k-1})\backslash\Gamma_-(\sigma_1,\tau_k)]\neq\phi\Big\}. \tag{4.12}$$

Then

$$\bigcup_{k=1}^{\infty}I_k=\{(\alpha,\beta)\in J_\delta^2\,|\cos(\alpha,\beta)\leqq\sigma_1'\} \tag{4.13}$$

for some $\sigma_1'\in(\sigma_1,0)$. Let $\psi\in C^\infty(R^N)$ satisfy

$$\psi(\xi)=\left\{\begin{array}{ll}1, & |\xi|\geqq S_1+1\\ 0, & |\xi|\leqq S_1,\end{array}\right. \tag{4.14}$$

for $S_1>2\max(R_1,3)$. We now define for $k\geqq 1$

$$\varphi_k(x,\xi)=\sum_{(\alpha,\beta)\in I_k}f_\alpha(x)f_\beta(\xi)\psi(x)\psi(\xi). \tag{4.15}$$

Then, for a sufficiently small $\delta\in(0,1)$, we have

$$\sum_{k\geqq 1}\varphi_k(x,\xi)=\psi(x)\psi(\xi) \quad \text{for} \quad \cos(x,\xi)\leqq\sigma_0, \tag{4.16}$$

and

$$\operatorname{supp}\varphi_k(x,\xi)\subset\{(x,\xi)\in R^{2N}\,|\cos(x,\xi)\leqq\sigma_1'\}. \tag{4.17}$$

We put

$$(4.18)\qquad \begin{cases} \tilde{g}_+^\infty(x,\xi)\sim \sum_\alpha \dfrac{1}{\alpha!}(\partial_\xi^\alpha D_x^\alpha g_+)(\xi,x), \\ g_+^\infty(x,\xi)=\tilde{g}_+^\infty(x,\xi)\psi(x)\psi(\xi), \end{cases}$$

and define

$$(4.19)\qquad \begin{cases} g_{-,k}(x,\xi)=(1-\tilde{g}_+^\infty(x,\xi))\varphi_k(x,\xi) \qquad (k\geqq 1), \\ g_{-,0}(x,\xi)=\psi(\xi)-\sum_{k\geqq 1} g_{-,k}(x,\xi)-g_+^\infty(x,\xi). \end{cases}$$

Then we have the following proposition.

PROPOSITION 4.1. *Let $\delta\in(0,1)$ be sufficiently small. Then we have*

$$(4.20)\qquad \begin{cases} \operatorname{supp} g_{-,k}\subset\{(x,\xi)\in R^{2N}\mid a_2\tau_k|\xi|\geqq|x|\geqq a_1\tau_k|\xi|, \quad \cos(x,\xi)\leqq\sigma_1'\} \\ \textit{for some } a_2>a_1>0 \textit{ and any } k\geqq 1; \end{cases}$$

$$(4.21)\qquad g_{-,0}(x,\xi)=\psi(\xi)(1-\psi(x));$$

and

$$(4.22)\qquad |\partial_x^\alpha\partial_\xi^\beta g_{-,k}(x,\xi)|\leqq C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-|\beta|}, \qquad k\geqq 0.$$

Proof is clear by definition.

We remark here that k's in the above specify the location of the particles and that τ_k can be interpreted as a kind of "incoming time" for the particles with the location "k". Practically we shall adopt $\tilde{\tau}_k=A\tau_k$ $(A\gg 1)$ instead of τ_k itself as our incoming time, because the behavior of the particles near the scatterer is too complicated to be described by the outgoing propagator $E_+(t)$.

Let $\tilde{\Phi}\in C_0^\infty(R^1)$ satisfy

$$(4.23)\qquad \tilde{\Phi}(\lambda)=\begin{cases} 1, & a_2\geqq\lambda\geqq a_1, \\ 0, & 2a_2\leqq\lambda \text{ or } \lambda\leqq a_1/2 \end{cases}$$

for the constants $a_2>a_1>0$ in (4.20), and set for $k\geqq 1$

$$(4.24)\qquad \Phi_k(\xi,y)=\tilde{\Phi}\Big(\frac{\langle y\rangle}{\tau_k\langle\xi\rangle}\Big).$$

For $k=0$, let $\Phi_0(\xi,y)\equiv 1$. Taking a real number R_2 and a function $\tilde{\chi}\in C_0^\infty(R^1)$ such that $R_2\gg\max\{S_1,a_2\}$ and

$$(4.25)\qquad \tilde{\chi}(\lambda)=\begin{cases} 1, & |\lambda|\leqq R_2, \\ 0, & |\lambda|\geqq 2R_2, \end{cases}$$

we set

$$(4.26)\qquad \begin{cases} \chi_k(\xi, y) = \tilde{\chi}\left(\dfrac{\langle y\rangle}{\tau_k\langle\xi\rangle}\right), & k \geqq 1, \\ \chi_0(\xi, y) = \tilde{\chi}\left(\dfrac{R_2\langle y\rangle}{S_1\langle\xi\rangle}\right), & \end{cases}$$

and define

$$(4.27)\qquad \chi_k = \chi_k(D, Y), \quad \Phi_k = \Phi_k(D, Y), \quad g_{-,k} = g_{-,k}(X, D), \quad k \geqq 0.$$

Now we can define the incoming approximate propagator $E_{-,k}(t)$. Let $\delta \in (0, 1)$ be sufficiently small. Let $k \geqq 0$ be an integer and A be a constant such that $A \geqq 2R_2 + 3$, and let $\varphi \in C^\infty(R^1)$ be the function defined by (3.4). Set $\tilde{\tau}_k = A\tau_k$; $\tilde{\tau}_0 = A$; and

$$(4.28)\qquad \tilde{U}(t) = \varphi(H_\rho^L) e^{-itH_\rho^L}.$$

Furthermore let $\Phi \in C_0^\infty([0, \infty))$ be the function satisfying

$$(4.29)\qquad \Phi(t) = \begin{cases} 1, & t \leqq 1/2, \\ 0, & t \geqq 1, \end{cases}$$

and let P_+ be defined by (2.38).

DEFINITION 4.2. i) For $t \in [0, \tilde{\tau}_k)$,

$$(4.30)\qquad \begin{aligned} E_{-,k}(t) = [&\tilde{U}(t)\chi_k + E_+(t)(1 - \varphi(H_\rho^L)\chi_k) \\ &+ \Phi(t)(I - P_+)(1 - \varphi(H_\rho^L)\chi_k)] g_{-,k}\Phi_k. \end{aligned}$$

ii) For $t \geqq \tilde{\tau}_k$,

$$(4.31)\qquad \begin{aligned} E_{-,k}(t) = [&E_+(t - \tilde{\tau}_k)\tilde{U}(\tilde{\tau}_k)\chi_k + \Phi(t - \tilde{\tau}_k)(I - P_+)\tilde{U}(\tilde{\tau}_k)\chi_k \\ &+ E_+(t)(1 - \varphi(H_\rho^L)\chi_k)] g_{-,k}\Phi_k. \end{aligned}$$

iii) For $t \geqq 0$, set

$$(4.32)\qquad E_-(t) = \sum_{k \geqq 0} E_{-,k}(t);$$

$$(4.33)\qquad \begin{cases} K_{-,k}^L(t) = (D_t + H_\rho^L) E_{-,k}(t), \\ K_-^L(t) = \sum_{k \geqq 0} K_{-,k}^L(t); \end{cases}$$

and

$$(4.34)\qquad \begin{cases} K_{-,k}(t)=(D_t+H)E_{-,k}(t), \\ K_-(t)=\sum_{k\geqq 0} K_{-,k}(t). \end{cases}$$

Then we have the following theorem.

THEOREM 4.3. *Let Assumption* (L) *be satisfied, and let* $k\geqq 0$ *be an integer. Then:*

i) $E_{-,k}(t)$ *is strongly continuous in* $t\geqq 0$ *as an operator in* $B(L^2)$, *hence as an operator in* $B(L^2_s, L^2_{-s})$ *for any* $s\geqq 0$.

ii) *We have for* $s\geqq 0$

$$(4.35)\qquad \|E_{-,k}(t)\|_{s\to 0}\leqq C_s\langle\tilde{\tau}_k\rangle^{-s},\qquad t\in[0,\tilde{\tau}_k),$$

$$(4.36)\qquad \|E_{-,k}(t)\|_{s\to -s}\leqq C_s\langle t-\tilde{\tau}_k\rangle^{-s}\langle\tilde{\tau}_k\rangle^{-s},\qquad t\geqq\tilde{\tau}_k,$$

where the constant $C_s>0$ *is independent of* $k\geqq 0$.

iii) *We have for* $s\geqq 0$ *and* $n\geqq 0$

$$(4.37)\qquad \|K^L_{-,k}(t)\|_{s\to s}\leqq\begin{cases} C_s\langle\tilde{\tau}_k\rangle^{-s}, & t\in[0,\tilde{\tau}_k), \\ C_{s,n}\langle t-\tilde{\tau}_k\rangle^{-n}\langle\tilde{\tau}_k\rangle^{-s}, & t\geqq\tilde{\tau}_k, \end{cases}$$

where C_s *and* $C_{s,n}$ *are independent of* $k\geqq 0$.

PROOF. i) is clear from the definition of $E_{-,k}(t)$.

ii) Since we have from (4.23) and (4.24)

$$(4.38)\qquad |\partial_\xi^\alpha\partial_y^\beta(\Phi_k(\xi,y)\langle y\rangle^{-s})|\leqq C_{\alpha\beta,s}\langle\tilde{\tau}_k\rangle^{-s}\langle\xi\rangle^{-s}$$

for $k\geqq 1$, we get by Calderón-Vaillancourt theorem ([2])

$$(4.39)\qquad \|\Phi_k\|_{s\to 0}\leqq C_s\langle\tilde{\tau}_k\rangle^{-s},\qquad s\geqq 0.$$

Thus we have (4.35) from (4.30) by $\sup_{t\geqq 0}\|U(t)\|_{0\to 0}<\infty$ and (2.39) with $s=0$. For $t\geqq\tilde{\tau}_k$, the first and third terms in the r. h. s. of (4.31) clearly satisfy (4.36) by (2.39). For estimating the remainder term in (4.31), we have only to prove

$$(4.40)\qquad \sup_{k\geqq 0}\|(I-P_+)\tilde{U}(\tilde{\tau}_k)\chi_k\|_{H^0_0\to H^m_m}<\infty$$

for any $m\geqq 0$. But by Theorem 3.2-(3.33), this is reduced to the estimate

$$(4.41)\qquad \sup_{k\geqq 0}\|\tilde{E}_-(-\tilde{\tau}_k)^*\chi_k\|_{H^0_0\to H^m_m}<\infty,\qquad m\geqq 0.$$

To prove this, we write $T=\tilde{\tau}_k$. Then we have

$$(4.42)\qquad (\tilde{E}_-(-T)^*\chi_k f)(x)=O_s\text{-}\iint e^{i(\phi^-(-T,x,\xi)-y\cdot\xi)}e_-(-T,\xi,x)\chi_k(\xi,y)f(y)dy\bar{d}\xi.$$

For (x,ξ,y) satisfying $\tilde{e}_-(-T,\xi,x)\chi_k(\xi,y)\neq 0$, we obtain from $A\geqq 2R_2+3$, (3.28) and the expression of $\tilde{e}_-(-T,\xi,x)$ (cf. (2.31)-(2.32)) that

$$(4.43)\qquad |\nabla_\xi\phi^-(-T,x,\xi)-y|\geqq c(|x|+T|\xi|)$$

for some $c>0$ independent of $k\geqq 0$. Setting

$$(4.44)\qquad L=\frac{1-i(\nabla_\xi\phi^-(-T,x,\xi)-y)\cdot\nabla_\xi}{\langle\nabla_\xi\phi^-(-T,x,\xi)-y\rangle^2},$$

we can write

$$(4.45)\qquad (\tilde{E}_-(-T)^*\chi_k f)(x)=O_s\text{-}\iint e^{i(\phi^-(-T,x,\xi)-y\cdot\xi)}({}^tL)^l \times\{\tilde{e}_-(-T,\xi,x)\chi_k(\xi,y)\}f(y)dy\bar{d}\xi.$$

The symbol function $a_{l,k}(T,x,\xi,y)=({}^tL)^l\{\tilde{e}_-(-T,\xi,x)\chi_k(\xi,y)\}$ satisfies by (4.43)

$$(4.46)\qquad |\partial_x^\alpha\partial_\xi^\beta\partial_y^\gamma a_{l,k}(T,x,\xi,y)|\leqq C_l\langle|x|+T|\xi|\rangle^{-l}$$

for some constant $C_l>0$ independent of $k\geqq 0$. Thus we obtain (4.41) by L^2-boundedness theorem ([1], [5]) for Fourier integral operators. We have proved ii).

iii) For $t\geqq\tilde{\tau}_k$, (4.37) follows from (4.39), (4.40) and Theorem 2.8-(2.43). For $t\in[0,\tilde{\tau}_k)$, we have

$$(4.47)\qquad K_{-,k}^L(t)=[K_+^L(t)(1-\varphi(H_\rho^L)\chi_k)-i\Phi'(t)(I-P_+)(1-\varphi(H_\rho^L)\chi_k) +\Phi(t)H_\rho^L(I-P_+)(1-\varphi(H_\rho^L)\chi_k)]g_{-,k}\Phi_k.$$

The first term on the r. h. s. of (4.47) clearly satisfies (4.37) for $t\in[0,\tilde{\tau}_k)$ by (4.39) and Theorem 2.8-(2.43). The estimate for the remainder terms follows from the estimate

$$(4.48)\qquad \sup_{k\geqq 0}\|(I-P_+)(1-\varphi(H_\rho^L)\chi_k)g_{-,k}\|_{H_0^0\to H_m^m}<\infty$$

for any $m\geqq 0$. To prove this, we note by Proposition 3.1, (4.14), (4.15), (4.19) and a direct calculation that

$$(4.49)\qquad \sup_{k\geqq 0}\|\varphi(H_\rho^L)\chi_k g_{-,k}-\chi_k g_{-,k}\|_{H_0^0\to H_m^m}<\infty,\qquad m\geqq 0.$$

On the other hand, from the definition of $g_{-,k}(x,\xi)$ and $\chi_k(\xi,y)$, we have

$$\sup_{k\geq 0}\|(I-P_+)(1-\chi_k)g_{-,k}\|_{H_0^0\to H_m^m}<\infty \tag{4.50}$$

for any $m\geq 0$. Thus from (4.49) and (4.50), we obtain (4.48). We have proved iii). □

The following proposition will be used in the next subsection in dealing with the short-range part $(V^L-V_\rho^L)+V^S$.

PROPOSITION 4.4. *Let Assumptions* (L) *and* (S) *be satisfied, and let* $\delta\in[0,1/2)$ *be the constant appeared in Assumption* (S). *Then:*

i) *For any* $s>1$, $\sigma\geq 0$ *and* $k\geq 0$, *one has*

$$\begin{aligned}&\|\langle x\rangle^{-\sigma}(H_0+1)^{\delta}E_{-,k}(t)\|_{s\to 0}\\&\leq C_{s,\sigma}\begin{cases}(t^{-2\delta}\vee 1)\langle\tilde{\tau}_k\rangle^{-s}, & t\in(0,\tilde{\tau}_k),\\(t-\tilde{\tau}_k)^{-2\delta}\langle t-\tilde{\tau}_k\rangle^{2\delta-\sigma}\langle\tilde{\tau}_k\rangle^{-s}, & t>\tilde{\tau}_k,\end{cases}\end{aligned} \tag{4.51}$$

where the constant $C_{s,\sigma}>0$ *is independent of* k, *and* $a\vee 1=\max\{a,1\}$.

ii) *For any* $s>1$, $n\geq 0$ *and* $k\geq 0$, *one has*

$$\|K_{-,k}(t)\|_{s\to s}\leq C_{s,n}\begin{cases}(t^{-2\delta}\vee 1)\langle\tilde{\tau}_k\rangle^{-s}, & t\in(0,\tilde{\tau}_k),\\(t-\tilde{\tau}_k)^{-2\delta}\langle t-\tilde{\tau}_k\rangle^{-n}\langle\tilde{\tau}_k\rangle^{-s}, & t>\tilde{\tau}_k,\end{cases} \tag{4.52}$$

where the constant $C_{s,n}>0$ *is independent of* k.

iii) *For any* $s>1$, $n\geq 0$, $k\geq 0$, *and* $\theta\geq 0$ *satisfying* $0\leq\theta+\delta<1/2$, *one has*

$$\begin{aligned}&\limsup_{h\downarrow 0}h^{-\theta}\|K_{-,k}(t+h)-K_{-,k}(t)\|_{s\to s}\\&\leq C_{s,n,\theta}\begin{cases}(t^{-2(\delta+\theta)}\vee 1)\langle\tilde{\tau}_k\rangle^{-s}, & t\in(0,\tilde{\tau}_k),\\(t-\tilde{\tau}_k)^{-2(\delta+\theta)}\langle t-\tilde{\tau}_k\rangle^{-n}\langle\tilde{\tau}_k\rangle^{-s}, & t>\tilde{\tau}_k,\end{cases}\end{aligned} \tag{4.53}$$

where $C_{s,n,\theta}$ *is independent of* k.

PROOF. i) Since $V_\rho^L(x)$ is uniformly bounded, we can easily see that for $\tilde{U}(t)=\varphi(H_\rho^L)e^{-itH_\rho^L}$

$$\sup_{t\geq 0}\|(H_0+1)\tilde{U}(t)-\tilde{U}(t)(H_0+1)\|_{0\to 0}<\infty. \tag{4.54}$$

Furthermore we have

$$\sup_{k\geq 0}\|(H_0+1)^{1/2}\chi_k g_{-,k}-\chi_k g_{-,k}(H_0+1)^{1/2}\|_{0\to 0}<\infty. \tag{4.55}$$

On the other hand, we have

$$(H_0+1)\Phi_k f(x)=\iint e^{i(x-y)\cdot\xi}\langle\xi\rangle^2\Phi_k(\xi,y)\langle y\rangle^{-s}\langle y\rangle^s f(y)dy\bar{d}\xi \tag{4.56}$$

and

$$|\partial_\xi^\alpha\partial_y^\beta(\langle\xi\rangle^2\Phi_k(\xi,y)\langle y\rangle^{-s})|\leqq C_{\alpha\beta,s}\langle\tilde{\tau}_k\rangle^{-s}\langle\xi\rangle^{2-s}. \tag{4.57}$$

Hence for $s>2$

$$\|(H_0+1)\Phi_k\|_{s\to 0}\leqq C_s\langle\tilde{\tau}_k\rangle^{-s}. \tag{4.58}$$

Interpolating this and $\sup_{k\geqq 0}\|\Phi_k\|_{0\to 0}<\infty$, we obtain

$$\|(H_0+1)^\delta\Phi_k\|_{s\to 0}\leqq C_s\langle\tilde{\tau}_k\rangle^{-s} \tag{4.59}$$

for $s>1$. Thus, the first term on the r. h. s. of (4.30) satisfies (4.51) for $t\in(0,\tilde{\tau}_k)$. The other terms in (4.30) clearly satisfies (4.51) for $t\in(0,\tilde{\tau}_k)$ by Proposition 2.9-i) and (4.48).

The estimate (4.51) for $t>\tilde{\tau}_k$ easily follows from Proposition 2.9-i), (4.40) and (4.39).

ii) now follows from i), Theorem 4.3-iii), and Assumption (S) by $K_{-,k}(t)=K_{-,k}^L(t)+(V^L-V_\rho^L+V^S)E_{-,k}(t)$.

iii) is proved in a way similar to i) by using Proposition 2.9-iii), (4.40) and (4.48). □

4.2. Total approximate propagators

Let

$$\begin{aligned}\tilde{g}_+(x,\xi)&=\iint e^{-iy\cdot\eta}g_+(\xi+\eta,x+y)dy\bar{d}\eta\\&\sim\sum_\alpha\frac{1}{\alpha!}(\partial_\xi^\alpha D_x^\alpha g_+)(\xi,x).\end{aligned} \tag{4.60}$$

Then $\tilde{g}_+(X,D)=g_+(D,Y)$. Let Ψ be the operator defined by

$$\begin{aligned}\Psi=1-\phi(D)+g_+^\infty(X,D)-\chi_0(RX)\tilde{g}_+(X,D)\\+\sum_{k\geqq 0}g_{-,k}(X,D)(1-\Phi_k(D,Y)).\end{aligned} \tag{4.61}$$

Then it is easily seen by (4.19), (4.20), (4.23) and (4.24) that Ψ satisfies for any $s,m\geqq 0$

$$\|\Psi\|_{H_s^0\to H_s^m}<\infty. \tag{4.62}$$

Let $\Phi(t)$ be the function in (4.29).

Now we can define the total approximate propagator $E(t)$.

DEFINITION 4.5. i) For $t\geqq 0$, we set

$$E_\infty(t)=\Phi(t)\Psi. \tag{4.63}$$

ii) For $t\geqq 0$, we define

$$E(t)=E_+(t)+\sum_{k\geqq 0}E_{-,k}(t)+E_\infty(t), \tag{4.64}$$

and

$$\begin{cases} K_\rho^L(t)=(D_t+H_\rho^L)E(t), \\ K(t)=(D_t+H)E(t). \end{cases} \tag{4.65}$$

Let $\{\mu_k\}_{k=0}^\infty=\{0\}\cup\{\tilde{\tau}_k\}_{k=0}^\infty$ satisfy

$$0=\mu_0<\mu_1<\mu_2<\cdots<\mu_k<\cdots\to\infty \qquad (\text{as } k\to\infty). \tag{4.66}$$

Then we have the following theorem.

THEOREM 4.6. *Let Assumptions* (L) *and* (S) *be satisfied, and let* $\delta\in[0,1/2)$ *be the constant appeared in Assumption* (S). *Then:*

i) $E(0)=I$;

ii) $\|E(t)\|_{s\to-s}\leqq C_{s,\varepsilon}\langle t\rangle^{-s+\varepsilon}$ *for* $s\geqq 0$, $\varepsilon>0$;

iii) $\|K_\rho^L(t)\|_{s\to s}\leqq C_{s,\varepsilon}\langle t\rangle^{-s+\varepsilon}$ *for* $s\geqq 0$, $\varepsilon>0$;

iv) $\|K(t)\|_{s\to s}\leqq C_{s,\varepsilon}((t-\mu_k)^{-2\delta}\vee 1)\langle t\rangle^{-s+\varepsilon}$ *for* $t\in(\mu_k,\mu_{k+1})$ $(k\geqq 0)$, $s>1$ *and* $\varepsilon>0$;

and

v) $\limsup_{h\downarrow 0} h^{-\theta}\|K(t+h)-K(t)\|_{s\to s}\leqq C_{s,\varepsilon,\theta}((t-\mu_k)^{-2(\delta+\theta)}\vee 1)\langle t\rangle^{-s+\varepsilon}$ *for* $t\in(\mu_k,\mu_{k+1})$ $(k\geqq 0)$, $s>1$, $\varepsilon>0$ *and* $\theta\geqq 0$ *with* $0\leqq\delta+\theta<1/2$.

PROOF. i) is obvious from the definitions of $E_+(t)$, $E_{-,k}(t)$ and $E_\infty(t)$. ii) follows from Theorem 2.8-i), Theorem 4.3-ii) and (4.62), since $\tilde{\tau}_k=A\tau_k=A(1+\delta)^{k-1}$ for $k\geqq 1$. iii) follows from Theorem 2.8-ii), Theorem 4.3-iii) and (4.62). iv) is an immediate consequence of Proposition 2.9-ii) and Proposition 4.4-ii). v) is a consequence of Proposition 2.9-iii) and Proposition 4.4-iii). □

§5. Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1.

We begin with the definition of the Fourier-Laplace transforms $\hat{E}(z)$ and $\hat{K}(z)$ ($\mathrm{Im}\, z>0$) of $E(t)$ and $K(t)$: For $f\in\mathcal{S}$,

$$\hat{E}(z)f=\int_0^\infty e^{itz}E(t)fdt, \qquad \hat{K}(z)f=\int_0^\infty e^{itz}K(t)fdt. \tag{5.1}$$

Then we have the following proposition.

PROPOSITION 5.1. *Let Assumptions* (L) *and* (S) *be satisfied, and let* $s>1$. *Then*

$$\begin{cases} \sup_{\mathrm{Im}\, z>0} \|\hat{E}(z)\|_{s\to -s}<\infty, \\ \sup_{\mathrm{Im}\, z>0} \|\hat{K}(z)\|_{s\to s}<\infty. \end{cases} \tag{5.2}$$

Furthermore, the boundary values $\hat{E}(\lambda)f=\hat{E}(\lambda+i0)f=\text{s-}\lim_{\varepsilon\downarrow 0}\hat{E}(\lambda+i\varepsilon)f$ *and* $\hat{K}(\lambda)f=\hat{K}(\lambda+i0)f=\text{s-}\lim_{\varepsilon\downarrow 0}\hat{K}(\lambda+i\varepsilon)f$ ($f\in L_s^2$) *exist and are continuous with respect to* λ, *in* L_{-s}^2 *and* L_s^2, *respectively.*

PROOF. Since $E(t)f$ and $K(t)f$ belong to $L^1([0,\infty); L_{-s}^2)$ and $L^1([0,\infty); L_s^2)$ for $f\in L_s^2$ and $s>1$ by Theorem 4.3-i), Theorem 4.6-ii) and iv), the assertions are obvious. □

Taking the Fourier-Laplace transforms of the both sides of the relation $(D_t+H)E(t)=K(t)$ $(t\geqq 0)$, and using the continuity of $E(t)$ in $t\geqq 0$, we can easily get

$$(H-z)\hat{E}(z)f=-i(I+i\hat{K}(z))f \tag{5.3}$$

for $f\in L_s^2$ $(s>1)$ and $\mathrm{Im}\, z>0$. From the well-known inequality:

$$\begin{aligned} \sup_{\mathrm{Im}\, z>0} \|\langle z\rangle^\theta \hat{K}(z)f\|_{L_s^2} \leqq C\int_0^\infty \|K(t)f\|_{L_s^2}dt + C\limsup_{h\downarrow 0} h^{-\theta} \\ \times\int_0^\infty \|(K(t+h)-K(t))f\|_{L_s^2}dt, \end{aligned} \tag{5.4}$$

and Theorem 4.6-iv) and v), we easily get the following proposition.

PROPOSITION 5.2. *Let Assumptions* (L) *and* (S) *be satisfied, and let* $\theta>0$ *satisfy* $0<\theta+\delta<1/2$ *for* $\delta\in[0,1/2)$ *appeared in Assumption* (S). *Let* $s>1$. *Then*

$$\sup_{\mathrm{Im}\, z>0} \|\langle z\rangle^\theta \hat{K}(z)\|_{s\to s}<\infty. \tag{5.5}$$

Let $s\geqq 0$ and $\varepsilon>0$ be fixed, and take $s_0>2$ so large that $s_0\varepsilon/2\geqq s$ holds.

Then there exists some $R_0=R_{0,s_0,\varepsilon}>1$ large enough such that for $|z|\geqq R_0$, $\operatorname{Im} z\geqq 0$, one has

$$\|\hat{K}(z)\|_{s_0\to s_0}<1/2. \tag{5.6}$$

From this and (5.3) we obtain

$$R(z)f=(H-z)^{-1}f=i\hat{E}(z)(I+i\hat{K}(z))^{-1}f \quad \text{in} \quad L^2_{-s_0} \tag{5.7}$$

for $|z|\geqq R_0$, $\operatorname{Im} z\geqq 0$ and $f\in L^2_{s_0}$. By Proposition 5.1, the r. h. s. of (5.7) are continuous in $|z|\geqq R_0$, $\operatorname{Im} z\geqq 0$. The similar thing can be shown to hold for $R(z)f$ with $|z|\geqq R_0$, $\operatorname{Im} z\leqq 0$ by using the approximate propagator $E(t)$ defined for $t\leqq 0$, which can be constructed in a way similar to that in sections 2-4. Thus we have for $\chi=\chi_{s_0,\varepsilon}$ in section 1 and for $f\in L^2_{s_0}$

$$\chi e^{-itH}f=\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{-it\lambda}\chi(\lambda)\{R(\lambda+i0)f-R(\lambda-i0)f\}d\lambda, \qquad t\in R^1. \tag{5.8}$$

For estimating this, we consider

$$\begin{aligned}\chi_{\pm}e^{-itH}f&=\int_{-\infty}^{\infty}e^{-it\lambda}\chi(\lambda)R(\lambda\pm i0)fd\lambda\\ &=i\int_{-\infty}^{\infty}e^{-it\lambda}\chi(\lambda)\hat{E}(\lambda\pm i0)(I+i\hat{K}(\lambda\pm i0))^{-1}fd\lambda.\end{aligned} \tag{5.9}$$

By (5.6) we can write

$$\begin{aligned}\chi_{+}e^{-itH}f=&\,i\sum_{l=0}^{\nu}i^l\int e^{-it\lambda}\chi(\lambda)\hat{E}(\lambda)\hat{K}(\lambda)^l fd\lambda\\ &+i^{\nu+2}\int e^{-it\lambda}\chi(\lambda)\hat{E}(\lambda)\hat{K}(\lambda)^{\nu+1}(I+i\hat{K}(\lambda))^{-1}fd\lambda\end{aligned} \tag{5.10}$$

for any $\nu\geqq 0$. We take and fix ν and θ such that

$$0<\theta+\delta<1/2, \qquad (\nu+1)\theta>1 \tag{5.11}$$

for $\delta\in[0,1/2)$ appeared in Assumption (S).

It follows from (5.1), (5.4) and Theorem 4.6-ii), iv), v) that

$$\begin{cases}\|\partial_\lambda^m\hat{E}(\lambda)\|_{s_0\to -s_0}\leqq C_{s_0,m},\\ \|\partial_\lambda^m\hat{K}(\lambda)\|_{s_0\to s_0}\leqq C_{s_0,m,\theta}\langle\lambda\rangle^{-\theta}\end{cases} \tag{5.12}$$

for $s_0>m+1$ $(m\geqq 0)$. Thus we can easily obtain

$$(5.13)\qquad \left\|\int e^{-it\lambda}\chi(\lambda)\hat{E}(\lambda)\hat{K}(\lambda)^{\nu+1}(I+i\hat{K}(\lambda))^{-1}fd\lambda\right\|_{L^2_{-s_0}} \leqq C_{s_0,\theta}\langle t\rangle^{-s_0+2}\|f\|_{L^2_{s_0}}.$$

For estimating the first summand on the r. h. s. of (5.10), we write for $0\leqq l\leqq\nu$

$$(5.14)\qquad \int e^{-it\lambda}\chi(\lambda)\hat{E}(\lambda)\hat{K}(\lambda)^l fd\lambda=\int e^{-it\lambda}\hat{E}(\lambda)\hat{K}(\lambda)^l fd\lambda + \int e^{-it\lambda}(\chi(\lambda)-1)\hat{E}(\lambda)\hat{K}(\lambda)^l fd\lambda.$$

The first term on the r. h. s. of (5.14) is equal to

$$(5.15)\qquad \int_0^t E(t-t_1)\int_0^{t_1} K(t_1-t_2)\cdots\int_0^{t_{l-1}} K(t_{l-1}-t_l)K(t_l)fdt_l\cdots dt_1$$

for $t\geqq 0$, and vanishes for $t<0$. By a straightforward calculation by using Theorem 4.6-ii), iv), the $L^2_{-s_0}$ norm of this is bounded by $C_{s_0,l}\langle t\rangle^{-s_0+1}\|f\|_{L^2_{s_0}}$. The second term on the r. h. s. of (5.14) can be estimated similarly by using $\chi(\lambda)-1\in C_0^\infty(R^1)$, and is bounded by $C_{s_0,l}\langle t\rangle^{-s_0+1}\|f\|_{L^2_{s_0}}$.

Summing up, we have proved

$$(5.16)\qquad \|\chi_+e^{-itH}\|_{s_0\to -s_0}\leqq C_{s_0}\langle t\rangle^{-s_0+2}$$

The same estimate can be shown to hold for χ_-e^{-itH} by using the approximate propagator $E(t)$ for $t\leqq 0$. Thus we get

$$(5.17)\qquad \|\chi e^{-itH}\|_{s_0\to -s_0}\leqq C_{s_0}\langle t\rangle^{-s_0+2}.$$

On the other hand, we have

$$(5.18)\qquad \|\chi e^{-itH}\|_{0\to 0}\leqq C_0$$

from (1.5). Thus, interpolating (5.17) and (5.18), we have proved

$$(5.19)\qquad \|\chi e^{-itH}\|_{s\to -s}\leqq C_{s,\varepsilon}\langle t\rangle^{-s+\varepsilon}$$

for $s\geqq 0$ and $\varepsilon>0$ fixed just after Proposition 5.2. This completes the proof of Theorem 1.1.

Appendix. Proof of Theorem 2.8

i) is obvious from the definition (2.37) of $E_+(t)$ and (2.36).

ii) By a straightforward calculation, we can easily prove the expression (2.41), and we see that $k_+^2(t,\xi,y)$ is derived from the sum of the terms which include $\partial_t^l\partial_x^\alpha\left[\chi_0\left(\frac{Rx}{\langle t\rangle}\right)\right]$ $(1\leqq l+|\alpha|\leqq 2)$, and that $k_+^1(t,\xi,y)$ is the symbol function of $K_+^1(t)=(D_t+H^L)E_+^1(t)$, where $E_+^1(t)$ is defined by (2.37) with deleting the factor $\chi_0\left(\frac{Rx}{\langle t\rangle}\right)$. Furthermore, we easily see that $\left|\partial_t^l\partial_x^\alpha\left[\chi_0\left(\frac{Rx}{\langle t\rangle}\right)\right]\right|\leqq C_{\alpha,l}\langle x\rangle^{-|\alpha|-l}$ for $l+|\alpha|\geqq 1$, and by Proposition 2.5–i), (2.17) and (2.35) that $\partial_t^l\partial_x^\alpha\left[\chi_0\left(\frac{Rx}{\langle t\rangle}\right)\right]\Big|_{x=\nabla_\xi\phi^+(t,y,\xi)}=0$ for $l+|\alpha|\geqq 1$ and (t,y,ξ) with $e_+(t,\xi,y)\neq 0$ if R is sufficiently large. The proof of (2.42) for $j=2$ is, therefore, quite similar to that for $j=1$. Hence we only prove (2.42) for $j=1$. To prove this, we first give two different estimates for $k_+^1(t,\xi,y)$ and then interpolate them to obtain (2.42). In the following we only prove the estimate (2.42) with $j=1$ for $|a|=|\alpha|=|\beta|=0$, since the other cases can be proved similarly:

$$\text{(A. 1)} \qquad |k_+^1(t,\xi,y)|\leqq C_n\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n}, \qquad n\geqq 0.$$

We remark that (2.43) follows from (2.42) by the L^2-boundedness theorem for Fourier integral operators (see Asada and Fujiwara [1], Kitada and Kumano-go [5], Kitada [4]).

1*st estimate*: Since $H_\rho^L=H_\rho^L(X,D)$ is symmetric, we can write

$$\text{(A. 2)} \qquad H_\rho^L E_+^1(t)f(x)=\iint e^{i(x-y)\cdot\xi}H_\rho^L(y,\xi)(E_+^1(t)f)(y)dy\bar{d}\xi.$$

Then we have

$$\text{(A. 3)} \qquad K_+^1(t)f(x)=O_s\text{-}\iint e^{i(x\cdot\xi-\phi^+(t,y,\xi))}\{s_1(t,\xi,y)+s_2(t,\xi,y)\}f(y)dy\bar{d}\xi,$$

where

$$\text{(A. 4)} \qquad \begin{cases} s_1(t,\xi,y)=O_s\text{-}\iint e^{i[z\cdot(\eta-\xi)+(\phi^+(t,y,\xi)-\phi^+(t,y,\eta))]}H_\rho^L(z,\xi)e_+(t,\eta,y)dz\bar{d}\eta, \\ s_2(t,\xi,y)=-\partial_t\phi^+(t,y,\xi)\cdot e_+(t,\xi,y)-i\partial_t e_+(t,\xi,y). \end{cases}$$

By Taylor's formula and Fourier's inversion formula, we get for any $L>n>2$

$$\text{(A. 5)} \qquad \begin{aligned} &s_1(t,\xi,y)\\ &=\sum_{l=0}^{\infty} i^l\sum_{|\alpha|\leqq 1}\frac{(-1)^{|\alpha|}}{\alpha!}D_\eta^\alpha[(\partial_x^\alpha H_\rho^L)(\nabla_\xi\phi^+(t;\eta+\xi,y,\xi),\xi)e_+(t,\eta+\xi,y)]\Big|_{\eta=0}\end{aligned}$$

$$+\sum_{2\leq|\alpha|<n}\frac{(-1)^{|\alpha|}}{\alpha!}D_\eta^\alpha[(\partial_x^\alpha H_\rho^L)(\nabla_\xi\phi^+(t;\eta+\xi,y,\xi),\xi)e_+(t,\eta+\xi,y)]\Big|_{\eta=0}$$
$$+B_{n,L}+C_L,$$

where

(A. 6)

$$\begin{cases} B_{n,L}=\sum_{n\leq|\alpha|<L}\frac{(-1)^{|\alpha|}}{\alpha!}D_\eta^\alpha[(\partial_x^\alpha H_\rho^L)(\nabla_\xi\phi^+(t;\eta+\xi,y,\xi),\xi)e_+(t,\eta+\xi,y)]\Big|_{\eta=0} \\ C_L=L\sum_{|\gamma|=L}\frac{(-1)^{|\gamma|}}{\gamma!}\int_0^1(1-\theta)^{|\gamma|-1}t_\gamma(\xi,y;\theta)d\theta, \\ t_\gamma(\xi,y;\theta)=O_s\text{-}\iint e^{i\eta\cdot z}D_\eta^\gamma\{(\partial_x^\gamma H_\rho^L)(\theta z+\nabla_\xi\phi^+(t;\eta+\xi,y,\xi),\xi)e_+(t,\eta+\xi,y)\}dz\,đ\eta, \end{cases}$$

and $\nabla_\xi\phi^+(t;\eta,y,\xi)$ is defined by (2.30). On the other hand, using Proposition 2.5-ii) and the transport equation (2.27), we have

(A. 7) $s_2(t,\xi,y)$

$$=-\sum_{l=0}^\infty i^l\sum_{|\alpha|\leq 1}\frac{(-1)^{|\alpha|}}{\alpha!}D_\eta^\alpha[(\partial_x^\alpha H_\rho)(t,\nabla_\xi\phi^+(t;\eta+\xi,y,\xi),\xi)e_+(t,\eta+\xi,y)]\Big|_{\eta=0}$$
$$-i\sum_{l=0}^\infty i^lB_l(t,\xi,y).$$

Thus, taking $\rho\in(0,1)$ so small that $\rho^{-1}\geqq\sup_{0\leq t\leq T}2\langle t\rangle/\varphi(t)$ for $T=T(\sigma_0'-\delta)$ (>1) appeared in Proposition 2.5-ii) (as for δ see (2.35)), and noting Proposition 2.1-iii), we obtain

$$k_+^1(t,\xi,y)=s_1(t,\xi,y)+s_2(t,\xi,y)$$
$$=A_n+B_{n,L}+C_L, \tag{A. 8}$$

where

(A. 9)
$$A_n=-i\sum_{l=0}^\infty i^lB_l(t,\xi,y)$$
$$+i\sum_{m=1}^\infty i^m\sum_{2\leq|\alpha|\leq\min(n-1,m+1)}\frac{1}{\alpha!}\partial_\eta^\alpha[(\partial_x^\alpha H_\rho^L)(\nabla_\xi\phi^+(t;\eta+\xi,y,\xi),\xi)$$
$$\times e_+^{m+1-|\alpha|}(t,\eta+\xi,y)]\Big|_{\eta=0}.$$

Also by Proposition 2.1-iii) and by (2.29), (2.35) and (2.36), A_n satisfies

$$|A_n|\leqq C_n\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n}, \tag{A. 10}$$

where for $t>0$ near 0 we used the inequalities:

$$\begin{cases} \langle |y|+t\langle\xi\rangle\rangle^{-1}|\partial_\xi^\alpha y^+(0,t;y,\xi)|\leqq C_\alpha\langle|y|+t\langle\xi\rangle\rangle^{-1}t\leqq C_\alpha\langle\xi\rangle^{-1}, \\ \langle|y|+t\langle\xi\rangle\rangle^{-1}\langle\xi\rangle^{-1}\leqq C\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-1}. \end{cases}$$

For $B_{n,L}$, we easily see by using these inequalities, (2.35) and (2.36) that

$$|B_{n,L}|\leqq C_{n,L}\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n}. \tag{A.11}$$

In order to estimate C_L, we put

$$q(z,\eta,y,\xi)=H_\rho^L(z+\nabla_\xi\phi^+(t;\eta,y,\xi),\xi)e_+(t,\eta,y). \tag{A.12}$$

Then

$$C_L=\sum_{|\gamma|=L}\frac{(-1)^{|\gamma|}}{\gamma!}\int_0^1(1-\theta)^{|\gamma|-1}O_s\text{-}\iint e^{i\eta\cdot z}(D_\eta^\gamma\partial_z^\gamma q)(\theta z,\eta+\xi,y,\xi)dz\bar{d}\eta d\theta. \tag{A.13}$$

For any multi-index α,β such that $|\alpha|,|\beta|\leqq M$ $(M>0)$, we have from the formula for the derivatives of composite functions (cf. e.g. Narita [10, Lemma 1.6])

$$\begin{aligned} &|(D_\eta^{\gamma+\alpha}\partial_z^{\gamma+\beta}q)(\theta z,\eta+\xi,y,\xi)| \\ &=\left|D_\eta^{\gamma+\alpha}\left[(\partial_z^{\gamma+\beta}V_\rho^L)\left(\theta z+\int_0^1 y^+(0,t;y,\xi+(1-\theta)\eta)d\theta\right)e_+(t,\eta+\xi,y)\right]\right| \\ &\leqq C\sum_{l+m=\gamma+\alpha}\sum_{\{\nu_\kappa\}_{0<\kappa\leqq l}\in M_l}\left|(D_z^{\Sigma\nu_\kappa+\gamma+\beta}V_\rho^L)\left(\theta z+\int_0^1 y^+(0,t;y,\xi+(1-\theta)\eta)d\theta\right)\right| \\ &\quad\times\prod_{0<\kappa\leqq l}\left|\left(\int_0^1(1-\theta)^{|\kappa|}(D_\eta^\kappa y^+)(0,t;y,\xi+(1-\theta)\eta)d\theta\right)^{\nu_\kappa}\right| \\ &\quad\times|(D_\eta^m e_+)(t,\eta+\xi,y)|, \end{aligned} \tag{A.14}$$

where

$$M_l=\left\{\{\nu_\kappa\}_{0<\kappa\leqq l}\,\middle|\,\nu_\kappa \text{ and } \kappa \text{ are } N\text{-dimensional multi-indices and } \sum_{0<\kappa\leqq l}|\nu_\kappa|k=l\right\}.$$

We easily see from (2.17) and (2.35) that

$$\left\langle\int_0^1 y^+(0,t;y,\xi+(1-\theta)\eta)d\theta\right\rangle^{-1}\leqq C\langle t\rangle^3\langle\eta\rangle^2\langle|y|+t\langle\xi\rangle\rangle^{-1} \tag{A.15}$$

for (t,y,ξ,η) satisfying $e_+(t,\eta+\xi,y)\neq0$. Thus for $t\geqq1$ we have from (A.14), (2.17), (2.36) and $|\gamma|=L$ that

$$\text{(A.16)}\qquad \begin{aligned} &|(D_\eta^{\gamma+\alpha}\partial_z^{\gamma+\beta}q)(\theta z, \eta+\xi, y, \xi)| \\ &\leqq C \sum_{\substack{l'+m'=\gamma \\ l''+m''=\alpha}} \sum_{\{\nu_\kappa\}\in M_{l'+l''}} \langle z\rangle^{|l'|}\langle t\rangle^{3|l'|}\langle\eta\rangle^{2|l'|}\langle |y|+t\langle\xi\rangle\rangle^{-|l'|} \\ &\quad\times \prod_{0<\kappa\leqq l'+l''} t^{|\nu_\kappa|}\langle\eta+\xi\rangle^{-|m'|}|\langle\eta+\xi\rangle^{|m'|}D_\eta^{m'+m''}e_+(t, \eta+\xi, y)| \\ &\leqq C\langle z\rangle^L\langle t\rangle^{3L}\langle\eta\rangle^{3L} \sum_{\substack{l'+m'=\gamma \\ l''+m''=\alpha}} \sum_{\{\nu_\kappa\}\in M_{l'+l''}} \langle t\langle\xi\rangle\rangle^{-|l'|}t^{|l'+l''|}\langle\xi\rangle^{-|m'|} \\ &\leqq C\langle z\rangle^L\langle\eta\rangle^{3L}\langle t\rangle^{3L+M}\langle\xi\rangle^{-L}. \end{aligned}$$

For proving a similar estimate for $0\leqq t\leqq 1$, we need the following lemma, whose proof will be given at the end of this Appendix.

LEMMA A.1. *For (t, y, ξ, η) satisfying $0\leqq t\leqq 1$ and $e_+(t, \eta+\xi, y)\neq 0$, and for $|\alpha|\geqq 2$ and $0\leqq\theta\leqq 1$, one has*

$$\text{(A.17)}\qquad |(\partial_\xi^\alpha y^+)(0, t; y, \xi+(1-\theta)\eta)|\leqq C_\alpha\langle\eta\rangle^{2\mu_\alpha}\langle\xi\rangle^{-\mu_\alpha}t$$

for any μ_α such that $0\leqq\mu_\alpha\leqq|\alpha|$.

From (A.14), (A.15), (A.17), (2.17) and (2.36), we have

$$\text{(A.18)}\qquad \begin{aligned} &|(D_\eta^{\gamma+\alpha}\partial_z^{\gamma+\beta}q)(\theta z, \eta+\xi, y, \xi)| \\ &\leqq C \sum_{\substack{l'+m'=\gamma \\ l''+m''=\alpha}} \sum_{\{\nu_\kappa\}\in M_{l'+l''}} \langle z\rangle^{|l'|}\langle t\rangle^{3|l'|}\langle\eta\rangle^{2|l'|}\langle |y|+t\langle\xi\rangle\rangle^{-|l'|} \\ &\quad\times \prod_{0<\kappa\leqq l'+l''} (\langle\eta\rangle^{2\mu_\kappa}\langle\xi\rangle^{-\mu_\kappa}t)^{|\nu_\kappa|}\langle\eta+\xi\rangle^{-|m'|}. \end{aligned}$$

Let $p=\sum_\kappa|\nu_\kappa|$ and $q=\sum_{|\kappa|=1}|\nu_\kappa|$. For the terms on the r.h.s. of (A.18) which satisfy $p\leqq|l'|$, we take μ_κ such that $\sum_\kappa|\nu_\kappa|\mu_\kappa=|l'|-p$ $(\geqq 0)$, which is possible since $\sum_\kappa|\nu_\kappa|\mu_\kappa$ can take any value in the interval

$$\Big[0, \sum_{|\kappa|\geqq 2}|\nu_\kappa||\kappa|\Big]=[0, |l|-q]\supset[0, |l'|-p].$$

Then such terms are bounded by

$$\text{(A.19)}\qquad \begin{aligned} &C\langle z\rangle^L\langle t\rangle^{3L}\langle\eta\rangle^{3L}\langle t\langle\xi\rangle\rangle^{-|l'|}t^p\langle\eta\rangle^{2(|l'|-p)}\langle\xi\rangle^{-|l'|+p}\langle\xi\rangle^{-|m'|} \\ &\leqq C\langle z\rangle^L\langle\eta\rangle^{5L}\langle t\rangle^{3L}\langle t\langle\xi\rangle\rangle^{p-|l'|}\langle t\langle\xi\rangle\rangle^{-p}t^p\langle\xi\rangle^{-L+p} \\ &\leqq C\langle z\rangle^L\langle\eta\rangle^{5L}\langle\xi\rangle^{-L}, \qquad 0\leqq t\leqq 1. \end{aligned}$$

For the terms in (A.18) which satisfy $p\geqq|l'|$, we take $\mu_\kappa=0$ for all κ. Then they are bounded by

$$\text{(A. 20)} \qquad \begin{aligned} &C\langle z\rangle^{L}\langle t\rangle^{3L}\langle\eta\rangle^{3L}\langle t\langle\xi\rangle\rangle^{-|l'|}t^{p}\langle\xi\rangle^{-|m'|}\\ &\leqq C\langle z\rangle^{L}\langle\eta\rangle^{3L}\langle t\rangle^{3L}\langle\xi\rangle^{-|l'|}t^{-|l'|+p}\langle\xi\rangle^{-|m'|}\\ &\leqq C\langle z\rangle^{L}\langle\eta\rangle^{3L}\langle\xi\rangle^{-L}, \qquad 0\leqq t\leqq 1. \end{aligned}$$

Therefore, combining (A. 18)-(A. 20), we get for $0\leqq t\leqq 1$

$$\text{(A. 21)} \qquad |(D_{\eta}^{\gamma+\alpha}\partial_{z}^{\gamma+\beta}q)(\theta z, \eta+\xi, y, \xi)|\leqq C\langle z\rangle^{L}\langle\eta\rangle^{5L}\langle\xi\rangle^{-L}.$$

Thus from (A. 16) and (A. 21) we get for any $t\geqq 0$

$$\text{(A. 22)} \qquad |(D_{\eta}^{\gamma+\alpha}\partial_{z}^{\gamma+\beta}q)(\theta z, \eta+\xi, y, \xi)|\leqq C\langle z\rangle^{L}\langle\eta\rangle^{5L}\langle t\rangle^{3L+M}\langle\xi\rangle^{-L},$$

where $|\gamma|=L$ and $|\alpha|, |\beta|\leqq M$.

Taking an even integer M such that $N+1+5L\leqq M\leqq N+2+5L$, we now obtain for $|\gamma|=L$

$$\text{(A. 23)} \qquad \begin{aligned} &\left|\iint e^{i\eta\cdot z}(D_{\eta}^{\gamma}\partial_{z}^{\gamma}q)(\theta z, \eta+\xi, y, \xi)dz đ\eta\right|\\ &=\left|\iint e^{i\eta\cdot z}\langle z\rangle^{-M}\langle D_{\eta}\rangle^{M}\langle\eta\rangle^{-M}\langle D_{z}\rangle^{M}(D_{\eta}^{\gamma}\partial_{z}^{\gamma}q)(\theta z, \eta+\xi, y, \xi)dz đ\eta\right|\\ &\leqq C\langle t\rangle^{8L+N+2}\langle\xi\rangle^{-L}. \end{aligned}$$

Combining (A. 8), (A. 23) and (A. 13), and writing $\tilde{A}_{n,L}=A_{n}+B_{n,L}$, we thus have

$$\text{(A. 24)} \qquad |k_{+}^{1}(t, \xi, y)-\tilde{A}_{n,L}|\leqq C_{L}\langle t\rangle^{8L+N+2}\langle\xi\rangle^{-L}.$$

Moreover by (A. 10)-(A. 11), we have

$$\text{(A. 25)} \qquad |\tilde{A}_{n,L}|\leqq C_{n}\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n}.$$

2nd estimate: We next prove the following estimate: For any integer $M>n$ (>2)

$$\text{(A. 26)} \qquad |k_{+}^{1}(t, \xi, y)-\tilde{A}_{n,M}|\leqq C_{M}\langle|y|+t\rangle^{-M}.$$

Let a real-valued C^{∞} function $\chi\in C_{0}^{\infty}(R^{N})$ satisfy

$$\text{(A. 27)} \qquad \chi(y)=\begin{cases} 1, & |y|\leqq\delta_{0},\\ 0, & |y|\geqq 2\delta_{0}\end{cases}$$

for some $\delta_{0}>0$ small enough. Using this χ and writing $\tau=\langle|y|+t\rangle$, we divide $s_{1}(t, \xi, y)$ as

$$
\begin{aligned}
\text{(A. 28)} \qquad s_1(t, \xi, y) &= p_1(t, \xi, y) + p_2(t, \xi, y) \\
&= \left[\iiint e^{i(\eta-\xi)\cdot(z-\nabla_\xi \phi^+(t;\eta,y,\xi))} V_\rho(z) e_+(t, \eta, y)(1-\chi(\tau^{-1}z - \tau^{-1}(y+t\eta)))\, dz d\!\!{}^-\eta\right] \\
&\quad + \left[\iiint e^{i(\eta-\xi)\cdot(z-\nabla_\xi \phi^+(t;\eta,y,\xi))} V_\rho(z) e_+(t, \eta, y)\chi(\tau^{-1}z - \tau^{-1}(y+t\eta))\, dz d\!\!{}^-\eta \right.\\
&\quad \left. -\frac{1}{2}|\xi|^2 e_+(t, \xi, y)\right].
\end{aligned}
$$

Then $p_1(t, \xi, y)$ is estimated in quite the same way as that for $p_{1,\pm}$ in the proof of Theorem 4.4 of Kitada and Yajima [6] as follows if δ_0 in (A. 27) and $\rho \in (0, 1)$ are taken small enough: For any $l \geqq 0$

$$
\text{(A. 29)} \qquad |p_1(t, \xi, y)| \leqq C_l \tau^{N-l}.
$$

Next, in order to estimate p_2, we set

$$
\begin{aligned}
\text{(A. 30)} \qquad \tilde{q}(z, \eta, y, \xi) &= V_\rho(z + \nabla_\xi \phi^+(t; \eta, y, \xi)) \\
&\quad \times e_+(t, \eta, y)\chi(\tau^{-1}\{z + \nabla_\xi \phi^+(t; \eta, y, \xi) - y - t\eta\}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\text{(A. 31)} \qquad & p_2(t, \xi, y) + \frac{1}{2}|\xi|^2 e_+(t, \xi, y) \\
&= \iint e^{i\eta\cdot z} \tilde{q}(z, \eta+\xi, y, \xi)\, dz d\!\!{}^-\eta \\
&= \sum_{|\alpha|<n} \frac{(-1)^{|\alpha|}}{\alpha!} (D_\eta^\alpha \partial_z^\alpha \tilde{q})(0, \xi, y, \xi) + \sum_{n \leqq |\alpha| < M} \frac{(-1)^{|\alpha|}}{\alpha!} (D_\eta^\alpha \partial_z^\alpha \tilde{q})(0, \xi, y, \xi) \\
&\quad + M \sum_{|\gamma|=M} \frac{(-1)^{|\gamma|}}{\gamma!} \int_0^1 (1-\theta)^{|\gamma|-1} \iint e^{i\eta\cdot z} (D_z^\gamma \partial_\eta^\gamma \tilde{q})(\theta z, \xi+\eta, y, \xi)\, dz d\!\!{}^-\eta d\theta.
\end{aligned}
$$

Noting by (2.17) and (A. 27) that

$$
\text{(A. 32)} \qquad (\partial_y^\beta \chi)(\tau^{-1}\{z + \nabla_\xi \phi^+(t; \eta+\xi, y, \xi) - y - t(\eta+\xi)\}) = 0
$$

for $|\beta| \geqq 1$, $z = \eta = 0$, and a sufficiently small $\rho \in (0, 1)$, we have

$$
\begin{aligned}
\text{(A. 33)} \qquad k_+^1(t, \xi, y) &= s_1(t, \xi, y) + s_2(t, \xi, y) \\
&= p_1(t, \xi, y) + (p_2(t, \xi, y) + s_2(t, \xi, y)) \\
&= p_1(t, \xi, y) + \tilde{A}_{n,M} + D_M.
\end{aligned}
$$

Here $\tilde{A}_{n,M} = A_n + B_{n,M}$ is defined by (A. 6) and (A. 9), and D_M is defined by

(A. 34) $$D_M=M\sum_{|\gamma|=M}\frac{(-1)^{|\gamma|}}{\gamma!}\int_0^1(1-\theta)^{M-1}\iint e^{i\eta\cdot z}(D_z^\gamma\partial_\eta^\gamma\tilde{q})(\theta z,\xi+\eta,y,\xi)dz\bar{d}\eta d\theta.$$

D_M is estimated in the same way as that for $R_\pm$ in the proof of Theorem 4.4 of Kitada and Yajima [6] as follows if ρ and δ_0 are taken small enough: For any integer $M\geqq 0$

(A. 35) $$|D_M|\leqq C_M\langle|y|+t\rangle^{-M}.$$

Thus combining (A. 29), (A. 33) and (A. 35), we have (A. 26).

Now we can prove (A. 1). Interpolating (A. 24) and (A. 26) with taking L and M as $M=8L+N+2$, we obtain

(A. 36) $$|k_+^1(t,\xi,y)-\tilde{A}_{n,L}|^{1/2}|k_+^1(t,\xi,y)-\tilde{A}_{n,M}|^{1/2}\leqq C\langle\xi\rangle^{-L/2}.$$

Since $M=8L+N+2>L$, we get from the definition of $\tilde{A}_{n,L}=A_n+B_{n,L}$

(A. 37) $$|\tilde{A}_{n,M}-\tilde{A}_{n,L}|=|B_{n,M}-B_{n,L}|\leqq C_L\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-L}.$$

Thus

(A. 38) $$\begin{aligned}&|k_+^1(t,\xi,y)-\tilde{A}_{n,L}|\\&\leqq|k_+^1(t,\xi,y)-\tilde{A}_{n,L}|^{1/2}(|k_+^1(t,\xi,y)-\tilde{A}_{n,M}|+|\tilde{A}_{n,M}-\tilde{A}_{n,L}|)^{1/2}\\&\leqq(|k_+^1(t,\xi,y)-\tilde{A}_{n,L}||k_+^1(t,\xi,y)-\tilde{A}_{n,M}|\\&\quad+|\tilde{A}_{n,M}-\tilde{A}_{n,L}||k_+^1(t,\xi,y)-\tilde{A}_{n,L}|)^{1/2}\\&\leqq(C\langle\xi\rangle^{-L}+C\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-L})^{1/2}\leqq C\langle\xi\rangle^{-L/2}.\end{aligned}$$

Again interpolating (A. 26) and (A. 38) with $M=L$, we then get

(A. 39) $$|k_+^1(t,\xi,y)-\tilde{A}_{n,L}|\leqq C_L\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-L/4}.$$

Therefore, using (A. 25) and taking $L>4n$ in (A. 39), we obtain

(A. 40) $$|k_+^1(t,\xi,y)|\leqq C_n\langle|y|+\langle t\rangle\langle\xi\rangle\rangle^{-n}$$

for any $n\geqq 0$. This completes the proof of (A. 1). There now remains to prove Lemma A. 1.

PROOF OF LEMMA A. 1. We give the proof only for the case $|\alpha|=2$. Then the case $|\alpha|\geqq 3$ can be easily proved by induction on $|\alpha|$. By Proposition 2.3-ii), we have

(A. 41) $$y^+(0,t;y,\zeta)=q(t,0;y,\eta^+(t,0;y,\zeta)),\qquad \zeta=\xi+(1-\theta)\eta.$$

Thus

$$\partial_\xi^2 y^+(0, t; y, \zeta) = \partial_\xi^2 q(t, 0; y, \eta^+(t, 0; y, \zeta)) \cdot \partial_\xi \eta^+(t, 0; y, \zeta)^2 + \partial_\xi q(t, 0; y, \eta^+(t, 0; y, \zeta)) \cdot \partial_\xi^2 \eta^+(t, 0; y, \zeta). \tag{A. 42}$$

Since we have from Proposition 2.3-i) that

$$\partial_\xi^2 \eta^+(t, 0; y, \zeta) = -\partial_\xi p(t, 0; y, \zeta)^{-1} \cdot \partial_\xi^2 p(t, 0; y, \eta^+(t, 0; y, \zeta)) \times \partial_\xi \eta^+(t, 0; y, \zeta)^2, \tag{A. 43}$$

we have only to consider the following equation

$$\begin{cases} \partial_\xi^2 q(t, 0; y, \eta^+) = \int_0^t \partial_\xi^2 p(\tau, 0; y, \eta^+) d\tau, \\ \partial_\xi^2 p(t, 0; y, \eta^+) = -\int_0^t \{\nabla_x^2 V_\rho(\tau, q(\tau, 0)) \cdot \nabla_\xi^2 q(\tau, 0) \\ \qquad + \nabla_x^3 V_\rho(\tau, q(\tau, 0)) \cdot \nabla_\xi q(\tau, 0)^2\} d\tau \end{cases} \tag{A. 44}$$

and estimate $(\partial_\xi^2 q, \partial_\xi^2 p)(t, 0; y, \eta^+)$, where $\eta^+ = \eta^+(t, 0; y, \zeta)$. But since we have by using (2.11), (2.14), (2.17) and $e_+(t, \eta+\xi, y) \neq 0$ that

$$\begin{aligned} &\int_0^t |\nabla_x^3 V_\rho(\tau, q(\tau, 0)) \cdot \nabla_\xi q(\tau, 0)^2| d\tau \\ &\leqq C\rho^{\varepsilon_0} \int_0^t \langle q(\tau, 0; y, \eta^+)\rangle^{-3-\varepsilon_0} \tau^2 d\tau \\ &\leqq C\rho^{\varepsilon_0} \int_0^t \langle |y+\tau\zeta| - C\rho^{\varepsilon_0}\tau\rangle^{-3-\varepsilon_0} \tau^2 d\tau \\ &\leqq C\rho^{\varepsilon_0} \int_0^t \langle |y| + \tau\langle\xi\rangle\rangle^{-\mu_2} \langle\eta\rangle^{2\mu_2} \langle\tau\rangle^{3\mu_2} \tau^2 d\tau \\ &\leqq C\rho^{\varepsilon_0} \langle\eta\rangle^{2\mu_2} \langle\xi\rangle^{-\mu_2} \int_0^t \tau^{2-\mu_2} d\tau \leqq C\rho^{\varepsilon_0} \langle\eta\rangle^{2\mu_2} \langle\xi\rangle^{-\mu_2} \end{aligned} \tag{A. 45}$$

if $0 \leqq t \leqq 1$ and $0 \leqq \mu_2 \leqq 2$, we can prove Lemma A. 1 for $|\alpha| = 2$ by successive approximation. The case $|\alpha| \geqq 3$ can now be proved by induction on $|\alpha|$.

The proof of Theorem 2.8 is now complete.

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(Received November 5, 1982)

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