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On asymptotic behaviors of the solution of a non-linear diffusion equation[*)]

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§ 1. Introduction

M. Kac [2] discovered the propagation of chaos for Kac's caricature of the Boltzmann equation for Maxwellian gas. In an analogy of this, H. P. McKean, Jr. [3] showed that a certain class of non-linear parabolic equations are derived from a system of n-particle diffusion processes through the propagation of chaos; if the initial distribution of the n-particle diffusion is $u_0^{\otimes n}$, then for any $m \in N$ and any $t>0$, the m-marginal distribution of the n-particle diffusion at time t converges to m-fold direct product of $u(t)$, where $u(t)$ is a weak solution of the non-linear parabolic equation with the initial data u_0.

In this paper we consider a system of some class of nd-dimensional diffusion processes $X^{(n)}$ $(n \in N)$ treated in H. P. McKean, Jr. [3]. For fixed $n \in N$, $X^{(n)}(t)=(X^{(n,1)}(t), \cdots, X^{(n,n)}(t))$ is described by the following stochastic differential equation:

$$
(1.1) \qquad \begin{cases} dX^{(n,i)}(t) = dB^{(n,i)}(t) - \operatorname{grad} \Phi_1(X^{(n,i)}(t))dt \\ \qquad\qquad + \beta \dfrac{1}{n} \displaystyle\sum_{j=1}^{n} \operatorname{grad}_1 \Phi_2(X^{(n,i)}(t), X^{(n,j)}(t))dt \quad (i=1, \cdots, n), \\ \text{the probability density of } X^{(n)}(0) = u_0^{\otimes n}, \end{cases}
$$

where $B^{(n)}(t)=(B^{(n,1)}(t), \cdots, B^{(n,n)}(t))$ $(n \in N)$ are nd-dimensional Brownian motions, β is a real constant and u_0 is a probability density on $\boldsymbol{R}^d$. We impose the following assumptions on the potentials Φ_1 and Φ_2:

ASSUMPTION. (i) $\Phi_1 = \dfrac{\alpha}{2}|x|^2 + \varphi_1(x)$, where $\alpha>0$ and $\varphi_1 \in \mathcal{S}(\boldsymbol{R}^d)$,

(ii) $\Phi_2(x, y) = \Phi_2(y, x)$

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*) This is a development of the author's Master's thesis, Department of Mathematics, University of Tokyo, 1982.

and

(iii) $\Phi_2(x, y) \in \mathcal{S}(\boldsymbol{R}^{2d})$ or $\Phi_2(x, y) = \varphi_2(x-y)$ with $\varphi_2 \in \mathcal{S}(\boldsymbol{R}^d)$.

We call the nd-dimensional diffusion process $X^{(n)}(t)$ an n-particle system. The result of H. P. McKean, Jr. [3] implies the following law of large numbers:

$$\lim_{m\to\infty} \frac{1}{n} \sum_{i=1}^{n} \delta_{\{X^{(n,i)}(t)\}} = U(t, \cdot) \qquad \text{in } \mathcal{P}(\mathcal{P}(\boldsymbol{R}^d)) \text{ for any } t>0$$

where the probability distribution $U(t, dx)$ on $\boldsymbol{R}^d$ has a density $u(t, x)$ and u is a solution of the following non-linear parabolic equation:

$$(1.2) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2}\Delta_x u(t, x) + \mathrm{div}_x(u(t, x)\,\mathrm{grad}\,\Phi_1(x)) \\ \qquad\qquad - \mathrm{div}_x\Big(u(t, x)\,\mathrm{grad}\cdot\beta\displaystyle\int_{\boldsymbol{R}^d} \Phi_2(x, y) u(t, y) dy\Big) \\ u(0, x) = u_0(x). \end{cases}$$

The first purpose of this paper is to introduce a free energy F following the idea of Donsker-Varadhan's variational principle for invariant measures of diffusion processes $X^{(n)}$ $(n \in N)$ (§ 3) and to give a characterization of stationary probability solutions of equation (1.2) in terms of F (§ 4). The next purpose is to show that there is a bifurcation point of a stationary probability solution of equation (1.2) for some Φ_2 (§ 4), and to investigate the order of the convergence of a unique solution of equation (1.2) to a stationary solution of equation (1.2) (§ 5).

The free energy F is a functional on $\mathcal{P}(\boldsymbol{R}^d)$ defined by:

$$F(\mu) = \begin{cases} \displaystyle\int_{\boldsymbol{R}^d} (\log \mu(x) + 2\Phi_1(x))\mu(x)dx - \beta\iint_{\boldsymbol{R}^{2d}} \Phi_2(x, y)\mu(x)\mu(y)dxdy, \\ \qquad \text{if } \mu(dx) \text{ has a density } \mu(x) \text{ and} \\ \qquad (\log \mu(x) + 2\Phi_1(x)) \in L^1(\boldsymbol{R}^d; \mu(x)dx), \\ +\infty, \qquad \text{otherwise.} \end{cases}$$

The main theorem in this paper is the following

THEOREM 5.1. *Let v_∞ be a stationary probability solution of equation* (1.2) *and let F be the free energy. Let X_{v_∞} be a Banach space defined by* $X_{v_\infty} = \Big\{\tilde{u}$, *measurable function on* $\boldsymbol{R}^d$; $\operatorname{ess.sup}_{x\in\boldsymbol{R}^d}\Big|\dfrac{\tilde{u}(x)}{v_\infty(x)}\Big| < \infty$, $\displaystyle\int_{\boldsymbol{R}^d} \tilde{u}(x)dx = 0\Big\}$. *If* $D^2F(v_\infty + \cdot)[\tilde{u}][\tilde{u}] > 0$ *for any non-zero* $\tilde{u} \in X_{v_\infty}$, *then* *there exist positive constants* a, b *and* λ *such that the unique* L^1*-solution* u *of equation* (1.2) *satisfies*

$$\|u(t)-v_\infty\|_{L^2(dx/v_\infty)} \leqq ae^{-\lambda t}$$

for any initial probability density u_0 *satisfying*

$$\|u_0-v_\infty\|_{L^2(dx/v_\infty)} \leqq b.$$

We prove Theorem 5.1 by the following procedure: first we construct the solution $\tilde{u}(t)$ of a certain integral equation for $u(t)-v_\infty$, and then, noting the uniqueness of the solution of equation (1.2), we check that $\tilde{u}(t)+v_\infty$ is a distribution solution of equation (1.2).

Let us illustrate the content of this paper with the following diagram.

$$\begin{array}{ccc}
P(X^{(n)}(t)\in dx_1\cdots dx_n) & \xrightarrow[n\nearrow\infty]{\S 2} & (u(t,x)dx)^{\otimes N} \\
 & & \Big\downarrow {\scriptstyle t\nearrow\infty}\ \S 5, \S 4 \\
\Big\downarrow {\scriptstyle t\nearrow\infty} & & (v_\infty(x)dx)^{\otimes N} \\
v^{(n)}(x_1,\cdots,x_n)dx_1\cdots dx_n & \xrightarrow{\S 3} & \int_{\mathcal{P}(\mathbf{R}^d)} Q(d\mu)\mu^{\otimes N}
\end{array}$$

(u: a solution of equation (1.2))

(v: a stationary solution of equation (1.2))

converge along subsequence of $\{n\}$

(Q is a probability measure on $\mathcal{P}(\mathbf{R}^d)$ and supp. $Q\subset K_0$),

where $v^{(n)}$ is a probability density on $\mathbf{R}^{nd}$ given by

$$(1.3)\qquad v^{(n)}(x_1,\cdots,x_n)=\frac{\exp\left(-2\sum_{i=1}^n \Phi_1(x_i)+\frac{\beta}{n}\sum_{i,j=1}^n \Phi_2(x_i,x_j)\right)}{\int_{\mathbf{R}^{nd}}\exp\left(-2\sum_{i=1}^n \Phi_1(x_i)+\frac{\beta}{n}\sum_{i,j=1}^n \Phi_2(x_i,x_j)\right)dx_1\cdots dx_n}$$

and

$$K_0=\{\mu\in\mathcal{P}(\mathbf{R}^d)\,;\ F \text{ attains the minimum at } \mu\}.$$

The above diagram is not necessarily commutative, but it is commutative if K_0 has only a single point.

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Notation.

$N_* = \{0, 1, 2, \cdots\}$.

$C_K^\infty(\boldsymbol{R}^d)$: the set of all infinitely differentiable functions on $\boldsymbol{R}^d$ with compact support.

$\boldsymbol{C}_K^\infty(\boldsymbol{R}^d) = \{(\varphi_1, \cdots, \varphi_d);\ \varphi_i \in C_K^\infty(\boldsymbol{R}^d),\ 1 \leqq i \leqq d\}$.

$\mathcal{S}(\boldsymbol{R}^d)$: the set of all rapidly decreasing functions on $\boldsymbol{R}^d$.

$W_d = C([0, \infty) \to \boldsymbol{R}^d)$.

Let X be a complete separable metric space.

$C_b(X)$: the set of all bounded and continuous functions on X.

$\mathcal{P}(X)$: the space of all probability measures on X with the weak topology.

Let X_1, X_2 and X_3 be Banach spaces and f a mapping from B into X_3, where B is an open set of $X_1 \times X_2$.

$D_{x_1} f(x_1, x_2)$: the Fréchet derivative of f with respect to the first variable at (x_1, x_2).

§ 2. Non-linear parabolic equation (1.2)

DEFINITION 2.1. (i) We say that a mapping u from $[0, \infty)$ into $\mathcal{S}^*(\boldsymbol{R}^d)$ is a distribution solution of non-linear equation (1.2) with an initial data u_0 if u satisfies that

(a) $u : [0, \infty) \to \mathcal{S}^*(\boldsymbol{R}^d)$ weak continuous

and

(b) for any $\varphi \in \mathcal{S}(\boldsymbol{R}^d)$,

$$\begin{aligned} {}_{\mathcal{S}(\boldsymbol{R}^d)}\langle \varphi, u(t) \rangle_{\mathcal{S}^*(\boldsymbol{R}^d)} &= {}_{\mathcal{S}(\boldsymbol{R}^d)}\langle \varphi, u_0 \rangle_{\mathcal{S}^*(\boldsymbol{R}^d)} \\ &\quad + \int_0^t {}_{\mathcal{S}(\boldsymbol{R}^d)}\left\langle \frac{1}{2}\Delta\varphi - \operatorname{grad} \Phi_1 \cdot \operatorname{grad} \varphi, u(s) \right\rangle_{\mathcal{S}^*(\boldsymbol{R}^d)} ds \\ &\quad + \int_0^t {}_{\mathcal{S}(\boldsymbol{R}^{2d})}\langle \operatorname{grad} \varphi(x) \cdot \operatorname{grad}_x \Phi_2(x, y),\ u(s) \otimes u(s) \rangle_{\mathcal{S}^*(\boldsymbol{R}^{2d})} ds. \end{aligned}$$

(ii) We say that u is an L^1-solution of non-linear equation (1.2) if

u satisfies that

(a) u is a distribution solution of equation (1.2)

and

(b) for any $t>0$, $u(t, \cdot) \in L^1(\boldsymbol{R}^d)$, and $\|u(t, \cdot)\|_{L^1(\boldsymbol{R}^d)}$ is locally bounded in t.

(iii) We say that u is a smooth solution of non-linear equation (1.2) if $u(t, x) \in C^\infty((0, \infty) \times \boldsymbol{R}^d)$ and satisfies (1.2).

Though the coefficients of equation (1.2) are not bounded, we can modify the results in H. P. McKean, Jr. [3] to get

THEOREM 2.1. (i) *A distribution solution of equation* (1.2) *is a smooth solution.*

(ii) *Let* $u_0 \in \mathcal{P}(\boldsymbol{R}^d)$ *with the second moment. Then there exists a unique* L^1*-solution of equation* (1.2).

(iii) *For any* $t>0$, $u(t, \cdot)$ *is equal to the probability density of the solution* $X(t)$ *of the following stochastic differential equation of the McKean type*:

$$(2.1) \quad \begin{cases} dX(t) = dB(t) - \operatorname{grad} \Phi_1(X(t))dt + \beta \operatorname{grad}_x \int_{\boldsymbol{R}^d} \Phi_2(X(t), y) u(t, y) dy dt, \\ u(t, \cdot) \text{ is the probability density of } X(t), \\ \text{the probability density of } X(0) = u_0. \end{cases}$$

REMARK 2.1. We can prove the existence and uniqueness of solutions for the stochastic differential equations of the McKean type whose coefficients have Lipschitz continuity with the growth condition of linear order with initial distributions having the second moment.

DEFINITION 2.2. Let X be a complete separable metric space. For any $n \in \boldsymbol{N}$, $U^{(n)} \in \mathcal{P}(X^n)$ and $\lambda \in \mathcal{P}(X)$ we define a probability measure $\underline{U^{(n)}}$ on X^N by

$$\underline{U^{(n)}}(dx_1, dx_2, \cdots) = U^{(n)}(dx_1, \cdots, dx_n) \prod_{j=n+1}^{\infty} \lambda(dx_j).$$

We call this $\underline{U^{(n)}}$ as an extension of $U^{(n)}$ with λ.

Next we quote the theorem on propagation of chaos from H. P. McKean, Jr. [3].

THEOREM 2.2 (H. P. McKean, Jr. [3]). *Let* $U^{(n)}$ *be a probability distribution on* W_d^n *of* n*-particle system* (1.1) *with an initial distribution*

$u_0^{\otimes n}$, where u_0 has the second moment. Let $\underline{U^{(n)}}$ be an extension of $U^{(n)}$ with an arbitrary $\lambda \in \mathcal{P}(\boldsymbol{R}^d)$. Then

$$\underline{U^{(n)}} \to U^{\otimes N} \quad in\ \mathcal{P}(W_d^N)\ as\ n \nearrow \infty,$$

where U is the probability distribution of the solution X of the stochastic differential equation (2.1) of the McKean type.

Remark 2.2. By estimating the moments and by Theorem 2.1, we can show the following law of large numbers (Y. Tamura; Master's thesis)

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{\{X^{(n,i)}\}} \longrightarrow U \quad \text{in law, as } n \nearrow \infty, \tag{2.2}$$

where $\{X^{(n,i)}; 1 \leqq i \leqq n\}$ is n-particle system (1.1) and U is the probability distribution of the solution of the stochastic differential equation (2.1) of the McKean type.

§ 3. Convergence of stationary solutions of n-particle system (1.1)

First we quote two results from S. R. S. Varadhan [5] and M. D. Donsker and S. R. S. Varadhan [1]. Let X be a complete separable metric space throughout this section.

Proposition 3.1 (S. R. S. Varadhan [5]). *Let $\{P_n; n \in N\} \subset \mathcal{P}(X)$ and H be a functional from X into $[0, \infty]$ such that*

(i) *H is lower semi-continuous,*
(ii) *for any positive k, $\{x \in X; H(x) \leqq k\}$ is a compact set of X,*
(iii) *for any closed subset C of X,*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log P_n(C) \leqq -\inf_{x\in C} H(x)$$

and

(iv) *for any open subset G of X,*

$$\underline{\lim_{n\to\infty}}\frac{1}{n}\log P_n(G) \geqq -\inf_{x\in G} H(G).$$

Then for any $\varphi \in C_b(X)$,

$$\lim_{n\to\infty}\frac{1}{n}\log\int_X \exp(n\varphi(x))P_n(dx) = \sup_{x\in X}[\varphi(x) - H(x)].$$

PROPOSITION 3.2. *Let $\{P_n; n\in N\}$ and H be as in Proposition* 3.1. *Furthermore we assume that $H\not\equiv\infty$. Let $\varphi\in C_b(X)$ and $K_{\varphi,0}=\{x\in X;$ $H-\varphi$ attains the minimum at $x\}$. For any $n\in N$, we define $Q_n\in\mathcal{P}(X)$ by*

$$Q_n(A)=\frac{\int_A \exp(n\varphi(x))P_n(dx)}{\int_X \exp(n\varphi(x))P_n(dx)}, \quad (A\in\mathcal{P}(X)).$$

Then (i) $\{Q_n; n\in N\}$ *is precompact in* $\mathcal{P}(x)$ *and*

(ii) *for any accumulating point Q of $\{Q_n; n\in N\}$,* supp. $Q\subset K_{\varphi,0}$.

PROOF. First we prove that for any open subset G of X which contains $K_{\varphi,0}$ and any $\varepsilon>0$, there exists n_0 such that for any $n\geqq n_0$,

$$Q_n(G^c)<\varepsilon. \tag{3.1}$$

Put a and $K_{\varphi,\delta}$ $(\delta>0)$ as follows:

$$-a=\inf_{x\in X}(H(x)-\varphi(x))$$

and

$$K_{\varphi,\delta}=\{x\in X; H(x)-\varphi(x)\leqq -a+\delta\}.$$

Since H satisfies (i) and (ii) in Proposition 3.1, there exists $\delta>0$ such that $G\supset K_{\varphi,\delta}$. Furthermore, since H satisfies Proposition 3.1 (iii),

$$\overline{\lim}\frac{1}{n}\log\int_{G^c}\exp(n\varphi(x))dP_n(x)\leqq -\inf_{x\in G^c}(H(x)-\varphi(x))\leqq a-\delta. \tag{3.2}$$

By Proposition 3.1 and (3.2), we can see that there exists n_0 such that for any $n\geqq n_0$,

$$\int_X \exp(n\varphi(x))dP_n(x)\geqq \exp(n(a-\delta/3))$$

and

$$\int_{G^c}\exp(n\varphi(x))dP_n(x)\leqq \exp(n(a-\delta+\delta/3)),$$

which implies (3.1). (ii) follows immediately from (3.1). In order to prove (i), it suffices to show that for any $\varepsilon>0$ and $\delta>0$, there exist balls $B_1, B_2, \cdots, B_k$ in X with the same radius δ such that

(3.3) $$Q_n\Big(\bigcup_{i=1}^{k} B_i\Big)>1-\varepsilon \quad \text{for any } n\in N.$$

We fix $\delta>0$ and $\varepsilon>0$. Then, since $K_{\varphi,0}$ is compact there exist balls $B_1, \cdots, B_m$ with radius δ such that $\bigcup_{i=1}^{m} B_i \supset K_{\varphi,0}$. Put $G=\bigcup_{i=1}^{m} B_i$. Then there exists n_0 such that for any $n\geqq n_0$ (3.1) holds. On the other hand, for any i $(1\leqq i\leqq n_0)$ there exists a compact set K_i such that $Q_i(K_i)>1-\varepsilon$. So there exist balls $B_{m+1}, \cdots, B_k$ with radius δ such that $\bigcup_{i=m+1}^{k} B_i \supset \bigcup_{i=1}^{n_0} K_i$. These $B_1, \cdots, B_k$ satisfies (3.3). Q.E.D.

COROLLARY 3.1. *If $K_{\varphi,0}=\{x_0\}$ then $Q_n \to \delta_{\{x_0\}}$ in $\mathcal{P}(X)$ as $n\nearrow\infty$.*

Following M. D. Donsker and S. R. S. Varadhan [1], we define an entropy functional H_λ.

DEFINITION 3.1. For fixed $\lambda\in\mathcal{P}(X)$, we define the functional H_λ on $\mathcal{P}(X)$ by

$$H_\lambda(\mu)=\begin{cases} \displaystyle\int_X \log\Big(\frac{d\mu}{d\lambda}\Big)d\mu, \text{ if } \mu \text{ is absolutely continuous with} \\ \qquad \text{respect to } \lambda \text{ and } \log\Big(\dfrac{d\mu}{d\lambda}\Big)\in L^1(d\mu), \\ +\infty, \quad \text{otherwise.} \end{cases}$$

PROPOSITION 3.3 (M. D. Donsker and S. R. S. Varadhan [1]). *Let $\{Y_n; n\in N\}$ be a family of independent X-valued random variables with common probability distribution λ. Let P_n be a probability distribution of $\frac{1}{n}\sum_{i=1}^{n}\delta_{\{Y_i\}}$ on $\mathcal{P}(X)$. Then the functional H_λ on $\mathcal{P}(X)$ satisfies* (i), (ii), (iii) *and* (iv) *in Proposition* 3.1 *replacing X by $\mathcal{P}(X)$.*

Now we return to non-linear equation (1.2) and define a free energy F for it.

DEFINITION 3.2. We define the functional F on $\mathcal{P}(\boldsymbol{R}^d)$ by

$$F(\mu)=\begin{cases} \displaystyle\int_{\boldsymbol{R}^d}(\log\mu(x)+2\Phi_1(x))\mu(x)dx-\beta\iint_{\boldsymbol{R}^{2d}}\Phi_2(x,y)\mu(x)\mu(y)dxdy, \\ \quad \text{if } \mu(dx) \text{ has a density } \mu(x) \text{ and } (\log\mu(x)+2\Phi_1(x))\in L^1(\mu(x)dx) \\ +\infty, \quad \text{otherwise.} \end{cases}$$

Convention. For any probability density $u(x)$ on $\boldsymbol{R}^d$, we denote by the same symbol u the probability measure $u(x)dx$.

REMARK 3.1. By a simple calculation, we can see that for any probability density u on $\boldsymbol{R}^d$ the following two conditions are equivalent:

(i) $\log u(x)+2\Phi_1(x)\in L^1(u(x)dx)$

and

(ii) $\log u(x)\in L^1(u(x)dx)$ and $2\Phi_1(x)\in L^1(u(x)dx)$.

Let

$$v_0(x)=Z_0^{-1}\exp(-2\Phi_1(x)),\quad \text{where } Z_0=\int_{\boldsymbol{R}^d}\exp(-2\Phi_1(x))dx. \tag{3.4}$$

Then, using an entropy functional H_{v_0} in Definition 3.1, we have

LEMMA 3.1.

$$F(\mu)=H_{v_0}(\mu)-\beta\iint_{\boldsymbol{R}^{2d}}\Phi_2(x, y)\mu(dx)\mu(dy)+\log Z_0.$$

Now we shall prove the next main theorem in this section.

THEOREM 3.1. *Let $v^{(n)}$ be the unique stationary probability distribution with density* (1.3) *of n-particle system* (1.1) *and let $\underline{v^{(n)}}$ be an extension of $v^{(n)}$ with an arbitrary $\lambda\in\mathcal{P}(\boldsymbol{R}^d)$. Let*

$$K_0=\{\mu\in\mathcal{P}(\boldsymbol{R}^d);\ F \text{ attains its minimum at } \mu\}.$$

Then (i) $\{\underline{v^{(n)}};n\in N\}$ *is precompact in* $\mathcal{P}((\boldsymbol{R}^d)^N)$

and

(ii) *for any accumulating point v of* $\{\underline{v^{(n)}};n\in N\}$.

there exists $Q\in\mathcal{P}(\mathcal{P}(\boldsymbol{R}^d))$ such that

$$v(d\underline{x})=\int_{\mathcal{P}(\boldsymbol{R}^d)}Q(d\mu)\mu^{\otimes N}(d\underline{x})$$

and

$$\text{supp.}\, Q\subset K_0.$$

PROOF. We define $L^{(n)}:(\boldsymbol{R}^d)^n\to\mathcal{P}(\boldsymbol{R}^d)$ by $L^{(n)}((x_1,\cdots,x_n))=\dfrac{1}{n}\sum_{i=1}^n\delta_{\{x_i\}}$.

Let P_n and Q_n be the induced probability measures on $\mathcal{P}(\boldsymbol{R}^d)$ of $v_0^{\otimes n}$ and $v^{(n)}$ by $L^{(n)}$, respectively, where v_0 is as in (3.4). We define $\varphi\in C_b(\mathcal{P}(\boldsymbol{R}^d))$ by

$$\varphi(\mu)=\beta\iint_{R^{2d}}\Phi_2(x, y)\mu(dx)\mu(dy).$$

Then, for any Borel subset A of $\mathcal{P}(\boldsymbol{R}^d)$,

$$\begin{aligned}Q_n(A)&=\int_{R^{nd}}\chi_A(L^{(n)}(x_1, \cdots, x_n))v^{(n)}(dx_1, \cdots, dx_n)\\&=\frac{\int_{R^{nd}}\chi_A(L^{(n)}(x_1, \cdots, x_n))\exp(n\varphi(L^{(n)}(x_1, \cdots, x_n)))v_0(x_1)\cdots v_0(x_n)dx_1\cdots dx_n}{\int_{R^{nd}}\exp(n\varphi(L^{(n)}(x_1, \cdots, x_n)))v_0(x_1)\cdots v_0(x_n)dx_1\cdots dx_n}\\&=\frac{\int_A\exp(n\varphi(\mu))P_n(d\mu)}{\int_{\mathcal{P}(R^d)}\exp(n\varphi(\mu))P_n(d\mu)}.\end{aligned}$$

By Propositions 3.2, 3.3 and Lemma 3.1, we see that $\{Q_n; n\in N\}$ is pre-compact in $\mathcal{P}(\boldsymbol{R}^d)$ and for any accumulating point Q of $\{Q_n; n\in N\}$, supp. Q is contained in K_0. Therefore, for the purpose of proving Theorem 3.1, we have only to show the following Lemma 3.2. Q.E.D.

LEMMA 3.2. *For $n\in N$ we define $L^{(n)}: X^n\to\mathcal{P}(X)$ by $L^{(n)}(x_1, \cdots, x_n)=\frac{1}{n}\sum_{i=1}^n\delta_{\{x_i\}}$. Let $u^{(n)}$ be a symmetric probability distribution on X^n, and Q_n an induced probability measure on $\mathcal{P}(X)$ of $u^{(n)}$ by $L^{(n)}$. Let $\underline{u}^{(n)}$ be an extension of $u^{(n)}$ with an arbitrary $\lambda\in\mathcal{P}(X)$. If there exists $Q\in\mathcal{P}(\mathcal{P}(X))$ such that $Q_n\to Q$ in $\mathcal{P}(\mathcal{P}(X))$ as $n\nearrow\infty$, then*

$$\underline{u}^{(n)}\longrightarrow\int_{\mathcal{P}(X)}Q(d\mu)\mu^{\otimes N}\quad\text{in }\mathcal{P}(X^N)\text{ as }n\nearrow\infty.$$

PROOF. It suffices to prove that for any $k\in N$ and $h\in C_b(X^k)$,

$$\begin{aligned}&\int_{X^n}h(x_1, \cdots, x_k)u^{(n)}(dx_1, \cdots, dx_n)\\&\longrightarrow\int_{\mathcal{P}(X)}Q(d\mu)\int_{X^k}h(x_1, \cdots, x_k)\mu(dx_1)\cdots\mu(dx_k), \quad\text{as } n\nearrow\infty.\end{aligned}\tag{3.5}$$

We fix $k\in N$ and $h\in C_b(X^k)$ and define $f\in C_b(\mathcal{P}(X))$ by

$$f(\mu)=\int_{X^k}h(x_1, \cdots, x_k)\mu(dx_1)\cdots\mu(dx_k).\tag{3.6}$$

Then it follows from definition that for any $n\geqq k$

$$\int_{\mathcal{P}(X)} f(\mu)Q_n(d\mu)=\frac{1}{n^k}\sum_{i_1,\cdots,i_k=1}^{n}\int_{X^n} h(x_{i_1},\cdots,x_{i_k})u^{(n)}(dx_1,\cdots,dx_n).$$

Therefore since $Q_n\to Q$ in $\mathcal{P}(\mathcal{P}(X))$,

$$\frac{1}{n^k}\sum_{i_1,\cdots,i_k=1}^{n}\int_{X^n} h(x_{i_1},\cdots,x_{i_k})u^{(n)}(dx_1,\cdots,dx_n)$$
$$\longrightarrow\int_{\mathcal{P}(X)} Q(d\mu)\int_{X^k} h(x_1,\cdots,x_k)\mu(dx_1)\cdots\mu(dx_k),\qquad\text{as } n\nearrow\infty.$$

On the other hand, since $u^{(n)}$ is symmetric, for any $n\geqq k$,

$$\int_{X^n} h(x_1,\cdots,x_k)u^{(n)}(dx_1,\cdots,dx_n)$$
$$=\frac{(n-k)!}{n!}\sum_{\substack{(i_1,\cdots,i_k):\\ \text{distinct from each other}}}\int_{X^n} h(x_{i_1},\cdots,x_{i_k})u^{(n)}(dx_1,\cdots,dx_n).$$

So we can obtain (3.5). Q.E.D.

§ 4. Stationary solutions of non-linear equation (1.2)

DEFINITION 4.1. (i) We say that v is a stationary distribution solution of non-linear equation (1.2) if $v\in\mathcal{S}^*(\boldsymbol{R}^d)$ and for any $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$,

$$_{\mathcal{S}(\boldsymbol{R}^d)}\left\langle\frac{1}{2}\Delta\varphi-\operatorname{grad}\Phi_1\operatorname{grad}\varphi,\ v\right\rangle_{\mathcal{S}^*(\boldsymbol{R}^d)}$$
$$+\beta_{\mathcal{S}(\boldsymbol{R}^{2d})}\langle\operatorname{grad}_x\Phi_2(x,y)\cdot\operatorname{grad}\varphi(x),\ v\otimes v\rangle_{\mathcal{S}^*(\boldsymbol{R}^{2d})}=0.$$

(ii) We say that v is a stationary probability solution of equation (1.2) if v is a probability density and v is a stationary distribution solution of equation (1.2).

REMARK 4.1. From Theorem 2.1, we see that stationary distribution solutions of equation (1.2) belong to $C^\infty(\boldsymbol{R}^d)$.

By regarding the constant β in equation (1.2) as a parameter in this section, we denote it by $(1.2)_\beta$. We define an operator $f: L^1(\boldsymbol{R}^d)\times\boldsymbol{R}\to L^1(\boldsymbol{R}^d)$ by

$$f(u,\beta)(x)=\frac{\exp\left(-2\Phi_1(x)+2\beta\int_{\boldsymbol{R}^d}\Phi_2(x,y)u(y)dy\right)}{\int_{\boldsymbol{R}^d}\exp\left(-2\Phi_1(x)+2\beta\int_{\boldsymbol{R}^d}\Phi_2(x,y)u(y)dy\right)dx}.\tag{4.1}$$

We note that $f(u,\beta)$ is a probability density for any $(u,\beta)\in L^1(\boldsymbol{R}^d)\times\boldsymbol{R}$.

We define a differential operator G_v for a probability density v by

$$G_v\varphi(x)=\frac{1}{2}\Delta\varphi(x)+\operatorname{div}\left(\varphi(x)\operatorname{grad}\left(\Phi_1(x)-\beta\int_{R^d}\Phi_2(x,y)v(y)dy\right)\right) \qquad \varphi\in C^2(\boldsymbol{R}^d). \tag{4.2}$$

Then the following Lemma is known.

LEMMA 4.1. *For any probability density v, there exists a unique solution of $G_v w=0$ among probability densities and furthermore w is expressed in the form*

$$w(x)=\frac{\exp\left(-2\Phi_1(x)+2\beta\int_{R^d}\Phi_2(x,y)v(y)dy\right)}{\int_{R^d}\exp\left(-2\Phi_1(x)+2\beta\int_{R^d}\Phi_2(x,y)v(y)dy\right)dx}.$$

THEOREM 4.1. (i) *For a fixed $\beta\in\boldsymbol{R}$, v is a stationary probability solution of equation $(1.2)_\beta$ if and only if $v=f(v,\beta)$.*

(ii) *There exists a stationary probability solution of equation $(1.2)_\beta$ for any $\beta\in\boldsymbol{R}$.*

(iii) *If $|\beta|$ is sufficiently small, there exists a unique stationary probability solution of equation $(1.2)_\beta$.*

PROOF. By noting that v is a stationary probability solution of $(1.2)_\beta$ if and only if $G_v v=0$, we obtain (i) by Lemma 4.1. Let $B_1=\{u\in L^1(\boldsymbol{R}^d);\ \|u\|_{L^1(\boldsymbol{R}^d)}\leqq 1\}$. We note that B_1 is a convex and bounded closed subset of $L^1(\boldsymbol{R}^d)$. We consider f in (4.1) as an operator from $B_1\times\boldsymbol{R}$ into B_1. We can see from the boundedness of Φ_2 that there exists a positive constant M such that

$$\begin{aligned}\int_{R^d}|f(u_1,\beta)-f(u_2,\beta)|dx \\ \leqq|\beta|M\int_{R^d}dx\left|\int_{R^d}\Phi_2(x,y)u_1(y)dy-\int_{R^d}\Phi_2(x,y)u_2(y)dy\right|, \\ \text{for } u_1,u_2\in B_1.\end{aligned} \tag{4.3}$$

Since the mapping on $L^1(\boldsymbol{R}^d)$: $u(x)\longmapsto\int_{R^d}\Phi_2(x,y)u(y)dy$ is compact, we obtain (ii) by (4.3) and Schauder's fixed point theorem. Furthermore, since for sufficiently small $|\beta|$, $f(\cdot,\beta)$ becomes a contraction mapping by (4.3), we obtain (iii). Q.E.D.

Next we shall show that there may be more than one stationary probability solution in general. Using v_0 in (3.4), we define a linear operator T in $L^1(\boldsymbol{R}^d)$ by

$$Tu(x)=2v_0(x)\int_{R^d}\Phi_2(x, y)u(y)dy. \tag{4.4}$$

We note that T is compact. Furthermore we define a mapping g from $L^1(\boldsymbol{R}^d)\times\boldsymbol{R}$ into $L^1(\boldsymbol{R}^d)$ by

$$g(u, \beta)=u-f(u, \beta). \tag{4.5}$$

THEOREM 4.2. *Let* (v_0, β_0) *be a point of* $L^1(\boldsymbol{R}^d)\times(\boldsymbol{R}\setminus\{0\})$ *such that*

(i) v_0 *is the same as in* (3.4)

and

(ii) β_0^{-1} *is an eigenvalue of the operator* T.

We assume that

(iii) $\int_{R^d}\Phi_2(x, y)v_0(y)dy=0$

and

(iv) $\dim\{v\in L^1(\boldsymbol{R}^d)\,;\, v=\beta_0 Tv\}=1$.

Then (v_0, β_0) *is a bifurcation point of* $g=0$,

i.e. (a) $g(v_0, \beta_0)=0$

and

(b) *for any neighborhood* B *of* (v_0, β_0) *in* $L^1(\boldsymbol{R}^d)\times\boldsymbol{R}$, *there exists a point* (v_1, β_1) *in* B *such that* $v_1\neq v_0$ *and* $g(v_1, \beta_1)=0$.

First we quote the next Lemma 4.2 which gives a sufficient condition under which bifurcation occurs (L. Nirenberg [4]).

LEMMA 4.2. *Let* X *and* Y *be Banach spaces and let* B *be a neighborhood of some point* (x_0, β_0) *in* $X\times\boldsymbol{R}$. *Let* g *be a mapping of* C^p *class* $(p\geqq 2)$ *from* B *into* Y *such that*

(i) $g(x_0, \beta_0)=0$,

(ii) $D_\beta g(x_0, \beta_0)=0$,

(iii) $\dim(\mathrm{Ker}\, D_x g(x_0, \beta_0))=1$,

(iv) $\mathrm{Im}\, D_x g(x_0, \beta_0)$ *is closed and* $\mathrm{codim}(\mathrm{Im}\, D_x g(x_0, \beta_0))=1$

and

(v) $D_\beta D_\beta g(x_0, \beta_0)\in \mathrm{Im}\, D_x g(x_0, \beta_0)$, $D_x D_\beta g(x_0, \beta_0)[x_2]\notin \mathrm{Im}\, D_x g(x_0, \beta_0)$,

where x_2 *is a non-zero element in* $\mathrm{Ker}\, D_x g(x_0, \beta_0)$.

Then (x_0, β_0) *is a bifurcation point of* $g=0$.

We can calculate the derivatives of f in (4.1) as follows.

LEMMA 4.3.

(i) $D_u f(u,\beta)[w](x)=2\beta f(u,\beta)(x)\int_{R^d}\Phi_2(x,y)w(y)dy$
$$-2\beta f(u,\beta)(x)\iint_{R^{2d}}\Phi_2(z,y)w(y)f(u,\beta)(z)dzdy$$

(ii) $D_\beta f(u,\beta)(x)=2f(u,\beta)(x)\int_{R^d}\Phi_2(x,y)u(y)dy$
$$-2f(u,\beta)(x)\iint_{R^{2d}}\Phi_2(z,y)u(y)f(u,\beta)(z)dzdy$$

(iii) $D_u D_\beta f(u,\beta)[w](x)=2f(u,\beta)(x)\int_{R^d}\Phi_2(x,y)w(y)dy$
$$-2f(u,\beta)(x)\iint_{R^{2d}}\Phi_2(z,y)w(y)f(u,\beta)(z)dzdy$$
$$+2D_u f(u,\beta)[w](x)\int_{R^d}\Phi_2(x,y)u(y)dy$$
$$-2D_u f(u,\beta)[w](x)\iint_{R^{2d}}\Phi_2(z,y)u(y)f(u,\beta)(z)dzdy$$
$$-2f(u,\beta)(x)\iint_{R^{2d}}\Phi_2(z,y)u(y)D_u f(u,\beta)[w](z)dzdy.$$

PROOF OF THEOREM 4.2. By the condition (iii) and Lemma 4.3 we note that

$$g(v_0,\beta)=0 \qquad \text{for any } \beta\in \mathbf{R} \tag{4.6}$$

and

$$D_u g(v_0,\beta_0)=I-\beta_0 T. \tag{4.7}$$

It suffices to check conditions (i), (ii), (iii), (iv) and (v) in Lemma 4.2. (i) and (ii) follow from (4.6) and (iii) follows from (4.7) and conditions (ii) and (iii) in Theorem 4.2. By (4.7) and Riesz-Schauder's theorem, we see that $\mathrm{Im} D_u g(v_0,\beta_0)=\mathrm{Ker}(I-\beta_0 T^*)^\perp$, and $\dim\mathrm{Ker}(I-\beta_0 T)=\dim\mathrm{Ker}(I-\beta_0 T^*)$, which implies (iv). By (4.6) we see that $D_\beta D_\beta g(v_0,\beta_0)=0$ and so it belongs to $\mathrm{Im}\, D_u g(v_0,\beta_0)$. Let v_2 be a non-zero element in $\mathrm{Ker}(I-\beta_0 T)$. Then we can see that $v_2/v_0\in\mathrm{Ker}(I-\beta_0 T^*)$. Since $D_u D_\beta g(v_0,\beta_0)[v_2]=\beta_0^{-1}v_2$ by Lemma 4.3 (iii), we obtain that

$${}_{L^1(R^d)}\langle D_u D_\beta g(v_0,\beta_0)[v_2],\, v_2/v_0\rangle_{L^\infty(R^d)}=\beta_0^{-1}\int_{R^d}|v_2(x)|^2dx\neq 0,$$

which implies $D_u D_\beta g(v_0,\beta_0)[v_2]\notin\mathrm{Ker}(I-\beta_0 T^*)^\perp$. Therefore we obtain (v).
Q.E.D.

Now we fix $\beta\in \boldsymbol{R}$ and investigate some properties of the free energy F introduced in Definition 3.2. Let w be an arbitrary probability density on $\boldsymbol{R}^d$ with $w>0$ (a.e.) and $F(w)<+\infty$. We define X_w by

$$(4.8)\quad X_w=\left\{\tilde{u},\ \text{measurable function on } \boldsymbol{R}^d;\ |||u|||\equiv \operatorname*{ess.sup}_{x\in \boldsymbol{R}^d}\left|\frac{\tilde{u}(x)}{w(x)}\right|<+\infty,\ \int_{\boldsymbol{R}^d}\tilde{u}(x)dx=0\right\}.$$

Then X_w is a Banach space with the norm $|||\cdot|||$. Furthermore we define $B_{1/2}$ by

$$(4.9)\qquad B_{1/2}=\{\tilde{u}\in X_w;\ |||\tilde{u}|||<1/2\},$$

and a functional $\tilde{F}_w$ from $B_{1/2}$ into $\boldsymbol{R}$ by

$$(4.10)\qquad \tilde{F}_w(\tilde{u})=F(w+\tilde{u}).$$

Then by a simple calculation we obtain

LEMMA 4.4. *$\tilde{F}_w$ is twice differentiable on $B_{1/2}$ and*

$$\text{(i)}\quad D\tilde{F}_w(0)[\tilde{u}]=\int_{\boldsymbol{R}^d}(\log w(x))\tilde{u}(x)dx+2\int_{\boldsymbol{R}^d}\Phi_1(x)\tilde{u}(x)dx -2\iint_{\boldsymbol{R}^{2d}}\Phi_2(x,y)\tilde{u}(x)w(y)dxdy,\quad \textit{for any } \tilde{u}\in X_w.$$

$$\text{(ii)}\quad D^2\tilde{F}_w(0)[\tilde{u}_1][\tilde{u}_2]=\int_{\boldsymbol{R}^d}\frac{\tilde{u}_1(x)\tilde{u}_2(x)}{w(x)}dx-2\beta\iint_{\boldsymbol{R}^{2d}}\Phi_2(x,y)\tilde{u}_1(x)\tilde{u}_2(y)dxdy,\quad \textit{for any } \tilde{u}_1, \tilde{u}_2\in X_w.$$

THEOREM 4.3. *Let v be any probability density on $\boldsymbol{R}^d$. Then v is a stationary probability solution of equation* $(1.2)_\beta$ *if and only if* (i) $v>0$ *(a.e.) and* (ii) $D\tilde{F}_v(0)=0$.

PROOF. First we assume that v is a stationary probability solution. Then by Theorem 4.1 and Lemma 4.4 (i), we have (i) and (ii). Conversely we assume (i) and (ii). Let

$$\tilde{\varphi}_v(x)=\varphi(x)v(x)-v(x)\int_{\boldsymbol{R}^d}\varphi(y)v(y)dy,\qquad \text{for any } \varphi\in C_K^\infty(\boldsymbol{R}^d).$$

Then $\tilde{\varphi}_v\in X_v$. By Lemma 4.4 (i)

$$0=D\tilde{F}_v(0)[\tilde{\varphi}_v]$$
$$=\int_{R^d}\varphi(x)\Big(v(x)\Big(\log v(x)+2\Phi_1(x)-2\beta\int_{R^d}\Phi_2(x,y)v(y)dy\Big)-cv(x)\Big)dx,$$
$$\text{for any } \varphi\in C_K^\infty(\boldsymbol{R}^d),$$
$$\text{where}\quad c=\int_{R^d}v(x)\Big(\log v(x)+2\Phi_1(x)-2\beta\int_{R^d}\Phi_2(x,y)v(y)dy\Big)dx.$$

As $v>0$ (a.e.),

$$v=\exp\Big(c-2\Phi_1(x)+2\beta\int_{R^d}\Phi_2(x,y)v(y)dy\Big),$$

which gives by Theorem 4.1 that v is a stationary probability solution of equation $(1.2)_\beta$. Q.E.D.

COROLLARY 4.1. *If F attains its minimum at a probability density v, then v is a stationary probability solution of equation* $(1.2)_\beta$.

PROOF. We claim that $v>0$ (a.e.). Let $A=\{x\in\boldsymbol{R}^d; v(x)=0\}$ and

$$\tilde{w}(x)=\begin{cases}-1/2v(x) & x\in A^c\\ av_0(x) & x\in A,\end{cases}$$

where v_0 is as in (3.4) and $a=(2v_0(A))^{-1}$. For small $\varepsilon>0$, $F(v+\varepsilon\tilde{w})<+\infty$ and by a direct calculation we have a constant c such that for any $\varepsilon\in(0,1)$,

$$\frac{1}{\varepsilon}(F(v+\varepsilon\tilde{w})-F(v))-a\log\varepsilon a\cdot v_0(A)\leqq c.$$

The first term is positive from our assumption. If $v_0(A)>0$ then the second term goes to $+\infty$ as ε tends to 0, hence $v_0(A)=0$. Q.E.D.

§ 5. Convergence of solutions of equation (1.2) to stationary solutions

Throughout this section we fix β and a stationary probability solution v_∞ of non-linear equation (1.2). Let F be the free energy defined in Definition 3.2, and let $\tilde{F}_{v_\infty}$ be as in (4.10). Let X_{v_∞} be as in (4.8). Then by Lemma 4.4, $\tilde{F}_{v_\infty}$ is twice differentiable on $B_{1/2}$ given in (4.9). Let

$$\mathscr{H}=L^2(\boldsymbol{R}^d; v_\infty^{-1}dx), \tag{5.1}$$

$$\mathscr{H}_0=\Big\{\tilde{u}\in\mathscr{H}; \int_{R^d}\tilde{u}(x)dx=0\Big\} \tag{5.2}$$

and

(5.3) $$\underline{\mathscr{H}}=\{(u_1,\cdots,u_d);\ u_i\in\mathscr{H},\ 1\leqq i\leqq d\}.$$

We shall prove the following Theorem 5.1.

THEOREM 5.1. *If*

(5.4) $$D^2\tilde{F}_{v_\infty}(0)[\tilde{u}][\tilde{u}]>0 \quad \textit{for any non-zero } \tilde{u}\in X_{v_\infty},$$

then there exist positive constants a, b *and* λ *such that the unique* L^1-*solution* u *of differential equation* (1.2) *satisfies*

$$\|u(t)-v_\infty\|_{\mathscr{H}}\leqq ae^{-\lambda t}$$

for any initial probability density u_0 *satisfying*

$$\|u_0-v_\infty\|_{\mathscr{H}}\leqq b.$$

REMARK 5.1. (5.4) means that F takes the "local strict minimum" at v_∞, and (5.4) is equivalent to (5.7) to be shown in Lemma 5.1.

REMARK 5.2. Since any probability density u_0 contained in $\mathscr{H}$ has the second moment, we have a uniqe L^1-solution of equation (1.2) by Theorem 2.1.

Let T be a linear operator on $\mathscr{H}_0$ by

(5.5) $$T\tilde{u}(x)=2v_\infty(x)\int_{R^d}\Phi_2(x,y)\tilde{u}(y)dy-2v_\infty(x)\iint_{R^{2d}}\Phi_2(z,y)v_\infty(z)\tilde{u}(y)dzdy.$$

Then from the assumptions on Φ_2, T is a compact symmetric operator on $\mathscr{H}_0$. Let ν_1 and ν_2 be the minimum and the maximum spectrum of T, respectively. Let

(5.6) $$\underline{\beta}=\begin{cases}\nu_1^{-1} & \text{if } \nu_1<0\\ -\infty & \text{if } \nu_1\geqq 0\end{cases},\quad \bar{\beta}=\begin{cases}\nu_2^{-1} & \text{if } \nu_2>0\\ +\infty & \text{if } \nu_2\leqq 0\end{cases}$$

LEMMA 5.1. *The condition* (5.4) *in Theorem* 5.1 *is equivalent to*

(5.7) $$\underline{\beta}<\beta<\bar{\beta}.$$

PROOF. By Lemma 4.4 (ii), the condition (5.4) means that for any non-zero $\tilde{u}\in X_{v_\infty}$, $\|\tilde{u}\|^2_{\mathscr{H}_0}-\beta(T\tilde{u},\tilde{u})_{\mathscr{H}_0}>0$. As from (5.6)

$$\beta(T\tilde{u},\tilde{u})_{\mathscr{H}_0}\leqq\begin{cases}\underline{\beta}^{-1}\beta\|\tilde{u}\|^2_{\mathscr{H}_0} & \beta<0\\ \bar{\beta}^{-1}\beta\|\tilde{u}\|^2_{\mathscr{H}_0} & \beta\geqq 0\end{cases}.$$

Then (5.7) implies (5.4). Conversely, since for any $\tilde{u}\in\mathscr{H}_0$, $T\tilde{u}\in X_{v_\infty}$, (5.4) implies (5.7). Q.E.D.

We introduce linear operators G and G_0 on $\mathcal{H}$ and $\mathcal{H}_0$, respectively. First we define a linear operator $G|_{C_K^\infty}$ on $C_K^\infty(\boldsymbol{R}^d)$ by

$$(5.8)\qquad G|_{C_K^\infty}\varphi(x)=\frac{1}{2}\operatorname{div}(v_\infty(x)\operatorname{grad}(\varphi(x)/v_\infty(x)))$$
$$=\frac{1}{2}\Delta\varphi(x)+\operatorname{div}\Big(\varphi(x)\operatorname{grad}\Big(\Phi_1(x)-\beta\int_{\boldsymbol{R}^d}\Phi_2(x,y)\,v_\infty(y)dy\Big)\Big),$$

which is equal to G_{v_∞} in (4.2). Since $G|_{C_K^\infty}$ is a non positive symmetric operator on $\mathcal{H}$, we obtain a Friedrichs' extension G of $G|_{C_K^\infty}$ with the domain

$$(5.9)\qquad \mathcal{D}(G)=\mathfrak{S}\cap\mathcal{D}(G|^*_{C_K^\infty}),$$

where $\mathfrak{S}\subset\mathcal{H}$ be a completion of $C_K^\infty(\boldsymbol{R}^d)$ by the norm

$$(\|\varphi\|^2_{\mathcal{H}}+\|v_\infty\operatorname{grad}(\varphi/v_\infty)\|^2_{\mathcal{H}})^{1/2}.$$

Let G_0 be a restriction of G on $\mathcal{H}_0$.

LEMMA 5.2. (i) *G has discrete non-positive eigenvalues, $0>-\lambda_1>-\lambda_2>\cdots$ in $\mathcal{H}$ with finite dimensional eigenspaces.*

(ii) *G_0 is a negative self-adjoint operator on $\mathcal{H}_0$ with eigenvalues, $-\lambda_1>-\lambda_2>\cdots$.*

PROOF. For any $\varphi\in C_K^\infty(\boldsymbol{R}^d)$, let

$$L|_{C_K^\infty}\varphi(x)=\frac{1}{2}\Delta\varphi(x)-\frac{1}{2}(|\operatorname{grad}\tilde{\Phi}_1(x)|^2-\Delta\tilde{\Phi}_1(x))\varphi(x)$$

where $\tilde{\Phi}_1(x)=\Phi_1(x)-\beta\int_{\boldsymbol{R}^d}\Phi_2(x,y)v_\infty(y)dy$. Then $L|_{C_K^\infty}$ is a non-positive symmetric operator on $L^2(\boldsymbol{R}^d)$. Let L be a Friedrichs' extension of $L|_{C_K^\infty}$ on $L^2(\boldsymbol{R}^d)$. We define a unitary operator U from $\mathcal{H}$ to $L^2(\boldsymbol{R}^d)$ by $Uu(x)=u(x)/\sqrt{v_\infty(x)}$. Then $UGU^{-1}=L$. Since $(-L+cI)^{-1}$ is a compact operator for some constant c from the assumptions on Φ_1 and Φ_2, L has a discrete spectrum in $L^2(\boldsymbol{R}^d)$. Therefore G has a discrete spectrum in $\mathcal{H}$. Let $u\in\mathcal{H}$ be a distribution solution of $Gu=0$. Since the transition probability density $p_{v_\infty}(t,y,x)$ of the G^*-diffusion satisfies $p_{v_\infty}(t,y,x)>0$ for any $t>0$ and almost all y and $x\in\boldsymbol{R}^d$, and u is an element of $L^1(\boldsymbol{R}^d)$, $u\geqq0$ (a.s.) or $u\leqq0$ (a.s.). Hence by Lemma 4.1 there exists a constant $c\in\boldsymbol{R}$ such that $u(x)=cv_\infty(x)$, which implies that G_0 is a self-adjoint operator on $\mathcal{H}_0$ with a negative discrete spectrum. Q.E.D.

Next we define linear operators D_1 and D_2 as follows;

(5.10) $$\mathcal{D}(D_1)=\{\boldsymbol{u}\in \underline{\mathcal{H}};\ \exists w\in\mathcal{H} \text{ such that } (w,\varphi)_{\mathcal{H}}=-(\boldsymbol{u}, v_\infty \operatorname{grad}(\varphi/v)_\infty)_{\mathcal{H}}\ \forall\varphi\in C_K^\infty(\boldsymbol{R}^d)\},$$

$$D_1\boldsymbol{u}=w \quad \text{for } \boldsymbol{u}\in\mathcal{D}(D_1),$$

$$\mathcal{D}(\boldsymbol{D}_2)=\{u\in\mathcal{H};\ \exists \boldsymbol{w}\in\underline{\mathcal{H}} \text{ such that } (\boldsymbol{w},\boldsymbol{\varphi})_{\mathcal{H}}=-(u,\operatorname{div}\boldsymbol{\varphi})_{\mathcal{H}}\ \forall\boldsymbol{\varphi}\in \boldsymbol{C}_K^\infty(\boldsymbol{R}^d)\},$$

$$\boldsymbol{D}_2u=\boldsymbol{w} \quad \text{for } u\in\mathcal{D}(\boldsymbol{D}_2).$$

REMARK 5.3. From (5.9), we see that $\mathcal{D}(G)\subset\mathcal{D}(\boldsymbol{D}_2)$.

LEMMA 5.3. (i) *For any* $u\in\mathcal{D}(G)$,

$$Gu=\frac{1}{2}D_1\boldsymbol{D}_2u.$$

(ii) *For any* $\tilde{u}\in\mathcal{D}(G_0)$,

$$\|\boldsymbol{D}_2\sqrt{-G_0^{-1}}\,\tilde{u}\|_{\mathcal{H}}=\sqrt{2}\,\|\tilde{u}\|_{\mathcal{H}_0}.$$

PROOF. Since for $u\in\mathcal{D}(G)$ and $\varphi\in C_K^\infty(\boldsymbol{R}^d)$, $-(\boldsymbol{D}_2u, v_\infty\operatorname{grad}(\varphi/v_\infty))_{\mathcal{H}}=2(u,G\varphi)_{\mathcal{H}}=2(Gu,\varphi)_{\mathcal{H}}$, we obtain (i). As $\sqrt{-G_0^{-1}}\tilde{u}\in\mathcal{D}(G_0)$ for any $\tilde{u}\in\mathcal{D}(G_0)$, by Lemma 5.2 (ii), (ii) follows from (i). Q.E.D.

Noting that for $\boldsymbol{\varphi}\in\boldsymbol{C}_K^\infty(\boldsymbol{R}^d)$, $D_1\boldsymbol{\varphi}\in\mathcal{D}(G_0)$, from Lemma 5.3 immediately we obtain

LEMMA 5.4. *For any* $\boldsymbol{\varphi}\in\boldsymbol{C}_K^\infty(\boldsymbol{R}^d)$,

$$\|\sqrt{-G_0^{-1}}\,D_1\boldsymbol{\varphi}\|_{\mathcal{H}_0}\leqq\sqrt{2}\,\|\boldsymbol{\varphi}\|_{\mathcal{H}}.$$

Since $\boldsymbol{C}_K^\infty(\boldsymbol{R}^d)$ is dense in $\mathcal{H}$, we can extend $\sqrt{-G_0^{-1}}D_1$ to a bounded linear operator A from $\mathcal{H}$ to $\mathcal{H}_0$, which satisfies

(5.11) $$\|A\boldsymbol{u}\|_{\mathcal{H}_0}\leqq\sqrt{2}\,\|\boldsymbol{u}\|_{\mathcal{H}} \qquad \text{for } \boldsymbol{u}\in\mathcal{H}.$$

Using T in (5.4), we define a bounded linear operator L_β and a bilinear form $(\ ,\ )_{\mathscr{H}_0}$ on $\mathcal{H}_0$ by

(5.12) $$L_\beta=I-\beta T,$$

(5.13) $$(\tilde{u},\tilde{v})_{\mathscr{H}_0}=(\tilde{u},L_\beta\tilde{v})_{\mathcal{H}_0} \quad (\tilde{u},\tilde{v}\in\mathcal{H}_0)$$

LEMMA 5.5. $(\mathcal{H}_0,(\ ,\ )_{\mathscr{H}_0})$ *is a Hilbert space with the norm equivalent to* $(\ ,\ )_{\mathcal{H}_0}$.

PROOF. It follows from Lemma 5.1 that $(\ ,\ )_{\tilde{\mathscr{H}}_0}$ is an inner product in $\mathscr{H}_0$. Furthermore, we see that

$$0<(1-\beta/\bar{\beta})\|u\|^2_{\mathscr{H}_0}\leqq\|u\|^2_{\tilde{\mathscr{H}}_0}\leqq(1-\beta/\underline{\beta})\|u\|^2_{\mathscr{H}_0} \tag{5.14}$$
$$(\text{resp. } 0<(1-\beta/\underline{\beta})\|u\|^2_{\mathscr{H}_0}\leqq\|u\|^2_{\tilde{\mathscr{H}}_0}\leqq(1-\beta/\bar{\beta})\|u\|^2_{\mathscr{H}_0})$$

if β is positive (resp. negative), which completes the proof. Q.E.D.
We denote by $\tilde{\mathscr{H}}_0$ this Hilbert space $(\mathscr{H}_0, (\ ,\ )_{\tilde{\mathscr{H}}_0})$.

We define a linear operator $\tilde{G}_0$ on $\mathscr{H}_0$ by

$$\mathscr{D}(\tilde{G}_0)=\{\tilde{u}\in\mathscr{H}_0;\ L_\beta\tilde{u}\in\mathscr{D}(G_0)\} \tag{5.15}$$
$$\tilde{G}_0\tilde{u}=G_0L_\beta\tilde{u}\qquad\text{for } \tilde{u}\in\mathscr{D}(G_0).$$

LEMMA 5.6. *$\tilde{G}_0$ is a self-adjoint operator on $\tilde{\mathscr{H}}_0$.*

PROOF. By the definition of $(\ ,\ )_{\tilde{\mathscr{H}}_0}$ and $\tilde{G}_0$, $\tilde{G}_0$ is a symmetric operator on $\tilde{\mathscr{H}}_0$. Since by Lemma 5.1, β^{-1} belongs to the resolvent set of T, we see that $\tilde{G}_0(\mathscr{D}(\tilde{G}_0))=\tilde{\mathscr{H}}_0$, which completes Lemma 5.6. Q.E.D.

Here we introduce a bilinear map f from $\mathscr{H}_0\times\mathscr{H}_0$ to $\mathscr{H}$ and a linear map B from $\mathscr{H}_0$ to $\mathscr{H}_0$ as follows:

$$f(\tilde{u},\tilde{v})(x)=\tilde{u}(x)\operatorname{grad}\int_{R^d}\Phi_2(x,y)\tilde{v}(y)dy \tag{5.16}$$

and

$$B\tilde{u}(x)=\operatorname{div}\Big(v_\infty(x)\operatorname{grad}\int_{R^d}\Phi_2(x,y)\tilde{u}(y)dy\Big). \tag{5.17}$$

Let u be a smooth L^1-solution of equation (1.2). Then we see that $\tilde{u}(t,x)=u(t,x)-v_\infty(x)$ satisfies the following non-linear differential equation:

$$\begin{aligned}\frac{\partial\tilde{u}}{\partial t}(t,x)=&\frac{1}{2}\Delta_x\tilde{u}(t,x)+\operatorname{div}_x(\tilde{u}(t,x)\operatorname{grad}\Phi_1(x))\\ &-\beta\operatorname{div}_x\Big(\tilde{u}(t,x)\operatorname{grad}_x\int_{R^d}\Phi_2(x,y)v_\infty(y)dy\Big)\\ &-\beta\operatorname{div}_x\Big(v_\infty(x)\operatorname{grad}_x\int_{R^d}\Phi_2(x,y)\tilde{u}(t,y)dy\Big)\\ &-\beta\operatorname{div}_x\Big(\tilde{u}(t,x)\operatorname{grad}_x\int_{R^d}\Phi_2(x,y)\tilde{u}(t,y)dy\Big).\end{aligned} \tag{5.18}$$

By modifying the integral equation corresponding to differential equation (5.18), we obtain the following integral equation on $\mathscr{H}_0$:

$$\tilde{u}(t)=\Psi(\tilde{u},u_0)(t) \tag{5.19}$$

where

$$\Psi(\tilde{u}, u_0)(t) = e^{t\tilde{G}_0}\tilde{u}_0 - \beta\int_0^t \sqrt{-G_0}\, e^{(t-s)G_0} A f(\tilde{u}(s), \tilde{u}(s))ds + \beta^2\int_0^t ds\int_0^{t-s} e^{(t-s-\sigma)\tilde{G}_0}\beta\sqrt{-G_0}\, e^{\sigma G_0} A f(\tilde{u}(s), \tilde{u}(s))d\sigma. \tag{5.20}$$

We define positive constants C_1 and γ by

$$C_1 = ((1-\beta/\underline{\beta})/(1-\beta/\bar{\beta}))^{1/2} \quad (\text{resp. } ((1-\beta/\bar{\beta})/(1-\beta/\underline{\beta}))^{1/2}) \tag{5.21}$$

and

$$\gamma = (1-\beta/\bar{\beta})/(1-\beta/\underline{\beta}) \quad (\text{resp. } (1-\beta/\underline{\beta})/(1-\beta/\bar{\beta})) \tag{5.22}$$

if β is positive (resp. negative). Further we define positive constants M_1, M_2 and C_2 by

$$M_1 = \Big(\sum_{i=1}^{d} \max_{x,y\in R^d}\Big|\frac{\partial}{\partial x_i}\Phi_2(x, y)\Big|\Big)^{1/2}$$

$$M_2 = \Big(2\max_{x,y\in R^d}|\Delta_x\Phi_2(x, y)| + 4M_1^2\Big(\int_{R^d}|\operatorname{grad}\Phi_1(x)|^2 v_\infty(x)dx + 4\beta^2 M_1^2\Big)\Big)^{1/2}$$

and

$$C_2 = \sqrt{2}\, M_1|\beta|\lambda_1^{-1/2}(1/\gamma + e^{(\gamma-1)/2}) + \sqrt{2}\, M_1 M_2 C_1|\beta|^2\lambda_1^{-3/2}(1/(\gamma(1-\gamma)) + e^{(\gamma-1)/2}), \tag{5.23}$$

where λ_1 is the same constant as in Lemma 5.2.

LEMMA 5.7. (i) *For any* $\tilde{u}$ *and* $\tilde{v}\in\mathcal{H}_0$,

$\|f(\tilde{u}, \tilde{v})\|_{\underline{\mathcal{H}}} \leqq M_1\|\tilde{u}\|_{\mathcal{H}_0}\|\tilde{v}\|_{\mathcal{H}_0}$,

$\|B\tilde{u}\|_{\mathcal{H}_0} \leqq M_2\|\tilde{u}\|_{\mathcal{H}_0}$.

(ii) *For any* $t>0$ *and* $\tilde{u}\in\mathcal{H}_0$,

$$\|\sqrt{-G_0}\, e^{tG_0}\tilde{u}\|_{\mathcal{H}_0} \leqq \begin{cases} (2e)^{-1/2}t^{-1/2}\|\tilde{u}\|_{\mathcal{H}_0} & \text{if } 0<t<(2\lambda_1)^{-1} \\ (\lambda_1)^{1/2}e^{-\lambda_1 t}\|\tilde{u}\|_{\mathcal{H}_0} & \text{if } (2\lambda_1)^{-1}\leqq t<+\infty. \end{cases}$$

(iii) *For any* $t>0$ *and* $\tilde{u}\in\mathcal{H}_0$,

$\|e^{t\tilde{G}_0}\tilde{u}\|_{\mathcal{H}_0} \leqq C_1 e^{-\gamma\lambda_1 t}\|\tilde{u}\|_{\mathcal{H}_0}$.

PROOF. By a simple calculation, we obtain (i). Let $(E(\lambda); \lambda\in[\lambda_1, +\infty))$

be the resolution of identity of the self-adjoint operator G_0 on $\mathscr{H}_0$. Since for any $\tilde{u}\in\mathscr{H}_0 \|\sqrt{-G_0}e^{tG_0}\tilde{u}\|^2_{\mathscr{H}_0}=\int_{\lambda_1}^{+\infty}e^{-2\lambda t}d\|E(\lambda)\tilde{u}\|^2_{\mathscr{H}_0}$, we obtain (ii). From (5.13) and (5.14), we see for any $\tilde{u}\in\mathscr{H}_0$, $(\tilde{G}_0\tilde{u},\tilde{u})_{\mathscr{H}_0}=(G_0L_\beta\tilde{u},L_\beta\tilde{u})_{\mathscr{H}_0}\leqq -\lambda_1\gamma\|\tilde{u}\|^2_{\mathscr{H}_0}$, therefore by Lemma 5.6, $\|e^{t\tilde{G}_0}\tilde{u}\|^2_{\mathscr{H}_0}\leqq e^{-2\gamma\lambda_1 t}\|\tilde{u}\|^2_{\mathscr{H}_0}$, so we obtain (iii). Q.E.D.

For any $t>0, \tilde{u}$ and $\tilde{v}\in\mathscr{H}_0$, we define $K_2(\tilde{u},\tilde{v})(t,\cdot)=K_2(u,v)(t,s)\in C([0,t)\to\mathscr{H}_0)$ and

$$K_3(\tilde{u},\tilde{v})(t,\cdot,\cdot)=K_3(\tilde{u},\tilde{v})(t,s,\sigma)\in C(\{(s,\sigma);0<\sigma\leqq t, 0\leqq s\leqq t-\sigma\}\to\mathscr{H}_0)$$

by

$$K_2(\tilde{u},\tilde{v})(t,s)=\sqrt{-G_0}e^{(t-s)G_0}Af(\tilde{u},\tilde{v}), \tag{5.24}$$

and

$$K_3(\tilde{u},\tilde{v})(t,s,\sigma)=e^{(t-s-\sigma)\tilde{G}_0}B\sqrt{-G_0}e^{\sigma G}Af(\tilde{u},\tilde{v}). \tag{5.25}$$

LEMMA 5.8. (i) *For any $t>0, \tilde{u}$ and $\tilde{v}\in\mathscr{H}_0$,*

$$\text{(a)}\quad \|K_2(\tilde{u},\tilde{v})(t,s)\|_{\mathscr{H}_0}\leqq\begin{cases} e^{-1/2}M_1(t-s)^{-1/2}\|\tilde{u}\|_{\mathscr{H}_0}\|\tilde{v}\|_{\mathscr{H}_0}, & t-(2\lambda_1)^{-1}\leqq s<t\\ \sqrt{2\lambda_1}M_1e^{-\lambda_1(t-s)}\|\tilde{u}\|_{\mathscr{H}_0}\|\tilde{v}\|_{\mathscr{H}_0}, & 0\leqq s<t-(2\lambda_1)^{-1},\end{cases}$$

$$\text{(b)}\quad \|K_3(\tilde{u},\tilde{v})(t,s,\sigma)\|_{\mathscr{H}_0}\leqq\begin{cases} e^{-1/2}M_1M_2C_1e^{-\gamma\lambda_1(t-s-\sigma)}\sigma^{-1/2}\|\tilde{u}\|_{\mathscr{H}_0}\|\tilde{v}\|_{\mathscr{H}_0} & 0<\sigma\leqq(t-s)\wedge(2\lambda_1)^{-1}\\ \sqrt{2\lambda_1}M_1M_2C_1e^{-\gamma\lambda_1(t-s-\sigma)}e^{-\lambda_1\sigma}\|\tilde{u}\|_{\mathscr{H}_0}\|\tilde{v}\|_{\mathscr{H}_0} & (t-s)\wedge(2\lambda_1)^{-1}<\sigma\leqq t-s.\end{cases}$$

(ii) *For any $\tilde{u}$ and $\tilde{v}\in C([0,\infty)\to\mathscr{H}_0)$ such that for some positive constants A and B,*

$$\|\tilde{u}(t)\|_{\mathscr{H}_0}\leqq Ae^{-\gamma\lambda_1 t} \text{ and } \|\tilde{v}(t)\|_{\mathscr{H}_0}\leqq Be^{-\gamma\lambda_1 t} \quad \text{for any } t>0, \tag{5.26}$$

we obtain

$$|\beta|\int_0^t\|K_2(\tilde{u}(s),\tilde{v}(s))(t,s)\|_{\mathscr{H}_0}ds+|\beta|^2\int_0^t ds\int_0^{t-s}\|K_3(\tilde{u}(s),\tilde{v}(s))(t,s,\sigma)\|_{\mathscr{H}_0}d\sigma \leqq C_2ABe^{-\gamma\lambda_1 t}.$$

PROOF. We obtain (i) (a) from Lemma 5.7 (i), (ii) and (5.11), and we obtain (i) (b) from Lemma 5.7 (i), (ii), (iii) and (5.11). For $\tilde{u}$ and $\tilde{v}\in C([0,\infty)\to\mathscr{H}_0)$ satisfying (5.26) and for $t>0$,

$$\int_0^t\|K_2(\tilde{u}(s),\tilde{v}(s))(t,s)\|_{\mathscr{H}_0}ds$$

$$\leqq\sqrt{2\lambda_1}M_1\int_0^{(t-(2\lambda_1)^{-1})\vee 0} e^{-\lambda_1(t-s)}ABe^{-2\gamma\lambda_1 s}ds$$

$$+e^{-1/2}M_1\int_{(t-(2\lambda_1)^{-1})\vee 0}^{t}(t-s)^{-1/2}ABe^{-2\gamma\lambda_1 s}ds$$

$$\leqq\sqrt{2}\,M_1(e^{(\gamma-1)/2}\lambda_1^{-1/2}+\gamma^{-1}\lambda_1^{-1/2})ABe^{-\gamma\lambda_1 t},$$

and similarly by (i) (b),

$$\int_0^t ds\int_0^{t-s}\|K_3(\tilde{u}(s),\tilde{v}(s))(t,s,\sigma)\|_{\mathcal{H}_0}d\sigma$$

$$\leqq e^{-1/2}M_1M_2C_1\int_0^t ds\int_0^{(t-s)\wedge(2\lambda_1)^{-1}} e^{-\gamma\lambda_1(t-s-\sigma)}\sigma^{-1/2}ABe^{-2\gamma\lambda_1 s}d\sigma$$

$$+\sqrt{2\lambda_1}M_1M_2C_1\int_0^t ds\int_{(t-s)\wedge(2\lambda_1)^{-1}}^{t-s} e^{-\gamma\lambda_1(t-s-\sigma)}e^{-\lambda_1\sigma}ABe^{-2\gamma\lambda_1 s}d\sigma$$

$$\leqq\sqrt{2}\,M_1M_2C_1(e^{(\gamma-1)/2}\gamma^{-1}\lambda_1^{-3/2}+\gamma^{-1}(1-\gamma)^{-1}\lambda_1^{-3/2})ABe^{-\gamma\lambda_1 t},$$

which implies (ii). Q.E.D.

We choose positive constants a and b satisfying

$$0<b<1/(4C_1C_2) \tag{5.27}$$

and

$$\frac{1-\sqrt{1-4C_1C_2b}}{2C_2}\vee\frac{1}{4C_2}<a<\frac{1}{2C_2}. \tag{5.28}$$

LEMMA 5.9. *For any* $\tilde{u}_0\in\mathcal{H}_0$ *with* $\|\tilde{u}_0\|_{\mathcal{H}_0}\leqq b$, *there exists a solution* $\tilde{u}\in C([0,\infty)\to\mathcal{H}_0)$ *of the integral equation* (5.19) *such that for any* $t>0$,

$$\|\tilde{u}(t)\|_{\mathcal{H}_0}\leqq\frac{a}{1-2aC_2}e^{-\gamma\lambda_1 t}.$$

PROOF. We choose $\tilde{u}_0\in\mathcal{H}_0$ to satisfy $\|\tilde{u}_0\|_{\mathcal{H}_0}\leqq b$. $\mathcal{C}$ denotes a complete separable metric space $C([0,\infty)\to\mathcal{H}_0)$ with usual metric. Then $\Psi(\cdot,\tilde{u}_0)$ in (5.20) is a map from $\mathcal{C}$ to $\mathcal{C}$. We define a sequence $\{\tilde{u}^{(n)};n\in N_*\}$ in $\mathcal{C}$ inductively by

$$\tilde{u}^{(0)}(t)=e^{t\tilde{G}_0}\tilde{u}_0,\quad \tilde{u}^{(n+1)}=\Psi(\tilde{u}^{(n)},\tilde{u}_0).$$

Let $q=2aC_2$. Then $0<q<1$. By induction, we shall prove the following (5.29) and (5.30):

$$\|\tilde{u}^{(n)}(t)\|_{\mathcal{H}_0}\leqq ae^{-\gamma\lambda_1 t}\qquad(n\in N_*) \tag{5.29$_n$}$$

and

$$(5.30)_{n+1} \qquad \|\tilde{u}^{(n+1)}(t)-\tilde{u}^{(n)}(t)\|_{\mathscr{H}_0}\leqq aq^{n+1}e^{-\gamma\lambda_1 t} \qquad (n\in N_*).$$

We obtain $(5.29)_0$ from Lemma 5.7 (iii), (5.27) and (5.28). We see that $(5.29)_n$ implies $(5.29)_{n+1}$ from Lemma (5.7) (iii) and Lemma 5.8 (ii). Next we prove (5.30). We obtain $(5.30)_1$ by Lemma 5.8 (ii). As K_2 and K_3 are bilinear, by Lemma 5.8, we see that under $(5.29)_n$ and $(5.30)_n$, for $t>0$

$$\begin{aligned}
&\|\tilde{u}^{(n+1)}(t)-\tilde{u}^{(n)}(t)\|_{\mathscr{H}_0}\\
&\leqq|\beta|\int_0^t\|K_2(\tilde{u}^{(n)}(s),\tilde{u}^{(n)}(s)-\tilde{u}^{(n-1)}(s))(t,s)\|_{\mathscr{H}_0}ds\\
&\quad+|\beta|\int_0^t\|K_2(\tilde{u}^{(n)}(s)-\tilde{u}^{(n-1)}(s),\tilde{u}^{(n-1)}(s))(t,s)\|_{\mathscr{H}_0}ds\\
&\quad+|\beta|^2\int_0^t ds\int_0^{t-s}\|K_3(\tilde{u}^{(n)}(s),\tilde{u}^{(n)}(s)-\tilde{u}^{(n-1)}(s))(t,s,\sigma)\|_{\mathscr{H}_0}d\sigma\\
&\quad+|\beta|^2\int_0^t ds\int_0^{t-s}\|K_3(\tilde{u}^{(n)}(s)-\tilde{u}^{(n-1)}(s),\tilde{u}^{(n-1)}(s))(t,s,\sigma)\|_{\mathscr{H}_0}d\sigma
\end{aligned}$$

which implies $(5.30)_{n+1}$.

By virtue of (5.30), we can define $\tilde{u}\in C$ by

$$\tilde{u}(t)=\tilde{u}^{(0)}(t)+\sum_{n=1}^{\infty}(\tilde{u}^{(n)}(t)-\tilde{u}^{(n-1)}(t)).$$

By (5.30), for any $t>0$ and $n\in N$,

$$\|\tilde{u}(t)\|_{\mathscr{H}_0}\leqq\frac{a}{1-q}e^{-\gamma\lambda_1 t},\quad \|\tilde{u}(t)-\tilde{u}^{(n)}(t)\|_{\mathscr{H}_0}\leqq\frac{q^{n+1}}{1-q}ae^{-\gamma\lambda_1 t}.$$

Therefore from Lemma 5.8 (ii), for any $t>0$ and $n\in N$,

$$\|\Psi(\tilde{u},\tilde{u}_0)(t)-\Psi(\tilde{u}^{(n)},\tilde{u}_0)(t)\|_{\mathscr{H}_0}\leqq\frac{a}{(1-q)^2}q^{n+2},$$

and so $\tilde{u}$ satisfies $\tilde{u}(t)=\Psi(\tilde{u},\tilde{u}_0)(t)$, which completes Lemma 5.9. Q.E.D.

The following lemmas are used to prove that the solution of integral equation (5.19) is a distribution solution of (5.18). First we introduce the next notation: for any $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$, put

$$(5.31) \qquad \tilde{\varphi}_\infty(x)=\varphi(x)v_\infty(x)-\int_{\boldsymbol{R}^d}\varphi(y)v_\infty(y)dy.$$

Then, by integration by parts we obtain

LEMMA 5.10. *For any* $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$ *and* $n\in\boldsymbol{N}$,

$$\tilde{\varphi}_\infty\in\mathcal{D}(G_0^n).$$

LEMMA 5.11. *For any* $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$, $\tilde{u}\in C([0,\infty)\to\mathcal{H}_0)$ *and* $t>0$,

$$\frac{d}{dt}\left(\tilde{\varphi}_\infty,\int_0^t K_2(\tilde{u}(s),\tilde{u}(s))(t,s)ds\right)_{\mathcal{H}_0}$$

$$=\left(G_0\tilde{\varphi}_\infty,\int_0^t K_2(\tilde{u}(s),\tilde{u}(s))(t,s)ds\right)_{\mathcal{H}_0}+(\sqrt{-G_0}\,\tilde{\varphi}_\infty, Af(\tilde{u}(t),\tilde{u}(t)))_{\mathcal{H}_0}.$$

PROOF. For any fixed $t>0$ and $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$, by Lemma 5.10, $(\tilde{\varphi}_\infty, K_2(\tilde{u}(s),\tilde{u}(s)))_{\mathcal{H}_0}=(\sqrt{-G_0}\,\tilde{\varphi}_\infty, e^{(t-s)G_0}Af(\tilde{u}(s),\tilde{u}(s)))_{\mathcal{H}_0}$ for $s\in[0,t)$, then $(\sqrt{-G_0}\,\tilde{\varphi}_\infty, e^{(t-s)G_0}Af(\tilde{u}(s),\tilde{u}(s)))_{\mathcal{H}_0}$ is bounded and continuous in $s\in[0,t]$. Similarly, as

$$\frac{\partial}{\partial t}(\tilde{\varphi}_\infty, K_2(\tilde{u}(s),\tilde{u}(s)))_{\mathcal{H}_0}=(G_0\sqrt{-G_0}\tilde{\varphi}_\infty, e^{(t-s)G_0}Af(\tilde{u}(s),\tilde{u}(s)))_{\mathcal{H}_0},$$

$\frac{\partial}{\partial t}(\tilde{\varphi}_\infty, K_2(\tilde{u}(s),\tilde{u}(s)))_{\mathcal{H}_0}$ is bounded and continuous in $s\in[0,t)$. Q.E.D.

LEMMA 5.12. *For any* $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$ *and* $\boldsymbol{u}\in\mathcal{H}$,

$$(\sqrt{-G_0}\,\tilde{\varphi}_\infty, A\boldsymbol{u})_{\mathcal{H}_0}=-(\boldsymbol{D}_2\tilde{\varphi}_\infty,\boldsymbol{u})_{\mathcal{H}}.$$

Let T and $\tilde{G}_0$ be the bounded linear operator and the closed linear operator on $\mathcal{H}_0$ as in (5.4) and (5.15), respectively. Noting Lemma 5.10, we obtain

LEMMA 5.13. (i) *For any* $\tilde{u}\in\mathcal{H}_0$,

$$B^*\tilde{u}(x)=v_\infty(x)\int_{\boldsymbol{R}^d}\mathrm{div}_y(v_\infty(y)\,\mathrm{grad}_y\,\Phi_2(y,x))\tilde{u}(y)\frac{dy}{v_\infty(y)}.$$

(ii) $\mathcal{R}(T)\subset\mathcal{D}(G_0^n)$, $\mathcal{R}(B)\subset\mathcal{D}(G_0^n)$ *and* $\mathcal{R}(B^*)\subset\mathcal{D}(G_0^n)$ *for any* $n\in\boldsymbol{N}$.

(iii) $\sqrt{-G_0}\,B^*$ *is bounded linear operator on* $\mathcal{H}_0$.

(iv) $G_0T=B$, $\tilde{G}_0=G_0-\beta B$ *and* $\tilde{G}_0^*=G_0-\beta B^*$.

(v) *For any* $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$ *and* $\tilde{u}\in\mathcal{H}_0$,

$$(\tilde{G}_0^*\tilde{\varphi}_\infty, \tilde{u})_{\mathscr{H}_0} = \int_{R^d}\Big(\frac{1}{2}\Delta\varphi(x) - \mathrm{grad}\,\Phi_1(x)\cdot\mathrm{grad}\,\varphi(x)\Big)\tilde{u}(x)dx$$
$$+\beta\iint_{R^{2d}}\mathrm{grad}_x\,\Phi_2(x, y)\cdot\mathrm{grad}\,\varphi(x)\tilde{u}(x)v_\infty(y)dxdy$$
$$+\beta\iint_{R^{2d}}\mathrm{grad}_x\,\Phi_2(x, y)\cdot\mathrm{grad}\,\varphi(x)v_\infty(x)\tilde{u}(y)dxdy.$$

LEMMA 5.14. *For any* $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$, $\tilde{u}\in C([0, \infty)\to\mathscr{H}_0)$ *and* $t>0$

$$\frac{d}{dt}\Big(\tilde{\varphi}_\infty, \int_0^t ds\int_0^{t-s}K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma)d\sigma\Big)_{\mathscr{H}_0}$$
$$=\Big(\tilde{G}_0^*\tilde{\varphi}_\infty, \int_0^t ds\int_0^{t-s}K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma)d\sigma\Big)_{\mathscr{H}_0}$$
$$+\Big(B^*\tilde{\varphi}_\infty, \int_0^t K_2(\tilde{u}(s), \tilde{u}(s))(t, s)ds\Big)_{\mathscr{H}_0}.$$

PROOF. Let $\mathscr{T}_t=\{(s, \sigma); 0<\sigma\leqq t, 0\leqq s\leqq t-\sigma\}$ for $t>0$. It suffices to show that for any fixed $t>0$ and $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$, $(\tilde{\varphi}_\infty, K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma))_{\mathscr{H}_0}$ and $\frac{\partial}{\partial t}(\tilde{\varphi}_\infty, K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma))_{\mathscr{H}_0}$ are bounded and continuous in $(s, \sigma)\in\mathscr{T}_t$. It follows from (5.25) and Lemma 5.7 (ii) that $(\tilde{\varphi}_\infty, K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma))_{\mathscr{H}_0}$ is continuous in $(s, \sigma)\in\mathscr{T}_t$. From Lemma 5.13, $(\tilde{\varphi}_\infty, K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma))_{\mathscr{H}_0}$ $=(e^{\sigma G_0}\sqrt{-G_0}B^*(e^{(t-s-\sigma)\tilde{G}_0})^*\tilde{\varphi}_\infty, Af(\tilde{u}(s), \tilde{u}(s)))_{\mathscr{H}_0}$ is bounded in $(s, \sigma)\in\mathscr{T}_t$. As $\frac{\partial}{\partial t}(\tilde{\varphi}_\infty, K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma))_{\mathscr{H}_0}=(e^{\sigma G_0}\sqrt{-G_0}B^*(e^{(t-s-\sigma)\tilde{G}_0})^*\tilde{G}_0^*\tilde{\varphi}_\infty, Af(\tilde{u}(s), \tilde{u}(s)))_{\mathscr{H}_0}$, this is bounded and continuous in $(s, \sigma)\in\mathscr{T}_t$. Q.E.D.

Now we are in a position to prove Theorem 5.1. Let $b=3/(16C_1C_2)$, $a=1/(2C_2)$, $\lambda=\gamma\lambda_1$ and u_0 be the probability density satisfying $\|u_0-v_\infty\|_{\mathscr{H}}\leqq b$. We put $\tilde{u}_0=u_0-v_\infty$ then we obtain the solution of integral equation (5.19) $\tilde{u}$ which satisfies $\|\tilde{u}(t)\|_{\mathscr{H}_0}\leqq ae^{-\lambda t}$ for any $t>0$. Then $\tilde{u}(t, \cdot)\in L^1(\boldsymbol{R}^d)$ whose L^1-norm is bounded in $t>0$.

By Lemmas 5.11, 5.12, 5.13 and 5.14, for any $\varphi\in\mathcal{S}(\boldsymbol{R}^d)$ and $t>0$,

$$\frac{d}{dt}{}_{\mathcal{S}}\langle\varphi, \tilde{u}(t)\rangle_{\mathcal{S}^*}=\frac{d}{dt}(\tilde{\varphi}_\infty, \tilde{u}(t))_{\mathscr{H}_0}$$
$$=(\tilde{G}_0^*\tilde{\varphi}_\infty, e^{t\tilde{G}_0}\tilde{u}_0)_{\mathscr{H}_0}-\beta\Big(G_0\tilde{\varphi}_\infty, \int_0^t K_2(\tilde{u}(s), \tilde{u}(s))(t, s)ds\Big)_{\mathscr{H}_0}$$
$$+\beta^2\Big(B^*\tilde{\varphi}_\infty, \int_0^t K_2(\tilde{u}(s), \tilde{u}(s))(t, s)ds\Big)_{\mathscr{H}_0}$$

$$+\beta^2\Big(\tilde{G}_0^*\tilde{\varphi}_\infty, \int_0^t ds \int_0^{t-s} K_3(\tilde{u}(s), \tilde{u}(s))(t, s, \sigma)d\sigma\Big)_{\mathscr{H}_0}$$

$$+\beta(\boldsymbol{D}_2\tilde{\varphi}_\infty, \boldsymbol{f}(\tilde{u}(s), \tilde{u}(s)))_{\underline{\mathscr{H}}}$$

$$=(\tilde{G}_0^*\tilde{\varphi}_\infty, \tilde{u}(t))_{\mathscr{H}_0}+\beta(\boldsymbol{D}_2\tilde{\varphi}_\infty, \boldsymbol{f}(\tilde{u}(s), \tilde{u}(s)))_{\underline{\mathscr{H}}}$$

$$=\int_{R^d}\Big(\frac{1}{2}\Delta\varphi(x)-\mathrm{grad}\,\Phi_1(x)\cdot\mathrm{grad}\,\varphi(x)\Big)\tilde{u}(t, x)dx$$

$$+\beta\iint_{R^{2d}}\mathrm{grad}_x\Phi_2(x, y)\cdot\mathrm{grad}\,\varphi(x)v_\infty(y)\tilde{u}(t, x)dxdy$$

$$+\beta\iint_{R^{2d}}\mathrm{grad}_x\Phi_2(x, y)\cdot\mathrm{grad}\,\varphi(x)\tilde{u}(t, y)v_\infty(x)dxdy$$

$$+\beta\iint_{R^{2d}}\mathrm{grad}_x\Phi_2(x, y)\cdot\mathrm{grad}\,\varphi(x)\tilde{u}(t, y)\tilde{u}(t, x)dxdy$$

$$={}_{S(R^d)}\Big\langle\frac{1}{2}\Delta\varphi-\mathrm{grad}\,\Phi_1\cdot\mathrm{grad}\,\varphi, \tilde{u}(t)\Big\rangle_{S^*(R^d)}$$

$$+{}_{S(R^{2d})}\langle\mathrm{grad}\,\varphi(x)\,\mathrm{grad}_x\Phi_2(x, y), v_\infty\otimes\tilde{u}(t)+\tilde{u}(t)\otimes v_\infty$$
$$+\tilde{u}(t)\otimes\tilde{u}(t)\rangle_{S^*(R^{2d})}.$$

Therefore, since v_∞ is a stationary distribution solution of differential equation (1.2), we see that $u(t)=\tilde{u}(t)+v_\infty$ is an L^1-solution of equation (1.2), which completes Theorem 5.1. Q.E.D.

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