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# *Gibbs measures for mean field potentials*

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## 1. Introduction.

Let $M$ be a separable complete metric space, $\mu$ be a probability measure on $M$ whose support is the whole space $M$ and $V: M\times M\to \boldsymbol{R}$ be a symmetric bounded continuous function. Lst $\nu_n$, $n\geqq 2$, be a symmetric probability measure on $M^\infty$ given by

$$\nu_n(d\underline{x})=Z_n^{-1}\exp\left(\frac{1}{n}\sum_{i,j=1}^{n} V(x_i, x_j)\right)\bigotimes_{i=1}^{\infty}\mu(dx_i), \tag{1.1}$$

where $\underline{x}=(x_1, x_2, \cdots)\in M^\infty$ and

$$Z_n=\int_{M^\infty}\exp\left(\frac{1}{n}\sum_{i,j=1}^{n} V(x_i, x_j)\right)\bigotimes_{i=1}^{\infty}\mu(dx_i)\,. \tag{1.2}$$

In the present paper, we shall study whether $\nu_n$ are convergent to a certain probability measure on $M^\infty$ as $n\to\infty$ and, if so, what is the limit probability measure.

A partial answer is given by the variational principle for $Z_n$. As has been studied in Donsker-Varadhan [1], we have the equality:

$$\lim_{n\to\infty}\frac{1}{n}\log Z_n=-\inf\{F(R)\,;\, R \text{ is a probability measure on } M\}, \tag{1.3}$$

where

$$F(R)=-\int_{M\times M} V(x, y)R(dx)R(dy)+\int_M \log\frac{dR}{d\mu}(x)R(dx)$$

if $R$ is absolutely continuous relative to $\mu$, and $F(R)=\infty$ otherwise. If $R_0$ is the unique probability measure on $M$ minimizing the function $F$, then we can conclude that $\nu_n$ are convergent to $R_0^{\otimes\infty}$. In the present

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paper, we shall study the case where finitely many or countably infinitely many probability measures minimize the function $F$, and we will show that the Hessian of $F$ plays an important role in this case.

Now let us show our main results in the present paper. Throughout this paper, we shall impose the following assumption (A) on the function $V: M\times M\to \boldsymbol{R}$.

(A) There exist a compact metric space $S$, a signed measure $\sigma$ on $S$ with bounded total variation, and a bounded continuous function $g: M\times S\to \boldsymbol{R}$ such that $V(x, y)=\int_S g(x, s)g(y, s)\sigma(ds)$.

For each probability measure $R$ on $M$, let $L_0^2(R)$ denote the Hilbert subspace $\left\{u\in L^2(M; dR); \int_M u\,dR=0\right\}$ of $L^2(M; dR)$, and let us define a symmetric bounded linear operator $D^2F(R)$ in $L_0^2(R)$ by

$$(1.4)\qquad (D^2F(R)u, v)_{L_0^2(R)} = -2\int_{M\times M} V(x, y)u(x)v(y)R(dx)\,R(dy)+\int_M u(x)v(x)R(dx),$$

$u, v\in L_0^2(R)$. Note that the operator $D^2F(R)$ can be considered the second order Fréchet differential of $F$ at $R$.

Then the following are our main Theorems.

THEOREM 1. *Suppose that the symmetric bounded operator $D^2F(R)$ in $L_0^2(R)$ is strictly positive definite for every probability measure $R$ on $M$ minimizing $F$. Then there exist only finitely many probability measures on $M$ minimizing $F$, say $R_1, \cdots, R_m$, and $\nu_n$ are convergent to the probability measure $\sum_{k=1}^m a_k R_k^{\otimes\infty}$ on $M^\infty$ as $n\to\infty$, where*

$$a_k=z^{-1}\det(D^2F(R_k))^{-1/2}\quad and \quad z=\sum_{k=1}^m \det(D^2F(R_k))^{-1/2}.$$

*Moreover $\lim_{n\to\infty} Z_n e^{nf}=z$, where $f$ is the minimum value of $F$.*

THEOREM 2. *Suppose that there exists a probability measure $R_0$ on $M$ minimizing $F$ for which the spectrum of the symmetric operator $D^2F(R_0)$ contains zero. Suppose moreover that the symmetric operator $D^2F(R)$ is strictly positive definite for each probability measure $R$ on $M$ minimizing $F$ except $R_0$. Then $\nu_n$ are convergent to $R_0^{\otimes\infty}$ as $n\to\infty$, and $\lim_{n\to\infty} Z_n e^{nf}=\infty$.*

We shall also show a certain kind of the central limit theorem, and we shall mention $r$-body potentials also, $r\geqq 3$, in Section 5.

In the last section, Section 6, we will show the central limit theorem for diffusion processes with mean field interaction as an application of our results. This kind of central limit theorems have been shown under much more general assumptions by Sznitman [3] and Tanaka [4]. But we still think that our proof is worth being noted down, because the central limit theorem follows immediately from the law of large numbers proved by McKean [2] and our results for 3-body potentials.

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## 2. Basic lemmas.

In this section, we shall prepare two lemmas for later use.

LEMMA 2.1. *Let $\{X_j; j=1, 2, \cdots\}$ be independently identically distributed $\boldsymbol{R}^d$-valued random variables, and $A_1$, $A_2$ and $A_3$ be positive constants. Assume that*

$$E[X_1]=0, \tag{2.1}$$

$$E[X_1\cdot {}^tX_1]\leqq A_1\cdot I_d, \tag{2.2}$$

*and*

$$E[\exp(A_2\|X_1\|)]\leqq A_3, \tag{2.3}$$

*where we consider $X_1$ a column vector, $I_d$ denotes the unit $d\times d$ matrix and we mean by ${}^tX_1$ the transposed matrix of $X_1$. Then for any $a<\frac{1}{2A_1}$, there exist $\varepsilon>0$ and $A_4>0$ such that*

$$E\left[\exp\left(a\cdot n\left\|\frac{1}{n}\sum_{j=1}^n X_j\right\|^2\right),\ \left\|\frac{1}{n}\sum_{j=1}^n X_j\right\|<\varepsilon\right]\leqq A_4, \qquad n\geqq 1. \tag{2.4}$$

*Here $\varepsilon$ is independent of the distribution of $X_1$ and the dimension $d$, and $A_4$ is independent of the distribution of $X_1$, i.e.*

$$\varepsilon=\varepsilon(A_1, A_2, A_3, a) \text{ and } A_4=A_4(d, A_1, A_2, A_3, a).$$

PROOF. Let $C$, $b_0$, and $\delta$ be constants given by

$$C=2\frac{A_3}{A_2^3}+\left(A_1+2\frac{A_3}{A_2^3}\right)^2, \tag{2.5}$$

(2.6) $$b_0=\min\left\{1,\ A_2,\ \frac{1}{2}\left(A_1+2\frac{A_3}{A_2^3}\right)^{-1/2}\right\},$$

and

(2.7) $$\frac{1+\delta}{1-\delta}a=\frac{1}{2}\left(a+\frac{1}{2A_1}\right),\qquad 0<\delta<1.$$

Now let $b_1$ be a constant satisfying that $0<b_1<b_0$ and

(2.8) $$a=\frac{1}{2}\cdot\frac{1-\delta}{1+3\delta}\cdot\frac{1}{A_1}<\frac{1}{2}\cdot\frac{1-\delta}{1+\delta}\cdot\frac{1}{A_1+Cb_1}.$$

Now let $\varepsilon=A_1b_1$. Observe that

$$\left\{x\in \boldsymbol{R}^d;\ \|x\|\leqq\frac{1}{1+\delta}\right\}=\cap\left\{\left\{x\in \boldsymbol{R}^d;\ (x,\xi)\leqq\frac{1}{1+\delta}\right\};\ \xi\in R^d,\ \|\xi\|=1\right\}.$$

Therefore there exists a finite subset $\{\xi_1,\cdots,\xi_N\}$ of $\boldsymbol{R}^d$ such that

(2.9) $$\|\xi_i\|=1,\qquad i=1,\cdots,N,$$

and

(2.10) $$\bigcap_{i=1}^{N}\left\{x\in \boldsymbol{R}^d;\ (x,\xi_i)\leqq\frac{1}{1+\delta}\right\}\subset\{x\in \boldsymbol{R}^d;\ \|x\|<1\}.$$

Note here that the number $N$ is determined only by $\delta$ and the dimension $d$.

Now let $X_n^i=(X_n,\xi_i)$, $i=1,\cdots,N$ and $n\geqq1$. Then we have for any $z,t\geqq0$

(2.11) $$\begin{aligned}&P\left[\left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|\geqq z\right]\\ &\leqq P\left[\max\left\{\left(\frac{1}{n}\sum_{j=1}^{n}X_j,\ \xi_i\right);\ i=1,\cdots,N\right\}\geqq\frac{z}{1+\delta}\right]\\ &\leqq\sum_{i=1}^{N}P\left[\sum_{j=1}^{n}X_j^i\geqq n\frac{z}{1+\delta}\right]\\ &\leqq\sum_{i=1}^{N}\exp\left(-n\frac{z}{1+\delta}t\right)E\left[\exp\left(t\sum_{j=1}^{n}X_j^i\right)\right]\\ &\leqq\sum_{i=1}^{N}\exp\left(n\left(-\frac{z}{1+\delta}t+\log\phi_i(t)\right)\right),\end{aligned}$$

where $\phi_i(t)=E[\exp(tX_1^i)]$, $i=1,\cdots,N$. Observe that

(2.12) $$v_i = E[(X_1^i)^2] \leqq A_1,$$

and

(2.13) $$\phi_i(t) = 1 + \frac{1}{2} v_i t^2 + \sum_{n=3}^{\infty} \frac{1}{n\,!} t^n E[(X_1^i)^n].$$

If $0 < t \leqq A_2$, then we have

(2.14) $$\left|\phi_i(t) - \left(1 + \frac{1}{2} v_i t^2\right)\right| \leqq \sum_{n=3}^{\infty} \frac{1}{n\,!} \frac{t^n}{A_2^n} E[A_2^n \|X_1^i\|^n]$$
$$\leqq t^3 \frac{1}{A_2^3} \sum_{n=0}^{\infty} \frac{1}{n\,!} E[A_2^n \|X_1^i\|^n]$$
$$\leqq t^3 \frac{A_3}{A_2^3}.$$

Observing that $\frac{1}{2} A_1 \cdot b_0^2 + \frac{A_3}{A_2^3} b_0^3 \leqq \frac{1}{2}$ from (2.6), we see from (2.5) and (2.6) that for $0 < t \leqq b_1 < b_0 \leqq 1$,

(2.15) $$\log \phi_i(t) \leqq \log \left\{1 + \left(\frac{1}{2} v_i + \frac{A_3}{A_2^3} b_1\right) \cdot t^2\right\}$$
$$\leqq \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left\{\left(\frac{1}{2} v_i + \frac{A_3}{A_2^3} b_1\right) \cdot t^2\right\}^n$$
$$\leqq \left(\frac{1}{2} A_1 + \frac{A_3}{A_2^3} b_1\right) \cdot t^2 \cdot \left[1 + \sum_{n=1}^{\infty} \left\{\left(\frac{1}{2} v_i + \frac{A_3}{A_2^3} b_1\right) t^2\right\}^n\right]$$
$$\leqq \left(\frac{1}{2} A_1 + \frac{A_3}{A_2^3} b_1\right) \cdot t^2 \cdot \left[1 + \left(\frac{1}{2} A_1 + \frac{A_3}{A_2^3}\right) \cdot b_1^2 \sum_{n=0}^{\infty} \frac{1}{2^n}\right]$$
$$\leqq \frac{1}{2} (A_1 + C \cdot b_1) t^2.$$

On the other hand, if $z \leqq \varepsilon$, we have $\frac{z}{A_1 + C b_1} \leqq \frac{A_1 b_1}{A_1 + C b_1} < b_1$. Thus letting $t = \frac{z}{A_1 + C b_1}$, we obtain from (2.11) and (2.15)

(2.16) $$P\left[\left\|\frac{1}{n} \sum_{j=1}^{n} X_j\right\| \geqq z\right] \leqq N \exp(-\tilde{\alpha} \cdot n \cdot z^2), \qquad z \leqq \varepsilon,$$

where $\tilde{\alpha} = \frac{1}{2} \cdot \frac{1-\delta}{1+\delta} \cdot \frac{1}{A_1 + C b_1} > \alpha$. Thus we have got

$$(2.17)\qquad E\left[\exp\left(a\cdot n\left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|^2\right),\ \left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|<\varepsilon\right]$$

$$=\int_0^{\varepsilon-}e^{n\cdot a\cdot x^2}\cdot P\left[\left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|\in dx\right]$$

$$=\int_0^{\varepsilon}2a\cdot n\cdot x\cdot e^{n\cdot a\cdot x^2}\cdot P\left[x\leqq\left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|<\varepsilon\right]dx$$

$$-\left[e^{n\cdot a\cdot x^2}\cdot P\left[x\leqq\left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|<\varepsilon\right]\right]_0^{\varepsilon}$$

$$\leqq 1+N\int_0^{\infty}2n\cdot a\cdot x\cdot\exp(-n(\tilde{a}-a)\cdot x^2)dx$$

$$=1+\frac{Na}{\tilde{a}-a}.$$

This completes the proof.

LEMMA 2.2. *Let $\{X_j\,;\,j=1,2,\cdots\}$ be independently identically distributed $\boldsymbol{R}^d$-valued random variables with mean 0 satisfying $E[\exp(b\|X_1\|)]<\infty$ for some $b>0$. Let $V$ be the covariance matrix of $X_1$, i.e. $V=E[X_1\cdot{}^tX_1]$, and let $C$ be a $d\times d$ symmetric matrix. If there exists $a_0>0$ for which $VCV\leqq a_0V$, then for any $a<\dfrac{1}{2a_0}$, there exists $\varepsilon>0$ such that*

$$\sup_n E\left[\exp\left(a\cdot n\cdot{}^t\left(\frac{1}{n}\sum_{j=1}^{n}X_j\right)\cdot C\cdot\left(\frac{1}{n}\sum_{j=1}^{n}X_j\right)\right),\ \left\|\frac{1}{n}\sum_{j=1}^{n}X_j\right\|<\varepsilon\right]<\infty.$$

PROOF. We regard the matrix $V$ as a linear operator in $\boldsymbol{R}^d$. Let $W$ be the image of the operator $V$. Then $W$ is a linear subspace of $\boldsymbol{R}^d$ and $X_j\in W$ with probability one, $j=1,2,\cdots$. Moreover the restricted map $V|_W$ of $V$ is a strictly positive definite symmetric linear operator in $W$. Let $Y_j=a_0^{1/2}\cdot(V|_W)^{-1/2}X_j\in W\subset\boldsymbol{R}^d$, $j=1,2,\cdots$. Then $\{Y_j;\,j=1,2,\cdots\}$ are independently identically distributed $\boldsymbol{R}^d$-valued random variables with mean 0 satisfying

$$(2.18)\qquad E[Y_1\cdot{}^tY_1]\leqq a_0\cdot I_d.$$

Note that

$$(2.19)\qquad X_j=a_0^{-1/2}\cdot V^{1/2}Y_j,\qquad j=1,2,\cdots,$$

$$(2.20)\qquad \|Y_1\|\leqq a_0^{1/2}\cdot\|(V|_W)^{-1/2}\|_{\text{operator}}\cdot\|X_1\|,$$

and

$$\left\| \frac{1}{n} \sum_{j=1}^{n} Y_j \right\| \leqq a_0^{1/2} \cdot \| (V|_W)^{-1/2} \|_{\text{operator}} \cdot \left\| \frac{1}{n} \sum_{j=1}^{n} X_j \right\|. \tag{2.21}$$

From (2.20) we see that there exists $b'>0$ such that

$$E[\exp(b' \| Y_1 \|)] < \infty. \tag{2.22}$$

Since $VCV \leqq a_0 V$, we have

$$V^{1/2} C V^{1/2} \leqq a_0 I_d. \tag{2.23}$$

Therefore we have from (2.19) and (2.23)

$${}^t\left(\frac{1}{n} \sum_{j=1}^{n} X_j\right) \cdot C \cdot \left(\frac{1}{n} \sum_{j=1}^{n} X_j\right) \leqq \left\| \frac{1}{n} \sum_{j=1}^{n} Y_j \right\|^2. \tag{2.24}$$

Thus we have our assertion from Lemma 2.1, (2.21) and (2.24).

## 3. Lemmas for local central limit theorem.

Now let us think of the situation in Introduction. For each $\underline{x}=(x_1, x_2, \cdots) \in M^\infty$ and $n \geqq 1$, we denote by $\rho_n(\underline{x})$ the probability measure on $M$ given by

$$\rho_n(\underline{x})(dy) = \frac{1}{n} \sum_{j=1}^{n} \delta_{x_j}(dy). \tag{3.1}$$

Let $P(M)$ denote the set of all probability measures on $M$. Then $P(M)$ is a metric space with the Prohorov metric.

Let us define a symmetric bounded operator $V_R$ in $L_0^2(R)$ for each $R \in P(M)$ by

$$(V_R u, v)_{L_0^2(R)} = \int_{M \times M} V(x, y) u(x) v(y) R(dx) R(dy), \qquad u, v \in L_0^2(R). \tag{3.2}$$

Then we have

$$D^2 F(R) = -2 V_R + I_R, \tag{3.3}$$

where $I_R$ denotes the identity map in $L_0^2(R)$.

LEMMA 3.1. *Let $R \in P(M)$. If the symmetric bounded operator $I_R-$*

*$V_R$ in $L_0^2(R)$ is strictly positive definite, then there exists $\varepsilon>0$ such that*

$$\sup_n E_{R^{\otimes\infty}}\left[\exp\left(\frac{n}{2}\int_{M\times M} V(y_1, y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\right),\right.$$
$$\left.\sup_{s\in S}\left|\int_M g(y, s)(\rho_n(\underline{x})-R)(dy)\right|<\varepsilon\right]<\infty.$$

*Here $E_{R^{\otimes\infty}}$ denotes the expectation with respect to $R^{\otimes\infty}(d\underline{x})$.*

PROOF. Let $g_0: M\times S\to \boldsymbol{R}$ be a continuous function given by

$$g_0(x, s)=g(x, s)-\int_M g(y, s)R(dy). \tag{3.4}$$

Then we have

$$\int_{M\times M} V(y_1, y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)=\int_S \sigma(ds)\left(\int_M g_0(y, s)\rho_n(\underline{x})(dy)\right)^2, \tag{3.5}$$

and

$$\int_M g(y, s)(\rho_n(\underline{x})-R)(dy)=\int_M g_0(y, s)\rho_n(\underline{x})(dy). \tag{3.6}$$

Let $K_{n,\varepsilon}$ denote the set $\left\{\underline{x}\in M^\infty;\ \sup_{s\in S}\left|\int_M g_0(y, s)\rho_n(\underline{x})(dy)\right|<\varepsilon\right\}$. Let $\{S_k^{(m)}\}_{k=1}^m$, $m=1, 2, \cdots$, be decompositions of $S$ satisfying

$$\lim_{m\to\infty}\max\{\text{diameter}(S_k^{(m)});\ k=1, \cdots, m\}=0. \tag{3.7}$$

Choose an element $s_k^{(m)}\in S_k^{(m)}$ for each $k=1, \cdots, m$ and $m=1, 2, \cdots,$ and let

$$g_0^{(m)}(y, s)=\sum_{k=1}^m \chi_{S_k^{(m)}}(s)g_0(y, s_k^{(m)}), \qquad y\in M, s\in S. \tag{3.8}$$

Then we have

$$d_m=\left\{\sup_{s\in S}\int_M |g_0(y, s)-g_0^{(m)}(y, s)|^2 R(dy)\right\}^{1/2}\longrightarrow 0, \tag{3.9}$$

as $m\to\infty$. Let $V_0: M\times M\to\boldsymbol{R}$ and $V_0^{(m)}: M\times M\to\boldsymbol{R}$ be functions given by

$$V_0(x, y)=\int_S \sigma(ds)g_0(x, s)g_0(y, s), \tag{3.10}$$

and

(3.11) $$V_0^{(m)}(x, y)=\int_S \sigma(ds) g_0^{(m)}(x, s) g_0^{(m)}(y, s).$$

Then we see that

(3.12) $$V_R u(x)=\int_M V_0(x, y) u(y) R(dy), \qquad u \in L_0^2(R),$$

and

(3.13) $$\int_{M\times M} |V_0(x, y)-V_0^{(m)}(x, y)|^2 R(dx) R(dy) \longrightarrow 0, \qquad m\to\infty.$$

Now let $V_R^{(m)}$ be the symmetric bounded operator in $L_0^2(R)$ given by

(3.14) $$V_R^{(m)} u(x)=\int_M V_0^{(m)}(x, y) u(y) R(dy), \qquad u \in L_0^2(R),$$

and let $\lambda$ be the maximum eigen value of $V_R$. Then we have $\lambda<1$ from the assumption, and so, from (3.13) and (3.14) we see that there exists an integer $m_1$ for which the maximum eigen value of $V_R^{(m)}$ is less than $\frac{1+2\lambda}{3}$ for $m \geqq m_1$. Observe that for any $\varepsilon>0$

(3.15) $$\log E_{R^{\otimes\infty}}\left[\exp\left(\frac{n}{2}\int_{M\times M} V_0(y_1, y_2)\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)\right),\ K_{n,\varepsilon}\right]$$
$$\leqq I_{n,m,\varepsilon}^{(1)}+I_{n,m,\varepsilon}^{(2)},$$

where

(3.16) $$I_{n,m,\varepsilon}^{(1)}$$
$$=\frac{1}{p}\log E_{R^{\otimes\infty}}\left[\exp\left(p\frac{n}{2}\int_{M\times M} V_0^{(m)}(y_1, y_2)\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)\right),\ K_{n,\varepsilon}\right],$$

(3.17) $$I_{n,m,\varepsilon}^{(2)}=\frac{1}{q}\log E_{R^{\otimes\infty}}\left[\exp\left(q\frac{n}{2}\int_{M\times M}(V_0(y_1, y_2)-V_0^{(m)}(y_1, y_2))\right.\right.$$
$$\left.\left.\times\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)\right),\ K_{n,\varepsilon}\right],$$

$p=\frac{3}{2+\lambda}$ and $q=\frac{p}{p-1}$.

Let $\sigma=\sigma_1-\sigma_2$ be the Jacobi decomposition of the signed measure $\sigma$ and let $\sigma_0=\sigma_1+\sigma_2$. Then $\sigma_0$, $\sigma_1$ and $\sigma_2$ are finite measures on $S$. Since we have

$$\int_{M\times M}(V_0(y_1, y_2)-V_0^{(m)}(y_1, y_2))\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)$$
$$=\int_S \sigma(ds)\left\{\left(\int_M g_0(y, s)\rho_n(\underline{x})(dy)\right)^2-\left(\int_M g_0^{(m)}(y, s)\rho_n(\underline{x})(dy)\right)^2\right\}$$
$$\leqq\int_S \sigma_0(ds)\left[\frac{1}{2}d_m\left\{\int_M (g_0(y, s)+g_0^{(m)}(y, s))\rho_n(\underline{x})(dy)\right\}^2\right.$$
$$\left.+\frac{1}{2}d_m^{-1}\left\{\int_M (g_0(y, s)-g_0^{(m)}(y, s))\rho_n(\underline{x})(dy)\right\}^2\right].$$

Therefore we obtain

$$I_{n,m,\varepsilon}^{(2)}\leqq\int_S \frac{1}{r}\sigma_0(ds)\left\{\frac{1}{2}\log E_{R^{\otimes\infty}}\left[\exp\left(nqr\cdot d_m\left\{\int_M (g_0(y, s)\right.\right.\right.\right.$$
$$\left.\left.\left.+g_0^{(m)}(y, s))^2\rho_n(\underline{x})(dy)\right\}\right), K_{n,\varepsilon}\right]$$
$$+\frac{1}{2}\log E_{R^{\otimes\infty}}\left[\exp\left(nqr\cdot d_m^{-1}\left\{\int_M (g_0(y, s)\right.\right.\right.$$
$$\left.\left.\left.\left.-g_0^{(m)}(y, s))^2\rho_n(\underline{x})(dy)\right\}\right), K_{n,\varepsilon}\right]\right\}, \tag{3.18}$$

where $r=\sigma_0(S)$. Observe that

$$\sup_{s\in S}\left\{qr\cdot d_m\int_M (g_0(y, s)+g_0^{(m)}(y, s))^2R(dy)\right\}\longrightarrow 0,$$
$$\sup_{s\in S}\left\{qr\cdot d_m^{-1}\int_M (g_0(y, s)-g_0^{(m)}(y, s))^2R(dy)\right\}\longrightarrow 0, \qquad m\to\infty,$$

and

$$\int_M (g_0(y, s)\pm g_0^{(m)}(y, s))\rho_n(\underline{x})(dy)=\frac{1}{n}\sum_{k=1}^n (g_0(x_k, s)\pm g_0^{(m)}(x_k, s)).$$

Then by virtue of Lemma 2.1 and (3.18), we see that there exist an integer $m_2$ and positive numbers $\varepsilon_m$, $m\geqq m_2$, such that

$$\sup_n I_{n,m,\varepsilon_m}^{(2)}<\infty, \qquad m\geqq m_2. \tag{3.19}$$

Now let $X_j(\underline{x})$ be $\boldsymbol{R}^m$-valued random variables given by

$$X_j(\underline{x})={}^t(g_0(x_j, s_1^{(m)}), \cdots, g_0(x_j, s_m^{(m)})),$$

and let $C^{(m)}$ be a symmetric $m\times m$ matrix given by

$$C^{(m)}=\begin{bmatrix}\sigma(S_1^{(m)}) & & 0\\ & \ddots & \\ 0 & & \sigma(S_m^{(m)})\end{bmatrix}.$$

Then we have

$$\int_{M\times M} V_0^{(m)}(y_1, y_2)\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)={}^t\Big(\frac{1}{n}\sum_{j=1}^n X_j\Big)C^{(m)}\Big(\frac{1}{n}\sum_{j=1}^n X_j\Big), \tag{3.20}$$

and

$$K_{n,\varepsilon}\subset\Big\{\underline{x}\in M^\infty\,;\,\Big\|\frac{1}{n}\sum_{j=1}^n X_j\Big\|<m\cdot\varepsilon\Big\}. \tag{3.21}$$

Let $V^{(m)}=E_{R^{\otimes\infty}}[X_1\cdot {}^tX_1]$. Then it is easy to see that for any $\xi=(\xi_1, \cdots, \xi_m)\in \boldsymbol{R}^m$

$$\begin{aligned} {}^t\xi\cdot V^{(m)}C^{(m)}V^{(m)}\xi &=\Big(\sum_{k=1}^m \xi_k g_0(\cdot, s_k^{(m)}),\ V_R^{(m)}\Big(\sum_{k=1}^m \xi_k g_0(\cdot, s_k^{(m)})\Big)\Big)_{L_0^2(R)} \\ &\leqq \frac{1+2\lambda}{3}\Big\|\sum_{k=1}^m \xi_k g_0(\cdot, s_k^{(m)})\Big\|^2_{L_0^2(R)} \\ &=\frac{1+2\lambda}{3}\cdot{}^t\xi V^{(m)}\xi. \end{aligned} \tag{3.22}$$

Then from Lemma 2.2, (3.16), (3.20), (3.21) and (3.22), there exists positive number $\varepsilon'_m$ for each $m\geqq m_1$ such that

$$\sup_n I^{(1)}_{n,m,\varepsilon'_m}<\infty. \tag{3.23}$$

From (3.15), (3.19) and (3.23), we have our assertion.

Lemma 3.2. *Let $R\in P(M)$. Suppose that the symmetric bounded operator $I_R-V_R$ in $L_0^2(R)$ is strictly positive definite. Then there exists an open neighborhood $G$ of $R$ in $P(M)$ such that for any open neighborhood $U$ of $R$ in $P(M)$ and $u\in L_0^2(R)$,*

$$\begin{aligned} &\lim_{n\to\infty} E_{R^{\otimes\infty}}\Big[\exp\Big(\sqrt{-1}\cdot n^{1/2}\int_M u(y)(\rho_n(\underline{x})-R)(dy) \\ &\quad+\frac{n}{2}\int_{M\times M} V(y_1, y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\Big),\ \{\underline{x}\in M^\infty;\rho_n(\underline{x})\in G\cap U\}\Big] \\ &=\det(I_R-V_R)^{-1/2}\exp\Big(-\frac{1}{2}(u, (I_R-V_R)^{-1}u)_{L_0^2(R)}\Big). \end{aligned}$$

Proof. Let us take a separable Hilbert space $H$ such that $L_0^2(R)$ is a dense linear subspace of $H$ and the inclusion map from $L_0^2(R)$ into $H$ is a Hilbert Schmidt operator. Then we can take an $H$-valued random variable $X$ such that

$$E[\exp(\sqrt{-1}{}_H\langle X, u\rangle_H^*)]=\exp\left(-\frac{1}{2}\|u\|_{L_0^2(R)}^2\right)$$

for any $u\in H^*\subset L_0^2(R)$ where $H^*$ denotes the dual space of $H$. Note that

$$E_{R^{\otimes\infty}}\left[n\cdot\left(\int_M u(y)(\rho_n(\underline{x})-R)(dy)\right)^2\right]=\|u\|_{L_0^2(R)}^2. \tag{3.24}$$

Thus we may regard $\rho_n(\underline{x})-R$, $n=1, 2, \cdots$, as $H$-valued random variables with respect to $R^{\otimes\infty}$. Therefore by virtue of the central limit theorem for independently identically distributed Hilbert space valued random variables, we see that

$$n^{1/2}(\rho_n(\underline{x})-R)\longrightarrow X, \qquad n\to\infty, \tag{3.25}$$

in distribution. Since $V_R$ is a nuclear operator in $L_0^2(R)$ from the assumption (A), we see from (3.24) and (3.25) that

$$(n^{1/2}(\rho_n(\underline{x})-R),\ n\cdot(\rho_n(\underline{x})-R,\ V_R(\rho_n(\underline{x})-R))_{L_0^2(R)})\longrightarrow(X,\ (X,\ V_RX)_{L_0^2(R)}) \quad \text{on } H\times\mathbf{R},\quad n\to\infty, \tag{3.26}$$

in distribution. Observe that

$$(\rho_n(\underline{x})-R,\ V_R(\rho_n(\underline{x})-R))_{L_0^2(R)}=\int_{M\times M}V(y_1, y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2). \tag{3.27}$$

By virtue of Lemma 3.1 and (3.27), we see that there exist $p>1$ and $\varepsilon>0$ such that

$$\sup_n E_{R^{\otimes\infty}}\left[\exp\left(p\cdot\frac{n}{2}\int_{M\times M}V(y_1, y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\right),\ \sup_{s\in S}\left|\int_M g(y, s)(\rho_n(\underline{x})-R)(dy)\right|<\varepsilon\right]<\infty. \tag{3.28}$$

Let $G=\left\{Q\in P(M);\ \sup_{s\in S}\left|\int_M g(y, s)(Q-R)(dy)\right|<\varepsilon\right\}$. Since $S$ is compact, $G$ is an open neighborhood of $R$ in $P(M)$. Therefore we see from (3.26) and (3.28) that

$$\lim_{n\to\infty}E_{R^{\otimes\infty}}\left[\exp\left(\sqrt{-1}n^{1/2}\cdot(u,\ \rho_n(\underline{x})-R)_{L_0^2(R)}\right.\right. \tag{3.29}$$

$$+\frac{n}{2}(\rho_n(\underline{x})-R,\ V_R(\rho_n(\underline{x})-R))_{L_0^2(R)}\Big),\ \rho_n(\underline{x})\in G\Big]$$

$$=E\Big[\exp\Big(\sqrt{-1}(u,\ X)_{L_0^2(R)}+\frac{1}{2}(X,\ V_RX)_{L_0^2(R)}\Big)\Big]$$

$$=\det(I_R-V_R)^{-1/2}\exp\Big(-\frac{1}{2}(u,\ (I_R-V_R)^{-1}u)_{L_0^2(R)}\Big).$$

Now let $U$ be an arbitrary open neighborhood of $R$ in $P(M)$. Then by the law of large numbers, we have

$$\lim_{n\to\infty} R^{\otimes\infty}[\rho_n(\underline{x})\in P(M)-U]=0. \tag{3.30}$$

From this and (3.28), we get

$$\lim_{n\to\infty} E_{R^{\otimes\infty}}\Big[\exp\Big(\frac{n}{2}(\rho_n(\underline{x})-R,\ V_R(\rho_n(\underline{x})-R))_{L_0^2(R)}\Big),\ \rho_n(\underline{x})\in G-U\Big]=0. \tag{3.31}$$

(3.29) and (3.31) lead us to our assertion.

LEMMA 3.3. *Let $R\in P(M)$. Suppose that the spectrum of the symmetric operator $I_R-V_R$ contains a non-negative number. Then for any open neighborhood $U$ of $R$ in $P(M)$,*

$$\lim_{n\to\infty} E_{R^{\otimes\infty}}\Big[\exp\Big(\frac{n}{2}\int_{M\times M}V(y_1,\ y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\Big),$$
$$\{\underline{x}\in M^\infty\ ;\ \rho_n(\underline{x})\in U\}\Big]=\infty.$$

PROOF. Let $H$ and $X$ be as in the proof of Lemma 3.2. Then (3.26), (3.27) and (3.30) hold also in this case. Thus by virtue of Fatou's lemma, we obtain

$$\varliminf_{n\to\infty} E_{R^{\otimes\infty}}\Big[\exp\Big(\frac{n}{2}\int_{M\times M}V(y_1,\ y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\Big),$$
$$\{\underline{x}\in M^\infty;\ \rho_n(\underline{x})\in U\}\Big]\geqq E\Big[\exp\Big(\frac{1}{2}(X,\ V_RX)_{L_0^2(R)}\Big)\Big]. \tag{3.32}$$

From the assumption on the spectrum of $I_R-V_R$, we easily see that the term of right hand side of (3.32) is infinity.

This completes the proof.

**4. The proof of Theorems 1 and 2.**

For each $R\in P(M)$, we define the entropy $h(R,\mu)$ of $R$ with respect to $\mu$ by $h(R,\mu)=\int_M \log\frac{dR}{d\mu}(x)R(dx)$, if $R$ is absolutely continuous relative to $\mu$ and $\int_M\left|\log\frac{dR}{d\mu}(x)\right|R(dx)<\infty$, and $h(R,\mu)=\infty$, otherwise. Then the following has been shown by Donsker-Varadhan [1] Theorem 4.5.

LEMMA 4.1. (1) *$h(\cdot,\mu)$ is a lower semi-continuous non-negative convex function on $P(M)$, and $\{R\in P(M); h(R,\mu)\leqq t\}$ is compact in $P(M)$ for any $t\geqq 0$.*

(2) $$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\mu^{\otimes\infty}(\{\underline{x}\in M^\infty;\rho_n(\underline{x})\in K\})\leqq-\inf\{h(R,\mu);R\in K\}$$

*for any closed set $K$ in $P(M)$.*

(3) $$\underline{\lim_{n\to\infty}}\frac{1}{n}\log\mu^{\otimes\infty}(\{\underline{x}\in M^\infty;\rho_n(\underline{x})\in G\})\geqq-\inf\{h(R,\mu);R\in G\}$$

*for any open set $G$ in $P(M)$.*

The function $F:P(M)\to\boldsymbol{R}$ in Introduction is described by

$$F(R)=-\int_{M\times M}V(x,y)R(dx)R(dy)+h(R,\mu).$$

Therefore, as a consequence of Lemma 4.1, we have

$$\lim_{n\to\infty}\frac{1}{n}\log E_{\mu^{\otimes\infty}}\left[\exp\left(n\int_{M\times M}V(y_1,y_2)\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)\right)\right]=-\inf\{F(R);R\in P(M)\}. \tag{4.1}$$

Now let $P_0$ denote the set of probability measures minimizing the function $F$ and $f$ denote the minimum value of $F$. Then we have the following.

PROPOSITION 4.2. (1) *$P_0$ is a compact subset of $P(M)$.*
(2) *For any open neighborhood $U$ of $P_0$,*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\nu_n(\{\underline{x}\in M^\infty;\rho_n(\underline{x})\in P(M)-U\})<0.$$

(3) *For any $R \in P_0$, we have*

$$\frac{dR}{d\mu}(x) = C_R \exp\left(2\int_M V(x, y) R(dy)\right) \qquad \textit{for } \mu\text{-a.e. } x, \tag{4.2}$$

*where* $C_R = \exp\left(f - \int_{M\times M} V(x, y) R(dx) R(dy)\right)$.

PROOF. The assertions (1) and (2) are obvious from Lemma 4.1 and the definition of $P_0$. Thus we shall show the assertion (3) only.

Suppose that $R \in P_0$. First we shall show that

$$\frac{dR}{d\mu}(x) > 0 \qquad \text{for } \mu\text{-a.e. } x. \tag{4.3}$$

Let $A = \left\{x \in M\,;\, \frac{dR}{d\mu}(x) = 0\right\}$ and let

$$R_t(dx) = (1-\mu(A)t) R(dx) + t \cdot \chi_A(x) \mu(dx), \qquad 0 \leqq t \leqq 1.$$

Then we have

$$\begin{aligned} h(R_t, \mu) &= \int_M \log\left((1-\mu(A)t)\frac{dR}{d\mu}(x) + t \cdot \chi_A(x)\right) R_t(dx) \\ &= (1-\mu(A)t)\log(1-\mu(A)t) + (1-\mu(A)t) h(R, \mu) + \mu(A)t \cdot \log t. \end{aligned}$$

Thus if $\mu(A) > 0$, then we see that

$$\overline{\lim_{t\to 0}} \frac{1}{t}(h(R_t, \mu) - h(R, \mu)) = -\infty,$$

and so we have

$$\overline{\lim_{t\to 0}} \frac{1}{t}(F(R_t) - F(R)) = -\infty.$$

But this contradicts the assumption that $R \in P_0$. Therefore we obtain $\mu(A) = 0$, which shows (4.3).

Now let

$$B = \left\{\lambda \cdot h \in L_0^2(\mu)\,;\, \lambda \in \boldsymbol{R},\ h \in L_0^2(\mu) \text{ and } |h(x)| \leqq \min\left\{1, \frac{dR}{d\mu}(x)\right\}\ \mu\text{-a.e. } x\right\}.$$

Then we see that for any $h \in B$,

$$0=\frac{d}{dt}F(R+t\cdot h(x)\mu(dx))\Big|_{t=0}$$
$$=-2\int_{M\times M}V(x,y)h(x)\mu(dx)R(dy)+\int_M\log\Big(\frac{dR}{d\mu}(x)\Big)h(x)\mu(dx).$$

Since $B$ is a dense subset in $L_0^2(\mu)$ by (4.3), we see that there exists a constant $C\in\boldsymbol{R}$ for which

$$\log\Big(\frac{dR}{d\mu}(x)\Big)-2\int_{M\times M}V(x,y)R(dy)=C, \qquad \mu\text{-a.e. } x. \tag{4.4}$$

Integrating both sides of (4.4) by $R(dx)$, we obtain

$$F(R)-\int_{M\times M}V(x,y)R(dx)R(dy)=C. \tag{4.5}$$

(4.4) and (4.5) lead us to our assertion (3).

Let $P_{00}=\{R\in P_0\,;$ The symmetric operator $D^2F(R)$ in $L_0^2(R)$ is strictly positive definite$\}$.

Now let $R\in P_0$. Then from Proposition 4.2 (3), we have

$$\begin{aligned}(4.6)\qquad Z_n\nu_n(d\underline{x})&=\exp\Big(n\int_{M\times M}V(y_1,y_2)\rho_n(\underline{x})^{\otimes 2}(dy_1\otimes dy_2)\\&\qquad -n\int_M\log\frac{dR}{d\mu}(y)\rho_n(\underline{x})(dy)\Big)\bigotimes_{k=1}^{n}R(dx_k)\bigotimes_{k=n+1}^{\infty}\mu(dx_k)\\&=\exp\Big(-n\cdot f+n\int_{M\times M}V(y_1,y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\Big)\\&\qquad\bigotimes_{k=1}^{n}R(dx_k)\bigotimes_{k=n+1}^{\infty}\mu(dx_k).\end{aligned}$$

Noting that $\rho_n(\underline{x})$ depends only on $(x_1,\cdots,x_n)$, we have got

$$\begin{aligned}(4.7)\qquad &e^{n\cdot f}Z_n\nu_n(\{\rho_n(\underline{x})\in G\})\\&=E_{R^{\otimes\infty}}\Big[\exp\Big(n\int_{M\times M}V(y_1,y_2)(\rho_n(\underline{x})-R)^{\otimes 2}(dy_1\otimes dy_2)\Big),\\&\qquad\{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in G\}\Big]\end{aligned}$$

for any open set $G$ in $P(M)$.

The following is an easy consequence of Lemma 3.2, (3.3) and (4.7).

LEMMA 4.3. *For any $R\in P_{00}$, there exists an open neighborhood $G_R$ of $R$ in $P(M)$ such that*

$$\lim_{n\to\infty} e^{n\cdot f} Z_n E_{\nu_n}\Big[\exp\Big(\sqrt{-1}\cdot n^{1/2}\cdot\int_M u(y)\rho_n(\underline{x})(dy)\Big),\ \{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in G_R\cap U\}\Big]$$
$$=\det(D^2F(R))^{-1/2}\exp\Big(-\frac{1}{2}(u,\,D^2F(R)^{-1}u)_{L_0^2(R)}\Big),$$

*for any* $u\in L_0^2(R)$ *and any open neighborhood* $U$ *of* $R$ *in* $P(M)$.

The following is an immediate consequence of Lemma 3.3 and (4.7).

LEMMA 4.4. *If* $R\in P_0\text{-}P_{00}$, *then for any open neighborhood* $U$ *of* $R$ *in* $P(M)$,

$$\lim_{n\to\infty} e^{n\cdot f} Z_n \nu_n(\{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in U\})=\infty.$$

Then we have the following.

PROPOSITION 4.5. *For each* $R\in P_{00}$, $R$ *is an isolated point in* $P_0$.

PROOF. Let $G_R$ be an open neighborhood of $R$ as in Lemma 4.3. It is sufficient to show that $G_R\cap P_0=\{R\}$. Suppose that $R_1\in G_R\cap P_0$, $R_1\neq R$. If $R_1\in P_0-P_{00}$, then we have from Lemma 4.4

$$\lim_{n\to\infty} e^{n\cdot f} Z_n \nu_n(\{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in G_R\})=\infty,$$

which contradicts our assumption for $G_R$. If $R_1\in P_{00}$, then there exists an open neighborhood $G_{R_1}$ of $R_1$ which is an open neighborhood for $R=R_1$ in Lemma 4.3. Let $U$ and $U_1$ be disjoint open neighborhood of $R$ and $R_1$ respectively. Then we have

$$\lim_{n\to\infty} e^{n\cdot f} Z_n \nu_n(\{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in G_R\cap U\})$$
$$=\lim_{n\to\infty} e^{n\cdot f} Z_n \nu_n(\{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in G_R\}),$$

and

$$\lim_{n\to\infty} e^{n\cdot f} Z_n \nu_n(\{\underline{x}\in M^\infty\,;\,\rho_n(\underline{x})\in G_R\cap G_{R_1}\cap U_1\})>0.$$

But this is impossible. Thus we obtain our assertion.

Now we are ready to give the proofs of Theorems 1 and 2.

First let us prove Theorem 1. Suppose that $P_0=P_{00}$. Then from Proposition 4.2 (1) and Proposition 4.5, we see that $P_{00}$ is a finite set. Let $P_0=P_{00}=\{R_1,\cdots,R_m\}$. Then from Proposition 4.2 (2) and Lemma 4.3, we see that

$$(4.8)\qquad \lim_{n\to\infty} e^{n\cdot f} Z_n = z = \sum_{k=1}^{m} \det(D^2F(R_k))^{-1/2},$$

and that there exist open neighborhoods $G_{R_k}$ of $R_k$, $k=1,\cdots,m$, such that $G_{R_k}$'s are mutually disjoint,

$$(4.9)\qquad \nu_n(\{\underline{x}\in M^\infty ; \rho_n(\underline{x})\in G_{R_k}\cap U\}) \longrightarrow a_k, \qquad n\to\infty,$$

for any open neighborhood $U$ of $R_k$, $k=1,\cdots,m$, and

$$(4.10)\qquad \nu_n(\{\underline{x}\in M^\infty ; \rho_n(\underline{x})\in P(M)-\bigcup_{k=1}^{m} G_{R_k}\}) \longrightarrow 0, \qquad n\to\infty.$$

Let $h: M^r\to \boldsymbol{R}$, $r\geqq 1$, be a bounded continuous function. Then by virtue of (4.9) and (4.10), we get

$$\begin{aligned}(4.11)\qquad &\lim_{n\to\infty}\int_{M^\infty} h(x_1,\cdots,x_r)\nu_n(d\underline{x})\\ &=\lim_{n\to\infty}\sum_{i_1,\cdots,i_r=1}^{n}\frac{1}{n^r}\int_{M^\infty} h(x_{i_1},\cdots,x_{i_r})\nu_n(d\underline{x})\\ &=\lim_{n\to\infty}\int_{M^\infty}\nu_n(d\underline{x})\Big(\int_{M^r} h(y_1,\cdots,y_r)\bigotimes_{i=1}^{r}\rho_n(\underline{x})(dy_i)\Big)\\ &=\lim_{n\to\infty}\sum_{k=1}^{m} E_{\nu_n}\Big[\int_{M^r} h(y_1,\cdots,y_r)\bigotimes_{i=1}^{r}\rho_n(\underline{x})(dy_i),\ \rho_n(\underline{x})\in G_{R_k}\Big]\\ &=\sum_{k=1}^{m} a_k\int_{M^\infty} h(x_1,\cdots,x_r)\bigotimes_{i=1}^{\infty} R_k(dx_i).\end{aligned}$$

This proves Theorem 1.

Now let us prove Theorem 2. Suppose that $\{R_0\}=P_0-P_{00}$. Then by virtue of Lemma 4.4, we see that

$$(4.12)\qquad \lim_{n\to\infty} e^{n\cdot f} Z_n = \infty.$$

This and Lemma 4.3 tell us that there exists an open neighborhood $G_R$ of $R$ for each $R\in P_{00}$ such that

$$(4.13)\qquad \lim_{n\to\infty}\nu_n(\{\underline{x}\in M^\infty ; \rho_n(\underline{x})\in G_R\})=0.$$

Therefore from Proposition 4.2 (2) we see that

$$(4.14)\qquad \lim_{n\to\infty}\nu_n(\{\underline{x}\in M^\infty ; \rho_n(\underline{x})\in U\})=1$$

for any open neighborhood $U$ of $R_0$ in $P(M)$. This and the argument for

(4.11) prove Theorem 2.

## 5. Remarks on $r$-body potentials.

In this section, we consider $r$-body potentials, $r\geqq 3$. Since the proof is almost the same as in the case of pair potentials, we will state results without proof.

Let $V:M^r\to\boldsymbol{R}$ be a symmetric bounded continuous function satisfying the assumption:

(A′) There exist a compact metric space $S$, a signed measure $\sigma$ on $S$ with bounded total variation and bounded continuous functions $f_i:M\times S\to\boldsymbol{R}$, $i=1,\cdots,r$, such that

$$V(x_1,\cdots,x_r)=\sum_{\{i_1,\cdots,i_r\}=\{1,\cdots,r\}}\int_S\prod_{k=1}^{m}f_{i_k}(x_k,s)\sigma(ds),\qquad (x_1,\cdots,x_r)\in M^r.$$

Let $\nu_n$, $n\geqq r$, be a probability measure on $M^\infty$ given by

$$\nu_n(d\underline{x})=Z_n^{-1}\exp\Big(\frac{1}{n^{r-1}}\sum_{i_1,\cdots,i_r=1}^{n}V(x_{i_1},\cdots,x_{i_r})\Big)\bigotimes_{j=1}^{\infty}\mu(dx_j),\tag{5.1}$$

where

$$Z_n=\int_{M^\infty}\exp\Big(\frac{1}{n^{r-1}}\sum_{i_1,\cdots,i_r=1}^{n}V(x_{i_1},\cdots,x_{i_r})\Big)\bigotimes_{j=1}^{\infty}\mu(dx_j).\tag{5.2}$$

Now let $F:P(M)\to\boldsymbol{R}$ be a function given by

$$F(R)=-\int_{M^r}V(x_1,\cdots,x_r)\bigotimes_{i=1}^{r}R(dx_i)+h(R;\mu),\qquad R\in P(M),\tag{5.3}$$

and for each $R\in P(M)$, let $D^2F(R)$ be a symmetric bounded operator in $L_0^2(R)$ given by

$$\begin{aligned}(D^2F(R)u,v)_{L_0^2(R)}=&-r(r-1)\int_{M^2}R(dx_1)R(dx_2)u(x_1)v(x_2)\\&\times\Big\{\int_{M^{r-2}}V(x_1,x_2,y_1,\cdots,y_{r-2})\bigotimes_{i=1}^{r-2}R(dy_i)\Big\}\\&+\int_M u(x)v(x)R(dx),\quad u,v\in L_0^2(R).\end{aligned}\tag{5.4}$$

Let $P_0$ denote the set of probability measures on $M$ minimizing $F$, $P_{00}$ denote the set $\{R\in P_0;D^2F(R)$ is strictly positive definite$\}$, and $f$ denote the minimum value of $F$ as in Section 4. Then we have the following.

PROPOSITION 5.1. (1) $\lim_{n\to\infty}\frac{1}{n}\log Z_n=-f$.

(2) *$P_0$ is a compact subset of $P(M)$.*

(3) *Each point of $P_{00}$ is an isolated point in $P_0$.*

(4) *For any open neighborhood $U$ of $P_0$,*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\nu_n(\{\underline{x}\in M^\infty;\ \rho_n(\underline{x})\in P(M)-U\})<0.$$

LEMMA 5.2. *For any $R\in P_{00}$, there exists an open neighborhood $G_R$ of $R$ in $P(M)$ such that*

$$\begin{aligned}&\lim_{n\to\infty}e^{n\cdot f}\cdot Z_n\\ &\quad\times E_{\nu_n}\Big[\exp\Big(\sqrt{-1}\cdot n^{1/2}\cdot\int_M u(y)\rho_n(\underline{x})(dy)\Big),\ \{\underline{x}\in M^\infty;\ \rho_n(\underline{x})\in G_R\cap U\}\Big]\\ &=\det(D^2F(R))^{-1/2}\exp\Big(-\frac{1}{2}(u,D^2F(R)^{-1}u)_{L_0^2(R)}\Big),\end{aligned}$$

*for any $u\in L_0^2(R)$ and any open neighborhood $U$ of $R$ in $P(M)$.*

LEMMA 5.3. *For any $R\in P_0-P_{00}$ and any open neighborhood $U$ of $R$ in $P(M)$,*

$$\lim_{n\to\infty}e^{n\cdot f}\cdot Z_n\cdot\nu_n(\{\underline{x}\in M^\infty;\ \rho_n(\underline{x})\in U\})=\infty.$$

THEOREM 3. *Suppose that $P_0=P_{00}$. Then $P_0$ is a finite set and $\nu_n$ are convergent to the probability measure $z^{-1}\sum_{R\in P_0}\det(D^2F(R))^{-1/2}R^{\otimes\infty}$, where $z=\sum_{R\in P_0}\det(D^2F(R))^{-1/2}$. Moreover $\lim_{n\to\infty}Z_n\cdot e^{n\cdot f}=z$.*

THEOREM 4. *Suppose that $\{R_0\}=P_0-P_{00}$. Then $\nu_n$ are convergent to the probability measure $R_0^{\otimes\infty}$, and $\lim_{n\to\infty}Z_n\cdot e^{n\cdot f}=\infty$.*

## 6. An application to the central limit theorem for diffusion processes with mean field interaction.

Let $(\Omega, B, P)$ be a complete probability space. Let $\Phi$ be a rapidly decreasing smooth function on $\boldsymbol{R}^d$ satisfying $\Phi(-z)=\Phi(z)$, $z\in\boldsymbol{R}^d$. Let $Y^j(\omega)$, $j=1,2,\cdots$, be independently identically distributed $\boldsymbol{R}^d$-valued random variables with $E[\|Y_1\|^2]<\infty$. Let us consider the following stochastic differential equation for each $n \geqq 1$:

$$(6.1)\quad \begin{cases} dX_n^i(t,\omega) = dB^i(t,\omega) + \dfrac{1}{n}\sum\limits_{j=1}^{n} \operatorname{grad} \Phi(X_n^i(t,\omega) - X_n^j(t,\omega))dt \\ X_n^i(0,\omega) = Y^i(\omega), \qquad i=1,\cdots,n, \end{cases}$$

where $B^i(t,\omega)$, $i=1,2,\cdots$, are independent $d$-dimensional Brownian motions.

Fix $T>0$, and let $M$ denote the complete metric space $C([0,T]\to \boldsymbol{R}^d)$. We define a random measure $R_n(dw;\omega)$ on $M$ by

$$(6.2)\quad \int_M f(w)R_n(dw;\omega) = \frac{1}{n}\sum_{i=1}^{n} f(X_n^i(\cdot,\omega))$$

for any bounded measurable function $f$ on $M$. Our concern is in asymptotic behaviour of the distribution of the random measure $R_n(dw;\omega)$ as $n\to\infty$.

Now let us consider a stochastic differential equation of McKean type:

$$(6.3)\quad \begin{cases} dX(t,\omega) = dB^1(t,\omega) + \left(\displaystyle\int_{R^d} \operatorname{grad} \Phi(X(t,\omega)-z)u_t(dz)\right)dt \\ X(0,\omega) = Y^1(\omega) \\ u_t(dz) \text{ is the probability distribution law of } X(t,\omega), \end{cases}$$

and let $R_0$ denote the probability distribution measure on $M$ induced by $X(t,\omega)$, $0\leqq t\leqq T$. Then the following has been known.

LEMMA 6.1 (McKean [2]). *For any bounded continuous function $f$ on $M^k$, $k\geqq 1$,*

$$\lim_{n\to\infty} E[f(X_n^1(\cdot,\omega),\cdots,X_n^k(\cdot,\omega))] = \int_{M^k} f(w^1,\cdots,w^k)\bigotimes_{i=1}^{k} R_0(dw^i).$$

Let $\mu$ be the probability distribution measure on $M$ induced by $Y^1(\omega)+B^1(t,\omega)$, $0\leqq t\leqq T$, and let $\tilde{\nu}_n$, $n\geqq 1$, be the probability distribution measure on $M^n$ induced by $(X_n^1(t,\omega),\cdots,X_n^n(t,\omega))$, $0\leqq t\leqq T$. Then by virtue of Cameron-Martin-Maruyama-Girsanov's formula, we have

$$(6.4)\quad \frac{d\tilde{\nu}_n}{d\mu^{\otimes n}}(w_n) = \exp\left(\frac{1}{n}\sum_{i,j=1}^{n}\int_0^T \operatorname{grad}\Phi(w^i(t)-w^j(t))dw^i(t) - \frac{1}{2n^2}\sum_{i=1}^{n}\int_0^T\left\|\sum_{j=1}^{n}\operatorname{grad}\Phi(w^i(t)-w^j(t))\right\|^2 dt\right)$$

where $w_n=(w^1, \cdots, w^n)\in M^n$ and the first integral in the term of the right hand side is Ito's stochastic integral.

Let $V_2: M^2\to \boldsymbol{R}$ and $V_3: M^3\to \boldsymbol{R}$ be given by

$$V_2(w^1, w^2) = \frac{1}{2}\int_0^T \operatorname{grad}\Phi(w^1(t)-w^2(t))(dw^1(t)-dw^2(t)) = \frac{1}{2}\Big\{\Phi(w^1(T)-w^2(T))-\Phi(w^1(0)-w^2(0)) - \int_0^T \Delta\Phi(w^1(t)-w^2(t))\,dt\Big\}, \tag{6.5}$$

and

$$V_3(w^1, w^2, w^3) = -\frac{1}{2}\int_0^T \operatorname{grad}\Phi(w^1(t)-w^2(t))\cdot\operatorname{grad}\Phi(w^1(t)-w^3(t))dt. \tag{6.6}$$

Let $V: M^3\to \boldsymbol{R}$ be a 3-body potential given by

$$V(w^1, w^2, w^3) = \frac{1}{3}\{V_2(w^1, w^2)+V_2(w^2, w^3)+V_2(w^3, w^1)\} + \frac{1}{3}\{V_3(w^1, w^2, w^3)+V_3(w^2, w^3, w^1)+V_3(w^3, w^1, w^2)\}. \tag{6.7}$$

Note that

$$\Phi(z)=\Big(\frac{1}{2\pi}\Big)^d\int_{R^d}\exp(\sqrt{-1}z\cdot\xi)\hat{\Phi}(\xi)d\xi,$$

where $\hat{\Phi}(\xi)$ is the Fourier transform of $\Phi$. Then it is easy to see that the 3-body potential $V$ satisfies the assumption (A′) in Section 5, observing that $\hat{\Phi}$ is a rapidly decreasing function. Let $D^2F(R)$, $R\in P(M)$, be the symmetric linear operator in $L_0^2(R)$ given by (5.4) for the 3-body potential $V$.

Then we have the following.

THEOREM 5. *For any* $u\in L_0^2(R_0)$,

$$\lim_{n\to\infty} E\Big[\exp\Big(\sqrt{-1}\cdot n^{1/2}\cdot\int_M u(w)(R(dw;\omega))\Big)\Big] = \exp\Big(-\frac{1}{2}(u, D^2F(R_0)^{-1}u)_{L_0^2(R_0)}\Big).$$

PROOF. Let $Z_n, \nu_n, F, f, P_0$ and $P_{00}$ be as in Section 5 for the 3-body potential $V$. Then we easily see that

(6.8) $$\nu_n(d\underline{w})=\tilde{\nu}_n(dw_n)\otimes\bigotimes_{i=n+1}^{\infty}\mu(dw^i).$$

Thus we see that $Z_n=1$, $n\geqq 1$, and $f=0$. Therefore we have

(6.9) $$\lim_{n\to\infty} Z_n e^{n\cdot f}=1.$$

On the other hand, we know from Lemma 6.1 and (6.8) that

(6.10) $$\nu_n(d\underline{w})\longrightarrow\bigotimes_{i=1}^{\infty}R_0(dw^i)\quad\text{in } P(M^\infty)\text{ as } n\to\infty.$$

It follows from Theorems 3, 4, (6.9) and (6.10) that $P_0=P_{00}=\{R_0\}$. Therefore from Lemma 5.2, Proposition 5.1 (4) and (6.8) we have our assertion.

This completes the proof.

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