Hodge filtrations on Gauss-Manin systems I

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Let Y be a complex manifold and let S be the unit open disk $\{t\in \boldsymbol{C} \mid |t|<1\}$. Let $f: Y\to S$ be a flat projective morphism such that f is smooth on $S^*=S-\{0\}$. The *Gauss-Manin system* of f is defined to be the complex of $\mathcal{D}_S$-Modules obtained by integrating $\mathcal{O}_Y$ as a holonomic system along the fibers of f, and is denoted by $\int_f \mathcal{O}_Y$ (cf. [7]). For any integer k, the k-th cohomology sheaf $\int_f^k \mathcal{O}_Y$ of $\int_f \mathcal{O}_Y$ is a regular holonomic system on S [5]. It is also called the (k-th) Gauss-Manin system on f, and is canonically endowed with the *Hodge filtration* $\mathcal{F}^\bullet$, which induces a variation of Hodge structures on S^* by the canonical isomorphism

$$\int_f^k \mathcal{O}_Y|_{S^*} \cong \mathcal{O}_{S^*} \underset{\boldsymbol{C}}{\otimes} (R^k f_* \boldsymbol{C}_Y|_{S^*}) \qquad \text{(cf. [1])}.$$

Let $\mathcal{L}^k$ be the canonical extension of $\mathcal{O}_{S^*} \underset{\boldsymbol{C}}{\otimes} (R^k f_* \boldsymbol{C}_Y|_{S^*})$ to S as a holomorphic vector bundle (i.e., a locally free $\mathcal{O}_S$-Module) with a connection ∇ such that the eigenvalues of res$(t\nabla)$ are in $(-1, 0]$ (cf. (2.1)). By a result of Schmid [13], $\mathcal{F}^\bullet|_{S^*}$ has an extension $\hat{\mathcal{F}}^\bullet$ to S such that $\hat{\mathcal{F}}^p$ is a holomorphic vector subbundle of $\mathcal{L}^k$ for each integer p (cf. [10]).

Our main theorem is as follows (cf. (2.5)).

THEOREM. *$\hat{\mathcal{F}}^p$ is a subsheaf of $\mathcal{F}^p$ for each integer p. More precisely, we have a natural inclution $\mathcal{L}^k \hookrightarrow \int_f^k \mathcal{O}_Y$ such that*

$$\hat{\mathcal{F}}^k(\mathcal{L}^k) = \mathcal{F}^k\Big(\int_f^k \mathcal{O}_Y\Big) \qquad \textit{for each } k,$$

and

$$\hat{\mathcal{F}}^p(\mathcal{L}^k) = \mathcal{L}^k \cap \mathcal{F}^p\Big(\int_f^k \mathcal{O}_Y\Big) \qquad \textit{for each } p \textit{ and } k.$$

From this, we can derive the following result of Scherk-Steenbrink [14] (see also [8] [9] [19]): the Hodge filtration of the mixed Hodge structure of Steenbrink [16] on the cohomology of the Milnor fiber of an isolated hypersurface singularity is determined by the Brieskorn lattice $\mathcal{H}^{(0)}_{X,0}=\Omega^{n+1}_{X,0}/df\wedge d\Omega^{n-1}_{X,0}$ (cf. Theorem (3.5)).

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§ 1. Holonomic systems and good filtrations

(1.1) Let Y be a complex manifold of dimension N, and let $\mathcal{M}$ be a *holonomic system* on Y, i.e., $\mathcal{M}$ is a coherent left $\mathcal{D}_Y$-Module such that $\mathcal{E}xt^i_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{D}_Y)=0$ for $i\neq N$.

By a result of Kashiwara [2], $DR_Y(\mathcal{M}):=\Omega^{\cdot}_Y\underset{\mathcal{O}_Y}{\otimes}\mathcal{M}$ has constructible cohomologies, and the functor DR_Y induces an equivalence $D^b_{rh}(\mathcal{D}_Y)\xrightarrow{\sim}D^b_c(\boldsymbol{C}_Y)$ (cf. [4] [6]). Here $D^b_{rh}(\mathcal{D}_Y)$ (resp. $D^b_c(\boldsymbol{C}_Y)$) is the full subcategory of $D(\mathcal{D}_Y)$ (resp. $D(\boldsymbol{C}_Y)$) consisting of bounded complexes with regular holonomic (resp. constructible) cohomologies, and $D(\mathcal{D}_Y)$ (resp. $D(\boldsymbol{C}_Y)$) is the derived category of $\mathcal{D}_Y$- (resp. $\boldsymbol{C}_Y$-) Modules on Y. Here we consider only left $\mathcal{D}_Y$-Modules.

(1.2) DEFINITION. a) An increasing filtration $\{\mathcal{M}_k\}_{k\in\boldsymbol{Z}}$ on a $\mathcal{D}_Y$-Module $\mathcal{M}$ is called a *good filtration*, if it satisfies the following conditions:

1) $\bigcup_{k\in\boldsymbol{Z}}\mathcal{M}_k=\mathcal{M}$ and $\mathcal{M}_k=0$ for $k\ll 0$ (locally),

2) $\mathcal{D}_Y(r)\mathcal{M}_k\subset\mathcal{M}_{r+k}$ for $r\in\boldsymbol{N}$ and $k\in\boldsymbol{Z}$, where $\mathcal{D}_Y(r):=\left\{\sum_{|\nu|\leq r}a_\nu\partial^\nu\right\}\subset\mathcal{D}_Y$,

3) $\mathrm{Gr}(\mathcal{M})$ is a coherent $\mathrm{Gr}(\mathcal{D}_Y)$-Module on Y.

b) A good filtration $\{\mathcal{M}_k\}_{k\in\boldsymbol{Z}}$ is called Cohen-Macaulay, if $\mathrm{Gr}(\mathcal{M})$ is a Cohen-Macaulay $\mathrm{Gr}(\mathcal{D}_Y)$-Module. We abbreviate it as "a good C.M. filtration".

c) For an increasing filtration $\{\mathcal{M}_k\}$ on a $\mathcal{D}_Y$-Module $\mathcal{M}$, we define the decreasing filtration $\{\mathcal{F}^p\}$ by $\mathcal{F}^p:=\mathcal{M}_{-p}$ for $p\in\boldsymbol{Z}$, and vice versa. We say that $\{\mathcal{F}^p\}$ is a good filtration (resp. a good C.M. filtration), if so is the corresponding increasing filtration $\{\mathcal{M}_k\}$.

d) Let $\mathcal{M}$ be a $\mathcal{D}_Y$-Module with a filtration $\{\mathcal{M}_k\}$. We denote by $\mathcal{M}\{n\}$ the $\mathcal{D}_Y$-Module with the filtration $\{\mathcal{M}\{n\}_k=\mathcal{M}_{k-n}\}$.

(1.3) DEFINITION. Let $DF(\mathcal{D}_Y)$ be the derived category of $\mathcal{D}_Y$-Modules with filtrations satisfying the first two conditions of Definition (1.2.a), and let $D^b_{grh}F(\mathcal{D}_Y)$ be the full subcategory of $DF(\mathcal{D}_Y)$ consisting of bounded complexes whose cohomologies are regular holonomic systems with good filtrations.

a) Let $\mathcal{M}^{\cdot}\in ObDF(\mathcal{D}_Y)$ be a complex of filtered $\mathcal{D}_Y$-Modules having a locally free filtered resolution $\mathcal{L}^{\cdot}\xrightarrow{\sim}\mathcal{M}^{\cdot}$ (i.e., $\mathcal{L}^p\cong\bigoplus_i\mathcal{D}_Y\{m_i\}$ locally). We define the dual filtered complex $\mathcal{M}^{\cdot *}\in ObDF(\mathcal{D}_Y)$ by

$$\mathcal{M}^{\cdot *}:=\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{L}^{\cdot}, \mathcal{D}_Y)\underset{\mathcal{O}_Y}{\otimes}(\Omega^N_Y)^{\otimes -1}\{N\}[N]$$

where $N := \dim Y$ (cf. [1]).

b) Let $f: Y \to X$ be a holomorphic map of complex manifolds and let $l = \dim Y - \dim X$. We define the integration of $\mathcal{M}^\bullet \in ObDF(\mathcal{D}_Y)$ by

$$\int_f \mathcal{M}^\bullet := \boldsymbol{R}f_*\Big(\mathcal{D}_{X \leftarrow Y} \overset{L}{\underset{\mathcal{D}_Y}{\otimes}} \mathcal{M}^\bullet\Big)[-l] \in ObD(\mathcal{D}_Y),$$

where $\mathcal{D}_{X \leftarrow Y} := \Omega_Y^N \underset{f^{-1}\mathcal{O}_X}{\otimes} f^{-1}(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} (\Omega_X^{N-l})^{\otimes -1})$ (cf. [3]).

We define a filtration on $\int_f \mathcal{M}^\bullet$ as follows (cf. [1]):

1) In the case where f is a closed immersion,

$$\mathcal{F}^p\Big(\int_f \mathcal{M}^\bullet\Big) := f_*\Big(\sum_k (\mathcal{D}_{X \leftarrow Y})_k \otimes \mathcal{F}^{p+k}(\mathcal{M}^\bullet\{-l\}[-l])\Big).$$

Here $(\mathcal{D}_{X \leftarrow Y})_k := \Omega_Y^N \otimes f^{-1}(\mathcal{D}_X(k)) \otimes (\Omega_X^{N-l})^{\otimes -1})$.

2) In the case where f is smooth[1], we define the filtration $\mathcal{F}^\bullet$ on $DR_{X/Y}(\mathcal{M}^\bullet) = \{DR_{Y/X}(\mathcal{M}^\bullet)^k\}_{k \in Z}$ to be

$$\mathcal{F}^p(DR_{Y/X}(\mathcal{M}^\bullet)^k) := \bigoplus_{i+j=k} \Omega_{Y/X}^i \otimes \mathcal{F}^{p-i}(\mathcal{M}^j),$$

and the filtration $\mathcal{F}^\bullet$ on $\int_f \mathcal{M}^\bullet \cong \boldsymbol{R}f_* DR_{Y/X}(\mathcal{M}^\bullet)$ to be the direct image of the filtered complex $(DR_{Y/X}(\mathcal{M}^\bullet), \mathcal{F}^\bullet) \in ObDF(f^{-1}\mathcal{D}_X)$.

3) The general case. We factor f as $f = p \circ i$, where i is a closed immersion and p is smooth. Then the filtration on $\int_f \mathcal{M}^\bullet \cong \int_p \Big(\int_i \mathcal{M}^\bullet\Big)$ is defined by 1) and 2). The does not depend on the decomposition $f = p \circ i$, if f is proper.

(1.4) LEMMA. a) (Kashiwara) *For a holonomic system $\mathcal{M}$ with a good C.M. filtration, $\mathcal{M}^* \in ObDF(\mathcal{D}_Y)$ is well-defined and is represented by a holonomic system with a good C.M. filtration.*

b) *For $\mathcal{M}^\bullet \in ObDF(\mathcal{D}_Y)$, we have $DR_X\Big(\int_f \mathcal{M}^\bullet\Big) \cong \boldsymbol{R}f_*(DR_Y(\mathcal{M}^\bullet))$ as filtered complexes, where $DR_X\Big(\int_f \mathcal{M}^\bullet\Big)$ and $DR_Y(\mathcal{M}^\bullet)$ have the filtrations defined in* (1.3.b.2) (*cf.* [1] [6]).

c) *Let $f: Y \to X$ and $g: Z \to Y$ be proper morphisms of complex manifolds. Then we have $\int_{f \circ g} \mathcal{M}^\bullet \cong \int_f \Big(\int_g \mathcal{M}^\bullet\Big)$ in $DF(\mathcal{D}_X)$ for $\mathcal{M}^\bullet \in ObDF(\mathcal{D}_Z)$* (*cf.* [1]).

d) *For a projective morphism $f: Y \to X$ and a regular holonomic system $\mathcal{M}$ with a good filtration, we have $\int_f \mathcal{M} \in ObD_{grh}^b F(\mathcal{D}_X)$* (*cf.* [1] [5]).

1) We have to assume that $Y = X \times W$ with a manifold W and $f: Y \to X$ is the projection.

e) *Let* $f: Y \to X$ *and* $g: X \to S$ *be projective morphisms of complex manifolds, and let* $\mathcal{M}^{\bullet}$ *be a bounded complex of* $\mathcal{D}_Y$*-Modules with good filtrations. Then, for every compact subset* $K \subset S$*, which has a Stein neighborhood in* S*, we have a neighborhood* X' *of* $g^{-1}(K)$ *such that*

$$\left(\int_{f'} \mathcal{M}^{\bullet}|_{Y'}\right)^{*} \cong \int_{f'} (\mathcal{M}^{\bullet}|_{Y'})^{*}\{l\}[2l]$$

as filtered complexes of $\mathcal{D}_{X'}$*-Modules, where* $Y' := f^{-1}(X')$, $f' := f|_{Y'}: Y' \to X'$ *and* $l = \dim Y - \dim X$ (*cf.* [6] [20]).

REMARKS. 1) Let $\mathcal{M}$ be a holonomic system with a good C.M. filtration. $\mathcal{M}$ is called a self dual system of weight n, if there is an isomorphism $\mathcal{M} \cong \mathcal{M}^{*}\{-n\}$ in $DF(\mathcal{D}_Y)$.

2) $\mathcal{O}_Y$ is a self dual system of weight 0. In fact it has the Hodge filtration: $\mathcal{F}^0\mathcal{O}_Y = \mathcal{O}_Y$ and $\mathcal{F}^1\mathcal{O}_Y = \{0\}$.

(1.5) PROPOSITION. *Let* $f: Y \to X$ *and* $g: X \to S$ *be projective morphisms of complex manifolds. We assume that* f *is a modification, i.e., there is an open dense subset* V *of* X *such that* f *induces an isomorphism* $f^{-1}(V) \to V$. *Then, for every compact subset* K *of* S*, which has a Stein neighborhood in* S*, we have a neighborhood* X' *of* $g^{-1}(K)$ *such that* $\mathcal{O}_{X'}$ *is a direct factor of* $\int_f \mathcal{O}_Y|_{X'}$ *as a filtered complex of* $\mathcal{D}_{X'}$*-Modules. Here* $\mathcal{O}_Y$ *and* $\mathcal{O}_X$ *have the Hodge filtrations* $\mathcal{F}^{\bullet}$ *stated in Remark* 2) *of Lemma* (1.4), *and* $\int_f \mathcal{O}_Y$ *has the induced filtration* $\mathcal{F}^{\bullet}$ *defined in* (1.3.b).

PROOF. A natural morphism $\boldsymbol{C}_X \to \boldsymbol{R}f_*\boldsymbol{C}_Y$ induces a morphism $f^{\#}: \mathcal{O}_X \to \int_f \mathcal{O}_Y$ in $DF(\mathcal{D}_X)$. Taking the dual complexes, we have the Gysin morphism $f_{\#}: \int_f \mathcal{O}_Y|_{X'} \to \mathcal{O}_{X'}$ on a neighborhood X' of $g^{-1}(K)$, since $\mathcal{O}_X \cong \mathcal{O}_X^{*}$, $\mathcal{O}_Y \cong \mathcal{O}_Y^{*}$ and $\left(\int_{f'} \mathcal{O}_{Y'}\right)^{*} \cong \int_{f'} (\mathcal{O}_{Y'}^{*})$ (cf. (1.4.e)). We can verify that $f_{\#} \circ f^{\#}$ is the identity on the open dense subset of X', where f induces an isomorphism. Hence $f_{\#} \circ f^{\#}$ is the identity on X'. This gives the desired result. Q.E.D.

§ 2. Limit Hodge filtration

(2.1) Let $f: Y \to S$ be a projective morphism of complex manifolds and let $n = \dim Y - \dim S$. We assume that Y is connected, S is a one-dimensional open disk, and f is surjective and smooth on $S^{*} := S - \{0\}$.

Let $H^k := R^k f_* \boldsymbol{C}_Y|_{S^*}$, which is a local system on S^*. As was proved by Deligne, H^k has a canonical extension to S as a holomorphic vector bundle $\mathcal{L}^k$

with a connection ∇, such that ∇ has a simple pole at the origin and the eigenvalues of $\mathrm{res}(t\nabla)$ are contained in $(-1, 0]$.

We set $\mathcal{L}^k(0) := \mathcal{L}_0^k/t\mathcal{L}_0^k$ and $H_\infty^k := \Gamma(U, \rho^* H^k)$, where $\rho: U \to S^*$ is a universal covering of S^*. We have an isomorphism $H_\infty^k \to \mathcal{L}^k(0)$ by $u \mapsto \exp(-(\log t/2\pi\sqrt{-1}) \log M)u$. Here M is the local monodromy of H^k, and the eigenvalues of $\log M$ are contained in $[0, 1)$. We regard $\mathcal{L}^k$ as a subsheaf of $j_*(\mathcal{O}_{S^*} \otimes_{\mathbf{C}} H^k)$, where $j: S^* \to S$ is the canonical inclusion.

(2.2) Let $\int_f \mathcal{O}_Y$ be the integration of $\mathcal{O}_Y$ defined in (1.3.b). $\int_f^k \mathcal{O}_Y := \mathcal{H}^k\left(\int_f \mathcal{O}_Y\right)$ is a regular holonomic system with a good filtration $\mathcal{F}^\bullet$ by Lemma (1.4.d). $\int_f^k \mathcal{O}_Y$ is called the *Gauss-Manin system* of f and $\mathcal{F}^\bullet$ is called the *Hodge filtration* on $\int_f^k \mathcal{O}_Y$ (cf. [1] [7] [8]).

The restriction $\mathcal{F}^\bullet|_{S^*}$ to $\int_f^k \mathcal{O}_Y|_{S^*} \cong \mathcal{O}_{S^*} \otimes H^k$ is a variation of Hodge structures in the sense of Griffiths and Schmid (cf. [13]), since $\mathcal{F}^\bullet$ is defined locally with respect to S and f is smooth on S^*.

By a result of Schmid, $\mathcal{F}^\bullet|_{S^*}$ can be extended to S as holomorphic vector subbundles $\hat{\mathcal{F}}^\bullet$ of $\mathcal{L}^k$, i.e., $\hat{\mathcal{F}}^\bullet = \mathcal{L}^k \cap j_*(\mathcal{F}^\bullet|_{S^*})$ (cf. [10] [13]). We shall construct $\mathcal{L}^k$ and $\hat{\mathcal{F}}^\bullet$ geometrically (cf. (2.4) (2.5)).

(2.3) Let $f = p \circ i$ be the canonical decomposition of f, i.e., $i: Y \to Z := Y \times S$ is the canonical inclusion and $p: Z \to S$ is the projection on the second factor.

We can verify that $\int_i \mathcal{O}_Y = \mathcal{B}_{Y|Z}\{1\}[1] \cong i_*(\mathcal{O}_Y[\partial_t])\{1\}[1]$ and $DR_{Z/S}\left(\int_i \mathcal{O}_Y\right) \cong i_*(\Omega_Y^\bullet[\partial_t])\{1\}[1]$ such that $\mathcal{F}^p\left(DR_{Z/S}\left(\int_i \mathcal{O}_Y\right)\right) = \bigoplus_{j \leq \cdot - p - 1} (i_* \Omega_Y^\bullet \partial_t^j)[1]$ and $d = d_Y - \partial_t df \wedge$.

We define a subcomplex $\mathcal{A}^\bullet[1]$ of $DR_{Z/S}\left(\int_i \mathcal{O}_Y\right)$ by $\mathcal{A}^k := \{w \in \Omega_Y^k : df \wedge w = 0\}$. Here we regard $\mathcal{A}^\bullet[1]$ and $DR_{Z/S}\left(\int_i \mathcal{O}_Y\right)$ as complexes of sheaves on $Y \subset Z$; thus i_* is omitted.

The Hodge filtration $\mathcal{F}^\bullet$ on $DR_{Z/S}\left(\int_i \mathcal{O}_Y\right)$ induces a stupid filtration on $\mathcal{A}^\bullet[1]$, i.e., $\mathcal{F}^p(\mathcal{A}^\bullet[1]) = \sigma_{\geq p}(\mathcal{A}^\bullet[1])$.

(2.4) Proposition. *We assume that $f^{-1}(0)$ is a divisor with normal crossings in Y, whose irreducible components are nonsingular. Then we have a canonical isomorphism*

$$\boldsymbol{R}^k f_*(\mathcal{A}^\bullet[1]) \cong \mathcal{L}^k \qquad \textit{for any } k,$$

which is $\mathcal{O}_S$-linear and commutes with the action of $\partial_t t = \nabla_{d/dt} t$. Furthermore, the spectral sequence

$$E_1^{pq}=R^q f_*\mathcal{A}^{p+1} \Longrightarrow R^{p+q}f_*(\mathcal{A}^\cdot[1])$$

degenerates at E_1, *and the induced filtration on* $\boldsymbol{R}^k f_*(\mathcal{A}^\cdot[1])\cong\mathcal{L}^k$ *coincides with the limit Hodge filtration* $\hat{\mathcal{F}}^\cdot=\mathcal{L}^k\cap j_*(\mathcal{F}^\cdot|_{S^*})$ (*cf.* (2.2)).

PROOF. Let $f^{-1}(0)=\cup E_i$ be the decomposition into irreducible components, and set $e:=LCM(\mathrm{mult}_{E_i}f^*t)$. Following Steenbrink [16], we take the normalization of a ramified e-fold covering of Y: i.e., $\pi:\tilde{S}\to S$ is a ramified e-fold covering such that $\pi^*t=\tilde{t}^e$ and $\tilde{Y}$ is the normalization of $Y\times_S\tilde{S}$, where t and $\tilde{t}$ are coordinates on S and $\tilde{S}$.

By assumption, we have $f^*t=x_0^{m_0}\cdots x_r^{m_r}$ locally, where $(x_0,\cdots,x_n)$ is a local coordinate system on Y. Then $w/x_0\cdots x_r$ is a logarithmic form for $w\in\mathcal{A}^{k+1}$, and we have an isomorphism $\mathcal{H}^k(\mathcal{A}^\cdot[1])|_{Y_0}\cong\boldsymbol{C}\{t\}\otimes_{\boldsymbol{C}}R^k\Psi\boldsymbol{C}$ as $\boldsymbol{C}\{t\}$-Modules with the connection ∇. Here ∇ is the canonical connection corresponding to the monodromy on $R^k\Psi\boldsymbol{C}$ such that the eigenvalues of $\mathrm{res}(\nabla t)$ are in $(0,1]$, and $R^k\Psi\boldsymbol{C}$ is the vanishing cohomology sheaf on Y_0, i.e., $R^k\Psi\boldsymbol{C}:=R^k\bar{j}_*\boldsymbol{C}_{Y_\infty}|_{Y_0}$, where $Y_\infty=Y\times_S U$, $Y_0=f^{-1}(0)$, $\bar{j}:Y_\infty\to Y$ is a natural morphism and $U\to S^*$ is a universal covering (cf. [21]).

Then we can obtain the canonical isomorphism $\boldsymbol{R}^k f_*(\mathcal{A}^\cdot[1])\cong\mathcal{L}^k$ for all k, in the same way as in [15].

For the second assertion, it suffices to show that $R^q f_*\mathcal{A}^{p+1}$ is a free $\mathcal{O}_S$-Module, since the spectral sequence degenerates on S^*.

Let $\tilde{\Omega}_{\tilde{Y}}^\cdot$ be the complex of holomorphic forms on $\tilde{Y}$ in the sense of [16]. We set $\tilde{\mathcal{A}}^\cdot:=\{w\in\tilde{\Omega}_{\tilde{Y}}^\cdot : d\tilde{f}\wedge w=0\}$, where $d\tilde{f}=\tilde{f}^*(d\tilde{t})$ and $\tilde{f}:\tilde{Y}\to\tilde{S}$ is the natural morphism.

We have an isomorphism $d\tilde{f}\wedge:\Omega_{\tilde{Y}/\tilde{S}}^\cdot(\log\tilde{Y}_0)\xrightarrow{\sim}\tilde{\mathcal{A}}^\cdot[1]$, which preserves the connections on cohomologies.

Let G be the covering transformation group of $\pi':\tilde{Y}\to Y$. Since $(\pi'_*\tilde{\mathcal{A}}^\cdot[1])^G\cong\mathcal{A}^\cdot[1]$, we can obtain the desired result in the same way as in [16]. Q.E.D.

(2.5) THEOREM. *Let* $f:Y\to S$ *be a one-parameter projective family as in* (2.1). *Then we have a canonical inclusion* $\mathcal{L}^k\hookrightarrow\int_f^k\mathcal{O}_Y$ *for every* k, *which is* $\mathcal{O}_S$-*linear and commutes with the action of* $\partial_t t=\nabla_{d/dt}t$. *Furthermore, we have*

$$\hat{\mathcal{F}}^k(\mathcal{L}^k)=\mathcal{F}^k\Big(\int_f^k\mathcal{O}_Y\Big)\quad\text{and}\quad\hat{\mathcal{F}}^p(\mathcal{L}^k)=\mathcal{L}^k\cap\mathcal{F}^p\Big(\int_f^k\mathcal{O}_Y\Big)$$

for any p *and* k (*cf.* (2.1) *and* (2.2)).

PROOF. The normal crossing case follows obviously from (2.4), except for the assertion that $\hat{\mathcal{F}}^k(\mathcal{L}^k)=\mathcal{F}^k\Big(\int_f^k\mathcal{O}_Y\Big)$. But we have a spectral sequence

$$E_1^{pq}=\boldsymbol{R}^{p+q}f_*\mathrm{Gr}_{\mathcal{F}}^{p}(\Omega_Y^{\cdot}[\partial_t][1])\Longrightarrow\int_f^{p+q}\mathcal{O}_Y$$

such that $E_1^{k,0}=\boldsymbol{R}^0f_*\mathrm{Gr}_{\mathcal{F}}^{k}(\Omega_Y^{\cdot}[\partial_t][k+1])\cong f_*\mathcal{A}^{k+1}$. This implies that $\mathcal{F}^k\left(\int_f^k\mathcal{O}_Y\right)\subset\mathcal{L}^k$ and hence $\hat{\mathcal{F}}^k(\mathcal{L}^k)=\mathcal{F}^k\left(\int_f^k\mathcal{O}_Y\right)$.

The general case is reduced to the normal crossing case as follows.

Let $g:\bar{Y}\to Y$ be a resolution of singularity of $f^{-1}(0)$, i.e., g is a projective morphism of complex manifolds which induces an isomorphism on $g^{-1}(Y-f^{-1}(0))$, and $g^{-1}f^{-1}(0)$ is a divisor with normal crossings in $\bar{Y}$, whose irreducible components are nonsingular.

By Proposition (1.5) and Lemma (1.4.c), we have the direct sum decomposition

$$\int_{fg}^k\mathcal{O}_{\bar{Y}}=\int_f^k\mathcal{O}_Y\oplus(\oplus\mathcal{B}_{\{0\}|S})$$

as filtered Modules. $\mathcal{L}^k\left(\subset\int_{fg}^k\mathcal{O}_{\bar{Y}}\right)$ is contained in the first component, because there is no connection preserving $\mathcal{O}_S$-linear map of $\mathcal{L}^k$ to $\mathcal{B}_{\{0\}|S}$ except the zero map. Thus we obtain the desired result, since $\mathcal{F}^{\cdot}$ is compatible with the decomposition. Q.E.D.

§ 3. Application

(3.1) Let $f:\boldsymbol{C}^{n+1},\ 0\to\boldsymbol{C},\ 0$ be a holomorphic function with an isolated singularity. We may assume that f is a polynomial and $\overline{f^{-1}(0)}\subset\boldsymbol{P}^{n+1}$ is smooth away from the origin. Here we can take the degree of f sufficiently large.

We set $Y:=\overline{\{f(x)=t\}}\subset\boldsymbol{P}^{n+1}\times S$, $S:=\{|t|<\eta\}$, $X:=Y\cap(B_\varepsilon\times S)$, $B_\varepsilon:=\{\|x\|<\varepsilon\}$, $Z:=\boldsymbol{P}^{n+1}\times S$, $p:Z\to S$ is the natural projection, $\bar{f}:=p|_Y$ and $f=\bar{f}|_X$.

For $1\gg\varepsilon\gg\eta>0$, $f:X\to S$ is a Milnor fibration and $\bar{f}:Y\to S$ is smooth away from the origin.

We set $Y_\infty:=Y\times_S U$, $X_\infty:=X\times_S U$, $H_{Y,\infty}:=\Gamma(U,\rho^*H_Y)$ and $H_{X,\infty}:=\Gamma(U,\rho^*H_X)$, where $\rho:U\to S^*$ is a universal covering. We have natural isomorphisms $H_{Y,\infty}\cong H^n(Y_\infty,\boldsymbol{C})$ and $H_{X,\infty}\cong H^n(X_\infty,\boldsymbol{C})$.

(3.2) Proposition (Schmid [13], Steenbrink [16], Scherk [11]).

a) *$H_{Y,\infty}$ and $H_{X,\infty}$ have canonical mixed Hodge structures with the Hodge filtration $F_S^{\cdot}$ and $F_{St}^{\cdot}$ such that the natural inclusion $i:X\to Y$ induces a morphism of mixed Hodge structures $i^*:H_{Y,\infty}\to H_{X,\infty}$ (in particular, $i^*F_S^{\cdot}=F_{St}^{\cdot}\cap i^*H_{Y,\infty}$)* [13] [16].

b) *If the degree of the polynomial f is sufficiently large, $i^*:H_{Y,\infty}\to H_{X,\infty}$ is surjective (hence $i^*F_S^{\cdot}=F_{St}^{\cdot}$)* [11].

(3.3) PROPOSITION ([7] [8] [9] [11]). *Let* $\mathcal{H}_Y := \int_{\bar{f}}^n \mathcal{O}_Y$ *and* $\mathcal{H}_X := \int_f^n \mathcal{O}_X$ *be the Gauss-Manin systems with the Hodge filtration* $\mathcal{F}^{\bullet}$.

a) $\mathcal{H}_Y$ *and* $\mathcal{H}_X$ *are regular holonomic systems on* S *such that* $\mathcal{H}^0(DR_S(\mathcal{H}_Y)) \cong R^n\bar{f}_*\boldsymbol{C}_Y$ *and* $DR_S(\mathcal{H}_X) \cong R^n f_*\boldsymbol{C}_X$.

b) *If the degree of* f *is sufficiently large,* $i^* : \mathcal{H}_Y \to \mathcal{H}_X$ *is surjective and strictly compatible with the Hodge filtration* $\mathcal{F}^{\bullet}$ *(i.e.,* $i^*\mathcal{F}^{\bullet}(\mathcal{H}_Y) = \mathcal{F}^{\bullet}(\mathcal{H}_X)$*). Furthermore,* Ker $(i^* : \mathcal{H}_Y \to \mathcal{H}_X)$ *is a free* $\mathcal{O}_S$*-Module of finite rank, and we have* $DR_S(\mathcal{H}_Y) \cong R^n\bar{f}_*\boldsymbol{C}_Y$.

(3.4) We assume from now on that the degree of f is sufficiently large so that Propositions (3.2.b) and (3.3.b) hold.

Let $H_{Y,\infty} = \bigoplus_{\lambda} H_{Y,\infty,\lambda}$ be the monodromy decomposition (i.e., $H_{Y,\infty,\lambda} := \{u \in H_{Y,\infty} : (M-\lambda)^{n+1}u = 0\}$), and let $\{\lambda_j = \exp(-2\pi\sqrt{-1}\,\alpha_j)\}_j$ be the eigenvalues of the monodromy M. We may assume that $\alpha_j \in (-1, 0]$ for all j and $\alpha_i < \alpha_j$ for $i < j$.

We define a decreasing filtration $V^{\bullet}$ on $H_{Y,\infty}$ by $V^j H_{Y,\infty} := \bigoplus_{i \geq j} H_{Y,\infty,\lambda_i}$.

Let $F^{\bullet}$ be a filtration on $H_{Y,\infty}$. We define a filtration $\mathrm{Gr}_V F^{\bullet}$ on $\mathrm{Gr}_V H_{Y,\infty}$ by $\mathrm{Gr}_V F^p := \bigoplus_j (F^p \cap V^j / F^p \cap V^{j+1})$. We can regard $\mathrm{Gr}_V F^{\bullet}$ as a filtration on $H_{Y,\infty}$ by the natural isomorphism $\mathrm{Gr}_V H_{Y,\infty} \cong H_{Y,\infty}$.

We define $\mathrm{Gr}_V F^{\bullet}$ similarly for a filtration $F^{\bullet}$ on $H_{X,\infty}$.

Let $\mathcal{L}_Y$ and $\mathcal{L}_X$ be the canonical extensions of H_Y and H_X such that $H_{Y,\infty} \cong \mathcal{L}_Y(0)$ and $H_{X,\infty} \cong \mathcal{L}_X(0)$ (cf. (2.1)). $\mathcal{L}_Y$ and $\mathcal{L}_X$ are naturally contained in $\mathcal{H}_Y$ and $\mathcal{H}_X$ respectively so that $(i^*)^{-1}(\mathcal{L}_X) = \mathcal{L}_Y$ (cf. (2.5) (3.3)).

We define the filtration $F_Y^{\bullet}$ (resp. $F_X^{\bullet}$) on $H_{Y,\infty}$ (resp. $H_{X,\infty}$) by

$$F_Y^{\bullet} := \mathrm{Im}\,(\mathcal{F}^{\bullet}(\mathcal{H}_Y) \cap \mathcal{L}_Y \longrightarrow \mathcal{L}_Y(0) \cong H_{Y,\infty})$$

$$(\text{resp. } F_X^{\bullet} := \mathrm{Im}\,(\mathcal{F}^{\bullet}(\mathcal{H}_X) \cap \mathcal{L}_X \longrightarrow \mathcal{L}_X(0) \cong H_{X,\infty}))\,,$$

where $\mathcal{F}^{\bullet}$ is the Hodge filtration on $\mathcal{H}_Y = \int_{\bar{f}}^n \mathcal{O}_Y$ $\left(\text{resp. } \mathcal{H}_X = \int_f^n \mathcal{O}_X\right)$. Then we can define $\mathrm{Gr}_V F_Y^{\bullet}$ and $\mathrm{Gr}_V F_X^{\bullet}$ as above.

(3.5) THEOREM. a) $\mathrm{Gr}_V F_Y^{\bullet} = F_S^{\bullet}$ (*cf.* (3.2)).
b) $\mathrm{Gr}_V F_X^{\bullet} = F_{St}^{\bullet}$ (*cf.* [8] [9] [14] [19]).

The first identity follows from Theorem (2.5) and the construction of Schmid (cf. [10] [13]). Then the second identity is obvious by Propositions (3.2) and (3.3).

REMARKS. 1) The first assertion is valid for any one-parameter projective family as in (2.1).

2) $\mathcal{F}^n(\mathcal{H}_{X,0})$ is isomorphic to $\mathcal{H}^{(0)}_{X,0} := \Omega^{n+1}_{X,0}/df\wedge d\Omega^{n-1}_{X,0}$ as an $\mathcal{O}_{S,0}$-module with a regular singular connection, so that $\mathcal{F}^{n-p}(\mathcal{H}_{X,0}) = \partial_t^p \mathcal{H}^{(0)}_{X,0}$ for $p \geqq 0$.

In [17], A. Varchenko defined the asymptotic Hodge filtration $F^{\bullet}_a$ on $H^n(X_\infty, \boldsymbol{C})$ using the asymptotic expansion of period integrals. $F^{\bullet}_a$ and $F^{\bullet}_{St}$ are different in general [12] [9, (3.5.2)], but they coincide for a plane curve singularity or a quasi-homogeneous singularity [17] [18]. Moreover, we have $\mathrm{Gr}^W F^{\bullet}_a = \mathrm{Gr}^W F^{\bullet}_{St}$ [19], where $W.$ is the monodromy weight filtration (cf. [16]), hence $F^{\bullet}_a$ and $W.$ form a mixed Hodge structure on $H^n(X_\infty, \boldsymbol{C})$.

In [14], J. Scherk and J. Steenbrink claimed that $F^{\bullet}_{St}$ is determined by $\mathcal{H}^{(0)}_{X,0}$, and F. Pham asserted that it is essential to use the Gauss-Manin system $\mathcal{H}_X = \int_f^n \mathcal{O}_X$ instead of $\mathcal{H}_X \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_S[t^{-1}]$ (cf. [8]). It is shown in [9] [10] that the unipotent base change (or equivalently, the graduation Gr_V (cf. (3.4))) is necessary in the formulation [9, (3.1)] (or (3.5.b)).

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