On Atiyah-Patodi-Singer η-invariant for S^1-bundles over Riemann surfaces

By Mutsumi KOMURO

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Introduction

In [2] Atiyah, Patodi and Singer defined an invariant $\eta(M)$ for a $(4k-1)$-dimensional closed oriented Riemannian manifold (M, g). When M bounds a compact manifold N, they proved the identity

$$\eta(M)=\int_N L(p)-\mathrm{Sign}\, N$$

where L is the Hirzebruch L-polynomial and p is the Pontrjagin form given by a metric $\tilde{g}$ on N which extends g and is product near the boundary M.

The behaviour of η for a finite Riemannian covering is well-known. But nothing about the behaviour of η for general fibre bundles is known. The main purpose of this paper is to give a formula of η for a circle bundle over an oriented closed Riemann surface (Theorem 4-1).

We shall explicitly compute the η-invariants of certain 3-manifolds. In particular, we get a formula of η-invariant for 3-dimensional complete intersections

$$X_a=\bigcap_{i=1}^{n-1} f_i^{-1}(0)\cap S^{2n+1},$$

where f_i is a homogeneous polynomial of degree a_i. We obtain

$$\eta(X_a)=\frac{1}{3}\left\{2\sum_{i=1}^{n-1} a_i-(2n+1)\right\}a_1a_2\cdots a_{n-1}+1\,.$$

In the case of a lens space, equating the η-invariants as computed by our method and by the method of Atiyah-Patodi-Singer (using finite coverings), we have an interesting formula of Dedekind sums;

$$\sum_{l=1}^{k}\cot^2\frac{l\pi}{k}=\frac{1}{3}(k-1)(k-2)\,.$$

The η-invariant is known to be closely related to the invariant of Chern and Simons. This is discussed in § 5, and we get a formula for Chern-Simons invariant of an S^1-bundle over a Riemann surface.

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§ 1. Definitions and some preliminary results

Let (X^{4k-1}, g) be a compact oriented Riemannian manifold. Consider the operator A on $\bigoplus_{p=1}^{2k} \Omega^{2p-1}$, the differential forms of odd degree, defined on Ω^{2p-1} by

$$A=(-1)^{k+p}(*d+d*),$$

where $*$ is the usual star operator defined by the Riemannian metric g. It can be easily verified that

i) A is self-adjoint,

ii) A^2 coincides with the Hodge Laplacian, Δ.

From i) and ii) it follows that A is diagonizable with real eigenvalues $\{\lambda_n\}$.

Definition 1-1 (Atiyah-Patodi-Singer). We define a function $\eta(s)$ by $\eta(s) = \sum_{\lambda_n \neq 0} \operatorname{sign} \lambda_n |\lambda_n|^{-s}$ which converges absolutely for Re s large and extends to a meromorphic function on the whole s-plane with a finite value at $s=0$. (For the further details see [2].) This finite value $\eta(0)$ is called the η-invariant of (X, g), and denoted by $\eta(X)$.

We now assume that (X, g) bounds a $4k$ dimensional Riemannian manifold $(Y, \tilde{g})$ in such a manner that $(Y, \tilde{g})$ is product near the boundary $\partial Y = X$. That is, in some collar neighbourhood $X \times I$, $(Y, \tilde{g})$ is isometric to the Riemannian product $(X \times I, g \times ds^2)$ where ds^2 is the standard flat metric of the interval I. In this situation using the local signature theorem of [1], Atiyah-Patodi-Singer proved the following

Theorem 1-2. $$\eta(X)=\int_Y L_k(p) - \operatorname{Sign} Y,$$

where Sign Y *is the signature of* Y, L_k *is the k-th Hirzebruch L-polynomial, and* p *is the Pontrjagin form defined by the Riemannian metric* $\tilde{g}$.

Note that $\eta(X)$ vanishes when (X, g) has an orientation reversing isometry φ. In fact, A changes its sign under φ.

We want to calculate the η-invariant for a principal S^1-bundle over a closed Riemann surface,

$$S(E) \longrightarrow M,$$

where $\pi: E \to M$ is the associated $\boldsymbol{C}^1$-bundle. To do this, we choose a fibre metric h on E, an h-preserving connection ∇ of E and a Riemannian metric $\hat{g}$ on M. Then the total space E becomes a Riemannian manifold $(E, \tilde{g})$ by assuming that

$$\tilde{g}\,|(\text{vertical spaces})=h$$

and

$$\tilde{g}\,|(\text{horizontal spaces})=\pi^*\hat{g}\,.$$

Take any point x of E and choose an orthonormal basis $\{e_I\}_{1\leq I\leq 4}$ of T_xE so that e_1, e_2 are vertical and e_3 and e_4 are horizontal. Let $f_i=\pi_*e_i$ $(i=3, 4)$. Then by the definition of $\tilde{g}$, $\{f_i\}_{i=3,4}$ is an orthonormal basis of $T_{\pi(x)}M$ with respect to $\hat{g}$. We may assume that $\{e_1, e_2\}$ is positive with respect to the given orientation of E, and that $\{f_3, f_4\}$ is positive with respect to the given orientation of M. Choose local orthonormal sections of E with respect to h. Then they naturally define a (global) coordinate system $\{u^1, u^2\}$ along the fibre of E. We may assume that $\partial/\partial u^\alpha=e_\alpha$ $(\alpha=1, 2)$. Define $\gamma_{i\mu}^\lambda$ $(i=3, 4,\ \lambda, \mu=1, 2)$ by

$$\nabla_{f_i}\partial/\partial u^\mu=\gamma_{i\mu}^\lambda\partial/\partial u^\lambda\,,$$

then it is easy to see that e_i and f_i $(i=1, 2)$ are related as follows. (We use the Greek indices as the vertical components and the Latin indices as the horizontal ones.)

LEMMA 1-3. $$e_i=f_i-\sum_{\alpha,\lambda}\gamma_{i\lambda}^\alpha u^\lambda\partial/\partial u^\alpha\,.$$

Let $\hat{\Omega}$ and $\tilde{\Omega}$ be the curvature forms of the Riemannian connections determined by $\hat{g}$ and $\tilde{g}$ respectively. Denote by $\hat{\Omega}_j^i$ and $\tilde{\Omega}_J^I$ their components with respect to $\{f_i\}_{i=3,4}$ and $\{e_I\}_{1\leq I\leq 4}$ respectively. Let R be the curvature tensor of the connection

$$\nabla:\Gamma(E)\longrightarrow\Gamma(T^*M\otimes E)\,.$$

Denote by $R_{\alpha\beta kl}$ $(1\leqq\alpha,\ \beta\leqq 2,\ 3\leqq k,\ l\leqq 4)$ its component with respect to $\{e_1, e_2\}$ and $\{f_3, f_4\}$, i.e.,

$$R_{\alpha\beta kl}=h(e_\alpha, ([\nabla_{f_k}, \nabla_{f_l}]-\nabla_{[f_k, f_l]})e_\beta)\,.$$

Let Ω be the corresponding curvature form and Ω_2^1 be its component with respect to $\{e_1, e_2\}$, i.e.,

$$\Omega_2^1=R_{1234}f_3^*\wedge f_4^*$$

where $\{f_i^*\}_{i=3,4}$ is the dual basis of $\{f_i\}_{i=3,4}$.

In this paper we will only consider a complex line bundle over a closed oriented Riemann surface. Hence, up to sign, each Ω, $\hat{\Omega}$ and R has only one non-zero component, that is Ω_2^1, $\hat{\Omega}_4^3$ and R_{1234}. For the brevity's sake we will simply write these components as Ω, $\hat{\Omega}$ and R respectively;

$$\Omega\equiv\Omega_2^1\,,\quad \hat{\Omega}\equiv\hat{\Omega}_4^3\,,\quad R\equiv R_{1234}\,.$$

$\tilde{\Omega}$ is described by the following

PROPOSITION 1-4.

(1-5) $$\tilde{\Omega}_2^1=\pi^*\Omega\,,$$

(1-6) $$\tilde{\Omega}_\alpha^j=\frac{1}{2}(-1)^\alpha\nabla_{f_j}\Omega u^\beta+\frac{1}{2}(-1)^{\alpha+j}Re_k^*\wedge\widetilde{du}^\beta$$
$$+\frac{1}{4}(-1)^{\alpha+1}R^2u^\beta e_j^*(u^2\widetilde{du}^1-u^1\widetilde{du}^2)\,,$$

(1-7) $$\tilde{\Omega}_4^3=\pi^*\hat{\Omega}-\frac{3}{4}R^2((u^1)^2+(u^2)^2)\pi^*\omega_M$$
$$+\frac{1}{2}\pi^*(*\delta^\nabla\Omega)\wedge(u^2\widetilde{du}^1-u^1\widetilde{du}^2)+R\widetilde{du}^1\wedge\widetilde{du}^2\,,$$

where $\{u^1, u^2\}$ *is a coordinate system along the fibre so that* $\{\partial/\partial u^1, \partial/\partial u^2\}=\{e_1, e_2\}$ *at* x, $\{\widetilde{du}^\alpha, e_i^*\}_{\alpha=1,2\, i=3,4}$ *is the dual basis of* $\{\partial/\partial u^\alpha, e_i\}_{\alpha=1,2\, i=3,4}=\{e_I\}_{1\le I\le 4}$, *and*

$$\delta^\nabla:\Gamma(\Lambda^2T^*M\otimes E\otimes E^*)\longrightarrow\Gamma(\Lambda^1T^*M\otimes E\otimes E^*)$$

is the formal adjoint of the covariant differential

$$d^\nabla:\Gamma(\Lambda^1T^*M\otimes E\otimes E^*)\longrightarrow\Gamma(\Lambda^2T^*M\otimes E\otimes E^*)$$

defined by the natural inner product induced by $\hat{g}$ *and* h. $*$ *is the tensor product of the star operator on* Λ^*T^*M *(defined by* $\hat{g}$*) and the identity map on* $E\otimes E^*$. *In* (1-6) $\alpha=1$ *or* 2, $j=3$ *or* 4 *and* $\beta=3-\alpha$, $k=7-j$. *In* (1-7) ω_M *is the volume form of* $(M, \hat{g})$, *i.e.*,

$$\omega_M=f_3^*\wedge f_4^*\,.$$

REMARK 1-8. i) By virtue of Lemma 1-3 e_i^* and $\widetilde{du}^\alpha$ ($i=3, 4$, $\alpha=1, 2$) are given by

(1-9) $$e_i^*=\pi^*f_i^*\,,$$

(1-10) $$\widetilde{du}^\alpha=du^\alpha+\sum_{\lambda i}\gamma_{i\lambda}^\alpha e_i^*u^\lambda\,.$$

ii) On $\Gamma(\Lambda^p\otimes E\otimes E^*)$ we have the relation

$$(-1)^{p+1}d^\nabla*=*\delta^\nabla\,.$$

If we use the polar coordinate system $\{r, \theta\}$, i.e., $r^2=(u^1)^2+(u^2)^2$, $\tan\theta=u^2/u^1$, then Proposition 1-4 is rewritten in the following form:

PROPOSITION 1-11. *The components of* $\tilde{\Omega}$ *with respect to the basis* $\{(1/r)(\partial/\partial\theta), \partial/\partial r, e_3, e_4\}$ *are given by the following*:

(1-12) $$\tilde{\Omega}_\theta^r=\pi^*\Omega\,,$$

$$\tilde{\Omega}^j_r=(-1)^{j+1}\frac{1}{2}Re^*_k\wedge r\widetilde{d\theta}\,, \tag{1-13}$$

$$\tilde{\Omega}^j_\theta=(-1)^j\frac{1}{2}Re^*_k\wedge dr+\frac{1}{2}(\nabla_{f_j})R\pi^*\omega_M+\frac{1}{4}r^2R^2e^*_j\wedge r\widetilde{d\theta}\,, \tag{1-14}$$

$$\tilde{\Omega}^3_4=\pi^*\hat{\Omega}-\frac{3}{4}r^2R^2\pi^*\omega_M+Rdr\wedge r\widetilde{d\theta}-\frac{1}{2}r^2\pi^*(*\delta^\nabla\Omega)\wedge\widetilde{d\theta}\,, \tag{1-15}$$

where the subscripts r and θ mean taking the components with respect to $\partial/\partial r$ and $(1/r)(\partial/\partial\theta)$ respectively, and $\{r\widetilde{d\theta}, dr, e^*_3, e^*_4\}$ is the dual basis of $\{(1/r)(\partial/\partial\theta), \partial/\partial r, e_3, e_4\}$. $j=3$ or 4 and $k=7-j$.

REMARK 1-16. Making use of (1-10), it is easy to see that $\widetilde{d\theta}$ is given by

$$\widetilde{d\theta}=d\theta-\sum_i\gamma^1_{i2}e^*_i\,.$$

For the use of later sections we describe here the connection form of $\tilde{g}$. In the same notation as in Proposition 1-11 we have the following

PROPOSITION 1-17. *The components of the connection form α of the Riemannian connection determined by $\tilde{g}$ are given by the following*:

$$\alpha^r_\theta=-\widetilde{d\theta}\,, \tag{1-18}$$

$$\alpha^r_j=0\,, \tag{1-19}$$

$$\alpha^\theta_j=(-1)^j\frac{1}{2}rRe^*_k\,, \tag{1-20}$$

$$\alpha^3_4=\pi^*\hat{\alpha}^3_4+\frac{1}{2}r^2R\widetilde{d\theta}\,, \tag{1-21}$$

where $\hat{\alpha}^3_4$ is the component of the connection form of $\hat{g}$ with respect to $\{f_3, f_4\}$, $j=3$ or 4 and $k=7-j$.

To emphasize the dependence on the radius r we will write the above α as $\alpha(r)$ also.

Here we sketch the proofs of Propositions 1-7, 1-11 and 1-17. In fact we can prove more general formulae. Namely let $E\to M$ be a vector bundle over a Riemannian manifold $(M, \hat{g})$ (without any assumption on $\dim M$ or $\operatorname{rank} E$). We assume that E has a fibre metric h and an h-preserving connection ∇. Then the total space E can be equipped with a Riemannian structure $(E, \tilde{g})$ just as in § 1. First we determine the Riemannian connection $\tilde{\nabla}$ of $\tilde{g}$. Let $\{f_i\}$ be a local orthonormal frame of M and $\{e_i\}$ be its horizontal lift with respect to ∇. Let $\{u^\alpha\}$ be a coordinate system along the fibre of E determined by local orthonormal sections with respect to h.

Let $\tilde{\theta}$ be the connection form of $\tilde{\nabla}$. Let $\tilde{\theta}^i_j$, $\tilde{\theta}^i_\alpha$ and $\tilde{\theta}^\alpha_\beta$ be its components with respect to the frame $\{\partial/\partial u^\alpha, f_i\}$. (We use the Greek indices as the vertical components and the Latin indices as the horizontal ones.) Then by the definition of $\tilde{\theta}$,

$$\tilde{\theta}^\alpha_\beta=\sum_i \tilde{g}(\partial/\partial u^\alpha,\ \tilde{\nabla}_{e_i}\partial/\partial u^\beta)e^*_i+\sum_\lambda \tilde{g}(\partial/\partial u^\alpha,\ \tilde{\nabla}_{\partial/\partial u^\lambda}\partial/\partial u^\beta)\widetilde{du}^\lambda\,,$$

where $\{e^*_i,\ \widetilde{du}^\lambda\}$ is the dual basis of $\{e_i,\ \partial/\partial u^\lambda\}$.

Using the identity

$$2\tilde{g}(X,\ \tilde{\nabla}_Z Y)=Z\tilde{g}(X,\ Y)+\tilde{g}(Z,\ [X,\ Y])+Y\tilde{g}(X,\ Z)+\tilde{g}(Y,\ [X,\ Z])$$
$$-X\tilde{g}(Y,\ Z)-\tilde{g}(X,\ [Y,\ Z])\,,$$

which holds for any three vector fields X, Y and Z, we can easily see that

$$\tilde{g}(\partial/\partial u^\alpha,\ \tilde{\nabla}_{e_i}\partial/\partial u^\beta)=h(\partial/\partial u^\alpha,\ \nabla_{f_i}\partial/\partial u^\beta)$$

and

$$\tilde{g}(\partial/\partial u^\alpha,\ \tilde{\nabla}_{\partial/\partial u^\lambda}\partial/\partial u^\beta)=0\,.$$

Hence we have $\tilde{\theta}^\alpha_\beta=\pi^*\theta^\alpha_\beta$, where θ is the connection form of ∇. We can determine the other components $\tilde{\theta}^i_j$ and $\tilde{\theta}^i_\alpha$ similarly, and we have the following proposition.

PROPOSITION 1-22. *The components of the connection form $\tilde{\theta}$ of $\tilde{\nabla}$ with respect to the frame $\{\partial/\partial u^\alpha, e_i\}$ are given by*

$$\tilde{\theta}^i_j=\pi^*\hat{\theta}^i_j+\sum_{\lambda,\alpha}\frac{1}{2}R_{\alpha\lambda ji}u^\lambda\widetilde{du}^\alpha\,,$$

$$\tilde{\theta}^\alpha_j=\sum_{\lambda,k}\frac{1}{2}R_{\alpha\lambda jk}e^*_k u^\lambda\,,$$

$$\tilde{\theta}^\alpha_\beta=\pi^*\theta^\alpha_\beta\,,$$

where $\hat{\theta}$ and θ are the connection forms of $\hat{g}$ and ∇ respectively.

Proposition 1-17 is easily deduced from this proposition.

Next a straightforward calculation using Proposition 1-22 will show the following.

PROPOSITION 1-23. *The components of the curvature form $\tilde{\Omega}$ of $\tilde{\nabla}$ with respect to the frame $\{\partial/\partial u^\alpha, e_i\}$ are given by*

$$\tilde{\Omega}^\alpha_\beta=\pi^*\Omega^\alpha_\beta-\frac{1}{4}\sum_{\lambda,\mu,j,k,l}R_{\alpha\lambda jk}R_{\beta\mu jl}u^\lambda u^\mu e^*_k\wedge e^*_l\,,$$

$$\tilde{\Omega}^j_\alpha=-\frac{1}{4}\sum_{k,i}(\nabla_{f_j}R)_{\alpha\lambda ik}u^\lambda e^*_i\wedge e^*_k$$

$$+\frac{1}{4}\sum_{\beta,l}(2R_{\alpha\beta jl}+\sum_{k,\lambda,\mu}R_{\alpha\mu kl}R_{\beta\lambda kj}u^{\lambda}u^{\mu})e_l^*\wedge\widetilde{du}^{\beta},$$

$$\tilde{\Omega}_j^i=\frac{1}{4}\sum_{k,l}(2K_{ijkl}-\sum_{\alpha,\lambda,\mu}R_{\alpha\lambda ij}R_{\alpha\mu kl}u^{\lambda}u^{\mu}+\sum_{\alpha,\lambda,\mu}R_{\alpha\lambda jk}R_{\alpha\mu il}u^{\lambda}u^{\mu})e_k^*\wedge e_l^*$$

$$+\frac{1}{2}\sum_{\alpha,\lambda,k}(\nabla_{f_k}R)_{\alpha\lambda ji}u^{\lambda}e_k^*\wedge\widetilde{du}^{\alpha}$$

$$+\frac{1}{4}\sum_{\alpha,\beta}(2R_{\alpha\beta ij}+\sum_{\lambda,\mu,k}R_{\alpha\lambda jk}R_{\beta\mu ik}u^{\lambda}u^{\mu})\widetilde{du}^{\alpha}\wedge\widetilde{du}^{\beta},$$

where K is the curvature tensor of the Riemannian connection defined by $\hat{g}$, and ∇R is the covariant derivative of the tensor $R\in\Gamma(T^*M\otimes T^*M\otimes E^*\otimes E^*)$ by the tensor product connection.

Proposition 1-4 is a special case of this proposition.

REMARK 1-24. i) In Proposition 1-23 it is to be noted that $\tilde{\Omega}$ is written down by R and K whenever $\nabla R=0$ occurs.

ii) Proposition 1-23 is considered to be a refinement of Theorems $1\sim3$ of [15] for the case of a vector bundle $E\to M$.

iii) An h-preserving connection ∇ on a complex line bundle $E\to M$ can be regarded as a connection on a principal S^1-bundle. In this sense Proposition 1-11 and 1-17 can be considered as generalizations of Propositions 1 and 2 of [9].

§2. Vanishing of ∇R

The aim of this paper is to study the η-invariant for a circle bundle over a Riemann surface. To do this we will write down the L-genus by curvature form and integrate it on the associated disc bundle. Although it may be very complicated, the same program can be applied to a general dimensional sphere bundle $S(E)\to M$ (associated with a vector bundle $E\to M$).

Since the connection form itself has not much geometrical meaning but the curvature form does, it may not too restrictive only to consider the case when L-genus is written down by R and K alone. By Remark 1-24 i), this actually occurs if $\nabla R=0$ holds.

REMARK 2-1. Although the term $\widetilde{du}^{\alpha}$ also contains the connection form, it causes no difficulty. In fact if one writes down the L-genus by forms, then the only part that contributes to the integration is of the form

$$\text{(a function on } D(E))\cdot\text{(volume form of } D(E))$$

$$=\pm\text{(a function on } D(E))du^1\wedge\cdots\wedge du^r\wedge\text{(volume form of } M)$$

where $r=\operatorname{rank}(E)$ and $D(E)$ is the associated disc bundle. But du^α appears only in the form of $\widetilde{du^\alpha}$ in Proposition 1-23.

In this section we shall give some sufficient conditions of vanishing of ∇R. We begin with an example in which $\nabla R=0$ holds naturally.

Example 2-2. Let $\gamma\to \boldsymbol{C}P^n$ be the Hopf bundle over the complex projective space. Its associated $\boldsymbol{C}^*$-bundle is the standard fibration

$$\boldsymbol{C}^{n+1}-\{0\} \longrightarrow \boldsymbol{C}P^n=(\boldsymbol{C}^{n+1}-\{0\})/\boldsymbol{C}^* .$$

Identifying

$$\gamma-(\text{zero cross-section})=\boldsymbol{C}^{n+1}-\{0\} ,$$

there is a Hermitian (fibre-)metric h defined as follows. For any local holomorphic section s and $z\in \boldsymbol{C}P^n$ we define

$$h(s, \bar{s})_z=s(z)\,{}^t\overline{s(z)}$$

where the right hand side is the standard Hermitian metric of $\boldsymbol{C}^{n+1}$. We will refer to h as the canonical Hermitian metric of the Hopf bundle. Let $(w^0 : w^1 : \cdots : w^n)$ be the homogeneous coordinates of $\boldsymbol{C}P^n$. And set $z^i=w^i/w^0$ on $U_0=\{(w^0:\cdots:w^n)\,;\,w^0\neq 0\}$. We take $((z^1, \cdots, z^n), U_0)$ as a local coordinate system. On U_0 we can take a local holomorphic section s of γ given by

$$s(z^1, \cdots, z^n)=(1, z^1, \cdots, z^n)\in \boldsymbol{C}^{n+1}-\{0\}$$

so that $h(s, \bar{s})=1+|z|^2$, where $|z|^2=\sum_{i=1}^{n}|z_i|^2$. For the sake of simplicity, the real valued function $1+|z|^2$ will be also denoted by h; $h=1+|z|^2$. Then the curvature tensor R of the Hermitian connection determined by h is given by

$$\begin{aligned} R^1_{1i\bar{j}}&=-h^{-1}\frac{\partial^2 h}{\partial z_i\partial\bar{z}_j}+h^{-2}\frac{\partial h}{\partial z_i}\frac{\partial h}{\partial \bar{z}_j} \\ &=-\frac{\delta_{ij}(1+|z|^2)-\bar{z}_i z_j}{(1+|z|^2)^2} \end{aligned}$$

where $R^1_{1i\bar{j}}$ is the component of R with respect to the basis s and $\{\partial/\partial z_i\}$. But up to sign this last expression agrees with that of the so-called Fubini-Study metric $\hat{g}$ on $\boldsymbol{C}P^n$; $R^1_{1i\bar{j}}=-\hat{g}_{i\bar{j}}$. (See [6] or [10].) From this relation, $\nabla R=0$ follows clearly.

Hence we have proved the first statement of the following

PROPOSITION 2-3. *Let $(\boldsymbol{C}P^n, \hat{g})$ be the complex projective space with the Fubini-Study metric. Let $\gamma\to \boldsymbol{C}P^n$ be the Hopf bundle with the canonical Hermitian metric h. Let ∇ be the Hermitian connection defined by h. Then the condition $\nabla R=0$ holds. Furthermore if we restrict the metric $\tilde{g}$ of γ (constructed from*

$\hat{g}$, h and ∇ as before) on $S_1(\gamma)$, it coincides with the standard metric of the unit sphere $S^{2n+1}=S_1(\gamma)$, where $S_1(\gamma)$ is the unit circle bundle associated with γ.

PROOF. We use the same notation as above. Let u be the coordinate of the fibre direction i.e., the points of $\gamma|_{U_0}$ are written in the form,

$$X=(u,\ uz_1,\ \cdots,\ uz_n).$$

Note that we can regard X as an element in $\boldsymbol{C}^{n+1}-\{0\}$ if $u\neq 0$.

Hence we have

$$\frac{\partial X}{\partial u}=(1,\ z_1,\ z_2,\ \cdots,\ z_n),$$

$$\frac{\partial X}{\partial z_i}=(0,\ \cdots,\ 0,\ u,\ 0,\ \cdots,\ 0).$$

(In the second equation u is in the $(i+1)$-st place.) It follows that

$$\frac{\partial X}{\partial u}\frac{{}^t\overline{\partial X}}{\partial u}=1+|z|^2,$$

$$\frac{\partial X}{\partial z_j}\frac{{}^t\overline{\partial X}}{\partial z_i}=|u|^2\delta_{ij},$$

$$\frac{\partial X}{\partial z_i}\frac{{}^t\overline{\partial X}}{\partial u}=\bar{z}_i u.$$

So the standard Hermitian metric ds^2 of $\boldsymbol{C}^{n+1}-\{0\}$ can be described as follows;

$$ds^2=(1+|z|^2)du\overline{du}+\sum_i \bar{z}_i u\overline{du}dz_i+\sum_i z_i\bar{u}du\overline{dz_i}+\sum_i |u|^2 dz_i\overline{dz_i}.$$

Define Γ_i by

$$\nabla_{\partial/\partial z_i}\partial/\partial u=\Gamma_i\partial/\partial u,$$

then by Lemma 1-3 the horizontal lift $\widetilde{\partial/\partial z_i}$ of $\partial/\partial z_i$ is given by

$$\widetilde{\partial/\partial z_i}=\partial/\partial z_i-\sum_i \Gamma_i u\partial/\partial u.$$

Define $\widetilde{du}$ by

$$\widetilde{du}=du+\sum_i \Gamma_i u dz_i,$$

then $\{dz_i,\ \widetilde{du}\}$ is the dual basis (over $\boldsymbol{C}$) of $\{\widetilde{\partial/\partial z_i},\ \partial/\partial u\}$. (See Remark 1-8.)

It follows that $\tilde{g}$ is given by

$$\begin{aligned} ds'^2 &= h\widetilde{du}\overline{\widetilde{du}}+\sum_{i,j}\hat{g}_{i\bar{j}}dz_i\overline{dz_j} \\ &= hdu\overline{du}+\sum_i h\overline{\Gamma_i}\bar{u}\overline{dz_i}du+\sum_i h\Gamma_i u\overline{du}dz_i \end{aligned}$$

$$+\sum_i \frac{dz_i\overline{dz_i}}{1+|z|^2}+\sum_{i,j}\Big(h|u|^2\Gamma_i\bar{\Gamma}_j-\frac{1}{(1+|z|^2)^2}\bar{z}_i z_j\Big)dz_i\overline{dz_j}\,.$$

Since ∇ is the Hermitian connection defined by $h=1+|z|^2$, Γ_i is given by

$$\Gamma_i=h^{-1}\frac{\partial h}{\partial z_i}=\frac{\bar{z}_i}{1+|z|^2}\,. \tag{2-4}$$

(See p. 183 of [10].) On $S_1(\gamma)$, we have $h|u|^2=1$, or equivalently

$$|u|^2=\frac{1}{1+|z|^2}\,. \tag{2-5}$$

Substituting (2-4) and (2-5), we have

$$ds'^2|S_1(\gamma)=ds^2|S^{2n+1}$$

as required.

REMARK 2-6. Sometimes $4\hat{g}$ is also called the Fubini-Study metric. It is to be noted that $\boldsymbol{C}P^1=S^2$ has volume π with our choice of the metric $\hat{g}$.

REMARK 2-7. The Hermitian connection determined by the canonical Hermitian metric is precisely the universal connection for the complex line bundles in the sense of Narasimhan and Ramanan [14].

The reason why $\nabla R=0$ holds in Proposition 2-3 is the relation $R^1_{1i\bar{j}}=$ constant$\cdot\hat{g}_{i\bar{j}}$. This relation still holds for a larger class of complex line bundles.

PROPOSITION 2-8. *Let (M, g) be a Kähler manifold and $E\to M$ be a holomorphic line bundle such that $c\Phi$ represents the cohomology class $c_1(E)$ for some $c\in\boldsymbol{R}$, where Φ is the fundamental 2-form $\sqrt{-1}\sum_{i,j} g_{i\bar{j}}dz_i\wedge\overline{dz_j}$ of M. Then we can choose a Hermitian metric h so that the condition $\nabla R=0$ is satisfied with the Hermitian connection ∇ defined by h.*

Proposition 2-8 is a direct consequence of the following well-known Lemma.

LEMMA 2-9. *Let ϕ be a (1, 1)-form on a Kähler manifold M, which is cohomologous to zero. Then there exists a C^∞ function f such that $\phi=(\sqrt{-1}/2\pi)\partial\bar{\partial}f$.*

For the proof of this Lemma, see [6] for example.

The following corollary of Proposition 2-3 will be used in § 4.

COROLLARY 2-10. *Let M be a complex submanifold of $\boldsymbol{C}P^n$. Let γ, h, $\hat{g}$ and ∇ be as in Proposition 2-3. Denote by $\gamma|M$, $h|M$, $\hat{g}|M$ and $\nabla|M$ the restrictions of γ, h, $\hat{g}$ and ∇ respectively. Then $\nabla|M$ and $g|M$ satisfy the condition $\nabla R=0$. Furthermore let $\tilde{g}$ be the metric constructed from $\hat{g}|M$, $h|M$ and $\nabla|M$ as before. Then the restriction $\tilde{g}|S_1(\gamma)$ coincides with the restriction of the standard metric of the unit sphere S^{2n+1} on $S_1(\gamma)\subset S^{2n+1}$.*

PROOF. In Example 2-2 we showed the relation

$$-R^1_{1i\bar{j}}=\hat{g}_{i\bar{j}}$$

for ∇ and $\hat{g}$. Restricting on M, we have the same relation for $\nabla|M$ and $\hat{g}|M$.

It follows that $\nabla R=0$ is satisfied with $\nabla|M$ and $\hat{g}|M$.

The latter half of the corollary is trivial.

REMARK 2-11. When both $\dim M$ and $\operatorname{rank} E$ are equal to 2, the condition $\nabla R=0$ is equivalent to the requirement that Ω is a Yang-Mills field. That is

$$\delta^{\nabla}\Omega=0\,.$$

In fact, by the dimensional restriction, both of these two conditions are equivalent to the requirement that $R=*\Omega$ is a constant function on M.

§ 3. Construction of boundary correction term

It is to be noted that in Theorem 1-2 the metric $\tilde{g}$ on Y is product near the boundary $\partial Y=X$, but that the metric $\tilde{g}$ on E defined in § 1 is not product near the associated circle bundle $S(E)$. In this section we will construct a boundary correction term TL_k such that (1-2) is rewritten as

$$\eta(X)=\int_Y L_k(p)+\int_X TL_k-\operatorname{Sign} Y\,. \tag{3-1}$$

This formulation of the correction term is due to Gilkey [5].

Let F be a vector bundle of rank r over N. Let ∇_0 ∇_1 be two connections of F. Let P be an invariant polynomial, i.e., a map from the space $M(r, \boldsymbol{C})$ of all the complex matrices to $\boldsymbol{C}$ such that $P(AB)=P(BA)$ for any $A, B\in M(r, \boldsymbol{C})$. Suppose that P is homogeneous of degree $2k$ and let $P(\ ,\ \cdots,\ )$ be its porlarized form. Denote by Ω_i, θ_i the curvature and the connection form of ∇_i ($i=0, 1$) respectively. Let $\theta=\theta_0-\theta_1$ and for each t ($0\leqq t\leqq 1$) we define a connection θ_t by

$$\theta_t=(1-t)\theta_0+t\theta_1\,.$$

Let Ω_t be the curvature form of θ_t.

We define

$$TP=2k\int_0^1 P(\theta, \Omega_t, \cdots, \Omega_t)dt\,.$$

Then it is well-known that

$$dTP=P(\Omega_0)-P(\Omega_1)\,.$$

(See Chapter XII of [10] for example.) We shall call this TP the boundary correction term connecting θ_1 to θ_0.

Let $N=X^{4k-1}\times[0, 1]$ and identify $X\times\{0\}$ with the boundary of Y. Extend $\tilde{g}$ on Y to a metric g_1 on $N\cup Y$ which is product near $X\times\{1\}$. Let g_0 be the product metric on N, i.e. $g_0=g\times ds^2$ where $g=\tilde{g}|X$ is the given metric on X and ds^2 is the standard flat metric on $[0, 1]$. Then two metrics g_0 and g_1 agree near $X\times\{1\}$. Take $P=L_k$. Since g_0 is flat in one direction, $L_k(g_0)=0$. Hence we have

$$\int_N L_k(g_1)=\int_N (L_k(g_1)-L_k(g_0))=-\int_N dTL_k \tag{3-2}$$

$$=-\int_{\partial N} TL_k=\int_{X\times\{0\}} TL_k\,.$$

It follows that

$$\eta(X)=\int_Y L_k(\tilde{g})+\int_N L_k(g_1)-\text{Sign}\,Y$$

$$=\int_Y L_k(\tilde{g})+\int_X TL_k-\text{Sign}\,Y$$

as desired.

Now let $X=S_r(E)$ be the associated circle bundle of radius r and let $Y=D_r(E)$ be the associated disc bundle of radius r. $D_r(E)-(0)$ can be identified with $S_r(E)\times(0, r]$ where (0) means the zero cross-section of E. Let g_r be the product metric on $D_r(E)-(0)$;

$$g_r=(\tilde{g}\,|\,S_r(E))\times ds^2$$

where ds^2 is the standard flat metric on $(0, r]$ and $\tilde{g}$ is the Riemannian metric on E constructed as before.

Let $\beta(r)$ be the connection form of the Riemannian connection determined by g_r. Assuming the results of Proposition 1–17, it is easy to see that the components of $\beta(r)$ are given by the following proposition.

PROPOSITION 3–3.

$$\beta(r)^r_\theta=0\,, \tag{3-4}$$

$$\beta(r)^r_j=0\,, \tag{3-5}$$

$$\beta(r)^\theta_j=(-1)^j\frac{1}{2}rRe^*_k\,, \tag{3-6}$$

$$\beta(r)^3_4=\pi^*\hat{\alpha}^3_4+\frac{1}{2}r^2R\widetilde{d\theta}\,, \tag{3-7}$$

where $j=3, 4$ and $k=7-j$.

Let $\omega_t(\rho, r)=t\alpha(\rho)+(1-t)\beta(r)$ with its curvature form $\Omega_t(\rho, r)$ $(\rho\leqq r)$. Then we have

(3–8) $$\omega_t(\rho, r)^r_j=0\,,$$

(3–9) $$\omega_t(\rho, r)^r_\theta=-t\widetilde{d\theta}\,,$$

(3–10) $$\omega_t(\rho, r)^\theta_j=(-1)^j\frac{1}{2}(t\rho+(1-t)r)Re^*_k\,,$$

(3–11) $$\omega_t(\rho, r)^3_4=\pi^*\hat{\alpha}^3_4+\frac{1}{2}(t\rho^2+(1-t)r^2)R\widetilde{d\theta}\,,$$

where $j=3$ or 4 and $k=7-j$.

A straightforward calculation will show the following

LEMMA 3-12. *The components of $\Omega_t(\rho, r)$ are given by*

(3–13) $$\Omega_t(\rho, r)^r_\theta=t\pi^*\Omega\,,$$

(3–14) $$\Omega_t(\rho, r)^r_j=(-1)^j\frac{1}{2}(t\rho+(1-t)r)Re^*_k\wedge\widetilde{d\theta}\,,$$

(3–15) $$\Omega_t(\rho, r)^\theta_j=(-1)^{j+1}\frac{1}{2}(t\rho+(1-t)r)\pi^*(*\delta^\nabla\Omega)\wedge e^*_k$$
$$-\frac{1}{4}(t\rho+(1-t)r)(t\rho^2+(1-t)r^2)R^2e^*_j\wedge\widetilde{d\theta}\,,$$

(3–16) $$\Omega_t(\rho, r)^3_4=\pi^*\hat{\Omega}-\frac{1}{2}(t\rho^2+(1-t)r^2)R\pi^*\Omega$$
$$-\frac{1}{2}(t\rho^2+(1-t)r^2)\pi^*(*\delta^\nabla\Omega)\wedge\widetilde{d\theta}$$
$$-\frac{1}{4}(t\rho+(1-t)r)^2R^2\pi^*\omega_M$$

where ω_M is the volume form of $\hat{g}$ on M, $j=3$ or 4 and $k=7-j$.

Let $TL_1(\rho, r)$ be the boundary correction term connecting $\alpha(\rho)$ to $\beta(r)$ $(\rho\leqq r)$. Then entirely the same arguement as in (3-2) shows that

(3–17) $$\int_{S(E)}TL_1(\rho, r)=\int_{D_r(E)}L_1(p)-\int_{D_\rho(E)}L_1(p)$$

where in the first integral $\int_{D_r(E)}L_1(p)$ the L-form is defined via the metric which is product near the boundary $S_r(E)$ but in the second integral $\int_{D_\rho(E)}L_1(p)$ the metric $\tilde{g}$ is not product near the boundary $S_\rho(E)$. Note that, in the case when $\rho=r$, $TL_1(\rho, r)$ agrees with the boundary correction term in (3-1). Letting $\rho\to 0$ in (3-17), since $L_1(p)$ is smooth near $D_0(E)=M$, we have

$$\int_{D_r(E)}L_1(p)=\lim_{\rho\to 0}\left\{\int_{S(E)}TL_1(\rho, r)+\int_{D_\rho(E)}L_1(p)\right\}$$

$$=\lim_{\rho\to 0}\int_{S(E)} TL_1(\rho, r).$$

But by the compactness of $S(E)$ the last term equals

$$\int_{S(E)}\lim_{\rho\to 0} TL_1(\rho, r).$$

Thus we have proved the following proposition.

PROPOSITION 3-18.

$$\int_{D_r(E)} L_1(p)=\int_{S(E)}\lim_{\rho\to 0} TL_1(\rho, r)$$

where in the left hand side $L_1(p)$ is the L_1-form defined via the metric which is product near the boundary $S_r(E)$.

REMARK 3-19. Consider the case when X is an oriented 3-dimensional manifold. As is well-known, X is parallelizable. Let σ be a framing of X, and θ_0 be the flat connection determined by σ. Let θ be another connection of TX. In this case the boundary correction term $TP(\theta_0, \theta)$ connecting θ_0 to θ is closely related to the invariant introduced by Chern and Simons. In fact take the inverse Pontrjagin polynomial as P, then $(1/2)TP(\theta_0, \theta)$ becomes the pull-back of $(1/2)TP_1(\theta)$ in [4] by σ.

$$\frac{1}{2}TP(\theta_0, \theta)=\sigma^*\Big(\frac{1}{2}TP_1(\theta)\Big). \tag{3-20}$$

That is, if we denote the Chern-Simons invariant of (TX, θ) as C.S.(X), then we have

$$\mathrm{C.S.}(X)\equiv\frac{1}{2}\int_X TP(\theta_0, \theta) \qquad (\mathrm{mod}\ \boldsymbol{Z}). \tag{3-21}$$

§4. Main theorem

In this section we will calculate the η-invariant of a circle bundle $S_r(E)$ of radius r which is associated with a complex line bundle over an oriented closed Riemann surface $E\to M$. To do this we may calculate either

$$\int_{D_r(E)} L_1(p)+\int_{S(E)} TL_1(r, r)-\mathrm{Sign}\, D(E)$$

or

$$\int_{S(E)}\lim_{\rho\to 0} TL_1(\rho, r)-\mathrm{Sign}\, D(E).$$

We will use the latter formula. Our main theorem is the following:

THEOREM 4-1. *Let $E\to M$ be a complex line bundle over an oriented, closed Riemann surface. Let $\hat{g}$ be a Riemannian metric on M, h be a fibre metric of E and ∇ be an h-preserving connection of E. Suppose that ∇ and $\hat{g}$ satisfy the condition $\nabla R=0$. Then the η-invariant of the Riemannian manifold $(S_r(E), \tilde{g})$ is given by*

$$\eta(S_r(E))=\frac{1}{3}c_1-\varepsilon+\frac{2}{3}c_1\left\{\frac{\pi r^2}{\mathrm{vol}\,M}\chi(M)-\left(\frac{\pi r^2}{\mathrm{vol}\,M}\right)^2 c_1^2\right\},$$

where $\tilde{g}$ is the metric constructed from $\hat{g}$, h and ∇ as in §1, $c_1=c_1(E)[M]$, $\mathrm{vol}\,M$ is the volume of M with respect to the metric $\hat{g}$, $\chi(M)$ is the Euler number of M and ε is defined by

$$\varepsilon=\begin{cases} 1 & \textit{if} \ \ c_1>0, \\ 0 & \textit{if} \ \ c_1=0, \\ -1 & \textit{if} \ \ c_1<0, \end{cases}$$

PROOF. Notation will be the same as in §3. Let $\omega=\lim_{\rho\to 0}(\beta(r)-\alpha(\rho))$, and $\Omega_t=\lim_{\rho\to 0}\Omega_t(\rho, r)$. The first L-polynomial is given by

$$L_1=-\frac{1}{3}\frac{1}{8\pi^2}\sum_{I,J}\tilde{\Omega}_J^I\wedge\tilde{\Omega}_I^J$$

so that we have

$$\lim_{\rho\to 0} TL_1(\rho, r)=-\frac{2}{3}\frac{1}{8\pi^2}\int_0^1 2\{\omega_4^3\wedge\Omega_{t3}^4+\omega_\theta^r\wedge\Omega_{tr}^\theta+\sum_{j=3,4}(\omega_j^\theta\wedge\Omega_{t\theta}^j+\omega_j^r\wedge\Omega_{tr}^j)\}\,dt\,.$$

By (1-16), (1-21), (3-7) and (3-16) we have

$$\begin{aligned}(4\text{-}2)\qquad \omega_4^3\wedge\Omega_{t3}^4&=\frac{1}{2}Rr^2\widetilde{d\theta}\wedge\Big\{-\pi^*\hat{\Omega}+\frac{1}{2}r^2(1-t)R\pi^*\Omega+\frac{1}{2}(1-t)r^2\pi^*(*\delta^\nabla\Omega)\wedge\widetilde{d\theta}\\ &\qquad+\frac{1}{4}(1-t)^2r^2R^2\pi^*\omega_M\Big\}\\ &=-\frac{1}{2}Rr^2d\theta\wedge\pi^*\hat{\Omega}+\frac{1}{8}R^2r^4(3-4t+t^2)d\theta\wedge\pi^*\Omega\,.\end{aligned}$$

Similarly we obtain

$$(4\text{-}3)\qquad \sum_{j=3,4}\omega_j^r\wedge\Omega_{tr}^j=0\,,$$

$$(4\text{-}4)\qquad \sum_{j=3,4}\omega_j^\theta\wedge\Omega_{t\theta}^j=\frac{1}{4}r^4(1-t)^2R^2d\theta\wedge\pi^*\Omega\,,$$

$$(4\text{-}5)\qquad \omega_\theta^r\wedge\Omega_{tr}^\theta=-t\,d\theta\wedge\pi^*\Omega\,.$$

Note that $\hat{\Omega}$ and Ω represent $2\pi e(M)$ and $2\pi c_1(E)$ respectively and that $R=*\Omega$, where $e(M)$ is the Euler class of M. Because Ω is a Yang-Mills field, $*\Omega$ is closed. (See Remark 2-11.) Hence R represents $2\pi *c_1(E)$. Thus, by (4-2)~(4-5), we have

(4-6) $$\int_{S(E)} \lim_{\rho\to 0} TL_1(\rho, r)$$
$$=\left\{\frac{1}{3}c_1(E)+\frac{2}{3}\pi r^2 e(M)\cup *c_1(E)-\frac{2}{3}(\pi r^2)^2 c_1(E)\cup(*c_1(E))^2\right\}[M].$$

Since $H^2(M;\boldsymbol{R})=\boldsymbol{R}$, we have

$$c_1(E)=\frac{c_1}{\mathrm{vol}\, M}[\omega_M]=\frac{c_1}{\mathrm{vol}\, M}[*1].$$

It follows that

(4-7) $$*c_1(E)=\frac{c_1}{\mathrm{vol}\, M}.$$

Substituting (4-7) to (4-6), we have

$$\int_{S(E)} \lim_{\rho\to 0} TL_1(\rho, r)=\frac{1}{3}c_1+\frac{2}{3}c_1\left\{\frac{\pi r^2}{\mathrm{vol}\, M}\chi(M)-\left(\frac{\pi r^2}{\mathrm{vol}\, M}\right)^2 c_1^2\right\}.$$

It remains only to prove Sign $D(E)=\varepsilon$. Sign $D(E)$ is, by definition, the signature of the quadratic form

$$\begin{array}{ccc} \hat{H}^2(D(E))\times\hat{H}^2(D(E)) & \longrightarrow & \boldsymbol{Z} \\ \Uparrow & & \Uparrow \\ (i^*x, i^*y) & \longmapsto & \{x\cup y\}[\mu] \end{array}$$

where $\mu\in H_4(D(E), S(E))$ is the fundamental class and $\hat{H}^2(D(E))$ is the image of the restriction map

$$i^*: H^2(D(E), S(E)) \longrightarrow H^2(D(E)).$$

On the other hand, by the Thom isomorphism, we have

$$\begin{array}{ccc} H^0(M) & \cong & H^2(D(E), S(E)) \\ \Uparrow & & \Uparrow \\ t & \longmapsto & t\omega \end{array}$$

where ω is the Thom class. Hence we see that Sign $D(E)$ equals the signature of the quadratic form

$$\begin{array}{ccc} H^0(M)\times H^0(M) & \longrightarrow & \boldsymbol{Z} \\ \Uparrow & & \Uparrow \\ (u, v) & \longmapsto & uv\{\omega\cup\omega\}[\mu]. \end{array}$$

But we have

$$uv\{\omega\cup\omega\}[\mu]=uv\{c_1(E)\cup\omega\}[\mu]=uv\langle c_1(E), [M]\rangle=uvc_1$$

since

$$c_1(E)\cup\omega=\omega\cup\omega\,.$$

This proves Sign $D(E)=\varepsilon$ and completes the proof of Theorem 4-1.

We shall give several applications of our theorem.

COROLLARY 4-8. *Let g_r be a left-invariant metric on $S^3=Sp(1)$ which is of the form*

$$\begin{pmatrix} r^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

on the tangent space T_eS^3 at the unit element, relative to the standard basis $\{i, j, k\}$ of the Lie algebra $\mathfrak{sp}(1)$. Then the η-invariant of the Riemannian manifold (S^3, g_r) is equal to $(2/3)(1-r^2)^2$.

REMARK 4-9. This is precisely the result of Hitchin [8].

PROOF. We use the same notation as in Proposition 2-3.

First note that $\partial/\partial\theta$ corresponds to the orbit of a maximal torus $S^1\subset S^3$. Hence it can be identified with the basis i.

There is a natural ($Sp(1)$-equivariant) diffeomorphism

$$\begin{array}{ccc} S_1(\gamma) & \longrightarrow & S_r(\gamma) \\ \Cup & & \Cup \\ ((w_0:w_1),\ u) & \longmapsto & ((w_0:w_1),\ ru)\,. \end{array}$$

Let z be a local coordinate of $\boldsymbol{C}P^1=S^2$. By Lemma 1-3, the horizontal lift $\widetilde{\partial/\partial z}\,|\,S_1(\gamma)$ is mapped to $\widetilde{\partial/\partial z}\,|\,S_r(\gamma)$ under the above diffeomorphism.

By definition, $\tilde{g}\,|\,S_1(\gamma)$ is of the form

$$\left(\begin{array}{c|c} 1 & \\ \hline & \hat{g} \end{array}\right)$$

relative to the basis $\{\partial/\partial\theta, \widetilde{\partial/\partial z}, \widetilde{\partial/\partial \bar{z}}\}$. On the other hand, by Proposition 2-3, $\tilde{g}\,|\,S_1(\gamma)$ is of the form

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

relative to $\{i, j, k\}$.

Now $\tilde{g}\,|\,S_r(\gamma)$ is of the form

$$\left(\begin{array}{c|c} r^2 & \\ \hline & \hat{g} \end{array}\right)$$

relative to $\{\partial/\partial\theta, \widetilde{\partial/\partial z}, \widetilde{\partial/\partial\bar{z}}\}$.

It follows that it is of the form

$$\begin{pmatrix} r^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

relative to $\{i, j, k\}$. This proves that $(S_r(\gamma), \tilde{g})$ is isometric to (S^3, g_r).

Therefore we obtain the value of the η-invariant by substituting $\chi(M)=2$, $c_1=-1$ and $\operatorname{vol} M=\pi$ to our theorem. (See Remark 2-6.)

COROLLARY 4-10 (3-dimensional lens spaces). *The η-invariant of the lens space $L(k\,;1)$ with the standard metric is given by*

$$\eta(L(k\,;1))=-\frac{1}{3k}(k-1)(k-2),$$

where $L(k\,;1)$ is the quotient space of $S^3\subset \boldsymbol{C}^2$ by the equivalence relation

$$(z_0, z_1)\sim(\lambda z_0, \lambda z_1)$$

for any $\lambda\in \boldsymbol{Z}_k\subset S^1$ and $(z_0, z_1)\in S^3$.

PROOF. Consider the k-th tensor product γ^k of the Hopf bundle $\gamma\to S^2$. γ^k has the natural tensor product metric h_k and the tensor product connection $\nabla^{(k)}$ derived from h and ∇ in Proposition 2-3. Let $\tilde{g}_k$ be the metric constructed from h_k, $\nabla^{(k)}$ and the Fubini-Study metric on S^2. $\nabla^{(k)}$ clearly satisfies the condition $\nabla R=0$.

Let s be a local cross-section of the Hopf bundle $\gamma\to \boldsymbol{C}P^1$. Let $s^k\in\Gamma(\gamma^k)$ be the k-th tensor product of s. Define $\varphi:\gamma\to\gamma^k$ by

$$\varphi(us)=\frac{1}{k}u^k s^k$$

for any $u\in \boldsymbol{C}$. Then φ is well-defined, and maps $S_1(\gamma)$ to $S_{1/k}(\gamma^k)$. Moreover, by a similar arguement (using Lemma 1-3) as in Corollary 4-8, we can show that φ is a local isometry.

Let f be the canonical identification of $S_1(\gamma)$ with S^3. Then f induces a diffeomorphism

$$\bar{f}: S_{1/k}(\gamma^k) \longrightarrow L(k\,;1)$$

such that the following diagram is commutative;

$$
\begin{array}{ccc}
S_1(\gamma) & \xrightarrow{\quad f \quad} & S^3 \\
\Big\downarrow{\scriptstyle\varphi} & & \Big\downarrow{\scriptstyle p} \\
S_{1/k}(\gamma) & \xrightarrow{\quad \bar{f} \quad} & L(k\,;1)\,,
\end{array}
$$

where p is the natural projection. In fact, φ is considered to be the quotient map of the fibrewise multiplication of $\boldsymbol{Z}_k \subset S^1$.

In the above diagram, p and φ are local isometries, and f is an isometry by Proposition 2-3. It follows that $\bar{f}$ is an isometry.

Therefore we obtain the value of the η-invariant by substituting $\chi(M)=2$, $c_1=-1$, and $\operatorname{vol} M=\pi$ to Theorem 4-1.

In [3] Atiyah-Patodi-Singer proved a formula of the η-invariants for (generalized) lens spaces. Their result asserts that the η-invariant of $L(k\,;1)$ is given by $-\frac{1}{k}\sum_{l=1}^{k-1}\cot^2\left(\frac{l}{k}\pi\right)$. Comparing these two results, we obtain a formula about Dedekind sums.

COROLLARY 4-11. $\sum_{l=1}^{k-1}\cot^2\left(\frac{l}{k}\pi\right)=\frac{1}{3}(k-1)(k-2)\,.$

REMARK 4-12. For a direct proof of this formula, the reader should consult Zagier [16].

COROLLARY 4-13 (complete intersections in S^{2n+1}). *Let $f_i(z_0, z_1, \cdots, z_n)\in \boldsymbol{C}[z_0, z_1, \cdots, z_n]$ be a homogeneous polynomial of degree a_i and set*

$$V_i=\{(z_0:z_1:\cdots:z_n)\in \boldsymbol{C}P^n\,;\,f_i(z_0, z_1, \cdots, z_n)=0\} \qquad (1\leqq i\leqq n-1)\,.$$

Suppose that V_i is non-singular with

$$V_a\equiv V_1\cap V_2\cap\cdots\cap V_{n-1}\neq\emptyset \qquad (a=(a_1, \cdots, a_{n-1}))$$

and that V_i and V_j intersect transversally $(1\leqq i,\, j\leqq n-1)$. Let X_a be the set

$$\{(z_0, z_1, \cdots, z_n)\in S^{2n+1}\,;\,f_i(z_0, z_1, \cdots, z_n)=0 \text{ for } 1\leqq i\leqq n-1\}\,.$$

Then the η-invariant of X_a with the standard metric is given by

$$\eta(X_a)=\frac{1}{3}\left\{2\sum_{i=1}^{n-1}a_i-(2n+1)\right\}a_1a_2\cdots a_{n-1}+1\,.$$

In particular the η-invariant of a hypersurface

$$X_N=\{(z_0, z_1, z_2)\in S^5\,;\,f(z_0, z_1, z_2)=0,\ f \text{ is homogeneous of degree } N\}$$

is given by

$$\eta(X_N)=\frac{1}{3}(N-1)(2N-3)\,.$$

PROOF. By Corollary 2-10, X_a is isometric to $(S_1(\gamma|V_a), \tilde{g})$, where $\tilde{g}$ is the metric constructed from $\hat{g}|V_a$, $h|V_a$ and $\nabla|V_a$ as before. Furthermore $\hat{g}|V_a$ and $\nabla|V_a$ satisfy the condition $\nabla R=0$. Therefore if we know the data $\chi(V_a)$, $\langle c_1(\gamma|V_a), [V_a]\rangle$ and $\mathrm{vol}\, V_a$, then we can get the value $\eta(X_a)$ by Theorem 4-1.

From the assumption the normal bundle $\nu(V_a)$ of V_a in $\boldsymbol{CP}^n$ is isometric to the direct sum of the normal bundle $\nu(V_i)$ of V_i $(1\leqq i\leqq n-1)$.

$$\nu(V_a)\cong\nu(V_1)\oplus\nu(V_2)\oplus\cdots\oplus\nu(V_{n-1}).$$

Hence the total Chern class of V_a is given by

$$c(V_a)=c(\boldsymbol{CP}^n)/c(\nu(V_1))\cup\cdots\cup c(\nu(V_{n-1})).$$

On the other hand we have

$$\nu(V_i)\cong(\gamma^*)^{a_i}|V_i,$$

where $(\gamma^*)^{a_i}$ is the a_i-th tensor product of the dual bundle γ^* of the Hopf bundle. So if we denote the standard generator of the cohomology ring $H^*(\boldsymbol{CP}^n;\boldsymbol{Z})$ by x (so that $c_1(\gamma)=-x$) we have

$$e(\nu(V_a))=a_1a_2\cdots a_{n-1}x^{n-1}|V_a$$

$$c(V_a)=(1+x)^{n+1}(1+a_1x)^{-1}\cdots(1+a_{n-1}x)^{-1}|V_a$$

where $e(\nu(V_a))$ denote the Euler class of $\nu(V_a)$. It follows that

(4-14) $$\chi(V_a)=(n+1-a_1-a_2-\cdots-a_{n-1})a_1a_2\cdots a_{n-1}$$

(4-15) $$\langle c_1(\gamma|V_a), [V_a]\rangle=-a_1a_2\cdots a_{n-1}.$$

Finally since $\deg V_a=a_1a_2\cdots a_{n-1}$, we have

(4-16) $$\mathrm{vol}(V_a)=\deg V_a\cdot\mathrm{vol}(S^2)=a_1a_2\cdots a_{n-1}\pi.$$

(See p. 89 of Mumford [13].)

Substituting (4-14), (4-15) and (4-16) to the formula of Theorem 4-1, we obtain

$$\eta(X_a)=1+\frac{a_1a_2\cdots a_{n-1}}{3}\left\{2\sum_{i=1}^{n-1}a_i-(2n+1)\right\}$$

as required.

§5. Chern-Simons invariant

Let X be an oriented 3-dimensional Riemannian manifold. First we remark that the Chern-Simons invariant can be defined for a connection θ on the stable tangent bundle $TX\oplus k$, where k denotes the trivial $\boldsymbol{R}^k$-bundle on X. Let φ_t be $\varphi_t=t\Omega+(t^2-t)\theta\wedge\theta$, where Ω is the curvature form of θ. For an invariant polynomial P of degree l, we define $TP(\theta)$ by

$$TP(\theta)=l\int_0^1 P(\theta,\ \varphi_t^{l-1})dt\,.$$

Let $F(X)\to X$ be the principal $SO(k+3)$-bundle associated with $TX\oplus k$. Then, since X is parallelizable, $F(X)$ has a cross-section σ.

Take the first Pontrjagin polynomial P_1 as P. Then $(1/2)TP_1(\theta)$ defines a class

$$\left\{\frac{1}{2}TP_1(\theta)\right\}\in H^3(F(X)\,;\boldsymbol{R})\,.$$

Furthermore it is known that

$$\left\{\frac{1}{2}TP_1(\theta)\right\}|F(X)_m\in H^3(F(X)_m\,;\boldsymbol{Z})\,,$$

where $F(M)_m\cong SO(k+3)$ is a fibre over $m\in X$. (See p. 63 of [4].)

We define $\mathrm{C.S.}(M,\ \theta)$ by

$$\mathrm{C.S.}(X,\ \theta)=\frac{1}{2}\int_\sigma TP_1(\theta) \qquad \mathrm{mod}\,\boldsymbol{Z}\,.$$

This is well-defined. In fact, let σ' be another section. Then, since the first and second betti numbers of $SO(k+3)$ vanish, we have (by the Künneth formula)

$$\sigma-\sigma'=1\times v+\text{torsion} \quad \text{in } H_3(F(X)\,;\boldsymbol{Z})\,,$$

where $v\in H_3(SO(k+3)\,;\boldsymbol{Z})$.

When θ is the Riemannian connection, we will write $\mathrm{C.S.}(X,\ \theta)$ simply as $\mathrm{C.S.}(X)$.

Let $\pi:E\to M$ be a complex line bundle over an oriented Riemann surface. Because the tangent bundle TM is stably trivial, the associated circle bundle $S(E)$ can be stably-framed.

More precisely we take a fibre metric h of E and an h-preserving connection ∇ as before. Then ∇ defines an isomorphism,

$$1\oplus TS(E)\cong\pi^*(TM\oplus E)\,|\,S(E)\,.$$

We fix, once for all, a framing of $TM\oplus 1$. Then $TS(E)\oplus 2$ is also framed by the above isomorphism. Let ∇_0 be the flat connection on $2\oplus TS(E)$ defined by this framing. Using the relation (3-21) and Theorem 4-1, we can get a result about the Chern-Simons invariant.

THEOREM 5-1. *The Chern-Simons invariant of the Riemannian manifold* $(S_r(E),\ \tilde{g})$ *is given by*

$$\mathrm{C.S.}(S_r(E))=c_1\left\{\frac{\pi r^2}{\mathrm{vol}\,M}\chi(M)-\left(\frac{\pi r^2}{\mathrm{vol}\,M}\right)^2 c_1^2\right\} \qquad (mod\,\boldsymbol{Z})$$

where $\tilde{g}$ is the metric of $S_r(E)$ constructed from a Riemannian metric $\hat{g}$ on M, a fibre metric h of E and an h-preserving connection ∇ as in § 1, vol M *is the volume of M with respect to $\hat{g}$, $\chi(M)$ is the Euler number of M and $c_1=\langle c_1(E), [M]\rangle$.*

REMARK 5-2. It is known that for any oriented 3-dimensional manifold X

$$\mathrm{C.S.}(X)-\frac{3}{2}\eta(X)\equiv\frac{1}{2}\sigma(X) \qquad (\mathrm{mod}\, \boldsymbol{Z})$$

where $\sigma(X)$ is the number of 2-primary summands in $H^2(X;\boldsymbol{Z})$. (See p. 426 of Atiyah-Patodi-Singer [3].) This formula, together with Theorem 4-1, implies Theorem 5-1 clearly. But here we will give another proof.

PROOF. In the proof of Theorem 4-1 we showed that

$$\frac{3}{2}\int_{S(E)}\lim_{\rho\to 0} TL_1(\rho, r)=\frac{1}{2}c_1+c_1\left\{\frac{\pi r^2}{\mathrm{vol}\, M}\chi(M)-\left(\frac{\pi r^2}{\mathrm{vol}\, M}\right)^2 c_1^2\right\}.$$

By virtue of (3-21), we need only to show the following: Let $TL_1(\nabla_0, \rho)$ be the boundary correction term connecting ∇_0 to $\alpha(\rho)$, then

$$\int_{S(E)}\lim_{\rho\to 0} TL_1(\nabla_0, \rho)=\frac{1}{3}c_1. \tag{5-3}$$

By Proposition 1-17 the components of $\lim_{\rho\to 0}\alpha(\rho)$ with respect to the basis $\{\partial/\partial r, (1/r)(\partial/\partial\theta), e_3, e_4, e_5\}$ are given by

(5-4) $\qquad \alpha_\theta^r=-\widetilde{d\theta}, \quad \alpha_j^i=\pi^*\hat{\alpha}_j^i \qquad (3\leqq i, j\leqq 5)$
and other components are all zero,

where $\{e_3, e_4, e_5\}$ is the horizontal lift of the framing $\{f_3, f_4, f_5\}$ of $TM\oplus 1$ with respect to

$\nabla\oplus$(trivial connection)

and $\hat{\alpha}_j^i$ is the component of

(Riemannian connection determined by g)$\oplus$(trivial connection)

with respect to $\{f_3, f_4, f_5\}$. We may assume that

(5-5) the components of ∇_0 with respect to f_3, f_4, f_5 are all zero.

Let $c_t=t\alpha+(1-t)\nabla_0$ with its curvature Ω_t. Then the components of Ω_t are given by

(5-6) $\qquad \Omega_{tj}^i=\pi^*\hat{\Omega}_j^i \quad (3\leqq i, j\leqq 5), \qquad \Omega_{t\theta}^r=t\pi^*\Omega$
and other components are zero.

(5-3) clearly follows from (5-4)$\sim$(5-6).

REMARK 5-7. The above arguement can be easily extended to the case when M is a (general dimensional) framed manifold, and gives the following result about the Adams e_C-invariant:

The Adams e_C-invariant of $S(E)$ (relative to the framing discussed above) is given by

$$e_C(S(E))=\begin{cases} \dfrac{1}{2} & \text{if } n=0\,, \\ 0 & \text{if } n\equiv 0 \pmod 4),\ n\neq 0\,, \\ (-1)^{k-1}\dfrac{B_k}{(2k)!}\langle c_1(E)^{2k-1}, [M]\rangle & \text{if } n=4k-2,\ k>0\,, \end{cases}$$

where $n=\dim M$, and B_k is the k-th Bernoulli number. This is precisely the result of Löffler-Smith [11].

In the same notation as in § 4 we have the following corollaries.

COROLLARY 5-8. *Let g_λ be the metric on $SO(3)$ induced from the left invariant metric g_r $(r=\lambda^{-1})$ on S^3 of Corollary 3-8. Then the Chern-Simon invariant of the Riemannian manifold $(SO(3), g_\lambda)$ is given by*

$$\text{C. S.}(SO(3))=-\frac{2\lambda^2-1}{2\lambda^4} \quad (mod\ \boldsymbol{Z})\,.$$

REMARK 5-9. Up to a sign convention, this agrees with the result of Chern-Simons [4].

COROLLARY 5-10. *The Chern-Simons invariant of a complete intersection X_a vanishes.*

REMARK 5-11. It is known that the Chern-Simons invariant of 3-dimensional manifold M vanishes whenever M admits a conformal immersion to $\boldsymbol{R}^4$. (Theorem 6-4 of [4]) The author does not know whether there is actually a conformal immersion $X_a\to\boldsymbol{R}^4$.

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan