On the existence of positive scalar curvature metrics on non-simply-connected manifolds

By Tetsuro MIYAZAKI

(Communicated by A. Hattori)

Introduction

In [2], Gromov and Lawson showed that a non-spin, simply connected, closed manifold of dimension greater than 4 admits a Riemannian metric of positive scalar curvature. Furthermore they showed that, for a simply connected closed manifold which is spin, the existence of such a metric depends only on the spin cobordism class and that the vanishing of the KO-characteristic number $\alpha(X)$ defined by Milnor is not necessary but almost sufficient for the existence of such a metric. Note that the number $\alpha(X)$ coincides essentially with the genus $\hat{A}(X)$ if the dimension of the manifold X is divisible by 4.

In [3], they proved the vanishing of the higher $\hat{A}$-genus for a closed spin manifold which admits a metric of positive scalar curvature but which is not necessarily simply connected.

This paper deals with the existence problem of a Riemannian metric of positive scalar curvature for an orientable manifold with some fundamental group and gives an almost complete answer when the fundamental group is $\boldsymbol{Z}_p$; p odd or $p=2$, $\boldsymbol{F}_m$ (a free group of m generators) or $\oplus_k \boldsymbol{Z}$, provided that the dimension of the non spin manifold is assumed to be greater than k.

In section 1, we extend the method in [2] and show that under some conditions, a bordism which preserves a fundamental group preserves the property that a manifold admits a metric of positive scalar curvature.

In section 2, we interpret the bordism which preserves the fundamental group from the standpoint of the G-bordism theory. Then the problem is reduced to seeking nice generators of the G-bordism group.

In the remainder of this paper, we shall give the almost final solution when the fundamental group G is isomorphic to $\boldsymbol{Z}_p$; p odd or $p=2$, $\oplus_k \boldsymbol{Z}$ or $\boldsymbol{F}_m$ (also when G is a finite group only for the case of spin manifolds). Here, we use the notion of "G-connected sums" instead of usual connected sum in the statement of our final result. In section 4, we consider the bordism invariant $\hat{\boldsymbol{\alpha}}$: $\Omega_*(X) \to H_*(\pi_1(X); \boldsymbol{Q})$ which replaces the higher $\hat{A}$-genus in our argument.

This paper is based on part of the author's Master's thesis presented to the

University of Tokyo in March, 1982.

I should like to express my gratitude to Professor Akio Hattori for suggesting this problem and giving many valuable suggestions and informations to me.

1. Bordisms preserving metrics of positive scalar curvature

In this section we prove two theorems concerning bordisms preserving metrics of positive scalar curvature. They are generalizations of Theorem B and Theorem C in [1].

THEOREM 1.1. *Let X_1 be a closed connected oriented spin manifold of dimension $\geqq 5$. If there are a closed oriented spin manifold X_2 with a metric of positive scalar curvature and a spin bordism W which satisfy two conditions:*

1) $\partial W = X_2 - X_1$

2) $\iota_{1*}: \pi_1(X_1) \cong \pi_1(W)$

where $\iota_1: X_1 \to W$ is the inclusion, then X_1 also carries a metric of positive scalar curvature. (Note that X_2 is not necessarily connected.)

PROOF. First recall the following.

THEOREM 1.2. (Gromov and Lawson [2], Schoen and Yau [4]). *Let X be a compact manifold which carries a Riemannian metric of positive scalar curvature. Then any manifold which can be obtained from X by performing surgeries in codimension $\geqq 3$ also carries a metric with positive scalar curvature.*

Consider a smooth triad $(W; X_1, X_2)$ as in Theorem 1.1 and a self indexing Morse function f. By definition, f has two properties:

1) $f(X_1) = -(1/2)$, $f(X_2) = n + (1/2)$ where $n = \dim W$.

2) $f(p) = \mathrm{index}(p)$ at each critical point p of f.

If we cancel index 0, index 1 and index 2 critical points of f, X_1 also carries positive scalar curvature from Theorem 1.2.

A pair (Y, X) is called r-connected if any map $g: (L, K) \to (Y, X)$ with L a CW complex of dimension $\leqq r$ and K a subcomplex, is homotopic as a map of pairs to a map taking L into X. Let $(B; M_-, M_+)$ be a bordism. We call (B, M_-) *geometrically r-connected* if there is a self indexing Morse function for the triad $(B; M_-, M_+)$ with no critical points of index $\leqq r$.

In [7], Wall proved the following theorem using the geometric cancellation theorem.

THEOREM 1.3. *Let $(B; M_-, M_+)$ be an n-dimensional bordism with (B, M_-) r-connected and $r \leqq n-4$. Then it is geometrically r-connected.*

In view of Theorem 1.3 and its proof, it is enough for us to show that it is possible to take W such that (W, X_1) is r-connected.

Since X_1 is connected, we may assume that W is connected. Then (W, X_1) is 0-connected. By the assumption, $\pi_1(W, X_1)$ is isomorphic to $\{1\}$ and (W, X_1) is 1-connected. Homotopy exact sequences show that $\pi_2(W, X_1)\cong\pi_2(W)/\iota_{1*}\pi_2(X_1)$. The Hurewicz homomorphism followed by the second Stiefel Whitney class $\pi_2(W)\to H_2(W)\to \boldsymbol{Z}_2$ detects the triviality of normal bundles of embedded 2-spheres. Since W is spin, we can kill all the elements of $\pi_2(W)$. Then $\pi_2(W, X_1)$ is isomorphic to $\{1\}$ and (W, X_1) is 2-connected.

This completes the proof of Theorem 1.1.

THEOREM 1.4. *Let X_1 be a closed connected oriented manifold of dimension $\geqq 5$ such that the universal covering $\tilde{X}_1$ of X_1 is not spin. If there are a closed oriented manifold X_2 with a metric of positive scalar curvature and an oriented bordism W which satisfy two conditions:*

1) $\partial W = X_2 - X_1$

2) $\iota_{1*}: \pi_1(X_1)\cong\pi_1(W)$

where $\iota_1: X_1\to W$ is inclusion, then X_1 also carries a metric of positive scalar curvature.

PROOF. The proof of the above theorem is similar to that of Theorem 1.1. The difference lies in the elimination of index 2 critical points.

Since $\tilde{X}_1$ is not spin, the Hurewicz isomorphism followed by the second Stiefel Whitney class of $\tilde{X}_1: \pi_2(\tilde{X}_1)\cong H_2(\tilde{X}_1)\to \boldsymbol{Z}_2$ is surjective. Hurewicz homomorphisms and Stiefel Whitney classes are natural. It follows that $\pi_2(X_1)\to H_2(X_1)\to \boldsymbol{Z}_2$ is surjective. On the one hand, we can eliminate the kernel of $\pi_2(W)\to H_2(W)\to \boldsymbol{Z}_2$ by surgeries because the generators are represented by 2-spheres with trivial normal bundles. Thus after surgeries, we may assume that $\pi_2(W)=0$ or $\boldsymbol{Z}_2$. We see $\iota_{1*}: \pi_2(X_1)\to\pi_2(W)$ is surjective by naturality and hence $\pi_2(W)/\iota_{1*}\pi_2(X_1)=0$. This shows that (W, X_1) is 2-connected. Then the result follows from Theorem 1.3.

2. The bordism preserving positive scalar curvature and the bordism with a free action

Throughout this section, X_1 denotes a closed connected oriented manifold with $\pi_1(X_1)$ isomorphic to a group G and $\tilde{X}_1$ denotes the universal covering of X_1. To use the G-bordism theory, we consider the following situations.

SITUATION A). There exists a compact oriented manifold W such that $\partial W = X_2 - X_1$ where X_2 is a closed oriented manifold (not necessarily connected), and that the inclusion $\iota_1 : X_1 \to W$ gives an isomorphism $\iota_{1*} : \pi_1(X_1) \to \pi_1(W)$.

SITUATIONS B). There exists a connected oriented manifold $\tilde{W}$ satisfying the following conditions:

1) $\tilde{W}$ admits an orientation preserving free G action such that $W = \tilde{W}/G$ is compact.

2) $\partial\tilde{W} = \tilde{X}_2 - \tilde{X}_1$ where $\tilde{X}_2$ is an oriented manifold.

3) On $\tilde{X}_1$, the G action agrees with the action of $\pi_1(X_1)$ as the covering transformation group.

THEOREM 2.1. *Situation* A) *and* B) *are equivalent.*

PROOF. Taking the universal covering spaces, it is easy to see A) implies B).

Conversely, we assume Situation B). Consider the homotopy exact sequences of $\pi : \tilde{W} \to W$ and $\pi|_{\tilde{X}_1} : \tilde{X}_1 \to X_1$. We get a commutative diagram:

2.2
$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\tilde{W}) & \xrightarrow{\pi_*} & \pi_1(W) & \xrightarrow{\partial} & G & \longrightarrow & 1 \\ & & & & \uparrow & & \| & & \\ & & 1 & \longrightarrow & \pi_1(X_1) & \longrightarrow & G & \longrightarrow & 1. \end{array}$$

If we eliminate $\pi_1(\tilde{W})$ by equivariant surgeries, A) follows.

Take an imbedding $\alpha : S^1 \to S_\alpha \subset W$ which represents an element of $\pi_*(\pi_1(\tilde{W})) \subset \pi_1(W)$. Then there exists a lift of α, $\beta : S^1 \to S_\beta \subset \tilde{W}$, such that $\pi|_{S_\beta} : S_\beta \to S_\alpha$ is a diffeomorphism. Since G acts transitively on each fiber of $\pi : \tilde{W} \to W$, $\pi^{-1}(S_\alpha) = \Sigma_{g \in G} gS_\beta$ (disjoint union). We can surgery S_α and all the gS_β at once without destroying Situation B).

Next, we show that we can eliminate $\pi_1(\tilde{W})$ after doing these surgeries finitely many times. If we consider G to be a covering transformation group, the rows in 2.2 are exact as group homomorphisms. Since X_1 is compact, G is a finitely generated group with finite relations. Let $\{c_k\}$ be a system of generators of G with relations $\{f_l(c_k)=1\}$. There are elements of $\pi_1(W)$ $\{d_k\}$ such that $\partial d_k = c_k$. Take elements of $\pi_1(W)$ $\{e_m\}$ so that $\{d_k, e_m\}$ are the generators of $\pi_1(W)$. If $\partial e_m = \Pi_i (c_{k_i})^{n_i}$, we replace e_m by $(\Pi_i (d_{k_i})^{n_i})^{-1} e_m$. So we may assume $\partial e_m = 1$.

ASSERTION 2.3. *$\pi_*(\pi_1(W))$ is generated by the set:*
$\{g^{-1} f_l(d_k) g,\ h^{-1} e_k h\ ;\ g,\ h$ are elements of the subgroup D generated by $\{d_k\}\}$.

PROOF. Take $g\in\pi_*(\pi_1(\widetilde{W}))\subset\pi_1(W)$. g is of the form $\prod_{i=1}^{p} b_i a_i$ where $a_i=a_i(d_k)\in D$ and b_i is the product of some elements of $\{e_m\}$. We transform this as follows.

$$g=a_1(a_1^{-1}b_1a_1)b_2a_2\cdots$$
$$=a_1a_2(a_2^{-1}a_1^{-1}b_1a_1a_2)(a_2^{-1}b_2a_2)\cdots$$

Iterating similar transformations, we get

$$g=\prod_{i=1}^{p} a_i \prod_{i=1}^{p}\Big(\prod_{j=i}^{p} a_j\Big)^{-1} b_i\Big(\prod_{j=1}^{p} a_j\Big)$$
$$=\prod_{i=1}^{p} a_i \prod_{i=1}^{q} h_i^{-1}e_{m_i}^{\pm1}h_i \qquad (h_i\in D)$$

Since $g\in\pi_*(\pi_1(\widetilde{W}))$, we have $\partial g=1$. Hence, if we think ∂ as a homomorphism from a free group generated by $\{d_k, e_m\}$ to a free group generated by $\{c_k\}$ defined by the correspondence $\partial d_k=c_k$ and $\partial e_m=1$, then $\partial g=\prod_{i=1}^{p} a(c_k)$ can be transformed into 1 using $\{f_l(c_k)=1\}$. Namely, we can write

$$\prod_{i=1}^{p} a_i(c_k)=\prod_{j=1}^{r} g_j(c_k)f_{l_j}(c_k)^{\pm1}g_j^{-1}(c_k)$$

where $g_j(c_k)$ is an element of the free group generated by $\{c_k\}$. Replacing c_k by d_k formally, we have

$$\prod_{i=1}^{p} a_i(d_k)=\prod_{j=1}^{r} g_j(d_k)f_{l_j}(d_k)^{\pm1}g_j^{-1}(d_k)\,.$$

We conclude that

$$g=\prod_{j=1}^{r} g_jf_{l_j}(d_k)^{\pm1}g_j^{-1}\prod_{i=1}^{q} h_i^{-1}e_m^{\pm1}h_i \qquad (g_j,\ h_i\in D)\,.$$

This completes the proof of Assertion 2.3.

By Assertion 2.3, we need the elimination of a finite set $\{f_l(d_k), e_m\}$ only. This completes the proof of Theorem 2.1.

3. G-connected sums

In this section, we consider only manifolds of dimension $\geqq 3$.

DEFINITION 3.1. Let M_1 and M_2 be closed connected spin manifolds whose fundamental groups are isomorphic to a group G. We call a closed connected spin manifold M_3 a *G-connected sum of M_1 and M_2* if it satisfies the following conditions:

1) There is a spin manifold B such that $\partial B = M_3 - (M_1 + M_2)$.

2) $\pi_1(M_3) \cong G$.

3) $\iota_{i*}: \pi_1(M_i) \cong \pi_1(B)$ where ι_i is the inclusion mapping $M_i \to B$; $i=1, 2, 3$.

We shall denote by $[M_1 \# M_2]_G$ the set of all G-connected sums of M_1 and M_2.

PROPOSITION 3.2. *We can construct a manifold M_3 satisfying the conditions of Definition 3.1 by surgerying 1-spheres imbedded in the connected sum of M_1 and M_2.*

PROOF. If $M_1 \# M_2$ denotes the usual connected sum of M_1 and M_2, then $\pi_1(M_1 \# M_2)$ is isomorphic to the free product $G*G$. Identifying the same elements of left and right G, we get a homomorphism $\rho: G*G \to G$. The composition of the classifying map of the universal covering of $M_1 \# M_2: M_1 \# M_2 \to K(G*G, 1)$ and the map $K(G*G, 1) \to K(G, 1)$ whose induced map: $\pi_1(K(G*G, 1)) \to \pi_1(K(G, 1))$ coincides with ρ gives a map $f: M_1 \# M_2 \to K(G, 1)$.

Let E be the universal G-bundle over $K(G, 1)$. Then, using the homotopy exact sequences of $p: f^*E \to M_1 \# M_2$ and $E \to K(G, 1)$, we easily see that the induced bundle f^*E is connected.

Note that in the homotopy exact sequences of $p: f^*E \to M_1 \# M_2$, $\partial: \pi_1(M_1 \# M_2) \to G$ coincides with $\rho: G*G \to G$. In fact, if we restrict f^*E on $M_i - D_i$; $i=1, 2$ where D_i is a disk embedded in M_i, $f^*E|_{M_i - D_i}$ can be seen as the universal covering space since $\pi_1(M_i) \cong \pi_1(M_i - D_i) \cong G \subset \pi_1(M_1 \# M_2)$ and $\rho|_{\pi_1(M_i - D_i)}$ is an isomorphism $G \to G$. Identifying G with the covering transformation group, ∂ coincides with ρ because they coincide on left and right G which generate $G*G$.

The argument analogous to those of section 2 shows that we can eliminate $p_*\pi_1(f^*E) \subset \pi_1(M_1 \# M_2)$ by equivariant surgeries preserving spin structures. We get a compact spin manifold with boundary B' and its connected G-covering $q: \tilde{B}' \to B'$ which has properties:

1) $\partial B' = M_3 - (M_1 \# M_2)$.

2) $\tilde{B}'|_{M_1 \# M_2} = f^*E$.

3) $q^{-1}(M_3)$ is 1-connected.

Killing $q_*\pi_1(\tilde{B}') \subset \pi_1(B')$ by the same technique as before, we may assume $G \cong \pi_1(M_3) \cong \pi_1(B')$. It is easily seen that $\pi_1(M_1 \# M_2) \to \pi_1(B') \cong \pi_1(M_3)$ is $\rho: G*G \to G$.

On the one hand, there is a spin bordism B'' such that $\partial B'' = M_1 \# M_2 - (M_1 + M_2)$. We can easily see that the manifold B obtained by patching B' and B'' along $M_1 \# M_2$ satisfies the desired conditions.

REMARK. Let X, Y be manifolds with their fundamental groups isomorphic to a group G. Let $f: X\to BG$ and $g: Y\to BG$ denote the classifying maps of the universal coverings $\tilde{X}\to X$ and $\tilde{Y}\to Y$ respectively. Then the set $[X\# Y]_G$ defines a unique class $[X, f]+[Y, g]$ in $\Omega_*(BG)$. We define inductively $[\#_l X]_G =[X\#[\#_{l-1}X]_G]_G$ and call this l-fold G-connected sums of X. Then $[\#_l X]_G =l[X, f]$ in $\Omega_*(BG)$.

4. The G-bordism invariant $\hat{\boldsymbol{a}}$

Let X be a topological space and let $\Omega_*(X)$ denote the bordism group of X as in [3].

DEFINITION 4.1. Consider a pair (M, f) where M is a closed oriented manifold and $f: M\to X$ is a continuous map. We define $\hat{\boldsymbol{a}}(M, f)\in H_*(X; \boldsymbol{Q})$

$$\hat{\boldsymbol{a}}(M, f)=f_*([M]\cap\hat{\mathcal{A}}(M))$$

where $[M]$ is a fundamental class and $\hat{\mathcal{A}}(M)$ is the characteristic class associated with a multiplicative sequence $(x/2)/\sinh(x/2)$.

PROPOSITION 4.2. 1) *(M, f) depends only on the bordism class of (M, f). So we get a map $\hat{\boldsymbol{a}}: \Omega_*(X)\to H_*(X; \boldsymbol{Q})$.*

2) *$\hat{\boldsymbol{a}}(\alpha+\beta)=\hat{\boldsymbol{a}}(\alpha)+\hat{\boldsymbol{a}}(\beta)$ where $\alpha, \beta\in\Omega_*(X)$.*

3) *$\hat{\boldsymbol{a}}(\alpha\xi)=\hat{\boldsymbol{a}}(\alpha)\hat{A}(\xi)$ where $\alpha\in\Omega_*(X)$ and $\xi\in\Omega_*$.*

4) *For a continuous map $\varphi: X\to Y$, $\hat{\boldsymbol{a}}\varphi_*=\varphi_*\hat{\boldsymbol{a}}$.*

The proof of Proposition 4.2 is straightforward, so we omit it.

Next, we shall see the relation of $\hat{\boldsymbol{a}}$ with the higher $\hat{A}$-genus $\hat{\mathfrak{A}}$. For the definition of $\hat{\mathfrak{A}}$, see [1], page 215.

Let G be a finitely generated group and let $\bar{G}$ denote the free abelian group $G/[G, G]/\mathrm{Tor}$. There is a map $\phi: K(G, 1)\to K(\bar{G}, 1)$ such that $\phi_*: \pi_1(K(G, 1))\to\pi_1(K(\bar{G}, 1))$ equals the quotient map $G\to\bar{G}$.

Suppose the rank of $\bar{G}$ is equal to N. Then $K(\bar{G}, 1)=T^N$ and $H_*(K(\bar{G}, 1); \boldsymbol{Q}) =H_*(T^N; \boldsymbol{Q})$.

Consider now a closed manifold M with its fundamental group isomorphic to G and the classifying map $f: M\to K(G, 1)$. $\phi\cdot f$ induces the isomorphism $(\phi\cdot f)_*: H_1(M; \boldsymbol{Q})\to H_1(T^N; \boldsymbol{Q})$. We identify $\wedge^* H_1(M; \boldsymbol{Q})$ with $H_*(T^N; \boldsymbol{Q})$ by this isomorphism. It is a routine matter to verify the following.

PROPOSITION 4.3. *Let G, M, f be as above. Moreover, we assume that M is spin. Then under the above identification,*

$$\phi_*\hat{\boldsymbol{a}}(M, f)=\hat{\mathfrak{A}}(M)\in\wedge^* H_1(M; \boldsymbol{Z})/\mathrm{Tor}\subset\wedge^* H_1(M; \boldsymbol{Q})=H_*(T^N; \boldsymbol{Q}).$$

DEFINITION 4.4. For M as above, we define $\hat{\boldsymbol{a}}(M)$ to be $\hat{\boldsymbol{a}}(M, f)$ where $f: M\rightarrow K(G, 1)$ is the classifying map. This is well defined since f is unique up to homotopy.

5. Application

Throughout this section, M denotes a closed connected manifold of dimension $n\geqq 5$ and $\tilde{M}$ denotes its universal covering space.

THEOREM 5.1. *If $\pi_1(M)$ is isomorphic to $\boldsymbol{Z}_p$ (p odd) and M is not spin, M has a Riemannian metric of positive scalar curvature.*

PROOF. As an Ω_*-module, $\Omega_*(\boldsymbol{Z}_p)$ is generated by $[\boldsymbol{Z}_p, \boldsymbol{Z}_p]$ and $[T_i, S^{2i-1}]$; $i=1, 2, 3, \cdots$ where $[\boldsymbol{Z}_p, \boldsymbol{Z}_p]$ is the class of the action of $\boldsymbol{Z}_p$ on itself and T_i is the standard $\boldsymbol{Z}_p$ action on the sphere S^{2i-1} (cf. [1], p. 90). Then in $\Omega_*(\boldsymbol{Z}_p)$,

$$[\pi_1(M), \tilde{M}]=a_0[N_0][\boldsymbol{Z}_p, \boldsymbol{Z}_p]+\sum_{i=1}^{(n+1)/2} a_i[N_i][T_i, S^{2i-1}]$$

where a_i denote integers and $[N_i]\in\Omega_{n-2i+1}$.

By Theorem 2.1, we have a compact oriented manifold W which satisfies the conditions:

$$\partial W=M-(a_0N_0+\sum_{i=1}^{(n+1)/2} a_iN_iS^{2i-1}/T_i)$$

$$\iota_*: \pi_1(M)\cong\pi_1(W).$$

Since T_i preserves the standard metric of S^{2i-1}, S^{2i-1}/T_i admits a metric of positive scalar curvature. Note that M is not spin. Then Theorem 1.4 shows that M carries a metric of positive scalar curvature.

THEOREM 5.2. *If $\pi_1(M)$ is isomorphic to $\boldsymbol{Z}_2$ and $\tilde{M}$ is not spin, M has a metric of positive scalar curvature.*

PROOF. In [5], K. Shibata determined the structure of $\Omega_*(\boldsymbol{Z}_2)$. In particular, as an Ω_*-module, $\Omega_*(\boldsymbol{Z}_2)$ has the following two types of generators.

1) $[a, S^0]$ and $[a, S^{2i+1}]$; $i=1, 2, 3, \cdots$, where a is an antipodal involution.

2) $E^{2i+1}(W_\omega)$; $i=1, 2, 3, \cdots$, $\omega\in\pi$ where π is the set of partitions of the form $(a_1, \cdots, a_r)$ with unequal parts a_i none of which is a power of 2.

W_ω is described as follows.

Let $\mathcal{W}_*=\boldsymbol{Z}_2[X_{2k-1}, X_{2k}; k\neq 2^i, (X_{2i})^2]$ be the polynomial subalgebra of $\mathfrak{N}_*$ defined by Wall in [6]. For an n-sphere S^n and an antipodal involution a, $\hat{\Omega}_*^+(S^n, a)$ (resp. $\hat{\Omega}_*^-(S^n, a)$) denotes the set of bordism classes of triples (N, μ, f) with N a closed oriented differentiable manifold, $\mu: N\rightarrow N$ a fixed point free differentiable involution which preserves (resp. reverses) the orientation, and $f:(N, \mu)\rightarrow(S^{n}, a)$ a continuous equivariant map. The map $\eta:\Omega_{*}^{-}(S^{1}, a)\rightarrow\mathfrak{N}_{*}$ sending $[N, \mu, f]$ to $[N/\mu]$ is a (ring) isomorphism onto $\mathcal{W}_{*}$. For each partition $\omega\in\pi$, $W_{\omega}\in\hat{\Omega}_{2|\omega|}^{-}(S^{1}, a)$ is given by $W_{\omega}=\eta^{-1}(X_{\omega})$ where $X_{\omega}=X_{2a_{1}}\cdots X_{2a_{r}}$ for $\omega=(a_{1}, \cdots, a_{r})$ and $|\omega|=a_{1}+\cdots+a_{r}$.

We shall give an explicit expression of $E^{2i+1}(W_{\omega})$.

Since $w_{1}(X_{\omega})$ comes from the integral class, we have an orientation bundle $\bar{X}_{\omega}\rightarrow X_{\omega}$ with an orientation reversing action μ_{ω} on $\bar{X}_{\omega}$ which is induced by the classifying map $f_{\omega}:X_{\omega}\rightarrow S^{1}$. Then $W=[\bar{X}_{\omega}, \mu_{\omega}, f_{\omega}]\in\hat{\Omega}_{*}^{-}(S^{1}, a)$.

LEMMA 5.3.

$$E^{2i+1}(W_{\omega})=[a\times 1, (S'^{2i+1}\times\bar{X}_{\omega})/\bar{a}\times\mu_{\omega}]$$

where

$$S'^{2i+1}=\{(z_{0}, \cdots, z_{i+1}, t)\in \boldsymbol{C}^{i+1}\times\boldsymbol{R}\,;\, z_{0} \text{ is real},$$
$$|z_{0}|^{2}+\cdots+|z_{i+1}|^{2}+t^{2}=1\}$$

and the action $\bar{a}$ on S'^{2i+1} is given by the correspondence

$$(z_{0}, \cdots, z_{i+1}, t)\longrightarrow(-z_{0}, \cdots, -z_{i+1}, t)$$

and a denotes the antipodal involution of S'^{2i+1}.

PROOF. In the proof of Lemma 5.3, we follow the notations of [5]. By the definition of E^{2i+1}, $E^{2i+2}(W_{\omega})\in\Omega_{*}^{-}(S^{2i+3}, a)$ has the representation

$$[S_{0}^{2i+2}\times\bar{X}_{\omega}/a\times\mu_{\omega},\ a\times 1,\ \hat{\mu}|_{(S_{0}^{2i+2}\times S^{1}/a\times a)}\cdot(id\times\bar{f}_{\omega}/a\times\mu_{\omega})]$$

where

$$S_{0}^{2i+2}=\{(z_{0}, \cdots, z_{i+2})\in S^{2i+3}\,;\, z_{0} \text{ is real}\}$$

and $\hat{\mu}:(S^{2i+3}\times S^{1}/a\times a, a\times 1)\rightarrow(S^{2i+3}, a)$ is given by the multiplication $\mu:S^{2i+3}\times S^{1}\rightarrow S^{2i+3}$ and $\bar{f}_{\omega}:\bar{X}_{\omega}\rightarrow S^{1}$ is the map which covers f_{ω}.

$E^{2i+1}:\hat{\Omega}_{*}^{-}(S^{1}, a)\rightarrow\hat{\Omega}_{*}^{+}(S^{2i+2}, a)$ is defined by $E^{2i+1}=\Delta\cdot E^{2i+2}$ where Δ is the Smith homomorphism.

For simplicity, we write only $\hat{\mu}$ or μ for $\hat{\mu}|_{(S_{0}^{2i+2}\times S^{1}/a\times a)}$ or $\mu|_{S_{0}^{2i+2}\times S^{1}}$. If we note f_{ω} is surjective, it can be seen that $\hat{\mu}\cdot(id\times\bar{f}_{\omega}/a\times\mu_{\omega})$ is transverse to $S^{2i+2}\subset S^{2i+3}$. By the definition, $\mu^{-1}(S^{2i+2})=A^{+}\cup A^{-}$ where

$$A^{\pm}=\{(z_{0}, \cdots, z_{i+1}, \pm\{1-(|z_{0}|^{2}+\cdots+|z_{i+1}|^{2})\}^{1/2}/x,\ x)$$
$$\in\boldsymbol{C}^{i+1}\times\boldsymbol{C}\,;\, z_{0} \text{ is real}, |z_{0}|^{2}+\cdots+|z_{i+1}|^{2}\leqq 1\}\,.$$

The diffeomorphism $S'^{2i+1}\times S^{1}\rightarrow\mu^{-1}(S^{2i+2})$ is given by the correspondence:

$$(z_{0}, \cdots, z_{i+1}, t, x)\longrightarrow(z_{0}, \cdots, z_{i+1}, t/x, x)$$

Then we have the diffeomorphism $S'^{2i+1}\times\bar{X}_{\omega}\rightarrow(id\times\bar{f})^{-1}\mu^{-1}(S^{2i+2})$ given by the correspondence from $S'^{2i+1}\times\bar{X}_\omega$ into $S_0^{2i+2}\times\bar{X}_\omega$.

$$\{(z_0, \cdots, z_{i+1}, t), y\} \longrightarrow \{(z_0, \cdots, z_{i+1}, t/\bar{f}(y)), y\}$$

We can see that the action $a\times 1$ (resp. $\bar{a}\times\mu_\omega$) on $S'^{2i+1}\times\bar{X}_\omega$ is induced by the action $a\times 1$ (resp. $\bar{a}\times\mu_\omega$) on $S_0^{i+2}\times\bar{X}_\omega$ and this completes the proof of the lemma.

Since X_ω is a product space of Dold manifolds, X_ω admits a metric of positive scalar curvature. $(S'^{2i+1}\times\bar{X}_\omega/\bar{a}\times\mu_\omega)/a\times 1$ also carries positive scalar curvature. Then Theorem 5.2 follows by the argument analogous to that of Theorem 5.1.

THEOREM 5.4. *Let M be a spin manifold whose fundamental group is a finite group G. Suppose $\hat{A}(M)=0$, then there exists $l\in\boldsymbol{N}$ such that each member of $[\#_l M]_G$ carries a metric of positive scalar curvature.*

PROOF. Analogously to $\Omega_*(G)$ (cf. [1], page 50), we can define $\Omega_*^{\mathrm{Spin}}(G)$. Namely, each manifold is endowed with a spin structure and each action is accompanied with a lifted action on the associated principal Spin $(*)$-bundle. For a pair of spaces (X, A), we can also define $\Omega_*^{\mathrm{Spin}}(X, A)$ when every manifold is spin.

LEMMA 5.5. 1) *$\Omega_*^{\mathrm{Spin}}(G)\cong\Omega_*^{\mathrm{Spin}}(BG)$ as Ω_*^{Spin}-modules. This isomorphism is given by $[T, N]\to[N/T, f]$ where f is the classifying map of $N\to N/T$.*

2) *As $\Omega_*^{\mathrm{Spin}}\otimes\boldsymbol{Q}$-modules, $\Omega_*^{\mathrm{Spin}}(X, A)\otimes\boldsymbol{Q}$ is isomorphic to $\Omega_*(X, A)\otimes\boldsymbol{Q}$ for any CW-pair(X, A).*

3) *$\Omega_*^{\mathrm{Spin}}(G)\otimes\boldsymbol{Q}\cong\Omega_*(G)\otimes\boldsymbol{Q}\cong\Omega_*\otimes\boldsymbol{Q}$.*

PROOF. The proof of 1) is referred to [1], page 51. 2) can be proved using the Eilenberg-Steenrod axioms for homology theory. Combining 1) and 2), we get 3).

Choose generators $\{M_j\}_{j=1}^\infty$ of the polynomial algebra $\Omega_*^{\mathrm{Spin}}\otimes\boldsymbol{Q}$ where M_j is a closed spin manifold of dimension $4j$ and $M_j=\boldsymbol{HP}^j$ for $j\geqq 2$. Since $[M_1]=c[\boldsymbol{CP}^2]$; $c\in\boldsymbol{Q}-\{0\}$, $\hat{A}(M_1)\neq 0$ and M_1 cannot carry a metric of positive scalar curvature.

Let $[G, G\times M_j]$ be the class where G acts on $G\times M_j$ by the left multiplication. Let P be the ideal in $\Omega_*\otimes\boldsymbol{Q}$ generated by $\{M_j\}_{j\geq 2}$ and $\tilde{P}$ the submodule of $\Omega_*^{\mathrm{Spin}}(G)\otimes\boldsymbol{Q}$ generated by $[G, G\times M_j]$; $j\geqq 2$. Then we get an isomorphism $\Omega_*^{\mathrm{Spin}}(G)\otimes\boldsymbol{Q}/\tilde{P}\to\Omega_*\otimes\boldsymbol{Q}/P$. Moreover, $\Omega_*\otimes\boldsymbol{Q}/P\cong\boldsymbol{Q}[M_1]$ (the polynomial algebra generated by $[M_1]$) and the $\hat{A}$-genus gives an isomorphism $\boldsymbol{Q}[M_1]\cong\boldsymbol{Q}$.

Since $\hat{A}(M)=0$, we have $[\pi_1(M), \tilde{M}]\in\tilde{P}$. So there is a number l such that

$$l[\pi_1(M), \tilde{M}]=\sum_{j\geq 2}[G, G\times M_j][N_j]$$

in $\Omega_*^{\mathrm{Spin}}(G)$ where $[N_j]\in\Omega_*^{\mathrm{Spin}}$. Take any $M'\in[\sharp_l M]_G$. Then the universal covering $\tilde{M}'$ of M' gives a pair $[\pi_1(M'), \tilde{M}']$ which determines the same class as $l[\pi_1(M), \tilde{M}]$ in $\Omega_*^{\mathrm{Spin}}(G)$. It is easily seen that Theorem 2.1 holds when every manifold is spin. Then together with Theorem 1.1, the result follows.

THEOREM 5.6. *Assume that $\pi_1(M)\cong\oplus_k\boldsymbol{Z}$. If M is not spin and $n>k$, then M carries a metric of positive scalar curvature. If M is spin and $\hat{\mathfrak{A}}(M)=0$, there exists $l\in\boldsymbol{N}$ such that each member of $[\sharp_l M]_{\oplus_k Z}$ carries a metric of positive scalar curvature.*

PROOF. Let N be an oriented manifold with an orientation preserving free G action T where G is a descrete group (not necessary finite). We consider only the case when N/G is compact. We can define the bordism class of such a pair (T, N). Denote the collection of all such classes by $\Omega c_*(G)=\sum_{n=0}^{\infty}\Omega c_n(G)$. Just like the case of the finite group, $\Omega c_*(G)$ has an Ω_*-module structure and $\Omega c_*(G)$ is isomorphic to $\Omega_*(BG)$ as Ω_*-modules.

First, note that $B(\oplus_k\boldsymbol{Z})=T^k$ (a k dimensional torus). We denote by S_i the 1-sphere at the i-th factor of T^k and its inclusion by $\iota_i: S_i\to T^k$; $i=1, \cdots, k$. For each multi-index $I=\{i_1, \cdots, i_p\}$ such that $1\leqq i_1<\cdots<i_p\leqq k$, let $\iota_I: T_I\to T^k$ be the inclusion where $T_I=S_{i_1}\times\cdots\times S_{i_p}$. If $I=\phi$, T_ϕ denotes one point in T^k and ι_ϕ is its inclusion.

LEMMA 5.7. *As an Ω_*-module, $\Omega_*(T^k)$ has a homogeneous base $\{[T_I, \iota_I]\}$.*

PROOF. $\{\iota_{I*}[T_I]\}$ forms an additive base for $H_*(T^k; \boldsymbol{Z})$. Then by [1], Theorem (18.1), the result follows.

The argument analogous to the proof of Theorem 5.1 together with Lemma 5.7 shows that there is a bordism which preserves the fundamental group $(W; M, \Sigma_I T_I\times M_I)$. Since $n>k$, $T_I\times M_I$ carries positive scalar curvature. By Theorem 1.4, the first half follows.

By Lemma 5.7 and Lemma 5.5, we have an isomorphism $\Omega_*^{\mathrm{Spin}}(T^k)\otimes\boldsymbol{Q}\cong\Sigma_I[T_I, \iota_I]\Omega_*\otimes\boldsymbol{Q}$.

We indentify $H_*(T^k; \boldsymbol{Q})$ with $\wedge^*(H_1(T^k; \boldsymbol{Z})\otimes\boldsymbol{Q})$. Denote $\iota_{i*}[S_i]\in H_1(T^k; \boldsymbol{Z})$ by x_i^*. We consider $\hat{\boldsymbol{a}}$ as a homomorphism $\Omega_*^{\mathrm{Spin}}(X)\otimes\boldsymbol{Q}\to H_*(X; \boldsymbol{Q})$ where X is any space. Since

$$\hat{\boldsymbol{a}}([T_I, id_{T_I}])=x_I^*\in H_*(T_I; \boldsymbol{Q}),$$

we get

$$\hat{\boldsymbol{a}}([T_I, \iota_I])=x_I^*\in H_*(T^k; \boldsymbol{Q})$$

where $x_I^*=x_{i_1}^*\wedge\cdots\wedge x_{i_p}^*$ for $I=\{i_1, \cdots, i_p\}$.

Let $\{M_i\}_{i=1}^{\infty}$ be the base for the polynomial algebra $\Omega_*^{\mathrm{Spin}}\otimes\boldsymbol{Q}$ as before. P

denotes an ideal generated by $\{M_i\}_{i\geq 2}$. Define the submodule $\tilde{P}\subset\Omega_*^{\mathrm{Spin}}(T^k)$ by $\tilde{P}=\Sigma_I[T_I, \iota_I]P$. Then we have isomorphisms:

$$\Omega_*^{\mathrm{Spin}}(T^k)\otimes\boldsymbol{Q}/\tilde{P}\cong\Sigma_I[T_I, \iota_I]\Omega_*^{\mathrm{Spin}}\otimes\boldsymbol{Q}/P$$
$$\cong\Sigma_I[T_I, \iota_I]\boldsymbol{Q}[M_1]$$

Since $\hat{A}(\xi)=0$ for $\xi\in P$, we have $\hat{\boldsymbol{a}}(\alpha)=0$ for $\alpha\in\tilde{P}$ by Proposition 4.2, 3). $\hat{\boldsymbol{a}}$ is reduced to

$$\hat{\boldsymbol{a}}: \Omega_*^{\mathrm{Spin}}(T^k)\otimes\boldsymbol{Q}/\tilde{P}\longrightarrow H_*(T^k).$$

LEMMA 5.8. *If $\hat{\boldsymbol{a}}(\alpha)=0$ for $\alpha\in\Omega_l^{\mathrm{Spin}}(T^k)\otimes\boldsymbol{Q}$, then α belongs to $\tilde{P}$.*

PROOF. Let $\bar{\alpha}$ be the class of α in $\Omega_*^{\mathrm{Spin}}(T^k)\otimes\boldsymbol{Q}/\tilde{P}$. $\bar{\alpha}$ is of the form $\Sigma_I a_I[T_I, \iota_I][M_I]^{(l-\dim T_I)/4}$; $a_I\in\boldsymbol{Q}$. Then we have

$$\hat{\boldsymbol{a}}(\alpha)=\Sigma_I a_I x_I^*\hat{A}(M_1)^{(l-\dim T_I)/4}.$$

On the one hand, $\hat{\boldsymbol{a}}(\alpha)=0$. So $a_I=0$. Then $\bar{\alpha}=0$. This implies $\alpha\in\tilde{P}$.

By Proposition 4.3, $\hat{\boldsymbol{a}}(M)=\hat{\mathfrak{A}}(M)=0$. Using Lemma 5.8, we see the class $[M, f]$ belongs to $\tilde{P}$ where f is the classifying map of the universal covering. The remainder of the proof is analogous to that of Theorem 5.4. The proof of Theorem 5.6 is completed.

THEOREM 5.9. *Assume that $\pi_1(M)\cong\boldsymbol{F}_m$. If M is not spin, M carries positive scalar curvature. If M is spin and $\hat{\mathfrak{A}}(M)=0$, there exists $l\in\boldsymbol{N}$ such that each member of $[\sharp_l M]_{\boldsymbol{F}_m}$ carries a metric of positive scalar curvature.*

PROOF. The classifying space for $\boldsymbol{F}_m$ is $S^1\vee\cdots\vee S^1$ (m-fold)$=\vee_m S^1$. Similarly to Lemma 5.7, we have the following.

LEMMA 5.10. *As an Ω_*-module, $\Omega_*(\vee_m S^1)$ has a homogeneous base*

$$\{[S_i, \iota_i]; i=1, \cdots, m, [x_0, \iota_0]\}$$

where S_i denotes the i-th 1-sphere of $\vee_m S^1$ with the inclusion mapping $\iota_i: S_i\to\vee_m S^1$ and x_0 is the point in $\vee_m S^1$ with the inclusion mapping $\iota_0: x_0\to\vee_m S^1$.

By the spectral sequence associated with the universal covering, we can see that $H^2(M; \boldsymbol{Z}_2)\to H^2(\tilde{M}; \boldsymbol{Z}_2)$ is injective. Thus $\tilde{M}$ is not spin. The remainder of the first half is analogous to that of Theorem 5.1.

There exists a continuous mapping $\phi: \vee_m S^1\to T^m$ such that the induced homomorphism $\pi_1(\vee_m S^1)\to\pi_1(T^m)$ coincides with $\boldsymbol{F}_m\to\boldsymbol{F}_m/[\boldsymbol{F}_m, \boldsymbol{F}_m]$. Then by Proposition 4.3, we have $\phi_*\hat{\boldsymbol{a}}(M)=\hat{\mathfrak{A}}(M)$. Since $\hat{\mathfrak{A}}(M)=0$ and ϕ_* is injective, we have $\hat{\boldsymbol{a}}(M)=0$. The remainder of the proof is analogous to that of Theorem 5.6.

References

[1] Conner, D. E. and E. E. Floyd, Differentiable periodic maps, Ergeb. Math. Grenzgeb. N. F., Band 33, Springer Verlag, Berlin, 1964.

[2] Gromov, M. and H. B. Lawson, Jr., The classification of simply connected manifolds of positive scalar curvature, Ann. of Math. **111** (1980), 423-434.

[3] Gromov, M. and H. B. Lawson, Jr., Spin and scalar curvature in the presence of a fundamental group I, Ann. of Math. **111** (1980), 209-230.

[4] Schoen, R. and S. T. Yau, The structure of manifolds with positive scalar curvature, Manuscripta Math. **28** (1979), 159-183.

[5] Shibata, K., Oriented and weakly complex bordism algebra of free periodic maps, Trans. Amer. Math. Soc. **177** (1973), 199-220.

[6] Wall, C. T. C., Determination of the cobordism ring, Ann. of Math. (2) **72** (1960), 292-311.

[7] Wall, C. T. C., Geometrical connectivity I, J. London Math. Soc. (2) **3** (1971), 597-604.

(Received September 7, 1982)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan