On the equation of nonstationary stratified fluid motion: Uniqueness and existence of the solutions

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§ 1. Introduction.

Let Ω be a bounded domain in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ with a smooth boundary. We consider an inhomogeneous viscous incompressible fluid occupying Ω. Let ρ, u, p be the mass density, the velocity vector and the pressure of the fluid, respectively. Then these quantities obey the following system of equations:

$$\frac{\partial \rho}{\partial t}+u\cdot\nabla\rho=0 \qquad (0<t,\ x\in\Omega), \tag{1.1}$$

$$\rho\left\{\frac{\partial u}{\partial t}+(u\cdot\nabla)u\right\}=\Delta u-\nabla p \qquad (0<t,\ x\in\Omega), \tag{1.2}$$

$$\operatorname{div} u=0 \qquad (0<t,\ x\in\Omega). \tag{1.3}$$

Here we have assumed for simplicity that the viscosity is unity and external force is absent. In what follows we solve (1.1), (1.2) and (1.3) under the initial-boundary conditions below:

$$u|_{\partial\Omega}=0 \tag{1.4}$$

$$u|_{t=0}=a(x), \qquad \rho|_{t=0}=\rho_0(x). \tag{1.5}$$

This system is a generalization of the Navier-Stokes system. In fact, if the initial value $\rho_0(x)$ is a positive constant, our system is reduced to the Navier-Stokes system because of the uniqueness of the solution (which will be proved in the context of the present paper).

The mathematical study for the initial-value problem (1.1), ⋯, (1.5) was initiated by Kazhikhov [11] and there he proved the existence of a weak solution of Hopf-type and also a classical solution. However, he did not show the uniqueness of the solution. Later Ladyzhenskaya and Solonnikov [13] proved the unique existence in the framework of the L^p-theory. They obtained the global solution for $n=2$ (n is the dimension of the domain Ω). However, they required that $p>n$. Furthermore they did not mention the global solution in the case of $n=3$. On the other hand, Lions [14] proved the existence of a weak solution different from Kazhikhov's without uniqueness even in the two dimensional

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problem. Marsden [15] dealt with the case of an inviscid inhomogeneous fluid, i.e., he solved the system in which the term Δu in (1.2) is dropped.

In this paper we employ the L^2-theory and prove the unique existence (local in time) of the solution. The main tool is the theory of linear evolution equations. This local existence theorem is used to prove the following global existence theorem:

I) In the two dimensional problem the solution always exists globally in time.

II) In the three dimensional problem the global solution is obtained, if the initial values are sufficiently small.

The important fact is that in the two dimensional problem we do not need to assume smallness of the initial values.

This paper consists of seven sections. In section 2 we give various function spaces and we formulate (1.1), $\cdots$, (1.5) in the framework of the theory of evolution equations. Main theorems are also stated in this section. Sections 3, 4 and 5 deal with the proof of the local existence theorem. The global existence of the solution of the two-dimensional problem is proved in section 6. The three dimensional case is considered in section 7. Here we use a technique inspired by Matsumura and Nishida [16].

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§ 2. Function spaces and abstract formulation of the problem.

We solve (1.1), $\cdots$, (1.5) by Fujita-Kato's method (see Fujita and Kato [4]). To this end we use the following function spaces:

$$C_{0,\sigma}^{\infty}(\Omega)=\{v=(v_1, \cdots, v_n)\in C_0^{\infty}(\Omega)^n ;\ \operatorname{div} v=0 \text{ in } \Omega\}$$

(n is 2 or 3 according as $\Omega\subset \boldsymbol{R}^2$ or $\Omega\subset \boldsymbol{R}^3$),

$$W^{m,q}(D)=\{f\in L^q(D) ;\ \|\partial^{\alpha}f\|_{L^q(D)}<\infty \quad (|\alpha|\leqq m)\}$$

($m=0, 1, 2, \cdots, 1\leqq q\leqq\infty$, $D=\Omega$ or $]0, T[\times\Omega$),

V; the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $H_0^1(\Omega)^n (=\{f\in W^{1,2}(\Omega)^n ;\ f|_{\partial\Omega}=0\})$,

H; the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $L^2(\Omega)^n$,

$L(H)$; the Hilbert space composed of all bounded linear operators in H.

We also use the following symbols:

P; the orthogonal projection from $L^2(\Omega)^n$ onto H,

A; the Stokes operator, i.e.,

$$A=-P\Delta, \qquad D(A) \text{ (the domain of } A)=H^2(\Omega)^n\cap V .$$

It is well-known that A is a positive definite self-adjoint operator in H and is characterized by the relation

$$(Aw, v)=(\nabla w, \nabla v) \qquad (w\in D(A),\ v\in V).$$

Here and in what follows we denote the inner-product in H (i.e., the L^2 inner-product) by $(\ ,\ )$. The L^2-norm is denoted by $\|\ \|$ and the L^∞-norm by $\|\ \|_\infty$.

The following assumptions on the initial data are employed throughout this paper.

(A.1) The initial value ρ_0 belongs to $W^{1,\infty}(\Omega)$ and satisfies

$$\min_{x\in\bar{\Omega}} \rho_0(x)>0.$$

Hereafter we put $m=\min_{x\in\bar{\Omega}} \rho_0(x)$ and $l=\max_{x\in\bar{\Omega}} \rho_0(x)$.

(A.2) The initial value a belongs to $D(A^\eta)$, where η is a number satisfying

$$n/4<\eta<1 \qquad (\Omega\subset \boldsymbol{R}^n,\ n=2 \text{ or } 3).$$

Now we formulate (1.1), $\cdots$, (1.5) as follows: Find $\rho\in W^{1,\infty}(]0, T[\times\Omega)$ and

$u\in C([0, T];\ H)\cap C^1(]0, T];\ H)\cap C(]0, T];\ D(A))$ such that

$$\frac{\partial\rho}{\partial t}+u\cdot\nabla\rho=0 \qquad (\text{a.e. } (t, x)\in]0, T[\times\Omega), \tag{2.1}$$

$$B_\rho(t)\frac{du}{dt}+Au(t)+F_\rho u(t)=0 \qquad (0<t<T), \tag{2.2}$$

$$u(0)=a,\ \rho(0, x)=\rho_0(x). \tag{2.3}$$

Here $B_\rho(t)$ is a bounded linear operator in H defined by

$$B_\rho(t)w=P\rho(t)w\equiv P\rho(t, \cdot)w \qquad (w\in H) \tag{2.4}$$

and F_ρ is a nonlinear operator defined by

$$F_\rho w=P\{\rho(t)(w\cdot\nabla)w\}. \tag{2.5}$$

Later we see that $F_\rho w$ is well-defined in H if $w\in D(A^{5/8})$ $(\Omega\subset\boldsymbol{R}^3)$ or if $w\in D(A^\eta)$ $(\Omega\subset\boldsymbol{R}^2,\ 1/2<\eta)$. The main results in this paper are stated as follows.

THEOREM 2.1. (*local existence theorem*). *We assume* (A.1) *and* (A.2). *The dimension of* Ω *may be two or three.*

i) *For any* $0<m_0<l_0<\infty$, $0<k_1$ *and* $0<k_2$ *there exists a positive constant* $T_0=T_0(\Omega, m_0, l_0, k_1, k_2)$ *such that a solution* $\{\rho, u\}$ *exists in* $[0, T_0]$ *provided that* $m_0<m$, $l<l_0$, $\|A^\eta a\|<k_1$ *and* $\|\nabla\rho_0\|_\infty<k_2$. *The solution is unique in* $W^{1,\infty}(]0, T_0[\times\Omega)\cap C([0, T_0];\ D(A^\eta))$.

ii) *Furthermore* u *and* ρ *satisfy the following inequalities*:

$$\|A^\alpha u(t)\|\leqq c_0\|A^\alpha a\| \qquad (\alpha=5/8,\ \eta;\ 0\leqq t\leqq T_0), \tag{2.6}$$

$$\|\nabla\rho(t)\|_\infty\leqq c_0\|\nabla\rho_0\|_\infty \exp(c_0\|A^\eta a\|) \qquad (0\leqq t\leqq T_0), \tag{2.7}$$

$$\left\| \frac{\partial \rho}{\partial t}(t) \right\|_\infty \leqq c_0 \|\nabla \rho_0\|_\infty \|A^\gamma a\| \exp(c_0 \|A^\gamma a\|) \qquad (0 \leqq t \leqq T_0). \tag{2.8}$$

Here c_0 is a positive constant depending only on Ω, m_0 and l_0.

THEOREM 2.2. *If Ω is a two dimensional domain, then the solution always exists in* $[0, \infty[$ *under the assumptions* (A.1) *and* (A.2).

THEOREM 2.3. *If Ω is a three dimensional domain, then there exists a positive constant $\varepsilon_1 = \varepsilon_1(\Omega, m_0, l_0)$ satisfying the following property: $\{\rho, u\}$ exists in $[0, \infty[$ so long as*

$$m_0 \leqq m, \ l \leqq l_0, \ \|A^\gamma a\| \leqq \varepsilon_1 \quad \text{and} \quad \|\nabla \rho_0\|_\infty \leqq \varepsilon_1 .$$

REMARK 2.1. Note that in THEOREM 2.1 we do not require smallness of $\|A^\gamma a\|$ or $\|\nabla \rho_0\|_\infty$.

The proof of THEOREM 2.1 is carried out by the successive approximation method, i.e., we put

$$u_1(t) \equiv e^{-tA} a \quad \text{and} \quad \rho_1(t, x) \equiv \rho_0(x).$$

And if u_{k-1} and ρ_{k-1} are given, we define ρ_k and u_k by the following linear equations, respectively.

$$\frac{\partial \rho_k}{\partial t} + u_{k-1} \cdot \nabla \rho_k = 0 \qquad (0 < t, \ x \in \Omega), \tag{2.9}$$

$$\rho_k(0, x) = \rho_0(x), \tag{2.10}$$

$$B_{\rho_{k-1}}(t) \frac{du_k}{dt} + A u_k(t) + F_{\rho_{k-1}} u_{k-1} = 0 \qquad (0 < t), \tag{2.11}$$

$$u_k(0) = a. \tag{2.12}$$

In the following sections we justify this procedure by making use of the theory of evolution equations. Here we summarize a general theory.

Let X be a Banach space with norm $\| \ \|$. The operator norm of a bounded linear operator in X is also denoted by $\| \ \|$. Assume that $\{Au(t)\}_{0 \leq t \leq T}$ is a family of densely defined closed linear operators in X satisfying the following properties:

(H-1) For any $t \in [0, T]$ the half plane $\{z \in \boldsymbol{C}; \operatorname{Re} z \leqq 0\}$ is contained in the resolvent set of $A(t)$ and we have

$$\|(z - A(t))^{-1}\| \leqq c/(1 + |z|) \qquad (\operatorname{Re} z \leqq 0) \tag{2.13}$$

with a constant c independent of t and z.

(H-2) The domain of $A(t)$ is independent of t. Furthermore we have for any $r, s, t \in [0, T]$

$$\|\{A(t) - A(s)\} A(r)^{-1}\| \leqq L |t - s|^\theta \tag{2.14}$$

with a constant $L > 0$, $\theta \in]0, 1]$ which are independent of r, s and t.

(H-3) There exists a positive constant c' such that for any $\alpha\in[0, 1]$ the following inequality holds true.

$$\|A(t)^{\alpha}A(s)^{-\alpha}\|\leqq c' \qquad (0\leqq t,\ s\leqq T).$$

Under these assumptions we have

THEOREM 2.4. (*Tanabe and Kato*). *There exists a unique evolution operator* $U(t, s)$ $(0\leqq s\leqq t\leqq T)$ *for the problem* $du/dt+A(t)u=0$ $(0<t<T)$. *Furthermore the following inequalities hold true.*

i) *For any* α *and* β *such that* $0\leqq\alpha<1+\theta$, $0\leqq\beta\leqq\min\{1, \alpha\}$ *we have*

$$\|A(t)^{\alpha}U(t, s)A(s)^{-\beta}\|\leqq c_1(t-s)^{-(\alpha-\beta)}+\phi(L)(t-s)^{-(\alpha-\beta)+\theta} \qquad (0<s<t<T). \tag{2.15}$$

ii) *For any* α, β *and* λ *such that* $0\leqq\beta\leqq\alpha<1$ *and* $0<\lambda<1-\alpha$ *we have*

$$\begin{aligned}&\|A(t)^{\alpha}\{U(t+h, s)-U(t, s)\}A(s)^{-\beta}\|\\ &\leqq c_2h^{\lambda}(t-s)^{-(\alpha-\beta)-\lambda}+\phi(L)h^{\lambda}(t-s)^{-(\alpha-\beta)-\lambda+\theta}\\ &\qquad (0<s<t\leqq t+h\leqq T).\end{aligned} \tag{2.16}$$

Here the constants c_1 *and* c_2 *depend only on* c, c', α, β, λ *and* T (*not on* θ *or* L). *The constant* $\phi(L)$ *is of the form*

$$\phi(L)=\tilde{c}L^2\exp(\tilde{c}L)$$

($\tilde{c}$ *is a constant independent of* L).

REMARK 2.2. From (2.15) we can estimate more roughly:

$$\|A(t)^{\alpha}U(t, s)A(s)^{-\beta}\|\leqq\phi(L)(t-s)^{-(\alpha-\beta)}. \tag{2.17}$$

For the proof of THEOREM 2.4, see Tanabe [18] page 127. The inequalities (2.15) and (2.16) do not seem to be well-known facts. However, one can show these inequalities by the same method as the proof of (2.17), which is a well-known inequality.

REMARK 2.3. To obtain $U(t, s)$, the assumption (H-3) is not necessary. But (H-3) is used to derive (2.15) and (2.16).

The next lemma is also useful.

LEMMA 2.1. *Let* T, α *and* β *be positive constants and* r *be a constant such that* $0<r<1$. *Then for any function* f; $[0, T]\to[0, \infty[$ *satisfying*

$$f(t)\leqq\alpha+\beta\int_0^t(t-s)^{-r}f(s)ds \qquad (0\leqq t\leqq T),$$

we have

$$f(t)\leqq c\alpha\exp\{c\beta^{1/(1-r)}t\} \qquad (0\leqq t\leqq T).$$

Here c *is a positive constant depending only on* r.

For the proof, see Okamoto [15].

§ 3. Preliminary linear problems.

Hereafter we consider the case where Ω is a three dimensional domain, since the two dimensional case goes similarly (even more easily). In this section, T is an arbitrary but fixed number. The following proposition will be used to solve (2.9) and (2.10).

PROPOSITION 3.1. *We assume* (A.1) *and* (A.2). *Let* u *be a continuous function on* $[0, T]\times\bar{\Omega}$ *such that*

$$\int_0^T \|\nabla u(t)\|_\infty dt < \infty . \tag{3.1}$$

Then there exists a unique $\rho\in W^{1,\infty}(]0, T[\times\Omega)$ *such that* $\partial\rho/\partial t+u\cdot\nabla\rho=0$ *for a.e.* $(t, x)\in]0, T[\times\Omega$ *and* $\rho(0, x)=\rho_0(x)$. *This* ρ *satisfies the following inequalities for any* $t\in]0, T[$;

$$m\leqq\rho(t, x)\leqq l \qquad (x\in\Omega), \tag{3.2}$$

$$\|\nabla\rho(t)\|_\infty\leqq c\|\nabla\rho_0\|_\infty \exp\left(\int_0^t\|\nabla u(s)\|_\infty ds\right), \tag{3.3}$$

$$\left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty\leqq c\|\nabla\rho_0\|_\infty\|u(t)\|_\infty \exp\left(\int_0^t\|\nabla u(s)\|_\infty ds\right), \tag{3.4}$$

with a constant c *depending only on* Ω.

We refer the proof to Ladyzhenskaya and Solonnikov [13]. It is, however, important to note that under the assumption (3.1) there exists the characteristic curve $\xi_{s,t}(x)$, i.e., $\xi_{s,t}(x)$ is a solution of

$$\xi_{s,t}(x)=x+\int_t^s u(r, \xi_{r,t}(x))dr \qquad (0\leqq s,\ t\leqq T,\ x\in\Omega). \tag{3.5}$$

The condition (3.1) ensures that (3.5) is uniquely solvable (see, Coddington and Levinson [3]). Note also that $\rho(t, x)=\rho_0(\xi_{0,t}(x))$, from which (3.2) is obvious.

To solve (2.11) and (2.12), consider $\rho\in W^{1,\infty}(]0, T[\times\Omega)$ and $v\in C([0, T]; D(A^\eta))$. We assume that

$$m\leqq\rho(t, x)\leqq l \qquad (0\leqq t\leqq T;\ x\in\Omega), \tag{3.6}$$

$$M_0\equiv\sup\left\{\left|\frac{\partial\rho}{\partial t}(t, x)\right|;\ 0\leqq t\leqq T,\ x\in\Omega\right\}<\infty \tag{3.7}$$

$$\|A^\alpha v(t)\|\leqq N \qquad (0<t\leqq T;\ \alpha=5/8,\ \eta), \tag{3.8}$$

$$\|Av(t)\|\leqq Nt^{-(1-\eta)} \qquad (0<t\leqq T), \tag{3.9}$$

$$\|A^{5/8}\{v(t)-v(s)\}\|\leqq N(t-s)^\theta s^{-\theta} \qquad (0<s<t<T). \tag{3.10}$$

Here N and $\theta\in]1/4, 3/8[$ are positive constants.

PROPOSITION 3.2. *We take γ and θ such that $1/4<\gamma<\theta<3/8$ and fix them. Under the conditions* (3.6), $\cdots$, (3.10) *there exists a unique* $u\in C([0, T]; D(A^{\eta}))\cap C(]0, T]; D(A))\cap C^{1}(]0, T]; H)$ *satisfying*

$$B_{\rho}(t)\frac{du}{dt}+Au(t)+F_{\rho}v(t)=0 \qquad (0<t<T), \tag{3.11}$$

$$u(0)=a. \tag{3.12}$$

Furthermore there exist positive constants $c_1=c_1(\Omega, m_0, l_0)$ and $c_2=c_2(\Omega, m_0, l_0, T)$ such that c_2 is increasing in T and we have the inequalities below if $m_0\leqq m$ and $l\leqq l_0$.

$$\|A^{\alpha}u(t)\|\leqq c_1\|A^{\alpha}a\|+t\phi(M_0)\|A^{\alpha}a\| +\phi(M_0)N^2t^{1-\alpha} \qquad (0\leqq t\leqq T;\ \alpha=5/8,\ \eta), \tag{3.13}$$

$$\|A^{5/8}\{u(t)-u(s)\}\|\leqq(t-s)^{\theta}\{c_1s^{-\theta}\|A^{5/8}a\| +\phi(M_0)\|A^{5/8}a\|+\phi(M_0)N^2\} \qquad (0<s<t<T), \tag{3.14}$$

$$\|Au(t)\|\leqq c_1t^{-(1-\eta)}\|A^{\eta}a\|+\phi(M_0)t^{\eta}\|A^{\eta}a\| +\phi(M_0)N^2 \qquad (0<t<T), \tag{3.15}$$

$$\|\nabla u(t)\|_{\infty}\leqq c_1t^{-(1+\gamma-\eta)}\|A^{\eta}a\|+\phi(M_0)t^{\eta-\gamma}\|A^{\eta}a\| +\phi(M_0)N^2t^{-\gamma} \qquad (0<t<T), \tag{3.16}$$

where we have put $\phi(M_0)=c_2(1+M_0)^3\exp(c_2M_0)$.

The proof of this proposition is divided into several parts.

First-step. If we put $A(t)\equiv B_{\rho}(t)^{-1}A$, then (3.11) is rewritten as

$$\frac{du}{dt}+A(t)u=-B_{\rho}(t)^{-1}F_{\rho}v(t). \tag{3.17}$$

In view of this equality we show that the family of operators $\{A(t)\}_{0\leq t\leq T}$ generates an evolution operator $U(t, s)$ $(0\leqq s\leqq t\leqq T)$. To start with, it is easy to see that $B_{\rho}(t)$ is a positive definite self-adjoint operator in H, whence $B_{\rho}(t)^{-1}$ is a bounded linear operator in H. Next we show that the family of operators $\{A(t)\}_{0\leq t\leq T}$ satisfies the hypotheses in THEOREM 2.4.

First it is obvious that $D(A(t))\equiv D(A)$. The inequality (2.4) follows from

$$\{A(t)-A(s)\}A(r)^{-1}=B_{\rho}(s)^{-1}\{B_{\rho}(s)-B_{\rho}(t)\}B_{\rho}(t)^{-1}B_{\rho}(r),$$

$$\|B_{\rho}(s)-B_{\rho}(t)\|\leqq\|\rho(s)-\rho(t)\|_{\infty}\leqq M_0|t-s|,$$

$$\|B_{\rho}(s)\|\leqq l_0 \quad\text{and}\quad \|B_{\rho}(s)^{-1}\|\leqq m_0^{-1}.$$

We can check (2.13) in the following way. Observe that

$$(z-A(t))^{-1}f=w \Longleftrightarrow B_{\rho}(t)f=zB_{\rho}(t)w-Aw$$

which is equivalent to

$$(\rho(t)f,\ v)=z(\rho(t)v,\ v)-(\nabla w,\ \nabla v) \qquad (v\in V). \tag{3.18}$$

If $\mathrm{Re}\,z\leqq 0$, then we have

$$\mathrm{Re}\,\{(\nabla v,\ \nabla v)-z(\rho(t)v,\ v)\}\geqq\|\nabla v\|^2 \qquad (v\in V).$$

Hence by the Lax-Milgram theorem we can assert that if $\mathrm{Re}\,z\leqq 0$ then we can find a unique $w\in V$ satisfying (3.18). This implies that $(z-A(t))^{-1}\in L(H)$. Substituting w into v in (3.18), we obtain $\|\nabla w\|^2-z(\rho(t)w,\ w)=-(\rho(t)f,\ w)$. From this equalily we have the inequality (2.13).

Now THEOREM 2.4 and REMARK 2.3 ensures the existence of an evolution operator $U(t,\ s)$ $(0\leqq s\leqq t\leqq T)$. Therefore we can write (3.11) and (3.12) as

$$u(t)=U(t,\ 0)a+\int_0^t U(t,\ s)\Phi(s)ds, \tag{3.19}$$

where we have put $\Phi(s)=-B_{\rho}(s)^{-1}F_{\rho}v(s)$. The hypotheses of PROPOSITION 3.2 and LEMMA 3.4 below assures that u defined by (3.19) belongs to $C([0,\ T];\ H)$.

Second-step. Here we study several properties of fractional powers of $A(t)$ and connections between $A(t)$ and $U(t,\ s)$.

LEMMA 3.1.

i) *For any* $t\in[0,\ T]$ *and* $w\in D(A)$ *we have*

$$\|A(t)w\|\leqq m_0^{-1}\|Aw\|, \quad \|Aw\|\leqq l_0\|A(t)w\|. \tag{3.20}$$

ii) *For any* $\alpha\in[0,\ 1]$ *we have* $D(A^{\alpha})=D(A(t)^{\alpha})$ $(0\leqq t\leqq T)$ *and the inequality*

$$\|A(t)^{\alpha}A^{-\alpha}\|\leqq c,\ \|A^{\alpha}A(t)^{-\alpha}\|\leqq c \quad (0\leqq t\leqq T) \tag{3.21}$$

with a constant c *depending only on* $\Omega,\ m_0,\ l_0$.

PROOF. The inequality (3.20) is obvious from (3.6). In order to show (3.21) we consider a Hilbert space H_t which is equal to H as a vector space but is equipped with an inner-product $(v,\ w)_t\equiv(\rho(t)v,\ w)$ $(v,\ w\in H_t)$. Denoting the norm in H_t by $\|\ \|_t$, we have

$$l_0^{-1/2}\|w\|_t\leqq\|w\|\leqq m_0^{-1/2}\|w\|_t \qquad (w\in H).$$

As is easily seen, $A(t)$ is a positive definite self adjoint operator in H_t. From these facts and the Heinz-Kato theorem we obtain (3.21) (see, e.g., Kato [8]).

Q.E.D.

LEMMA 3.2. *Let* α *and* β *be fixed numbers satisfying* $0\leqq\beta\leqq\alpha<2$ *and* $0\leqq\beta\leqq 1$. *Then there exist positive constants* $c_3=c_3(\Omega,\ m_0,\ l_0,\ \alpha)$ *and* $c_4=c_4(\Omega,\ m_0,\ l_0,\ \alpha,\ T)$ *such that for any* $0\leqq s<t\leqq T$ *we have*

(3.22) $$\|A(t)^{\alpha}U(t,\ s)A(s)^{-\beta}\| \leqq c_3(t-s)^{-(\alpha-\beta)} + \phi(M_0)(t-s)^{-(\alpha-\beta)+1}.$$

Here we have put $\phi(M_0)=c_4(1+M_0)^2\exp(c_4M_0)$.

REMARK 3.1. The above $\phi(M_0)$ is different from that in the statement of PROPOSITION 3.2. However, we denote quantities of the form $c(1+M_0)^k\exp(cM_0)$ ($k=1, 2, 3$, $c=c(\Omega, m_0, l_0, T)$) by $\phi(M_0)$ or $\phi'(M_0)$ without distinction. On the other hand, c, c', c_j etc. mean constants depending only on Ω, m_0, l_0 and T (not on M_0), even when we do not explain them.

REMARK 3.2. From (3.22) we can estimate more roughly;

(3.23) $$\|A(t)^{\alpha}U(t,\ s)A(s)^{-\beta}\| \leqq \phi(M_0)(t-s)^{-(\alpha-\beta)}.$$

LEMMA 3.3. *Let α, β and λ be fixed numbers satisfying $0\leqq\beta\leqq\alpha<1$ and $0<\lambda<1-\alpha$. Then there exist positive constants $c_5=c_5(\Omega, m_0, l_0, \alpha, \lambda)$ and $c_6=c_6(\Omega, m_0, l_0, \alpha, \lambda, T)$ such that we have*

(3.24) $$\|A^{\alpha}\{U(t+h,\ s)-U(t,\ s)\}A(s)^{-\beta}\| \leqq c_6h^{\lambda}(t-s)^{-(\alpha-\beta)-\lambda}+\phi(M_0)h^{\lambda} \qquad (0<s<t\leqq t+h\leqq T)$$

so long as $m_0\leqq m$, $l\leqq l_0$. Here $\phi(M_0)=c_6(1+M_0)\exp(c_6M_0)$.

These two lemmas are direct consequences of THEOREM 2.4 and LEMMA 3.1 (the case of $\theta=1$ in THEOREM 2.4).

LEMMA 3.4. *There exists a positive constant $c_7=c_7(\Omega)$ such that we have for any w, $v\in D(A^{5/8})$*

(3.25) $$\|(w\cdot\nabla)v\| \leqq c_7\|A^{5/8}w\|\|A^{5/8}v\|.$$

We refer the proof to Inoue and Wakimoto [7].

The inequality (3.25) shows that the mapping F_{ρ}; $D(A^{5/8})\to H$ is locally Lipschitz continuous. This fact, together with (3.8) and (3.10), implies that $\Phi(t)$ is locally Hölder continuous in t. Therefore it is concluded that $u\in C(]0, T]; D(A))\cap C^1(]0, T]; H)$ and that u satisfies (3.11) and (3.12) (see Fujita and Kato [4]).

Third-step. Here we prove (3.13) and (3.14). For $\alpha=5/8$, η we have

$$\|A^{\alpha}u(t)\| \leqq \|A^{\alpha}U(t,\ 0)a\| + \int_0^t \|A^{\alpha}U(t,\ s)\Phi(s)\|ds.$$

By LEMMAS 3.1 and 3.2 the first term of the right hand side is majorized by $c(1+t\phi(M_0))\|A^{\alpha}a\|$. It is easy to see that $\|\Phi(s)\|\leqq cN^2$. Now (3.13) immediately follows from (3.23) and this inequality.

Next we show (3.14). We start with the identity

$$A^{5/8}\{u(t+h)-u(t)\} = A^{5/8}\{U(t+h,\ 0)-U(t,\ 0)\}a$$

$$+\int_t^{t+h} A^{5/8}U(t+h,\ s)\Phi(s)ds$$

$$+\int_0^t A^{5/8}\{U(t+h,\ s)-U(t,\ s)\}\Phi(s)ds$$

$$\equiv I_1+I_2+I_3.$$

By LEMMA 3.3 we have

$$\|I_1\|\leqq\{ch^{\theta}t^{-\theta}+\phi(M_0)h^{\theta}\}\|A^{5/8}a\|.$$

By (3.23) we have

$$\|I_2\|\leqq\phi(M_0)\int_t^{t+h}(t+h-s)^{-5/8}\|A^{5/8}v(s)\|^2ds$$

$$\leqq\phi'(M_0)N^2h^{3/8}\leqq\phi''(M_0)N^2h^{\theta}.$$

I_3 is estimated by (3.24) as

$$\|I_3\|\leqq\phi(M_0)h^{\theta}\int_0^t(t-s)^{-\theta-5/8}cN^2ds\leqq\phi'(M_0)N^2h^{\theta}.$$

From these inequalities we obtain (3.14).

Fourth-step. Here we prove (3.15) and (3.16). To this end we consider the following equality:

$$A(t)u(t)=A(t)U(t,\ 0)a \tag{3.26}$$

$$+A(t)\int_0^t U(t,\ s)\{\Phi(s)-\Phi(t)\}ds$$

$$+A(t)\int_0^t U(t,\ s)\Phi(t)ds$$

$$\equiv J_1+J_2+J_3.$$

By LEMMA 3.2 we have

$$\|J_1\|\leqq c\{t^{-(1-\eta)}+t^{\eta}\phi(M_0)\}\|A^{\eta}a\|, \tag{3.27}$$

$$\|J_2\|\leqq\phi(M_0)\int_0^t(t-s)^{-1}\|\Phi(s)-\Phi(t)\|ds. \tag{3.28}$$

On the other hand, it follows from LEMMA 3.4 and the hypotheses of PROPOSITION 3.2 that

$$\|\Phi(s)-\Phi(t)\|\leqq\|B_{\rho}(s)^{-1}\{F_{\rho}v(s)-F_{\rho}v(t)\}\| \tag{3.29}$$

$$+\|\{B_{\rho}(s)^{-1}-B_{\rho}(t)^{-1}\}F_{\rho}v(t)\|$$

$$\leqq m_0^{-1}\|\{\rho(s)-\rho(t)\}(v(s)\cdot\nabla)v(s)\|$$

$$+m_0^{-1}\|\rho(t)\{(v(s)\cdot\nabla)v(s)-(v(t)\cdot\nabla)v(t)\}\|$$

$$+m_0^{-2}M_0(t-s)\|F_\rho v(t)\|$$
$$\leqq\phi(M_0)(t-s)N^2+cN^2(t-s)^\theta s^{-\theta}.$$

By (3.28) and (3.29) we obtain

$$\|J_2\|\leqq\phi(M_0)N^2. \tag{3.30}$$

With the aid of integration by parts we can rewrite J_3 as

$$\begin{aligned}J_3&=A(t)\int_0^t\left\{\frac{\partial}{\partial s}U(t,\ s)\right\}A(s)^{-1}\Phi(t)ds\\&=\Phi(t)-A(t)U(t,\ 0)A(0)^{-1}\Phi(t)\\&\quad -A(t)\int_0^t U(t,\ s)A^{-1}P\frac{\partial\rho}{\partial s}(s)\Phi(t)ds.\end{aligned} \tag{3.31}$$

Hence we easily obtain $\|J_3\|\leqq\phi(M_0)N^2$. From this inequality, (3.27) and (3.30), we have (3.15).

Before entering on the proof of (3.16), we note the equality

$$\begin{aligned}\frac{du}{dt}=&-A(t)U(t,\ 0)a-A(t)\int_0^t U(t,\ s)\{\Phi(s)-\Phi(t)\}\,ds\\&+A(t)U(t,\ 0)A(0)^{-1}\Phi(t)+A(t)\int_0^t U(t,\ s)A^{-1}P\frac{\partial\rho}{\partial s}(s)\Phi(t)ds,\end{aligned} \tag{3.32}$$

which follows from (3.11), (3.26) and (3.31). We recall that $1/4<\gamma<\theta<3/8$. Consequently we have the continuous imbeddings:

$$D(A^\gamma)\subset W^{2\gamma,2}(\Omega)^3\subset L^q(\Omega)^3 \qquad (1/q=1/2-2\gamma/3).$$

Here the first imbedding is well-known (see Fujita and Morimoto [5]). The second one is a consequence of Sobolev's imbedding theorem. Since q satisfies the inequality $3<q<4$, we have

$$\|\nabla u(t)\|_\infty\leqq c\|u(t)\|_{W^{2,q}(\Omega)^3}\leqq c'\|Au(t)\|_{L^q}. \tag{3.33}$$

On the other hand, we obtain

$$\begin{aligned}\|Au(t)\|_{L^q}&\leqq\left\|B_\rho(t)\frac{du}{dt}\right\|_{L^q}+\|F_\rho v(t)\|_{L^q}\\&\leqq l_0\left\|\frac{du}{dt}\right\|_{L^q}+l_0\|(v\cdot\nabla)v\|_{L^q}\\&\leqq c\left\|A^\gamma\frac{du}{dt}(t)\right\|+c'\|A^\eta v(t)\|\|Av(t)\|.\end{aligned} \tag{3.34}$$

The second term of the last side is majorized by $cN^2t^{-\gamma}$ in virtue of (3.8) and (3.9). Therefore we obtain (3.16) if we have shown that

$$\left\| A^{\gamma}\frac{du}{dt}(t)\right\| \leqq ct^{-(1+\gamma-\eta)}\|A^{\eta}a\|+\phi(M_0)t^{\eta-\gamma}\|A^{\eta}a\|+\phi(M_0)N^2t^{-\gamma} \quad (0<t<T). \tag{3.35}$$

To show this inequality we note an inequality obvious from (3.32):

$$\begin{aligned}\left\| A^{\gamma}\frac{du}{dt}(t)\right\| \leqq & \|A^{\gamma}A(t)U(t, 0)a\| \\ & +\int_0^t \|A^{\gamma}A(t)U(t, s)\|\|\Phi(s)-\Phi(t)\|ds \\ & +\|A^{\gamma}A(t)U(t, 0)A(0)^{-1}\|\|\Phi(t)\| \\ & +\int_0^t \|A^{\gamma}A(t)U(t, s)A^{-1}\|\left\|P\frac{\partial\rho}{\partial s}(s)\right\|\|\Phi(t)\|ds \\ \equiv & K_1+K_2+K_3+K_4.\end{aligned}$$

It is clear from LEMMAS 3.1 and 3.2 that

$$K_1 \leqq \{ct^{-(1+\gamma-\eta)}+\phi(M_0)t^{\eta-\gamma}\}\|A^{\eta}a\| \qquad (0<t<T).$$

K_2 is estimated by means of (3.23) and (3.29) as

$$K_2 \leqq \phi(M_0)N^2t^{-\gamma} \qquad (0<t<T).$$

From (3.23) and the inequality $\|\Phi(t)\| \leqq c\|A^{5/8}v(t)\|^2 \leqq cN^2$, we obtain

$$K_3 \leqq \phi(M_0)N^2t^{-\gamma} \qquad (0<t<T).$$

The same inequality for K_4 is derived in a similar way. From the above inequalities we obtain (3.35). Thus the proof of PROPOSITION 3.2 is completed.

Q.E.D.

§ 4. Construction of approximate solutions and their boundedness.

We still assume that Ω is a three dimensional domain. Let T be a fixed number such that $0<T<1$. Then one can easily verify that $u_1(t)\equiv e^{-tA}$ a satisfies the hypotheses of PROPOSITION 3.1 and that $\rho_1(t, x)\equiv\rho_0(x)$ and $u_1(t)$ satisfy those of PROPOSITION 3.2. Consequently we obtain $\{\rho_2, u_2\}$ in $[0, T]$ by means of the formulas (2.9), $\cdots$, (2.12). Successive use of PROPOSITIONS 3.1 and 3.2 ensures that $\{\rho_n, u_n\}$ defined by the schemes (2.9), $\cdots$, (2.12) exists in $[0, T]$ $(n=1, 2, \cdots)$.

To show boundedness of the approximate solutions we put

$$\begin{aligned} N_n(t)\equiv\max\Big(& \sup_{0<s<t}\|A^{5/8}u_n(s)\|, \sup_{0<s<t}\|A^{\eta}u_n(s)\|, \\ & \sup_{0<s<t}s^{1-\eta}\|Au_n(s)\|, \sup_{0<r<s<t}(s-r)^{-\theta}r^{\theta}\|A^{5/8}\{u_n(s)-u_n(r)\}\|\Big)\end{aligned}$$

and

$$M_n(t)\equiv\max\Big(\sup_{0<s<t}\|\nabla\rho_n(s)\|_\infty,\ \sup_{0<s<t}\Big\|\frac{\partial\rho_n}{\partial s}(s)\Big\|_\infty\Big).$$

Then we have the following

PROPOSITION 4.1. *For sufficiently small* $T_1\in]0, T]$, *the sequences* $\{N_n(T_1)\}_n$ *and* $\{M_n(T_1)\}_n$ *are bounded.*

PROOF. In the first place we have by PROPOSITION 3.2

$$\|A^\alpha u_{n+1}(s)\|\leqq c_1\{1+s\phi(M_n(t))\}\|A^\alpha a\| + \phi(M_n(t))N_n(t)^2 s^{1-\alpha}\qquad (0<s<t) \tag{4.1}$$

for $\alpha=5/8,\ \eta$,

$$s^{1-\eta}\|Au_{n+1}(s)\|\leqq c_1\{1+s\phi(M_n(t))\}\|A^\eta a\| + \phi(M_n(t))N_n(t)^2 s^{1-\eta}\qquad (0<s<t), \tag{4.2}$$

$$(s-r)^{-\theta}r^\theta\|A^{5/8}\{u_{n+1}(s)-u_{n+1}(r)\}\|\leqq c_1\|A^{5/8}a\| + \phi(M_n(t))r^\theta\|A^{5/8}a\|+\phi(M_n(t))N_n(t)^2 r^\theta\qquad (0<r<s<t). \tag{4.3}$$

Furthermore, by (3.16) and PROPOSITION 3.1, we obtain

$$\|\nabla\rho_{n+1}(s)\|_\infty\leqq c_2\|\nabla\rho_0\|_\infty\exp(c_1 s^{\eta-\gamma}\{1+s\phi(M_n(t))\}\|A^\eta a\| + \phi(M_n(t))N_n(t)^2 s^{1-\gamma})\qquad (0<s<t), \tag{4.4}$$

$$\begin{aligned}\Big\|\frac{\partial\rho_{n+1}}{\partial s}(s)\Big\|_\infty\leqq{}& c_2\|\nabla\rho_0\|_\infty(c_1\{1+s\phi(M_n(t))\}\|A^\eta a\| \\ &+\phi(M_n(t))N_n(t)^2 s^{1-\eta}) \\ &\times\exp(c_1 s^{\eta-\gamma}\|A^\eta a\|\{1+s\phi(M_n(t))\}+\phi(M_n(t))N_n(t)^2 s^{1-\gamma}) \\ &\qquad (0<s<t).\end{aligned} \tag{4.5}$$

Then we define K as the maximum of $N_1(T)$, $M_1(T)$, $3c_1\|A^{5/8}a\|$, $3c_1\|A^\eta a\|$, $c_2\|\nabla\rho_0\|_\infty\exp(3c_1\|A^\eta a\|)$ and $c_2\|\nabla\rho_0\|_\infty 3c_1\|A^\eta a\|\exp(3c_1\|A^\eta a\|)$. Then it is easy to take a $T_1\in]0, T]$ such that we have $N_{n+1}(T_1)\leqq K$ and $M_{n+1}(T_1)\leqq K$ so long as $N_n(T_1)\leqq K$, $M_n(T_1)\leqq K$. Since $N_1(T_1)\leqq K$ and $M_1(T_1)\leqq K$ hold true, we obtain the desired results. Q.E.D

COROLLARY 4.2. *The following inequalities hold true for all* n;

$$\|A^\alpha u_n(t)\|\leqq 3c_1\|A^\alpha a\|\qquad (0<t<T_1;\ \alpha=5/8,\ \eta), \tag{4.6}$$

$$\|\nabla\rho_n(t)\|_\infty\leqq c_2\|\nabla\rho_0\|_\infty\exp(3c_1\|A^\eta a\|)\qquad (0<t<T_1), \tag{4.7}$$

$$\|\nabla u_n(t)\|_\infty\leqq 3c_1 t^{-(1+\gamma-\eta)}\|A^\eta a\|\qquad (0<t<T_1). \tag{4.8}$$

PROOF. The inequalities (4.6) and (4.7) are shown in the proof of the preceding proposition. The inequality (4.8) is derived from (3.16). Q.E.D.

§ 5. Proof of Theorem 2.1.

Putting $\sigma_n \equiv \rho_n - \rho_{n-1}$, we have

$$\frac{\partial \sigma_n}{\partial t} + u_n \cdot \nabla \sigma_n = -(u_n - u_{n-1}) \cdot \nabla \rho_{n-1} \qquad (0<t<T_1,\ x \in \Omega), \tag{5.1}$$

$$\sigma_n(0,\ x) = 0. \tag{5.2}$$

Therefore we prepare a lemma useful for estimating σ_n.

LEMMA 5.1. *Assume that* $v \in C([0,\ T_1] \times \bar{\Omega})$ *satisfies*

$$\int_0^{T_1} \|\nabla v(t)\|_\infty dt < \infty .$$

Let $r \in W^{1,\infty}(]0,\ T_1[\times \Omega)$ *satisfy*

$$\frac{\partial r}{\partial t} + v \cdot \nabla r = g \in L^1(]0,\ T_1[\,;\ L^\infty(\Omega)).$$

Then we have

$$\|r(t)\|_\infty \leqq \|r(0)\|_\infty + \int_0^t \|g(s)\|_\infty ds \qquad (0<t<T_1). \tag{5.3}$$

PROOF. (This lemma is essentially due to Bardos [2]). As is noted after PROPOSITION 3.1, there exists, under the above hypotheses, a unique characteristic curve $\xi_{s,t}(x)$, i.e.,

$$\frac{d}{ds} \xi_{s,t}(x) = v(s,\ \xi_{s,t}(x)) \qquad (0<s<T_1),$$

$$\xi_{t,t}(x) = x .$$

Then it is easily verified that

$$\frac{d}{dt} r(t,\ \xi_{t,0}(x)) = g(t,\ \xi_{t,0}(x)).$$

Consequently we have

$$r(t,\ \xi_{t,0}(x)) = r(0,\ x) + \int_0^t g(s,\ \xi_{s,0}(x)) ds. \tag{5.4}$$

On the other hand, it is well-known that for a fixed $t \in [0,\ T_1]$ the mapping $x \mapsto \xi_{t,0}(x)$ is an isomorphism from Ω onto itself (see, e.g., Kato [9]). From this fact and (5.4) we obtain (5.3). Q.E.D.

Applying LEMMA 5.1 to (5.1) and (5.2), we obtain by means of (4.7)

$$\|\sigma_n(t)\|_\infty \leqq K \int_0^t \|u_n(s) - u_{n-1}(s)\|_\infty ds .$$

If we put $w_n(t) \equiv u_n(t) - u_{n-1}(t)$ and $h_n(t) \equiv \sup_{0 \leqq s \leqq t} \|A^\gamma w_n(s)\|$, we have

(5.5) $$\|\sigma_n(t)\|_\infty \leqq cth_n(t)$$

by Sobolev's imbedding theorem.

Next we note that

(5.6) $$B_n(t)\frac{dw_{n+1}}{dt}+Aw_{n+1}=-\{F_n u_n(t)-F_{n-1}u_{n-1}\}$$
$$-\{B_n(t)-B_{n-1}(t)\}\frac{du_n}{dt},$$
$$w_{n+1}(0)=0,$$

which follows from (2.11). Here we have employed abbreviations $B_n(t)$ for $P\frac{\partial\rho_n}{\partial t}(t)$, and $F_n u_n(t)$ for $P\{\rho_n(t)(u_n\cdot\nabla)u_n\}$. The family of operators $A_n(t)\equiv B_n(t)^{-1}A$ $(0\leqq t\leqq T_1)$ generates an evolution operator. Denoting this operator by $U_n(t, s)$, we have

$$\|A_n(t)^\alpha U_n(t, s)A_n(s)^{-\beta}\|\leqq c(t-s)^{-(\alpha-\beta)}$$

for any $0<s<t<T_1$ and $0\leqq\beta\leqq\alpha<1$. Here c does not depend on n in virtue of PROPOSITION 4.1 and COROLLARY 4.2. Using this inequality, we estimate $w_{n+1}(t)$ which is represented by (5.6) as

$$w_{n+1}(t)=\int_0^t U_n(t, s)B_n(s)^{-1}\{F_{n-1}u_{n-1}(s)-F_n u_n(s)\}ds$$
$$+\int_0^t U_n(t, s)B_n(t)^{-1}\{B_{n-1}(s)-B_n(s)\}\frac{du_n}{ds}(s)ds.$$

In the first place we have

$$\|A^\eta w_{n+1}(t)\|\leqq c\int_0^t(t-s)^{-\eta}\|F_{n-1}u_{n-1}(s)-F_n u_n(s)\|ds$$
$$+c\int_0^t(t-s)^{-\eta}h_n(s)s^\eta ds$$

by making use of PROPOSITION 4.1 and (5.5). On the other hand, it holds by PROPOSITION 4.1 that

$$\|F_{n-1}u_{n-1}(s)-F_n u_n(s)\|\leqq\|\rho_n(s)\{(u_{n-1}\cdot\nabla)u_{n-1}-(u_n\cdot\nabla)u_n\}\|$$
$$+\|\{\rho_{n-1}(s)-\rho_n(s)\}(u_{n-1}\cdot\nabla)u_{n-1}\|$$
$$\leqq c\|A^\eta w_n(s)\|+c\|\sigma_n(s)\|_\infty\leqq c'(1+s)h_n(s).$$

Consequently there exists a positive constant $c=c(\Omega, m_0, l_0)$ such that

$$\|A^\eta w_{n+1}(t)\|\leqq c\int_0^t(t-s)^{-\eta}h_n(s)ds \qquad (0<t<T_1).$$

Since $h_n(s)$ is increasing, the right hand side is increasing in t. Hence we have

$$h_{n+1}(t) \leqq c\int_0^t (t-s)^{-\eta} h_n(s)ds \qquad (0<t<T_1)$$

By induction we obtain for $n=3, 4, \cdots$,

$$\begin{aligned} h_n(t) &\leqq \{ct^{1-\eta}\} B(1-\eta, 1)B(1-\eta, 2-) \cdots B(1-\eta, 1+(n-3)(1-\eta))h_2(t) \\ &= \frac{\{ct^{1-\eta}\Gamma(1-\eta)\}^{n-2}}{\Gamma(1+(n-2)(1-\eta))} h_2(t) \qquad (0<t<T_1), \end{aligned}$$

where Γ and B are Euler's gamma function and beta function, respectively. Therefore we have

$$\sum_{n=2}^{\infty} h_n(T_1) < \infty,$$

which implies that the series $\sum \|A^\eta w_n(t)\|$ converges uniformly on $[0, T_1]$ and that $\sum \|\sigma_n(t)\|_\infty$ converges uniformly on $[0, T_1]$. Consequently there exist $\rho \in C([0, T_1] \times \bar{\Omega})$ and $u \in C([0, T_1]; D(A^\eta))$ such that

$$\rho_n \longrightarrow \rho \quad \text{as} \quad n \longrightarrow \infty \qquad \text{uniformly on} \quad [0, T_1] \times \bar{\Omega}, \tag{5.7}$$

$$\|A^\eta\{u_n(t)-u(t)\}\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \qquad \text{uniformly on} \quad [0, T_1]. \tag{5.8}$$

We shall show this $\{\rho, u\}$ is a solution. First we obtain by (4.8)

$$\nabla u(t) \in L^\infty(\Omega)^9 \ (0<t<T_1), \ \int_0^{T_1} \|\nabla u(t)\|_\infty dt < \infty. \tag{5.9}$$

Clearly ρ satisfies the initial condition $\rho(0, x)=\rho_0(x)$. To show (2.1) we take an arbitrary smooth function ϕ on $[0, T_1] \times \bar{\Omega}$ satisfying $\phi(T_1, x) \equiv 0$. Multiplying $\partial\rho_n/\partial t + u_{n-1} \cdot \nabla \rho_n = 0$ by $\phi(t, x)$, we integrate it on $[0, T_1] \times \Omega$. Then we have

$$0 = \int_\Omega \rho_0(x)\phi(0, x)ds + \int_0^{T_1}\int_\Omega \rho_n(t, x)\left\{\frac{\partial\phi}{\partial t}(t, x) + u_{n-1} \cdot \nabla\phi(t, x)\right\} dx dt$$

since $\rho_n(0, x)=\rho_0(x)$ and $\operatorname{div} u_{n-1}=0$. Letting n tend to infinity, we obtain

$$0 = \int_\Omega \rho_0(x)\phi(0, x)ds + \int_0^{T_1}\int_\Omega \rho(t, x)\left\{\frac{\partial\phi}{\partial t}(t, x) + u \cdot \nabla\phi(t, x)\right\} dx dt.$$

Hence by the following lemma we can conclude that $\rho \in W^{1,\infty}(]0, T_1[\times \Omega)$ and ρ satisfies (2.1).

LEMMA 5.2. *Let* $\omega_0 \in W^{1,\infty}(\Omega)$, $\omega \in L^1(]0, T_1[\times \Omega)$ *and* $v \in C([0, T_1] \times \bar{\Omega})$ *be functions satisfying*

$$\int_0^{T_1} \|\nabla v(t)\|_\infty dt < \infty \quad \textit{and}$$

$$0=\int_{\Omega}\omega_0(x)\phi(0, x)dx+\int_0^{T_1}\int_{\Omega}\omega(t, x)\left\{\frac{\partial\phi}{\partial t}(t, x)+v(t, x)\cdot\nabla\phi(t, x)\right\}dxdt$$

for any smooth function ϕ *such that* $\phi(T_1, x)\equiv 0$. *Then* ω *belongs to* $W^{1,\infty}(]0, T_1[\times\Omega)$ *and satisfies*

$$\frac{\partial\omega}{\partial t}+v\cdot\nabla\omega=0,\ \omega(0, x)=\omega_0(x).$$

For the proof, see Ladyzhenskaya and Solonnikov [13].

Next we show that $u(t)$ satisfies (2.2). Since $\rho\in W^{1,\infty}(]0, T_1[\times\Omega)$ and $m\leqq\rho(t, x)\leqq l$, the family of operators $\{B_\rho(t)^{-1}A\}_{0\leqq t\leqq T_1}$ generates an evolution operator $U(t, s)$ $(0\leqq s\leqq t\leqq T_1)$. On the other hand, we already know that $\|B_n(t)^{-1}\|$, $\|F_n u_n(t)\|$ and $\|U_n(t, s)\|$ are bounded. Therefore if

$$U_n(t, s)\longrightarrow U(t, s)\ \text{(strong convergence)}\ \text{as}\ n\longrightarrow\infty \tag{5.10}$$

is true, then we have

$$u(t)=U(t, 0)-\int_0^t U(t, s)B_\rho(s)^{-1}F_\rho u(s)ds \tag{5.11}$$

by letting n tend to infinity in

$$u_{n+1}(t)=U_n(t, s)a-\int_0^t U_n(t, s)B_n(s)^{-1}F_n u_n(s)ds.$$

From PROPOSITION 4.1 it follows that

$$\|A^{5/8}\{u(t)-u(s)\}\|\leqq c(t-s)^\theta s^{-\theta}\qquad (0<s<t<T_1).$$

This inequality allows us to conclude that the solution $u(t)$ of (5.11) belongs to $C(]0, T_1]; D(A))\cap C^1(]0, T_1]; H)$ and satisfies (2.2) (the proof is the same as that in Fujita and Kato [4]).

There remains only the verification of (5.10). However, this is easily done in virtue of

$$(z-A_n(t))^{-1}\longrightarrow(z-A(t))^{-1}\quad\text{(strong convergence)} \tag{5.12}$$

$$\text{as}\ n\longrightarrow\infty\qquad\text{for}\ Re\, z\leqq 0,\ 0\leqq t\leqq T_1$$

(see the construction of $U_n(t, s)$ or $U(t, s)$ in Tanabe [18]). (5.12) follows from the uniform convergence of ρ_n to ρ. We omit the details.

Finally we prove the uniqueness. Let $\{\rho, u\}$ be the solution which we have just proved to exist and $\{\sigma, v\}$ be another solution in $W^{1,\infty}(]0, T_2[\times\Omega)\times C([0, T_2]; D(A^\gamma))$. Putting $T_3\equiv\min\{T_1, T_2\}$, we prove that $\rho\equiv\sigma$ and $u\equiv v$ in $[0, T_3]$, which show the uniqueness. To this end, put $w=u-v$ and $\pi=\rho-\sigma$, then we have

$$\frac{\partial\pi}{\partial t}+u\cdot\nabla\pi=-w\cdot\nabla\sigma, \tag{5.13}$$

(5.14) $$\pi(0, x)=0,$$

(5.15) $$B_\rho(t)\frac{dw}{dt}+Aw(t)=-F_\rho u(t)+F_\sigma v(t)+\{B_\sigma(t)-B_\rho(t)\}\frac{dv}{dt}$$

(5.16) $$w(0)=0.$$

Hence we obtain for $0\leqq t\leqq T_3$

$$w(t)=\int_0^t U_\rho(t, s)\Big(F_\sigma v(s)-F_\rho u(s)+\{B_\sigma(s)-B_\rho(s)\}\frac{dv}{ds}(s)\Big)ds,$$

where $U_\rho(t, s)$ is the evolution operator generated by $\{B_\rho(t)^{-1}A\}$. Since we can easily obtain $\|A^{5/8}v(t)\|\leqq c$ and $\left\|\frac{dv}{dt}(t)\right\|\leqq ct^{-(1-\eta)}$, it follows that

$$\|A^\eta w(t)\|\leqq c\int_0^t (t-s)^{-\eta}\{\|\sigma(s)-\rho(s)\|_\infty(1+s^{-(1-\eta)})$$
$$+\|A^\eta w(s)\|\}ds.$$

Applying LEMMA 5.1 to (5.13), we have

$$\|\sigma(s)-\rho(s)\|_\infty\leqq\int_0^s\|A^\eta w(r)\|\|\nabla\sigma(r)\|_\infty dr.$$

Therefore $h(t)\equiv\sup_{0\leq s\leq t}\|A^\eta w(s)\|$ satisfies

$$h(t)\leqq c\int_0^t(t-s)^{-\eta}h(s)ds \qquad (0<t<T_3).$$

From this inequality we obtain $h(t)\equiv 0$ $(0<t<T_3)$ (see the proof of the convergence of $\sum\|A^\eta w_n(t)\|$). Hence we have $u\equiv v$ and $\rho\equiv\sigma$. Thus THEOREM 2.1 is proved. Q.E.D.

§6. Global existence in the two-dimensional problem.

To prove THEOREM 2.2 we derive in this section several a priori estimates, assuming existence of the solution. Hence we assume that T is an arbitrary positive constant and that $\{\rho, u\}$ exists in $[0, T[$.

LEMMA 6.1 *We have for any* $t\in[0, T[$

(6.1) $$m\|u(t)\|^2+2\int_0^t\|\nabla u(s)\|^2ds\leqq l\|a\|^2.$$

PROOF. Taking the inner-product of (2.2) and $u(t)$, we have

$$\Big(\rho(t)\frac{du}{dt}, u\Big)+(\nabla u(t), \nabla u(t))+(\rho(t)(u\cdot\nabla)u, u)=0.$$

On the other hand, we obtain

$$\left(\rho(t)\frac{du}{dt},\ u\right)=\frac{1}{2}\frac{d}{dt}(\rho(t)u,\ u)-(\rho(t)(u\cdot\nabla)u,\ u)$$

by means of (2.1) and integration by parts. Consequently it holds that

$$\frac{d}{dt}(\rho(t)u,\ u)+2\|\nabla u(t)\|^2=0 \qquad (0<t<T).$$

Integrating this inequality on $[0,\ T]$, we have (6.1) because of $m\leqq\rho(t,\ x)\leqq l$.

LEMMA 6.2. *There exists a positive constant* c *depending only on* Ω, m, l *and* $\|A^{1/2}a\|$ *such that*

$$\|\nabla u(t)\|^2+\int_0^t\|Au(s)\|^2ds\leqq c \qquad (0<t<T). \tag{6.2}$$

PROOF. Taking the inner-product of (2.2) and $B_\rho(t)^{-1}A$, we have

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u(t)\|^2+(Au(t),\ B_\rho(t)^{-1}Au(t))$$
$$\leqq|(F_\rho u(t),\ B_\rho(t)^{-1}Au(t))|.$$

Since $B_\rho(t)^{-1}$ is positive definite, we obtain

$$\frac{d}{dt}\|\nabla u(t)\|^2+\|Au(t)\|^2\leqq c\|(u\cdot\nabla)u\|\|Au(t)\|$$

with a constant $c=c(\Omega,\ m,\ l)$. (Note that $c\|\nabla u\|\leqq\|A^{1/2}u\|\leqq c'\|\nabla u\|$). On the other hand, the following inequality holds:

$$\|(u\cdot\nabla)u\|\leqq c\|u\|_{L^4}\|\nabla u\|_{L^4}$$
$$\leqq c'\|u\|^{1/2}\|\nabla u\|\|Au\|^{1/2}.$$

Therefore it follows that

$$\frac{d}{dt}\|\nabla u(t)\|^2+\|Au(t)\|^2\leqq c\|a\|^{1/2}\|\nabla u(t)\|\|Au(t)\|^{3/2} \tag{6.3}$$
$$\leqq\frac{1}{2}\|Au(t)\|^2+c'\|a\|^2\|\nabla u(t)\|^4.$$

Here use has been made of (6.1) and Young's inequality. If we put $f(t)=\|\nabla u(t)\|^2$ and integrate the above inequality, we obtain

$$f(t)\leqq\|\nabla a\|^2+c\|a\|^2\int_0^t\|\nabla u(s)\|^2f(s)ds \qquad (0<t<T).$$

Applying Gronwall's lemma, we have $f(t)\leqq c$ $(0<t<T)$. This inequality, together with (6.1) and (6.3), implies (6.2). Q.E.D.

LEMMA 6.3. *There exist positive constants* $\theta\in]0,\ 1[$ *and* L *depending only on* T, Ω, m, l, $\|A^{1/2}a\|$, $\|\nabla\rho_0\|_\infty$ *such that*

(6.4) $$\sup_{x\in\Omega}|\rho(t, x)-\rho(s, x)|\leqq L|t-s|^{\theta} \qquad (s, t\in[0, T]).$$

REMARK 6.1. This lemma is essentially due to [13]. But we give the proof for the completeness.

PROOF. From now on we denote by L or L' various constants depending only on T, Ω, m, l, $\|A^{1/2}a\|$, $\|\nabla\rho_0\|_\infty$. We start with the equality $\rho(t, x)=\rho_0(\xi_{0,t}(x))$ where $\xi_{s,t}(x)$ is determined by (3.5). In view of

(6.5) $$|\rho(t, x)-\rho(s, x)|\leqq\|\nabla\rho_0\|_\infty|\xi_{0,t}(x)-\xi_{0,s}(x)|$$

we estimate $Y(\tau)\equiv|\xi_{\tau,t}(x)-\xi_{\tau,s}(x)|$ $(0\leqq\tau\leqq s)$. Since we now deal with the two dimensional problem, we have

$$|u(s, z)-u(s, w)|\leqq c\|Au(s)\|\phi(|z-w|),$$

where $\phi(t)\equiv t(1+|\log t|)$ $(0<t)$ and $c=c(\Omega)$. Now it is easily obtained by (3.5) that

$$|\xi_{\tau,t}(x)-\xi_{\tau,s}(x)|\leqq\int_\tau^s|u(r, \xi_{r,s}(x))-u(r, \xi_{r,t}(x))dr$$
$$+\int_s^t|u(r, \xi_{r,t}(x))|dr.$$

Hence we obtain

(6.6) $$\begin{aligned}Y(\tau)&\leqq|t-s|^{1/2}\Big(\int_s^t\|u(r)\|_\infty^2dr\Big)^{1/2}\\&\quad+c\int_\tau^s\|Au(r)\|\phi(Y(r))dr\\&\leqq L|t-s|^{1/2}+c\int_\tau^s\|Au(r)\|\phi(Y(r))dr \qquad (0<\tau<s).\end{aligned}$$

Here the second inequality follows from $\|u\|_\infty\leqq c\|Au\|$ and (6.2). We put the last side of (6.6) as $Z(\tau)$. Then we have

$$-\frac{d}{d\tau}Z(\tau)=c\|Au(\tau)\|\phi(Y(\tau))\leqq c\|Au(\tau)\|\phi(Z(\tau))$$

by the monotonicity of ϕ. Putting $f(r)=\{\mathrm{sign}(\log r)\}\times\log(1+|\log r|)$, we can rewrite the above inequality as

$$-\frac{d}{d\tau}f(Z(\tau))=-\Big\{\frac{d}{d\tau}Z(\tau)\Big\}\Big/\phi(Z(\tau))\leqq c\|Au(\tau)\|.$$

Hence it follows that

(6.7) $$-f(Z(s))+f(Z(\tau))\leqq c\int_\tau^s\|Au(r)\|dr\leqq cT^{1/2}\Big(\int_0^t\|Au(r)\|^2dr\Big)\equiv L'.$$

From this inequality we can easily derive the following assertion:

(6.8) $$L|t-s|^{1/2}\leqq\min\{1, \exp(1-e^{L'})\} \quad \text{implies} \quad Z(\tau)\leqq 1.$$

From (6.7) and (6.8) we obtain

(6.9) $$\log Z(\tau)\leqq 1-e^{-L'}+\log(L''|t-s|^{\theta})$$

if $L|t-s|^{1/2}\leqq\min\{1, \exp(1-e^{L'})\}$. Here we have put $\theta=\frac{1}{2}e^{-L'}$, whence θ satisfies $0<\theta<1$. Now (6.6) and (6.9) imply

$$|\xi_{\tau,t}(x)-\xi_{\tau,s}(x)|\leqq L'''|t-s|^{\theta} \qquad (0\leqq\tau\leqq s\leqq t).$$

From this inequality and (6.5) we obtain (6.4). Q.E.D.

LEMMA 6.4. *The following inequalities hold true:*

(6.10) $$\|A^{\eta}u(t)\|\leqq L \qquad (0<t<T),$$

(6.11) $$\int_0^T\|\nabla u(t)\|_{\infty}dt\leqq L.$$

Here and hereafter we denote by L or L′ various constants depending only on T, Ω, m, l, $\|A^{\eta}a\|$ *and* $\|\nabla\rho_0\|_{\infty}$.

PROOF. We have by THEOREM 2.4 and LEMMA 6.3

$$\begin{aligned}\|A^{\eta}u(t)\|&\leqq c\|A(t)^{\eta}U(t, 0)A(0)^{-\eta}\|\|A^{\eta}a\|\\&\quad+c\int_0^t\|A(t)^{\eta}U(t, s)\|\|\Phi(s)\|ds\\&\leqq L+L\int_0^t(t-s)^{-\eta}\|(u\cdot\nabla)u\|ds\\&\leqq L+L'\int_0^t(t-s)^{-\eta}\|A^{1/2}u(s)\|\|A^{\eta}u(s)\|ds.\end{aligned}$$

The last inequality is a consequence of Sobolev's imbedding theorem. Therefore we have

$$\|A^{\eta}u(t)\|\leqq L+L\int_0^t(t-s)^{-\eta}\|A^{\eta}u(s)\|ds.$$

Applying LEMMA 2.1 to this inequality, we obtain (6.10).

To show (6.11) we note that

$$\begin{aligned}\|\nabla u(t)\|_{\infty}&\leqq c\|Au(t)\|_{L^q}\\&\leqq c'\left\|A^{\varepsilon}\frac{du}{dt}(t)\right\|+c'\|F_{\rho}u(t)\|_{L^q}\\&\leqq c'\left\|A^{\varepsilon}\frac{du}{dt}(t)\right\|+c''\|A^{\eta}u(t)\|^2.\end{aligned}$$

Here we have chosen the constants q and ε such that $1/q=1/2-\varepsilon$, $0<\varepsilon<\min\{1/4, \theta, \eta-1/2\}$, θ being the constant appearing in LEMMA 6.3. Hence it suffices to show

$$\left\|A^{\varepsilon}\frac{du}{dt}(t)\right\|\leqq Lt^{-(1+\varepsilon-\eta)} \qquad (0<t<T).$$

For this purpose we use the equality

$$\begin{aligned} A^{\varepsilon}\frac{du}{dt} &= -A^{\varepsilon}A(t)U(t, 0)a \\ &\quad -A^{\varepsilon}A(t)\int_0^t U(t, s)\{\Phi(s)-\Phi(t)\}ds \\ &\quad -A^{\varepsilon}A(t)\int_0^t U(t, s)\Phi(t)ds+A^{\varepsilon}\Phi(t) \\ &\equiv J_1+J_2+J_3 \end{aligned}$$

(see (3.26) and (2.2)).

The inequality

$$\|J_1\|\leqq Lt^{-(1+\varepsilon-\eta)} \qquad (0<t<T) \tag{6.12}$$

evidently holds by THEOREM 2.4 and LEMMA 6.3. THEOREM 2.4 also allow us to estimate

$$\|J_2\|\leqq L\int_0^t(t-s)^{-1-\varepsilon}\|\Phi(s)-\Phi(t)\|ds.$$

In a way similar to the proof of (3.29), we obtain

$$\|\Phi(s)-\Phi(t)\|\leqq L\|\rho(s)-\rho(t)\|_{\infty}+L\|A^{\eta}\{u(s)-u(t)\}\|.$$

The first term is majorized by $L'(t-s)^{\theta}$. Estimating in a way similar to the proof of (3.14), we see that the second term is majorized by $L'(t-s)^{\theta}s^{-\theta}$. Consequently we have

$$\|J_2\|\leqq Lt^{-\varepsilon} \qquad (0<t<T) \tag{6.13}$$

in virtue of the inequality $\varepsilon<\theta$.

We next rewrite J_3 as follows.

$$\begin{aligned} J_3 &= -A^{\varepsilon}A(t)\int_0^t U(t, s)A(s)\{A(s)^{-1}-A(t)^{-1}\}\Phi(t)ds \\ &\quad -A^{\varepsilon}A(t)\int_0^t U(t, s)A(s)A(t)^{-1}\Phi(t)ds+A^{\varepsilon}\Phi(t) \\ &\equiv J_4+J_5. \end{aligned}$$

Then we have

(6.14) $$\|J_4\| \leqq L\int_0^t (t-s)^{-1-\varepsilon}(t-s)^{\theta}ds \leqq L' t^{\theta-\varepsilon}.$$

Integration shows that

$$J_5 = -A^{\varepsilon}A(t)\int_0^t \left\{\frac{\partial}{\partial s}U(t,\ s)\right\}A(t)^{-1}\Phi(t)ds + A^{\varepsilon}\Phi(t)$$
$$= A^{\varepsilon}A(t)U(t,\ 0)A(t)^{-1}\Phi(t).$$

Hence we have

(6.15) $$\|J_5\| \leqq Lt^{-\varepsilon}.$$

From (6.12), $\cdots$, (6.15) we obtain (6.11). Q.E.D.

PROOF OF THEOREM 2.2. First we note that for any $T>0$ there exists a positive constant L such that

(6.16) $$\|\nabla\rho(t)\|_{\infty} \leqq L \qquad (0<t<T),$$

(6.17) $$\|A^{\eta}u(t)\| \leqq L \qquad (0<t<T).$$

This is evident from LEMMA 6.4 and the inequality (3.3). Now let $[0,\ T^*[$ be the maximal interval where the solution $\{\rho,\ u\}$ exists. If T^* is finite, then we denote by L^* the constant appearing in the right hand side of (6.16) and (6.17) for $T=T^*$. The local existence theorem (THEOREM 2.1) ensures the existence of a positive constant T_0 satisfying the following property: $\{\rho,\ u\}$ exists in $[\tau,\ \tau+T_0]$ if

$$\|A^{\eta}u(\tau)\| \leqq L^* \quad \text{and} \quad \|\nabla\rho(\tau)\|_{\infty} \leqq L^*$$

are satisfied. Since $\|A^{\eta}u(T^*-T_0/2)\| \leqq L^*$ and $\|\nabla\rho(T^*-T_0/2)\|_{\infty} \leqq L^*$ are known, the solution must be continued to $[0,\ T^*+T_0/2[$. This is a contradiction, since $[0,\ T^*[$ is the maximal interval. Thus we have established THEOREM 2.2.

Q.E.D.

§ 7. Global existence in the three-dimensional problem.

THEOREM 2.3 is derived from the following a priori estimate.

THEOREM 7.1. *There exist positive constants* $\delta=\delta(\Omega,\ l_0)$, $\varepsilon_3=\varepsilon_3(\Omega,\ m_0,\ l_0)$ *and* $\hat{c}_j=\hat{c}_j(\Omega,\ m_0,\ l_0)$ $(j=1,\ 2)$ *such that the assertion below holds so long as* $m_0 \leqq m$, $l \leqq l_0$: *Take an arbitrary positive number* T *and suppose that*

$$\sup_{0\leq t\leq T}\left\|\frac{\partial\rho}{\partial t}(t)\right\|_{\infty} \leqq \varepsilon_3, \quad \sup_{0\leq t\leq T} e^{\delta t}\|A^{5/8}u(t)\| \leqq \varepsilon_3$$

and $\|A^{\eta}a\| \leqq 1$. *Then we have for any* $t \in]0,\ T]$

(7.1) $$\|A^{\alpha}u(t)\| \leqq \hat{c}_1\|A^{\alpha}a\|e^{-\delta t} \qquad (\alpha=5/8,\ \eta),$$

(7.2) $$\|\nabla\rho(t)\|_\infty \leqq \hat{c}_2\|\nabla\rho_0\|_\infty, \quad \left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty \leqq \hat{c}_2\|\nabla\rho_0\|_\infty.$$

Admitting this theorem, we now carry out the proof of THEOREM 2.3. In the first place we note that THEOREM 2.1 ensures the existence of a positive constant $T_0=T_0(\Omega, m_0, l_0)$ such that $\{\rho, u\}$ exists in $[0, T_0]$ so long as $\|A^\eta u(0)\| \leqq \hat{c}_1$, $\|\nabla\rho(0)\|_\infty \leqq \hat{c}_2$, $m_0 \leqq m$ and $l \leqq l_0$. Taking a constant k such that $\|A^{5/8}a\| \leqq k\|A^\eta a\|$ $(v \in D(A^\eta))$, we put

(7.3) $$\varepsilon_1=\min\{1,\ \varepsilon_3/c_0 k \exp(\delta T_0),\ \varepsilon_3/c_0\hat{c}_1 k \exp(\delta T_0),\ \hat{c}_1,\ \hat{c}_2,\ \varepsilon_3/c_0\exp(c_0),\ \varepsilon_3/c_0\hat{c}_1\hat{c}_2\exp(c_0\hat{c}_1)\}.$$

Here c_0 is a constant appearing in THEOREM 2.1. We then show that for this ε_1 the statement of THEOREM 2.3 is valid. So assume that $\|A^\eta a\| \leqq \varepsilon_1$ and $\|\nabla\rho\|_\infty \leqq \varepsilon_1$. In virtue of the definition of T_0, the solution $\{\rho, u\}$ exists in $[0, T_0]$ and satisfies

$$\|A^{5/8}u(t)\| \leqq c_0\|A^{5/8}a\| \leqq c_0 k\|A^\eta a\| \qquad (0<t<T_0),$$

$$\left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty \leqq c_0\|\nabla\rho_0\|_\infty\|A^\eta a\|\exp(c_0\|A^\eta a\|) \leqq c_0\exp(c_0)\|\nabla\rho_0\|_\infty \qquad (0<t<T_0).$$

Consequently it holds that

(7.4) $$\sup_{0\leq t\leq T_0} e^{\delta t}\|A^{5/8}u(t)\| \leqq c_0 k \exp(\delta T_0)\|A^\eta a\| \leqq \varepsilon_3,$$

(7.5) $$\sup_{0\leq t\leq T_0}\left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty \leqq c_0\exp(c_0)\|\nabla\rho_0\|_\infty \leqq \varepsilon_3.$$

Hence we can apply THEOREM 7.1. We have for $0 \leqq t \leqq T_0$

(7.6) $$\|A^\alpha u(t)\| \leqq \hat{c}_1\|A^\alpha a\|e^{-\delta t} \qquad (\alpha=5/8,\ \eta),$$

(7.7) $$\|\nabla\rho(t)\|_\infty \leqq \hat{c}_2\|\nabla\rho_0\|_\infty,$$

(7.8) $$\left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty \leqq \hat{c}_2\|\nabla\rho_0\|_\infty.$$

In particular we have $\|A^\eta u(T_0)\| \leqq \hat{c}_1$ and $\|\nabla\rho(T_0)\|_\infty \leqq \hat{c}_2$. Again from the definition of T_0, the solution $\{\rho, u\}$ exists in $[T_0, 2T_0]$. Furthermore it holds that

$$\|A^{5/8}u(t)\| \leqq c_0\|A^{5/8}u(T_0)\| \leqq c_0\hat{c}_1\|A^{5/8}a\|\exp(-\delta T_0),$$

$$\left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty \leqq c_0\|\nabla\rho(T_0)\|_\infty\|A^\eta u(T_0)\|\exp(c_0\|A^\eta u(T_0)\|) \leqq c_0\hat{c}_1\hat{c}_2\|\nabla\rho_0\|_\infty\|A^\eta a\|\exp(c_0\hat{c}_1\|A^\eta a\|)$$

for any $t \in [T_0, 2T_0]$. Therefore we have for any $t \in [T_0, 2T_0]$

$$e^{\delta t}\|A^{5/8}u(t)\|\leqq c_0\hat{c}_1 e^{\delta T_0}\|A^{5/8}a\|\leqq c_0\hat{c}_1 k e^{\delta T_0}\|A^{\gamma}a\|\leqq \varepsilon_3,$$

$$\left\|\frac{\partial\rho}{\partial t}(t)\right\|_\infty\leqq c_0\hat{c}_1\hat{c}_2\exp(c_0\hat{c}_1)\|\nabla\rho_0\|_\infty\leqq\varepsilon_3.$$

These inequalities also holds true in $[0, T_0]$ (see (7.4) and (7.5)). Hence we can apply THEOREM 7.1. Consequently the inequalities (7.6), (7.7) and (7.8) hold in $[0, 2T_0]$. In particular we obtain $\|A^{\gamma}u(2T_0)\|\leqq\hat{c}_1$ and $\|\nabla\rho(2T_0)\|\leqq\hat{c}_2$. Therefore $\{\rho, u\}$ exists in $[2T_0, 3T_0]$. Repeating these procedures, we can conclude that the solution exists in $[0, nT_0]$ for any natural number n. Thus we have proved THEOREM 2.3. Q.E.D.

The remaining part of the present paper is devoted to the proof of THEOREM 7.1. From now on we do not mention the assumption $m_0\leqq m$, $l\leqq l_0$. We prepare some general theorems. Let $\sigma\in W^{1,\infty}(]0, T[\times\Omega)$ be a given function satisfying $m_0\leqq\sigma(t, x)\leqq l_0$ $(0<t<T,\ x\in\Omega)$ and

$$\sup_{0\leqq t\leqq T}\left\|\frac{\partial\sigma}{\partial t}(t)\right\|_\infty\leqq M_0.$$

We denote by $U(t, s)$ the evolution operator generated by $B_\sigma(t)^{-1}A\equiv A(t)$. Then the following proposition holds true.

PROPOSITION 7.1. *There exists a positive constant δ depending only on Ω and l_0 such that*

i) *For any $\alpha\in[0, 1[$ and $\beta\in[0, \alpha]$ we have*

$$\text{(7.9)}\qquad \|A(t)^{\alpha}U(t, s)A(s)^{-\beta}\| \leqq c_{4,\alpha}(1+M_0)(1+t-s)(t-s)^{-(\alpha-\beta)}\exp\{(-\delta+c_3M_0)(t-s)\},$$

where $c_3=c_3(\Omega, m_0, l_0)$ and $c_{4,\alpha}=c_{4,\alpha}(\Omega, m_0, l_0, \alpha)$ are positive constants:

ii) *For any $\gamma\in[0, 1[$ and $\beta\in[0, 1]$ we have*

$$\text{(7.10)}\qquad \|A(t)^{1+\gamma}U(t, s)A(s)^{-\beta}\| \leqq c_{5,\gamma}(1+M_0)^2(1+t-s)^2(t-s)^{-(1+\gamma-\beta)}\exp\{(-\delta+c_3M_0)(t-s)\} \quad (0<s<t<T),$$

where c_3 is the same as in i) *and $c_{5,\gamma}=c_{5,\gamma}(\Omega, m_0, l_0, \gamma)>0$:*

iii) *For any α, β and θ such that $0\leqq\beta\leqq\alpha<1$, $0<\theta<1-\alpha$, we have*

$$\text{(7.11)}\qquad \|A^{\alpha}\{U(t+h, s)-U(t, s)\}A(s)^{-\beta}\| \leqq c_{\alpha,\theta}h^{\theta}(1+M_0)(t-s)^{-(\alpha-\beta)-\theta}\exp\{(-\delta+c_3M_0)(t-s)\} \times\{1+t-s+h^{1-\alpha-\theta}(t-s)^{\alpha-\beta+\theta}(t+h-s)^{\beta}\exp[(-\delta+c_3M_0)(t-s)]\} \quad (0<s<t<t+h<T).$$

Here $c_{\alpha,\theta}=c_{\alpha,\theta}(\Omega, m_0, l_0, \alpha, \theta)>0$.

COROLLARY 7.2. *If M_0 satisfies the inequality $c_3M_0 \leqq \delta/3$, then we have*

$$\|A^{\alpha}\{U(t+h, s)-U(t, s)\}A(s)^{-\beta}\| \leqq c'_{\alpha,\theta}h^{\theta}(1+M_0)(1+t-s)(t-s)^{-(\alpha-\beta)-\theta}\exp\{-2\delta(t-s)/3\}. \tag{7.12}$$

REMARK 7.1. Various constants appearing in PROPOSITION 7.1 and COROLLARY 7.2 do not depend on T.

To show PROPOSITION 7.1 we have only to estimate various operators arising in Tanabe [18], page 118-127. Although it needs careful calculations, we can carry out that procedure in an elementary way. Hence we omit the proof.

Now we prove that the conclusion (7.1) in THEOREM 7.1 holds for $\alpha=5/8$. Hereafter constants depending only on Ω, m_0, l_0 are denoted by c, c' or c''. Put

$$E=\max\Big(\sup_{0\leq t\leq T}\Big\|\frac{\partial\rho}{\partial t}(t)\Big\|_{\infty},\ \sup_{0\leq t\leq T}e^{\delta t/3}\|A^{5/8}u(t)\|\Big).$$

Then we have in a usual way

$$\|A^{5/8}u(t)\|\leqq c(1+E)(1+t)\|A^{5/8}a\|\exp\{-(\delta-c_3E)t\}$$
$$+c(1+E)\int_0^t(1+t-s)(t-s)^{-5/8}\|A^{5/8}u(s)\|^2\exp\{-(\delta-c_3E)(t-s)\}\,ds.$$

Here we have made use of PROPOSITION 7.1. If E is so small that $E\leqq \min\{1, \delta/3c_3\}$, then we obtain

$$\|A^{5/8}u(t)\|\leqq c(1+t)e^{-2\delta t/3}\|A^{5/8}a\|$$
$$+c\int_0^t(1+t-s)e^{-2\delta(t-s)/3}(t-s)^{-5/8}\|A^{5/8}u(s)\|^2ds.$$

Putting $K(t)\equiv \sup_{0\leq s\leq T} e^{\delta s/3}\|A^{5/8}u(s)\|$, we can rewrite the above inequality as

$$K(t)\leqq c\|A^{5/8}a\|+cK(t)^2e^{-\delta t/3}\int_0^t(1+t-s)(t-s)^{-5/8}ds$$
$$\leqq c\|A^{5/8}a\|+c'EK(t).$$

Consequently we have $K(t)\leqq 2c\|A^{5/8}a\|$ if $E\leqq 1/2c'$. Thus (7.1) for $\alpha=5/8$ is verified by putting $\varepsilon_3=\min\{1, 1/3c_3, 1/2c'\}$ and by rewriting $\delta/3$ as δ.

Proof of (7.1) *for* $\alpha=\eta$. We assume that $E\leqq\varepsilon_3$. Then it is clear that

$$\|A^{\eta}u(t)\|\leqq c(1+t)e^{-2\delta t/3}\|A^{\eta}a\|$$
$$+c\int_0^t(1+t-s)e^{-2\delta(t-s)/3}(t-s)^{-\eta}\|A^{5/8}u(s)\|^2ds$$
$$\leqq c'e^{-\delta t/3}\|A^{\eta}a\|$$
$$+c'\int_0^t(1+t-s)e^{-2\delta(t-s)/3}(t-s)^{-\eta}e^{-2\delta s/3}ds\|A^{5/8}a\|^2.$$

Since $\|A^{5/8}a\| \leqq \varepsilon_3 \leqq 1$, the above inequality implies (7.1) for $\alpha = \eta$.

Next we note that under the condition $E \leqq \varepsilon_3$, it holds that for any $0 \leqq t \leqq t+h \leqq T$

$$e^{\delta t/3}\|A^{5/8}\{u(t+h)-u(t)\}\|t^{\theta} \leqq ch^{\theta}\|A^{5/8}a\|. \tag{7.13}$$

Here θ is the constant appearing in section 3. This inequality can be shown in the same way as the proof of (3.14) (use COROLLARY 7.2). Hence we omit the proof.

We next prove that

$$t^{1-\eta}\|Au(t)\| \leqq ce^{-\delta t/3}\|A^{\eta}a\| \qquad (0<t<T), \tag{7.14}$$

so long as $E \leqq \varepsilon_3$. To this end we put $\Phi(t) \equiv -B_{\rho}(t)^{-1}F_{\rho}u(t)$. Then we have by (3.26) and (3.31)

$$\begin{aligned}\|Au(t)\| &\leqq \|AU(t, 0)a\| \\ &\quad + \int_0^t \|AU(t, s)\|\|\Phi(s)-\Phi(t)\|ds \\ &\quad + \|AA(t)^{-1}\Phi(t)\| + \|AU(t, 0)A(0)^{-1}\|\|\Phi(t)\| \\ &\quad + \int_0^t \|AU(t, s)A^{-1}\|\left\|P\frac{\partial\rho}{\partial s}(s)\right\|\|\Phi(t)\|ds \\ &\equiv J_1+J_2+J_3+J_4.\end{aligned}$$

By PROPOSITION 7.1 we obtain

$$\begin{aligned}\|J_1\| &\leqq c(1+t)e^{-2\delta t/3}t^{-(1-\eta)}\|A^{\eta}a\| \\ &\leqq c'e^{-\delta t/3}t^{-(1-\eta)}\|A^{\eta}a\|.\end{aligned}$$

On the other hand, it follows in a way similar to the proof of (3.29) that

$$\|\Phi(s)-\Phi(t)\| \leqq c\|A^{5/8}a\|e^{-2\delta/3}\{(t-s)^{\theta}s^{-\theta}+(t-s)\}.$$

Here we have used (7.13). From this inequality we obtain

$$\|J_2\| \leqq ce^{-\delta t/3}\|A^{\eta}a\|t^{-(1-\eta)}.$$

The inequality $\|J_3\| \leqq ce^{-\delta t/3}\|A^{\eta}a\|t^{-(1-\eta)}$ is proved similarly. J_4 is estimated as

$$\begin{aligned}\|J_4\| &\leqq c\int_0^t (1+t-s)^2 e^{-2\delta(t-s)/3}e^{-2\delta s/3}ds\|A^{5/8}a\|^2 \\ &\leqq c'e^{-\delta t/3}\|A^{\eta}a\|.\end{aligned}$$

Here we have used (7.1), $\|A^{\eta}a\| \leqq 1$ and $\|A^{5/8}a\| \leqq k\|A^{\eta}a\|$. From the above inequalities for J_i $(i=1, 2, 3, 4)$ we obtain (7.14).

Our next task is to show that

$$\|\nabla u(t)\|_{\infty} \leqq c\|A^{\eta}a\|e^{-\delta t/3}t^{-(1+\gamma-\eta)} \qquad (0<t<T), \tag{7.15}$$

so long as $E \leqq \varepsilon_3$ and $\|A^\gamma a\| \leqq 1$. However, this inequality is proved in a way similar to the proof of (3.16). The only difference is to use PROPOSITION 7.1 and (7.14) instead of LEMMA 3.2 and (3.15), respectively. Hence we omit the proof.

Now (7.2) follows from (7.15) and the inequalities (3.3) and (3.4). Thus we have completed the proof of THEOREM 7.1. Q.E.D.

REMARK 7.2. The argument in this section is suggested by Matsumura and Nishida [16].

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