Fractionally logarithmic canonical rings of algebraic surfaces

By Takao FUJITA

Introduction

Let S be an algebraic surface defined over an algebraically closed field $\mathfrak{K}$ of any characteristic. Given an effective $\boldsymbol{Q}$-divisor D on S, we consider the graded $\mathfrak{K}$-algebra $G(S, K+D)=\oplus_{t\geq 0}H^0(S, tK+\underline{tD})$, where K is the canonical bundle of S and $\underline{tD}$ is the largest usual divisor such that $tD-\underline{tD}$ is effective. Of course, in general, this algebra is not finitely generated. One of the main purpose of this paper is to prove the following

THEOREM. *$G(S, K+D)$ is finitely generated if D is reduced, that means, the coefficient of each prime component of D is not greater than one.*

Actually we will prove the following

THEOREM. *Let D be a reduced effective $\boldsymbol{Q}$-divisor on S such that $K+D$ is pseudo-effective. Then the semipositive part of the Zariski decomposition of $K+D$ is semiample.* (*See* (1.4)).

These results are reduced to classical theory when $D=0$. Kawamata [K] considered the problem in case $\mathfrak{K}=\boldsymbol{C}$. Although he apparently assumed that D is a usual divisor with no singularities other than nodes, his method involves a study of the problem in general situation. Later Sakai [S] generalized his results for any reduced curve D. Sakai considered the problem partly in positive characteristic cases too. Miyanishi [Mi] studied another trouble in positive characteristic cases (compare § 2 of this paper). Since we consider the most general case, I hope, our argument is easier to understand because what was implicit in the preceding works is explicit here. Among others a new notion of minimality due to Sakai is very powerful in the framework of our method. Finally we remark that our results here is an important step of our proof of the fact that the canonical ring of an algebraic threefold of Kodaira dimension two is finitely generated if char($\mathfrak{K}$)=0.

The author would like to express his hearty thanks to Professors E. Horikawa and F. Sakai for their valuable comments on the first version of this paper.

§ 1. Preliminaries

(1.1) DEFINITION and NOTATION. A *$\boldsymbol{Q}$-divisor* on S is a formal linear combination $D=\sum \mu_i D_i$ of prime divisors D_i on S with coefficients μ_i being rational numbers. It is said to be *effective* (resp. *reduced*) if $\mu_i \geqq 0$ (resp. $0 \leqq \mu_i \leqq 1$) for each coefficient μ_i. Given a $\boldsymbol{Q}$-divisor D, we define $\bar{D}$ (resp. $\underline{D}$), called the *upper* (resp. *lower*) *integral hull* of D, to be the least (resp. greatest) usual divisor such that $\bar{D}-D$ (resp. $D-\underline{D}$) is effective. D is said to be *integral* if $D=\underline{D}=\bar{D}$, namely, if D is a usual divisor.

A *$\boldsymbol{Q}$-bundle* on S is an element of $\mathrm{Pic}(S)\otimes\boldsymbol{Q}$. The $\boldsymbol{Q}$-bundle defined by a $\boldsymbol{Q}$-divisor D is denoted by $[D]$, or just by D when there is no danger of confusion. Given $\boldsymbol{Q}$-bundles L_1 and L_2 on S, we define the intersection number $L_1L_2 \in \boldsymbol{Q}$ in the obvious way. A $\boldsymbol{Q}$-bundle L is said to be *semipositive* if $LC \geqq 0$ for any curve C in S. A $\boldsymbol{Q}$-bundle F is said to be *pseudo-effective* if $FH \geqq 0$ for any semipositive $\boldsymbol{Q}$-bundle H on S.

Given a $\boldsymbol{Q}$-bundle F, take a positive integer m and a line bundle L on S such that $mF=L$ in $\mathrm{Pic}(S)\otimes\boldsymbol{Q}$. It is easy to see that the L-dimension $\kappa(L, S)$ (cf. [I] and [F2]) is determined uniquely by F. So this will be denoted by $\kappa(F)$. If we can find (m, L) as above such that $\mathrm{Bs}|L|=\emptyset$, F is said to be *semiample*. A line bundle L is semiample (as a $\boldsymbol{Q}$-bundle) in the above sense if and only if the invertible sheaf $\mathcal{O}(mL)$ is generated by its global sections for some positive integer m.

(1.2) THEOREM. *For any pseudo-effective $\boldsymbol{Q}$-bundle F on S, there is an effective $\boldsymbol{Q}$-divisor N with the following properties.*

a) *$H=F-N$ is a semipositive $\boldsymbol{Q}$-bundle.*

b) *Let $C_1, \cdots, C_r$ be prime components of N. Then $HC_i=0$ for each i and the (r, r)-matrix (C_iC_j) is negative definite unless $N=0$.*

For a proof, see [F1]. Moreover, one can show that such a $\boldsymbol{Q}$-divisor N is determined uniquely by the numerical equivalence class of F. Therefore N (resp. H) will be called the negative (resp. semipositive) part of (the Zariski decomposition of) F.

(1.3) PROPOSITION. *Let F be a pseudo-effective usual line bundle on S and let $F=N+H$ be the Zariski decomposition as above. Then $H^0(S, tF)\cong H^0(S, tF-\overline{tN})$ for any positive integer t.*

Proof is easy and is essentially due to Zariski.

(1.4) MAIN THEOREM. *Let D be a reduced $\boldsymbol{Q}$-divisor such that $K+D$ is pseudo-effective* (*here K denotes the canonical bundle of S*). *Then the semipositive* *part of $K+D$ is semiample.*

This will be proved in the sequel. One of the main consequence of this result is the following

(1.5) COROLLARY. *Let things be as in* (1.4). *Then the graded algebra* $G(S, K+D)=\bigoplus_{t\geq 0} H^0(S, tK+\underline{tD})$ *is finitely generated.*

PROOF. Let $K+D=N+H$ be the Zariski decomposition of $K+D$ and take a positive integer m such that both mD and mN are integral. Then $L_m=mK+mD-mN=mH$ is semiample as a $\boldsymbol{Q}$-bundle. So, replacing m by its multiple if necessary, we may assume that $Bs|L_m|=\emptyset$. Setting $L_t=tK+\underline{t(D-N)}$ for $t\geqq 0$, we infer $H^0(S, tK+\underline{tD})\cong H^0(S, L_t)$ from (1.3). By virtue of [F3; (1.7)], $G_m=\bigoplus_{s\geq 0} H^0(S, sL_m)$ is a finitely generated algebra and $\bigoplus_{s\geq 0} H^0(S, L_{sm+j})$ is a finitely generated G_m-module for each $j=0, 1, \cdots, m-1$. Therefore $G(S, K+D)$ is finitely generated.

§2. Beginning of the proof of the Main Theorem

We first prove $\kappa(K+D)\geqq 0$ and then consider the problem separately according to the value of $\kappa(K+D)$.

(2.1) In order to show $\kappa(K+D)\geqq 0$, we may clearly assume $\kappa(K)<0$. If S is rational, the argument in [F1; (2.8)] works well. So we may assume that S is an irrational ruled surface. By the pseudo-effectivity we have $(K+D)Y\geqq 0$ for any general fiber Y of the Albanese fibration $\alpha: S\to C$. Hence it suffices to prove the following

(2.2) THEOREM. *Let $\alpha: S\to C$ be the ruling of an irrational ruled surface S and let X be a reduced $\boldsymbol{Q}$-divisor on S such that $(K+X)Y=0$ for any general fiber Y of α. Then $\kappa(K+X)\geqq 0$.*

In order to prove this, we may clearly assume that X has no vertical component. Now we recall several lemmas.

(2.3) LEMMA. *Let $f: V\to W$ be a surjective morphism between normal varieties and let L be a line bundle on W. If f^*L is a torsion in* Pic(V), *then so is L in* Pic(W).

For a proof, consult [F2; (3.17)] or [F3; (1.20)].

(2.4) LEMMA. *Let $\alpha: S\to C$ be a $\boldsymbol{P}^1$-bundle over a curve C and suppose that*

α has two sections Z_1, Z_2 disjoint with each other. Then the relative canonical sheaf $\omega_{S/C}$ is isomorphic to $\mathcal{O}_S[-Z_1-Z_2]$.

Proof is easy and well-known.

(2.5) COROLLARY. *Let α be as above and suppose that there are three sections Z_1, Z_2 and Z_3 of α disjoint with each other. Then $S\cong \boldsymbol{P}^1\times C$ and Z_j's are fibers of the projection onto $\boldsymbol{P}^1$.*

PROOF. (2.4) implies $\mathcal{O}_S[Z_1+Z_2]=\mathcal{O}_S[Z_2+Z_3]=\mathcal{O}_S[Z_3+Z_1]$. Hence Z_j's are linearly equivalent to each other and $|Z_j|$ defines a morphism onto $\boldsymbol{P}^1$. The assertion is now obvious.

(2.6) Proof of (2.2), in case α has no singular fiber.

Since $(K+X)Y=0$, we have a $\boldsymbol{Q}$-bundle L on C such that $K+X=\alpha^*L$. We may assume $\deg L\leqq 0$ because otherwise the assertion is obvious.

If there is a component Z of X such that $Z^2<0$, set $X=\mu Z+X'$ where X' is the linear combination of other components. Then we have $0\geqq Z\alpha^*L=Z(K+X)\geqq Z(K+\mu Z)\geqq Z(K+Z)=2h^1(\mathcal{O}_Z)-2\geqq 0$. So we must have equalities and hence $\deg L=X'Z=0$, $\mu=1$ and Z is a normal elliptic curve. Then $L_Z=[K+Z]_Z=\mathcal{O}_Z$. Therefore L is a torsion in Pic(C) by (2.3). This implies $\kappa(K+X)=\kappa(L)=0$.

Thus we may assume that $Z^2\geqq 0$ for every component Z of X. Then we have $0\leqq X^2=(\alpha^*L-K)^2\leqq K^2=8(1-h^1(\mathcal{O}_C))\leqq 0$. Therefore $X^2=0$, C is an elliptic curve and $\deg L=0$ because $K\cdot\alpha^*L=-2\deg L$. Moreover $X^2=0$ implies that $Z^2=0$ and that Z is disjoint from other components of X. Hence $(K+Z)Z=(K+X)Z=Z\cdot\alpha^*L=0$. So every component Z of X is an elliptic curve.

Suppose that the natural mapping $Z\to C$ is not separable for some component Z of X. Let $\tilde{S}$ be the fiber product $S\times_C Z$. Then $\tilde{S}$ is a $\boldsymbol{P}^1$-bundle over Z since so is S over C. Moreover, the inclusion $Z\to S$ gives a section $\tilde{Z}$ of $\tilde{S}$. The multiplicity of $\tilde{Z}$ in the pull-back of Z to $\tilde{S}$ is equal to p^e, the purely inseparable degree of the mapping $Z\to C$. The canonical bundle $\tilde{K}$ of $\tilde{S}$ is the pull-back of K, because both are relative canonical bundles of $\tilde{S}\to Z$ and $S\to C$ respectively. We have $\mathcal{O}_Z=\omega_Z=[K+Z]_Z$, hence $[\tilde{K}+p^e\tilde{Z}]_{\tilde{Z}}=0$. On the other hand we have $[\tilde{K}+\tilde{Z}]_{\tilde{Z}}=\omega_{\tilde{Z}}=\mathcal{O}_{\tilde{Z}}$. So $(p^e-1)\tilde{Z}=0$ in Pic($\tilde{Z}$). So $[Z]$ is a torsion in Pic(Z) by (2.3). Then $L_Z=[K+X]_Z=[K+\mu Z]_Z=(\mu-1)[Z]_Z=0$ in Pic$(Z)\otimes\boldsymbol{Q}$. So $\kappa(L)=0$ by (2.3).

Thus we may assume that the mapping $Z\to C$ is separable for every component Z of X. Let n be the number of set-theoretic intersection points of X and a general fiber Y of α. Clearly $n\geqq 2$ because $XY=-KY=2$. Since $Z\to C$ is étale for every component Z, there is an étale covering $\tilde{C}\to C$ such that the pull-back $\tilde{X}$ of X to $\tilde{S}=S\times_C\tilde{C}$ consists of n disjoint components which are sections of $\tilde{S} \to \tilde{C}$. The canonical bundle $\tilde{K}$ of $\tilde{S}$ is the pull-back of K. Hence, in $\mathrm{Pic}(\tilde{S}) \otimes \boldsymbol{Q}$, we have $[K+X]_{\tilde{S}} = \tilde{K}+\tilde{X}=0$ by (2.5) if $n \geqq 3$. If $n=2$, X must be integral and $\tilde{K}+\tilde{X}=0$ by (2.4). In either case we infer that $K+X$ is a torsion in $\mathrm{Pic}(S)$ by (2.3). Thus we complete the proof when α is a $\boldsymbol{P}^1$-bundle.

(2.7) Proof of (2.2), general case. We use the induction on the Picard number of S. By (2.6), we may assume that α has a singular fiber F. For any component F_i of F which is not an exceptional curve, we have $F_i^2 \leqq -2$ and $F_i K \geqq 0$. Hence $F_i(K+X) \geqq 0$ since X has no vertical component. Therefore, F must contain an exceptional curve E with $E(K+X) \leqq 0$, since $F(K+X)=0$. Let $\pi: S \to S'$ be the contraction of E to a smooth point and let X' (resp. K') be the image $\boldsymbol{Q}$-divisor of X on S' (resp. the canonical bundle of S'). Then $K+X= \pi^*(K'+X')+\mu E$, where $\mu=-E(K+X) \geqq 0$. X' is clearly reduced and hence $\kappa(K'+X', S') \geqq 0$ by the induction hypothesis. So $\kappa(K+X, S) \geqq 0$.

(2.8) As a corollary of (2.2), one obtains the following

THEOREM (Miyanishi [Mi]). *Let S be an irrational (non-complete) surface of negative logarithmic Kodaira dimension. Then S admits an $\boldsymbol{A}^1$-ruling.*

§ 3. The case $\kappa=2$

(3.1) Take a positive integer m such that mH is a usual line bundle. Then $\mathrm{SBs}(mH) = \bigcap_{t \geq 1} \mathrm{Bs}|tmH|$ is independent of m and is determined by the $\boldsymbol{Q}$-bundle H. So this is denoted by $\mathrm{SBs}(H)$. By [F3; (1.18)], it suffices to show $\mathrm{SBs}(H)=\emptyset$.

(3.2) Take an effective $\boldsymbol{Q}$-divisor Y such that $D-Y$ and $N-Y$ are effective and have no common component. Then $K+(D-Y)=(N-Y)+H$ is the Zariski decomposition of $K+D-Y$. So, replacing D by $D-Y$ if necessary, we may assume that D and N have no common component.

(3.3) DEFINITION. An exceptional curve E on S is said to be *redundant* if $HE=0$. The pair (S, D) is said to be *Sakai-minimal* if there is no redundant exceptional curve.

(3.4) Suppose that S contains a redundant exceptional curve E. Let $f: S \to S'$ be the contraction of E to a smooth point, let $D'=f_*D$ be the image of D and let K' be the canonical bundle of S'. We claim that $K'+D'$ is pseudo-effective and $H=f^*H'$ for the semipositive part H' of $K'+D'$.

Indeed, $f^*(K'+D') \equiv K+D \bmod E$. So, for any semipositive line bundle L on S', we have $(K'+D')L=(K+D)f^*L \geqq 0$ since f^*L is semipositive. Thus $K'+D'$ is pseudo-effective. Let $K'+D'=N'+H'$ be the Zariski decomposition.

Clearly $f^*N'+mE$ satisfies the condition (1.2; b) for any $m>0$. Therefore f^*H' is the semipositive part of $f^*(K'+D')+mE$. On the other hand, we have $H^2>0$ since $\kappa(K+D)=\kappa(H)=2$. So, by the index theorem, $N+\mu E$ satisfies the condition (1.2; b) for any $\mu>0$. Therefore H is the semipositive part of $K+D+\mu E$. Combining them we infer $H=f^*H'$.

REMARK. $H=f^*H'$ is not always true if $\kappa(K+D)<2$.

(3.5) Repeating the above argument, we reduce the problem to the case in which (S, D) is Sakai-minimal. Then we have $N=0$.

Indeed, if not, there is a component C of N such that $NC<0$. Then $(K+D)C=(N+H)C=NC<0$ and $DC\geqq 0$ by the assumption (3.2). So $KC<0$. Since $C^2<0$, this implies that C is an exceptional curve, which is redundant by definition. Thus (S, D) is not Sakai-minimal.

(3.6) LEMMA. *Any irreducible component Z of* SBs(H) *is a curve with $HZ=0$.*

For a proof, use [Z; Theorem 9.1].

(3.7) In view of the above lemma, we will study curves C such that $HC=0$. Since $C^2<0$ by the index theorem, these curves are numerically independent with each other. So there are at most finitely many such curves. We will study the structure of a connected component Σ of the union of all such curves.

(3.8) Suppose that $\Sigma\not\subset D$. Take a prime component Z of Σ such that $Z\not\subset D$. Then $(K+D)Z=HZ=0$ and $KZ=-DZ\leqq 0$. Since $Z^2<0$ and Z is not an exceptional curve by the Sakai-minimality, we infer $KZ=DZ=0$. So any component of Σ meeting Z is not contained in D, hence is of the same type as Z. Thus we infer that Σ does not meet D. Moreover, by the theory of Artin [A], Σ can be contracted to a rational double point. So, by [F3; (1.19)], $\Sigma\cap\mathrm{SBs}(H)=\emptyset$.

In the following we consider the case in which $\Sigma\subset D$.

(3.9) Suppose that $Z\not\cong \boldsymbol{P}^1$ for some component Z of Σ. Write $D=\delta Z+D^*$, where D^* is a linear combination of components other than Z. We have $0=HZ=(K+D)Z\geqq(K+Z)Z-(1-\delta)Z^2\geqq(K+Z)Z\geqq 0$ because $Z^2<0$. So $\delta=1$, $D^*Z=0$ and $(K+Z)Z=0$. Hence Z is an isolated component of D with arithmetic genus one. In particular $\Sigma=Z$. Now, by the vanishing theorem [F3; (7.5)], we have $H^1(S, K+tH)=0$ for $t\gg 0$ provided tH is integral. So the mapping $H^0(S, K+tH+Z)\rightarrow H^0(Z, \omega_Z[tH])$ is surjective. Since $H_Z=(K+Z)_Z=\omega_Z\cong\mathcal{O}_Z$, this implies that $X\cap Z=\emptyset$ for any general member X of $|K+tH+Z|$. Take a positive integer m such that mD^* is integral. Then $mX+mD^*\in|(t+1)mH|$. Hence $Z\not\subset\mathrm{SBs}(H)$.

(3.10) Suppose that Σ is a rational tree. This means that all the components $Z_1, \cdots, Z_r$ of Σ are rational normal curves and the dual graph of Σ is a tree. Since the matrix (Z_iZ_j) is negative definite by the index theorem, Σ can be contracted to a normal point. We claim that this singularity is rational.

We will prove $H^1(\mathcal{O}_X)=0$ for any effective divisor $X=\sum \xi_i Z_i$ supported in Σ by the induction on $\xi=\sum \xi_i$. If $\xi_i \leqq 1$ for each i, we have $H^1(\mathcal{O}_X)=0$ since Σ is a rational tree. Hence we may assume $\xi_i \geqq 2$ for some i. Write $X-D=E_1-E_2$ for effective $\boldsymbol{Q}$-divisors E_1, E_2 such that E_1 and E_2 have no common component. $E_1 \neq 0$ by the above assumption. Take a component C of E_1 such that $E_1C<0$, and set $Y=X-C$. Clearly C is a component of X and hence Y is effective. So $H^1(\mathcal{O}_Y)=0$ by the induction hypothesis. We have also $h^1(C, -Y)=h^0(C, \omega_C[Y])=0$ since $\deg \omega_C[Y]=(K+C+Y)C=(K+X)C=(X-D)C=(E_1-E_2)C \leqq E_1C<0$. Using the exact sequence $0 \to \mathcal{O}_C[-Y] \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, we obtain $H^1(\mathcal{O}_X)=0$, as desired.

It follows that some multiple of H is the pull-back of a line bundle on the normal variety obtained by the contraction of Σ. Therefore, similarly as in (3.8), we infer $\Sigma \cap \mathrm{SBs}(H)=\emptyset$.

(3.11) If Σ is not of the types considered above, every component Z of Σ must be a rational normal curve contained in D. Moreover, since Σ is not a rational tree, one of the following conditions is satisfied.

a) $Z_1Z_2 \geqq 2$ for some components Z_1, Z_2 of Σ.

b) Σ has components $Z_1, Z_2, \cdots, Z_r$ $(r \geqq 3)$ such that $Z_iZ_j>0$ if $|i-j| \equiv 1$ modulo r.

In case a), we write $D=\delta_1 Z_1+\delta_2 Z_2+D^*$ where D^* is a linear combination of other components. By symmetry we may assume $\delta_1 \leqq \delta_2$. Note that $Z_i^2 \leqq -2$ because of the Sakai-minimality. Therefore $2=HZ_1-(K+Z_1)Z_1=(D-Z_1)Z_1=(\delta_1-1)Z_1^2+\delta_2 Z_1Z_2+D^*Z_1 \geqq 2(1-\delta_1)+2\delta_2 \geqq 2$. So we must have equalities and hence $\delta_1=\delta_2$, $D^*Z_1=0$ and $Z_1Z_2=2$. Moreover $Z_1^2=-2$ unless $\delta_1=1$. Since $\delta_1=\delta_2$, we have $D^*Z_2=0$ by a similar argument. Moreover, if $\delta_1=\delta_2<1$, we would have $Z_1^2=Z_2^2=-2$ and $(Z_1+Z_2)^2=0$, which contradicts the index theorem. Hence $\delta_1=\delta_2=1$. Thus we have $\Sigma=Z_1+Z_2$, $D=\Sigma+D^*$. Note that $\Sigma \cap D^*=\emptyset$ and $\omega_\Sigma \cong \mathcal{O}_\Sigma$.

In case b), we write $D=\sum_{i=1}^{r} \delta_i Z_i+D^*$ similarly as above. We may assume $\delta_1=\mathrm{Min}_i(\delta_i)$. Then $2=HZ_1-(K+Z_1)Z_1=(D-Z_1)Z_1=(\delta_1-1)Z_1^2+D^*Z_1+\sum_{i=2}^{r} \delta_i Z_i Z_1 \geqq 2(1-\delta_1)+\delta_2+\delta_r \geqq 2$. So $D^*Z_1=0$, $\delta_1=\delta_2=\delta_r$, $Z_1Z_2=Z_1Z_r=1$ and $Z_iZ_1=0$ for $2<i<r$. Moreover $Z_1^2=-2$ unless $\delta_1=1$. Since $\delta_2=\delta_1$, this argument applies to Z_2 too. Repeating similarly we obtain $\delta_1=\cdots=\delta_r$. Moreover this is equal to one because otherwise $Z_i^2=-2$ for every i and $(Z_1+\cdots+Z_r)^2 \geqq 0$, contradicting the index theorem. Thus we infer $D=\Sigma+D^*$ and $\Sigma=Z_1+\cdots+Z_r$. Note that $\Sigma \cap D^*=\emptyset$ and $\omega_\Sigma \cong \mathcal{O}_\Sigma$.

In both cases a) and b), by a similar reasoning as in (3.9), we have $X\cap\Sigma=\emptyset$ for any general member X of $|K+tH+\Sigma|$, provided tH is integral and $t\gg 0$. Therefore $\Sigma\cap\mathrm{SBs}(H)=\emptyset$.

(3.12) The preceding arguments altogether prove $\mathrm{SBs}(H)=\emptyset$. So $\mathrm{Bs}|mH|=\emptyset$ for some positive m. Moreover, the rational mapping defined by $|mH|$ contracts only those curves as in (3.7). Thus we have proved the following

(3.13) THEOREM. *Let D be a reduced $\boldsymbol{Q}$-divisor on S such that $\kappa(K+D, S)=2$. Then there is a birational morphism $\pi: S\to V$ onto a normal variety V which has only rational or elliptic singularities, together with an ample $\boldsymbol{Q}$-bundle A on V such that π^*A is the semipositive part of $K+D$. In particular $G(S, K+D)$ is a finitely generated $\mathfrak{K}$-algebra and $V\cong\mathrm{Proj}(G(S, K+D))$. Furthermore, if the coefficient of each component of D is less than one, then V has only rational singularities.*

§ 4. The case $\kappa=1$

As for this case, we have the following result.

(4.1) THEOREM (Zariski [Z]). *Let H be a semipositive line bundle on a normal surface S with $\kappa(H, S)=1$. Then H is semiample.*

PROOF. Taking a non-singular model and applying [F3; (1.20)], we reduce the problem to the case in which S is non-singular. Take a positive integer m such that $|mH|$ gives an Iitaka fibration (cf. [F2; § 3]). Replacing S by the graph of $\rho_{|mH|}$ if necessary, we obtain a morphism $\rho: S\to C$ onto a curve C in $\boldsymbol{P}^N$ such that $mH=E+\rho^*\mathcal{O}_C(1)$ in $\mathrm{Pic}(S)$ for some effective divisor E on S. Set $X=\rho^*\mathcal{O}_C(1)\in\mathrm{Pic}(S)$. Then $0\leqq XE=X(X+E)=mXH\leqq m(X+E)H=m^2H^2=0$, since $H^2>0$ would imply $\kappa(H)=2$. Hence $XE=HE=0$. So E is contained in fibers of ρ. Then $X_E=0$ and $E_E=mH_E$ is semipositive. From this we infer that any connected component E_i of E is proportional to the Cartier divisor $\rho^*[\rho(E_i)]$. Hence E_i is semiample, and so is $mH=X+\sum_i E_i$.

(4.2) COROLLARY. *Let things be as in* (4.1) *and let Y be an effective Cartier divisor on S such that $HY=0$. If Y is semipositive, then Y is semiample.*

PROOF. Replacing S by a suitable birational model if necessary, we may assume that there is a morphism $\rho: S\to C$ as in (4.1). We may assume that Y is connected. Since $HY=0$, $\rho(Y)$ is a point on C. By the semipositivity we infer that Y is proportional to the divisor $\rho^*(\rho(Y))$. Hence Y is semiample.

(4.3) REMARK. In our particular case, H is the semipositive part of $K+D$. Since H is semiample, $|mH|$ gives a morphism $\rho: S\to C$ as in (4.1) for some $m>0$. $HN=0$ implies that any component of N is contained in a fiber of ρ. So, for any general fiber F of ρ, we have $(K+D)F=(N+H)F=0$. Hence $(K+F)F=KF=-DF\leqq 0$. Thus $F\cong \boldsymbol{P}^1$ unless $DF=0$.

This argument works even if D is not reduced, provided D is effective.

§5. The case $\kappa=0$

(5.1) We have $H^2=0$ because $H^2>0$ would imply $\kappa(H)=2$. By assumption there is a member Z of $|mH|$ for some positive integer m such that mD is integral (hence so is mH). Then, for every positive integer t, tZ is the unique member of $|tmH|$. We will derive a contradiction assuming $Z\neq 0$.

(5.2) Similarly as in (3.2), we may assume that N and D have no common component. In the sequel we first consider the case in which (S, D) is Sakai-minimal. The general case will be treated in (5.10).

(5.3) As in (3.5), we infer $N=0$ from the Sakai-minimality. So $H=K+D$.

(5.4) Let $Z=\sum \xi_i Z_i$ be the prime decomposition of Z. We have $HZ=mH^2=0$ while $HZ_i\geqq 0$ for each i. So $HZ_i=0$. This implies $Z_i^2\leqq 0$. We now claim $KZ_i\geqq 0$ for every i.

Indeed, if $KZ_i<0$, then $(K+Z_i)Z_i<0$ and hence $Z_i\cong \boldsymbol{P}^1$. If $Z_i^2\geqq 0$, we would have $\kappa(H)=\kappa(Z)\geqq\kappa(Z_i)>0$. If $Z_i^2<0$, then Z_i would be an exceptional curve, contradicting the Sakai-minimality.

This claim implies $KH\geqq 0$. On the other hand we have $DH\geqq 0$ and $(K+D)H=H^2=0$. Therefore $KH=DH=0$. So $KZ_i=0$ for every i by the claim. Then $DZ_i=(H-K)Z_i=0$. Thus we get:

(5.5) LEMMA. *$KZ_i=DZ_i=HZ_i=ZZ_i=0$ for every component Z_i of Z if (S, D) is Sakai-minimal.*

(5.6) *Let μY be a connected component of Z, where μ is the greatest common divisor of the coefficients of prime components of Y in Z. Then $\omega_Y\cong \mathcal{O}_Y$.*

Indeed, by (5.5), Y is indecomposable of canonical type in the sense of Mumford [Mu; §2]. So his Corollary 1 in p. 333 applies.

(5.7) Assume that $\kappa(K)\geqq 0$. Take $t>0$ such that $|tmK|\neq\emptyset$. For any member E of $|tmK|$, we have $E+tmD\in|tmH|$. So $E+tmD=tZ$. By (5.5) we have $KE_i=0$ for any component E_i of E. Therefore E does not contain an exceptional curve and S must be minimal. In particular K is semipositive. So we can apply the argument in [Mu; §2, Step(II) and (III)] to infer $\kappa(Z)>0$. Thus this case is ruled out.

(5.8) Assume that $H^1(S, \mathcal{O}_S)=0$. Take Y as in (5.6). By the exact sequence $0\rightarrow\omega_S\rightarrow\mathcal{O}_S[K+Y]\rightarrow\omega_Y\rightarrow 0$ we infer that $H^0(S, K+Y)\rightarrow H^0(Y, \omega_Y)$ is surjective. So, by (5.6), $X\cap Y=\emptyset$ for a general member X of $|K+Y|$. Set $Z^*=Z-\mu Y$. Then $m\mu X+\mu mD+mZ^*=\mu m(K+Y)+\mu mD+mZ^*=\mu m(K+D)+mZ=(\mu+m)Z$ in Pic(S). Since $\kappa(Z)=0$, this must be an equality between divisors. Compare the multiplicities of a component of Y in them. Since $Y\cap Z^*=\emptyset$, that of the left side $\leqq\mu m$, while that of the right side $\geqq(\mu+m)\mu$. Thus we get a contradiction.

(5.9) By (5.7) and (5.8) we may assume that S is an irrational ruled surface. Let $\alpha: S\rightarrow C$ be the Albanese fibration. By (5.5) we infer that any connected component μY of Z cannot be contained in a fiber of α. So Y has a component Y_0 such that $\alpha(Y_0)=C$. Write $\mu Y=\eta Y_0+Y^*$, where Y^* is a linear combination of other components. Then, by (5.5), we have $0=ZY_0=\mu YY_0\geqq\eta Y_0^2$ and hence $0\geqq(K+Y_0)Y_0=2h^1(\mathcal{O}_{Y_0})-2\geqq 2g(C)-2$. So we must have equalities and $Y_0^2=Y_0Y^*=0$, $g(Y_0)=g(C)=1$. $Y_0Y^*=0$ implies $Y^*=0$ since Y is connected. So $Y=Y_0$ and $\mu=\eta$. Moreover Y is a non-singular elliptic curve. Now we claim that $[Y]_Y$ is a torsion in Pic(Y).

To see this, let δ be the multiplicity of Y in D and set $D^*=D-\delta Y$. Then $D^*Y=(D-\delta Y)Y=0$ by (5.5). So, in Pic(Y), we have $\mu Y=Z=mH=mK+mD=mK+m\delta Y$, while $K+Y=\omega_Y=0$. So $(\mu+m-m\delta)Y=0$ in Pic(Y). Since $m+\mu-m\delta>m(1-\delta)\geqq 0$, this proves our claim.

Let r be the order of $[Y]$ in Pic(Y). Then $H^1(S, K+tY)=0$ for any integer t with $1\leqq t\leqq r$.

We prove this assertion by induction on t. We have an exact sequence $0\rightarrow\omega_S\rightarrow\mathcal{O}_S[K+Y]\rightarrow\omega_Y\rightarrow 0$ and $\omega_Y=\mathcal{O}_Y$. If $\varphi: H^0(S, K+Y)\rightarrow H^0(Y, \omega_Y)$ is not a zero mapping, then there is a member X of $|K+Y|$ such that $X\cap Y=\emptyset$. From this we can derive a contradiction as in (5.8). Hence $\varphi=0$ and $H^0(Y, \omega_Y)\rightarrow H^1(S, \omega_S)$ is injective. Since $h^1(S, \omega_S)=h^1(S, \mathcal{O}_S)=g(C)=1$, this mapping is bijective. Therefore $H^1(S, K+Y)\rightarrow H^1(Y, \omega_Y)$ is injective. On the other hand, we have $h^2(S, K+Y)=h^0(S, -Y)=0$. This implies that $H^1(Y, \omega_Y)\rightarrow H^2(S, \omega_S)$ is surjective, while $h^1(Y, \omega_Y)=1=h^2(S, \omega_S)$ by Serre duality. So this mapping is bijective. Combining these observations we infer $H^1(S, K+Y)=0$, proving the assertion in case $t=1$. When $1<t\leqq r$, we have an exact sequence $H^1(S, K+(t-1)Y)\rightarrow H^1(S, K+tY)\rightarrow H^1(Y, \omega_Y[(t-1)Y])$. The first term vanishes by the induction hypothesis and the last term vanishes because $(t-1)Y$ is a non-trivial torsion in Pic(Y). Therefore $H^1(S, K+tY)=0$, as desired.

From this assertion for $t=r$, we infer that $H^0(S, K+(r+1)Y)\rightarrow H^0(Y, \omega_Y[rY])$

is surjective. Hence $X\cap Y=\emptyset$ for any general member X of $|K+(r+1)Y|$. Set $Z^*=Z-\mu Y$ as in (5.8). Then $m\mu X+\mu mD+m(r+1)Z^*=(\mu+m(r+1))Z$ in $\mathrm{Pic}(S)$, so they must be the same divisor. Comparing the coefficients of Y on both sides we obtain a contradiction.

Thus we complete the proof of (5.1) in case (S, D) is Sakai-minimal.

(5.10) We consider now the general case, using the induction on the Picard number of S.

We may assume that there exists a redundant exceptional curve E on S. Let $\pi: S\to S'$ be the contraction of E to a smooth point, $D'=\pi_*D$ and $N'=\pi_*N$ be the images of the $\boldsymbol{Q}$-divisors D and N respectively, K' be the canonical line bundle on S' and H' be the $\boldsymbol{Q}$-bundle on S' such that $\pi^*H'=H$. Since the pull-backs of K', D' and N' are K, D and N modulo E, we have $K'+D'=N'+H'$ in $\mathrm{Pic}(S')\otimes\boldsymbol{Q}$. Clearly H' is semipositive and $H'N'=HN=0$.

Let $N'=N''+H''$ be the Zariski decomposition of N'. If $H''=0$, then $N'=N''$ and $K'+D'=N''+H'$ is the Zariski decomposition of $K'+D'$. So $\kappa(K'+D')=\kappa(H')=\kappa(H)=\kappa(K+D)=0$. Hence H' is semiample by the induction hypothesis. This implies $H=\pi^*H'=0$.

Even if $H''\neq 0$, we have $H'N''=H'H''=0$ since both are non-negative and $H'(N''+H'')=H'N'=0$. So $N''(H'+H'')=0$. From this we infer that $H^*=H'+H''$ is the semipositive part of $K'+D'$. If $(H'')^2>0$, then H' is numerically equivalent to zero by the index theorem because $H'H''=0$. So $H'=0$ as a $\boldsymbol{Q}$-bundle since $\kappa(H')=\kappa(H)=0$. Therefore we may assume that $(H'')^2=0$. Then $(H^*)^2=0$ and hence $\kappa(H^*)\leqq 1$.

If $\kappa(H^*)=0$, we obtain $H^*=0$ by the induction hypothesis. This implies $H'=0$. If $\kappa(H^*)=1$, we apply (4.2) to infer that H' is semiample, because $H'H^*=0$.

Thus, in any case, H' is semiample and hence so is H.

(5.11) REMARK. It is really possible that $H''\neq 0$ in the above situation. For example, let C be an elliptic curve and let $S'=C\times\boldsymbol{P}^1$. Let D_0' be a fiber of $S'\to C$ and let D_1', D_2' be two fibers of $S'\to\boldsymbol{P}^1$. Let S be the blowing-up of S' at the point $x=D_0'\cap D_1'$ and let E be the exceptional curve lying over x. Let D_i be the strict transform of D_i' and set $D=D_0+D_1+D_2$. Then we have $K+D=D_0$ in $\mathrm{Pic}(S)$ and hence $\kappa(K+D)=0$ because $D_0^2=-1$. E is redundant with respect to (S, D) and $K'+D'=D_0'$. Clearly D_0' is semiample and $\kappa(K'+D')=1$.

(5.12) Any way, we have thus proved the following

THEOREM. *Let D be a reduced $\boldsymbol{Q}$-divisor such that $\kappa(K+D)=0$. Then the semipositive part of $K+D$ is zero in* $\mathrm{Pic}(S)\otimes\boldsymbol{Q}$.

References

[A1] Artin, M., Some numerical criteria for contractibility of curves on algebraic surfaces, Amer. J. Math. 84 (1962), 485–496.

[A2] Artin, M., On isolated rational singularities of surfaces, Amer. J. Math. 88 (1966), 129–137.

[F1] Fujita, T., On Zariski problem, Proc. Japan Acad. **55** (1979), 106–110.

[F2] Fujita, T., On L-dimension of coherent sheaves, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), 215–236.

[F3] Fujita, T., Semipositive line bundles, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), 353–378.

[F4] Fujita, T., Canonical rings of algebraic varieties, in Classification of Algebraic and Analytic Manifolds, ed. K. Ueno, Progr. Math. **39**, Birkhäuser, 1983.

[I] Iitaka, S., On D-dimensions of algebraic varieties, J. Math. Soc. Japan **23** (1971), 356–373.

[K] Kawamata, Y., On the classification of non-complete algebraic surfaces, in Algebraic Geometry (Proceedings Copenhagen 1978), Lecture Notes in Math. 732, Springer, 1979.

[Mi] Miyanishi, M., On affine-ruled irrational surfaces, Invent. math. **70** (1982), 27–43.

[Mu] Mumford, D., Enriques' classification of surfaces in char p: I, in Global Analysis (Papers in honor of K. Kodaira), Princeton Univ. Press, 1969, pp. 325–339.

[S] Sakai, F., D-dimensions of algebraic surfaces and numerically effective divisors, Compositio Math. 48 (1983), 101–118.

[Z] Zariski, O., The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. **76** (1962), 560–615, with an Appendix "The canonical ring of an algebraic surface" by D. Mumford.

(Received April 14, 1983)

Department of Mathematics
College of General Education
University of Tokyo
Komaba, Meguro, Tokyo
153 Japan