Notes on Fourier multipliers for H^p, BMO *and the Lipschitz spaces*

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ABSTRACT. (i) General mapping properties of Fourier multipliers between the spaces H^p, BMO and $\tilde{\Lambda}_s$ (the Lipschitz space) are summarized; inhomogeneous versions of these spaces (h^p, bmo and Λ_s) are also considered. (ii) Mapping properties of the multipliers $|\xi|^{-b}\exp(i|\xi|^{a})$ *etc.* are determined. (iii) A converse to the assertion for (i) is obtained, which asserts that we can construct a Fourier multiplier with "arbitrarily" given mapping properties. (iv) It is shown that the H^1-boundedness of a convolution operator does not imply the weak (1, 1)-boundedness, and *vice versa*.

NOTATION. Fourier transform $\mathcal{F}$ and the inverse Fourier transform $\mathcal{F}^{-1}$ are defined by

$$\mathcal{F}f(\xi)=(2\pi)^{-n/2}\int_{\mathbf{R}^n}f(x)e^{-i\xi\cdot x}dx\,,$$

$$\mathcal{F}^{-1}f(\xi)=\mathcal{F}f(-\xi)\,,\qquad \xi\in\mathbf{R}^n.$$

ψ and ϕ denote fixed functions with the following properties: ψ and ϕ are smooth functions on $[0, \infty)$, $\psi(x)=0$ for $x\leqq 1$, $\psi(x)=1$ for $x\geqq 2$ and $\phi(x)=1-\psi(x)$. If $A(x)$ and $B(x)$ are nonnegative functions, $A(x)\approx B(x)$ means that there exists a positive number C independent of x such that $C^{-1}A(x)\leqq B(x)\leqq CA(x)$; we shall refer to the relation "$A(x)\approx B(x)$" as "inequalities". If E is a measurable subset of $\mathbf{R}^n$, $|E|$ denotes the Lebesgue measure of E. For $\alpha=(\alpha_1, \cdots, \alpha_n)\in(\mathbf{N}\cup\{0\})^n$, $|\alpha|$ is defined by $|\alpha|=\alpha_1+\cdots+\alpha_n$ and the differential operator D^α by

$$(D^\alpha f)(x)=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}f(x)\,,\quad x=(x_1, \cdots, x_n)\in\mathbf{R}^n\,.$$

For $s\in\mathbf{R}$, $[s]$ denotes the integer part of s; $[s]$ is an integer and $[s]\leqq s<[s]+1$. The letter C will denote a constant which may be different in each occasion.

§1. General mapping properties of Fourier multipliers.

Let X and Y be subspaces of $\mathcal{S}'(\mathbf{R}^n)$ or $L^p(\mathbf{R}^n)$, $p<1$, equipped with norms (or semi-norms or quasi-norms) $\|\ \|_X$ and $\|\ \|_Y$. Then the classes of Fourier multipliers for (X, Y) are defined as follows: $\dot{\mathcal{M}}(X, Y)$ is the class of all $m\in\mathcal{D}'(\mathbf{R}^n\setminus\{0\})$ such that

$$\|m\|_{\dot{\mathcal{M}}(X,Y)}=\sup\{\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_Y/\|f\|_X \mid f\in\mathcal{S}_0\cap X,\ \|f\|_X\neq 0\}<\infty\,,$$

where $\mathcal{S}_0$ is the set of all $f \in \mathcal{S}(\mathbf{R}^n)$ such that $\mathcal{F} f \in C_0^\infty(\mathbf{R}^n \setminus \{0\})$; $\mathcal{M}(X, Y)$ is the class of all $m \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|m\|_{\mathcal{M}(X,Y)} = \sup\{\|\mathcal{F}^{-1}(m\mathcal{F} f)\|_Y / \|f\|_X \mid f \in \mathcal{S} \cap X,\ \|f\|_X \neq 0\} < \infty .$$

The spaces of functions and distributions considered in this paper are the following ones. H^p, $0<p<\infty$, is the space of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$f^+(x) = \sup_{0<t<\infty} |(t^{-n}\eta(\cdot/t)*f)(x)| \in L^p(\mathbf{R}_x^n),$$

where $*$ denotes the convolution and η is a fixed element of $\mathcal{S}$ with $\mathcal{F}\eta(0) \neq 0$; the norm (or quasi-norm) in H^p is defined by $\|f\|_{H^p} = \|f^+\|_{L^p}$. h^p, $0<p<\infty$, is the space of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$f^{+,1}(x) = \sup_{0<t<1} |(t^{-n}\eta(\cdot/t)*f)(x)| \in L^p(\mathbf{R}_x^n)$$

with η as above; the norm (or quasi-norm) in h^p is defined by $\|f\|_{h^p} = \|f^{+,1}\|_{L^p}$. Note that $H^p = h^p = L^p$ for $1<p<\infty$. As for H^p and h^p, see [5] and [6]. BMO is the space of all locally integrable functions f on $\mathbf{R}^n$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_B \left\{ |B|^{-1} \int_B |f(x) - f_B| \, dx \right\} < \infty ,$$

where $f_B = |B|^{-1} \int_B f(x) dx$ and the sup is taken over all balls B. bmo is the space of all locally integrable functions f on $\mathbf{R}^n$ such that

$$\|f\|_{\mathrm{bmo}} = \|f\|_{\mathrm{BMO}} + \sup_{B; |B|>1} \left\{ |B|^{-1} \int_B |f(x)| \, dx \right\} < \infty ,$$

where the sup is taken over all balls B with $|B|>1$. If $s=k+\varepsilon$ with k non-negative integer and $0<\varepsilon<1$, then $\tilde{\Lambda}_s$ is the space of all functions $f \in C^k(\mathbf{R}^n)$ such that

$$\|f\|_{\tilde{\Lambda}_s} = \sum_{|\alpha|=k} \sup_{x \neq y} \{|D^\alpha f(x) - D^\alpha f(y)| / |x-y|^\varepsilon\} < \infty ;$$

if $s=k+1$ is a positive integer, then $\tilde{\Lambda}_s$ is the space of all functions $f \in C^k(\mathbf{R}^n)$ such that

$$\|f\|_{\tilde{\Lambda}_s} = \sum_{|\alpha|=k} \sup_{x \neq y} \{|D^\alpha f(x) - 2D^\alpha f((x+y)/2) + D^\alpha f(y)| / |x-y|\} < \infty .$$

Λ_s, $s>0$, is the space of all functions $f \in C^k(\mathbf{R}^n)$, k being the largest integer less than s, such that

$$\|f\|_{\Lambda_s} = \|f\|_{\tilde{\Lambda}_s} + \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty} < \infty .$$

REMARK 1.1. We shall briefly summarize some relations between the above spaces and the *spaces of Triebel-Lizorkin type*. The latter spaces are defined as follows. Let Ψ be an element of $\mathcal{S}$ such that support$(\mathcal{F}\Psi) \subset \{1/2 \leq |\xi| \leq 2\}$ and $\sum_{j=-\infty}^{\infty} \mathcal{F}\Psi(2^j \xi) = 1$ for $\xi \neq 0$, and set $\Psi_j(x) = 2^{jn}\Psi(2^j x)$ and

$$\Theta=\mathcal{F}^{-1}\Big(1-\sum_{j=1}^{\infty}\mathcal{F}\Psi(2^{-j}\xi)\Big);$$

then $\dot{F}^{s}_{p,q}$, $s\in\mathbf{R}$, $0<p$, $q\leqq\infty$, is the space of all $f\in\mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{\dot{F}(s,p,q)}=\left\|\left\{\sum_{j=-\infty}^{\infty}|2^{js}(\Psi_j*f)(x)|^q\right\}^{1/q}\right\|_{L^p(\mathbf{R}^n_x)}<\infty$$

and $F^{s}_{p,q}$, $s\in\mathbf{R}$, $0<p$, $q\leqq\infty$, is the space of all $f\in\mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{F(s,p,q)}=\left\|\left\{\sum_{j=1}^{\infty}|2^{js}(\Psi_j*f)(x)|^q\right\}^{1/q}+|(\Theta*f)(x)|\right\|_{L^p(\mathbf{R}^n_x)}<\infty,$$

where $\{\sum(\cdots)^q\}^{1/q}$ shall be replaced by $\sup(\cdots)$ if $q=\infty$. (It is customary to denote the spaces $\dot{F}^{s}_{\infty,\infty}$ and $F^{s}_{\infty,\infty}$ by $\dot{B}^{s}_{\infty,\infty}$ or $B^{s}_{\infty,\infty}$ respectively.) The following inequalities hold for $f\in\mathcal{S}'(\mathbf{R}^n)$ with $\mathrm{support}(\mathcal{F}f)\not\ni 0$:

$$\|f\|_{H^p}\approx\|f\|_{\dot{F}(0,p,2)},\qquad 0<p<\infty;$$
$$\|f\|_{\tilde{\Lambda}_s}\approx\|f\|_{\dot{F}(s,\infty,\infty)},\qquad s>0;$$
$$C^{-1}\|f\|_{\dot{F}(0,\infty,\infty)}\leqq\|f\|_{\mathrm{BMO}}\leqq C\|f\|_{\dot{F}(0,\infty,2)}.$$

The following inequalities hold for all $f\in\mathcal{S}'(\mathbf{R}^n)$:

$$\|f\|_{h^p}\approx\|f\|_{F(0,p,2)},\qquad 0<p<\infty;$$
$$\|f\|_{\Lambda_s}\approx\|f\|_{F(s,\infty,\infty)},\qquad s>0;$$
$$C^{-1}\|f\|_{F(0,\infty,\infty)}\leqq\|f\|_{\mathrm{bmo}}\leqq C\|f\|_{F(0,\infty,2)}.$$

As for these facts, see [14], [16], [17], [13; § 6.1] and/or [2; § 6].

REMARK 1.2. It can be shown that if X and Y are H^p, BMO, L^1, L^∞ or $\tilde{\Lambda}_s$ and if $m\in\mathcal{M}(X, Y)$, then m $(\in\mathcal{D}'(\mathbf{R}^n\setminus\{0\}))$ can be extended to $\mathbf{R}^n$ as a tempered distribution, *i.e.*, there exists an $\tilde{m}\in\mathcal{S}'(\mathbf{R}^n)$ which coincides with m on $\mathbf{R}^n\setminus\{0\}$.

Now we define $\tilde{X}_\rho$ and X_ρ as follows:

$$\tilde{X}_\rho=\tilde{\Lambda}_{-n\rho}\ \text{ if }\ \rho<0,\ \ =\mathrm{BMO}\ \text{ if }\ \rho=0,\ \ =H^{1/\rho}\ \text{ if }\ \rho>0;$$
$$X_\rho=\Lambda_{-n\rho}\ \text{ if }\ \rho<0,\ \ =\mathrm{bmo}\ \text{ if }\ \rho=0,\ \ =h^{1/\rho}\ \text{ if }\ \rho>0.$$

For $m\in\mathcal{D}'(\mathbf{R}^n\setminus\{0\})$, we define

$$\dot{D}(m)=\{(\rho,\sigma)\mid\ \rho\geqq 0,\ \sigma\in\mathbf{R},\ m\in\mathcal{M}(\tilde{X}_\rho,\tilde{X}_\sigma)\};$$

for $m\in\mathcal{S}'(\mathbf{R}^n)$, we define

$$D(m)=\{(\rho,\sigma)\mid\ \rho\geqq 0,\ \sigma\in\mathbf{R},\ m\in\mathcal{M}(X_\rho, X_\sigma)\}.$$

Then the following theorems hold.

THEOREM 1.1. *If $m\in\mathcal{D}'(\mathbf{R}^n\setminus\{0\})$ and $m\neq 0$, then the set $K=\dot{D}(m)$ has the following properties*:

(A) *there exists a set $\tilde{K}\subset\mathbf{R}^2$ which is convex and symmetric with respect to*

$\{\rho+\sigma=1\}$ *such that* $K=\tilde{K}\cap\{\rho\geqq 0,\ \rho\geqq\sigma\}$;
(B) *if* $(\rho_0, \sigma_0)\in K$, $\rho_0>1$, $\sigma_0<0$, *then* K *contains all the points* (ρ, σ) *such that* $\rho>1$, $\sigma<0$ *and* $\rho-\sigma=\rho_0-\sigma_0$.

THEOREM 1.2. *If* $m\in\mathcal{D}'(\mathbf{R}^n\setminus\{0\})$, $m\neq 0$ *and if* support(m) *is a bounded set of* $\mathbf{R}^n$, *then the set* $K=\dot{D}(m)$ *has, in addition to the properties* (A) *and* (B), *the following property*:

(C) *if* $(\rho_0, \sigma_0)\in K$, *then* K *contains all the points* (ρ, σ_0) *with* $\rho\geqq\rho_0$.

THEOREM 1.3. *If* $m\in\mathcal{S}'(\mathbf{R}^n)$ *and* $m\neq 0$, *then the set* $K=D(m)$ *has the properties* (A), (B) *and the following property*:

(D) *if* $(\rho_0, \sigma_0)\in K$ *and* $\rho_0>1$, *then* K *contains all the points* (ρ, σ_0) *with* $\max\{1, \sigma_0\}\leqq\rho\leqq\rho_0$.

THEOREM 1.4. *If* $m\in\mathcal{S}'(\mathbf{R}^n)$, $m\neq 0$ *and if* m *has a compact support, then the set* $K=D(m)$ *has the properties* (A), (B), (C) *and* (D).

THEOREM 1.5. *If* $m\in\mathcal{S}'(\mathbf{R}^n)$, $m\neq 0$ *and if* $\mathcal{F}^{-1}m$ *has a compact support, then the set* $K=D(m)$ *has, in addition to the properties* (A), (B) *and* (D), *the following property*:

(E) *if* $(\rho_0, \sigma_0)\in K$ *and* $\rho_0>\sigma_0$, *then* K *contains all the points* (ρ_0, σ) *with* $\rho_0\geqq\sigma\geqq\sigma_0$.

PROOF OF THEOREM 1.1. 1°) $\dot{D}(m)\subset\{\rho\geqq\sigma\}$ *if* $m\neq 0$. See [7; Theorem 1.1] and [13; Theorem 3.1].

2°) $\dot{D}(m)\cap\{\sigma\leqq 1\}$ *is symmetric with respect to* $\{\rho+\sigma=1\}$. This fact is due to the following duality inequalities:

$$\sup\{|\langle f, g\rangle| \mid g\in\mathcal{S}_0\cap\tilde{X}_\rho,\ \|g\|_{\tilde{X}_\rho}\leqq 1\}\approx\|f\|_{\tilde{X}_{1-\rho}}, \qquad \rho\geqq 0,$$

which are valid for $f\in\mathcal{S}'(\mathbf{R}^n)$ with support$(\mathcal{F}f)\not\ni 0$. Cf. [5; Theorem 2], [3; II, Theorems 2.1 and 2.5], [17; §2.5] and [4; Theorem (4.1), p. 638].

3°) $\dot{D}(m)$ *itself is convex.* This fact is shown by using the interpolation. Let ρ_0, $\rho_1\geqq 0$, σ_0, $\sigma_1\in\mathbf{R}$, $0<\theta<1$, $\rho=\rho_0(1-\theta)+\rho_1\theta$ and $\sigma=\sigma_0(1-\theta)+\sigma_1\theta$. In order to prove the relation $\mathcal{M}(\tilde{X}_{\rho_0}, \tilde{X}_{\sigma_0})\cap\mathcal{M}(\tilde{X}_{\rho_1}, \tilde{X}_{\sigma_1})\subset\mathcal{M}(\tilde{X}_\rho, \tilde{X}_\sigma)$, it is sufficient to show that

$$\tilde{X}_\rho\subset[\tilde{X}_{\rho_0}, \tilde{X}_{\rho_1}]_\theta \tag{1.1}$$

and

$$[\tilde{X}_{\sigma_0}, \tilde{X}_{\sigma_1}]_\theta\subset\tilde{X}_\sigma, \tag{1.2}$$

where $[\ ,\ ]_\theta$ denotes the complex intermediate space. (The assertions (1.1) and (1.2) are formal ones since some of the spaces $\tilde{X}_\rho$ are not Banach spaces. However precise formulation and rigorous proof can be given since all the spaces $\tilde{X}_\rho$

are subspaces of $\mathcal{S}'(\mathbf{R}^n)$; follow the example of [3; II, Theorem 3.1 and 3.3].) The inclusion (1.1) for ρ_0, $\rho_1>0$ can be found in [3; II, Theorem 3.3]. We shall not go into (1.1) in the case $\rho_0=0<\rho_1$ but shall explain below how we can get through with this case (the inclusion (1.1) is certainly true for $\rho_0=0\leqq\rho_1<1$, but the present author cannot verify it in the case $\rho_0=0$ and $\rho_1\geqq 1$). The inclusion (1.2) for σ_0, $\sigma_1\geqq 0$ can be found in [3; II, Theorem 3.1]. In the case σ_0, $\sigma_1\leqq 0$, (1.2) can be found in [2; Chapter 6] (the theorems in [2; Chapter 6] deal with the spaces $\dot{F}^s_{\infty,\infty}$ for all real s; hence, with the aid of Remark 1.1, we see that (1.2) holds in the case σ_0, $\sigma_1\leqq 0$). We shall show (1.2) in the case $\sigma_0<0<\sigma_1$; suppose that $\sigma_0=-s/n$ and $\sigma_1=1/p$. By the complex interpolation theorem for mixed L^p-spaces (see [1; §7], the theorems in which can be extended to the case $p<1$), we have

$$[\dot{F}^s_{\infty,\infty},\ \dot{F}^0_{p,2}]_\theta\subset\dot{F}^{s(1-\theta)}_{p/\theta,2/\theta}.$$

Then, applying the imbedding theorems for the spaces $\dot{F}^s_{p,q}$ ([16; §2.3.3, p. 87], [17; §2.4.1, pp. 100–104]) and using Remark 1.1, we obtain (1.2) in the case $\sigma_0=-s/n<0$ and $\sigma_1=1/p>0$.

The above results for the intermediate spaces show that $\dot{D}(m)\cap\{\rho>0\}$ is convex. We shall prove that the whole $\dot{D}(m)$ is convex. Suppose that (ρ_0,σ_0) and $(\rho_1,\sigma_1)\in\dot{D}(m)$; we shall show that the line segment joining these two points is also contained in $\dot{D}(m)$. Consider the following cases separately: (i) $\rho_0>0$ and $\rho_1>0$; (ii) $\sigma_0<1$ and $\sigma_1<1$; (iii) $\rho_0=\sigma_0=0$ and $\rho_1=\sigma_1=1$; (iv) $\rho_0=0>\sigma_0$ and $\rho_1=\sigma_1=1$; (v) $\rho_0=0\geqq\sigma_0$, $\rho_1\geqq\sigma_1\geqq 1$ and $\rho_1>1$. The case (i) has already been settled; the case (ii) is reduced to the case (i) by duality or the symmetry of $\dot{D}(m)\cap\{\sigma\leqq 1\}$, and hence is also settled. The case (iv) is reduced to the case (v) by duality. Hence it is sufficient to treat the cases (iii) and (v). In the case (iii), we can prove that $(1/2,1/2)\in\dot{D}(m)$ (see [13; Theorem 3.5]); thus the case (iii) is reduced to the cases (i) and (ii), and hence is settled. In the case (v), we shall appeal to the atomic characterization of H^1 to show first that $\dot{D}(m)$ contains the point $(1,\sigma)$ which is on the line segment joining (ρ_0,σ_0) and (ρ_1,σ_1). (As for the atomic characterization of H^p, see [10] and [11; Remark 1, p. 395].) Let f be a 1-atom which is orthogonal to polynomials of order $\leqq[n(\rho_1-1)]$; we assume that support$(f)\subset\{|x-x_0|\leqq r\}$ and $\|f\|_{L^\infty}\leqq r^{-n}$. If we set

$$f_z(x)=r^{-n\rho_1 z+n}f(x),\quad z\in\mathbf{C},\quad 0\leqq\mathrm{Re}(z)\leqq 1,$$

then f_z is a holomorphic function of z, $f_{1/\rho_1}=f$ and

$$\max\left\{\sup_{y\in\mathbf{R}}\|f_{iy}\|_{L^\infty},\ \sup_{y\in\mathbf{R}}\|f_{1+iy}\|_{1/\rho_1}\right\}\leqq C,$$

where $\|\ \|_{1/\rho_1}$ denotes the quasi-norm in H^{1/ρ_1} (observe that f_{1+iy} is a $(1/\rho_1)$-atom). Hence, applying (1.2) to the analytic family $\mathcal{F}^{-1}(m\mathcal{F}f_z)$, we obtain $\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_{\tilde{X}_\sigma}\leqq C$, which, by the atomic characterization of H^1, implies that $m\in\mathcal{M}(H^1,\tilde{X}_\sigma)$ or $(1,\sigma)\in\dot{D}(m)$. Thus the case (v) is reduced to the cases (i), (ii) and (iii), which have already been treated. Thus we have proved that $\dot{D}(m)$ is convex.

4°) In order to prove the property (A), we shall use 1°), 2°), 3°) and the following

LEMMA 1.1. (i) *If* $m\in\mathcal{M}(H^{p_0}, H^{p_0})$ *with* $0<p_0\leqq 1$, *then* $m\in\mathcal{M}(H^p, H^p)$ *for all* $p\geqq p_0$ *and* $m\in\mathcal{M}(\mathrm{BMO}, \mathrm{BMO})$.

(ii) *If* $m\in\mathcal{M}(H^{p_0}, H^{q_0})$ *with* $0<p_0<q_0\leqq 1$, *then*: $m\in\mathcal{M}(H^p, H^q)$ *for all* (p, q) *such that* $1/p-1/q=1/p_0-1/q_0$ *and* $1/p_0\geqq 1/p>1/p_0-1/q_0$; $m\in\mathcal{M}(H^r, \mathrm{BMO})$ *with* $1/r=1/p_0-1/q_0$; $m\in\mathcal{M}(H^p, \tilde{\Lambda}_s)$ *for all* (p, s) *such that* $1/p+s/n=1/p_0-1/q_0$ *and* $1/p_0-1/q_0>1/p>0$; *and* $m\in\mathcal{M}(\mathrm{BMO}, \tilde{\Lambda}_t)$ *with* $t/n=1/p_0-1/q_0$.

PROOF. For the proof of (i), see [13; Theorem 3.5]. We give a proof of (ii). If $q_0=1$, then the claims are consequences of 2°) and 3°) in the proof of Theorem 1.1. Hence we assume that $q_0<1$. If $a=n/q_0-n$, $b=n/p_0-n$ and $b-a=n/p-n$, then

$$\begin{aligned}
\mathcal{M}(H^{p_0}, H^{q_0}) &\subset \mathcal{M}(\tilde{\Lambda}_a, \tilde{\Lambda}_b) && \text{(by duality)}\\
&=\mathcal{M}(\dot{F}^0_{\infty,\infty}, \tilde{\Lambda}_{b-a}) &&\\
&\subset \mathcal{M}(L^\infty, \tilde{\Lambda}_{b-a}) && (\text{since } L^\infty\subset\dot{F}^0_{\infty,\infty})\\
&=\mathcal{M}(H^p, L^1) && \text{(by duality)}\\
&=\mathcal{M}(H^p, H^1);
\end{aligned}$$

as for the first equality, see *e.g.* [15; II], and as for the last equality, see *e.g.* [13; Theorem 3.4]. Now, duality and interpolation (the assertions 2°) and 3°) in the proof of Theorem 1.1) give all the other assertions in (ii). This completes the proof of Lemma 1.1.

Proof of the property (A). Set $H=\dot{D}(m)\cap\{\rho+\sigma\geqq 1\}$ and let H' be the reflected image of H with respect to $\{\rho+\sigma=1\}$. Set $\tilde{K}=H\cup H'$. $\tilde{K}$ is certainly symmetric with respect to $\{\rho+\sigma=1\}$, $\tilde{K}\subset\{\rho\geqq\sigma\}$ by 1°), and $\dot{D}(m)=\tilde{K}\cap\{\rho\geqq 0\}$ by 2°). By using 3°) and Lemma 1.1, we can easily show that $\tilde{K}$ is convex. This completes the proof of the property (A).

5°) The property (B) is a restatement of the following

LEMMA 1.2. *If* $0<p, q<1$, $s, t>0$ *and* $1/p+s/n=1/q+t/n$, *then* $\mathcal{M}(H^p, \tilde{\Lambda}_s) =\mathcal{M}(H^q, \tilde{\Lambda}_t)$.

PROOF. (Cf. also [8; Theorem 7].) Suppose that $s<t$. It holds that $|\xi|^{-(t-s)} \in\mathcal{M}(H^p, H^q)$ (fractional integration; see *e.g.* [3; II, Theorem 4.1]) and $|\xi|^{t-s} \in\mathcal{M}(\tilde{\Lambda}_t, \tilde{\Lambda}_s)$ (see *e.g.* [2; Lemma 6.2.1]). Hence, if $m\in\mathcal{M}(H^q, \tilde{\Lambda}_t)$, then

$$m=|\xi|^{t-s}\cdot m\cdot|\xi|^{-(t-s)}\in\mathcal{M}(H^p, \tilde{\Lambda}_s);$$

thus $\mathcal{M}(H^q, \tilde{\Lambda}_t)\subset\mathcal{M}(H^p, \tilde{\Lambda}_s)$. In order to prove the reverse inclusion, take r, v,

r' and v' such that $0<r<p<q<n/(n+v)=r'$, $0<v<s<t<n/r-n=v'$ and $1/r+v/n=1/r'+v'/n=1/p+s/n=1/q+t/n$. Then, by the inclusion relation proved above, $\mathcal{M}(H^p, \tilde{\Lambda}_s)\subset\mathcal{M}(H^r, \tilde{\Lambda}_v)$; hence, by duality and interpolation (the property (A)),

$$\mathcal{M}(H^p, \tilde{\Lambda}_s)\subset\mathcal{M}(H^r, \tilde{\Lambda}_v)\cap\mathcal{M}(H^{r'}, \tilde{\Lambda}_{v'})\subset\mathcal{M}(H^q, \tilde{\Lambda}_t).$$

This completes the proof of Lemma 1.2.

Thus we have proved Theorem 1.1.

PROOF OF THEOREM 1.2. Proof of the property (C). Suppose that $m\in\mathcal{M}(\tilde{X}_{\rho_0}, \tilde{X}_{\sigma_0})$, $\rho>\rho_0$ and that m has a compact support. Take a function $f\in C_0^\infty(\mathbf{R}^n)$ which is equal to 1 in a neighborhood of support(m) and set $\lambda=n(\rho-\rho_0)$. Then $|\xi|^{-\lambda}\in\mathcal{M}(\tilde{X}_\rho, \tilde{X}_{\rho_0})$ (fractional integration) and $|\xi|^\lambda f(\xi)\in\mathcal{M}(\tilde{X}_{\sigma_0}, \tilde{X}_{\sigma_0})$ (by the Hörmander-Mihlin criterion; see *e.g.* [17; §2.1.3] and [12; Theorems 1 and 2]), and hence

$$m=|\xi|^\lambda f(\xi)\cdot m\cdot|\xi|^{-\lambda}\in\mathcal{M}(\tilde{X}_\rho, \tilde{X}_{\sigma_0}).$$

This completes the proof.

PROOF OF THEOREM 1.3. Proof of the properties (A) and (B) is similar to that for Theorem 1.1. We shall prove the property (D). The atomic characterization of h^p, $0<p\leqq1$, asserts that, if $\rho\geqq1$ and $\rho\geqq\sigma$, a linear operator T maps $X_\rho=h^{1/\rho}$ into X_σ boundedly if and only if $\sup\{\|Tf\|_{X_\sigma}\mid f\in\mathcal{A}(1/\rho, N)\}<\infty$ for some $N\geqq[n\rho-n]$, where $\mathcal{A}(1/\rho, N)$ is the set of all functions f with the following properties: $\|f\|_{L^\infty}\leqq1$ and support$(f)\subset B$ for some ball B with $|B|=1$, or support$(f)\subset B$, $\|f\|_{L^\infty}\leqq|B|^{-\rho}$ and $\int_B f(x)x^\alpha dx=0$ for $|\alpha|\leqq N$ for some ball B with $|B|<1$ (cf. [6; Lemma 5], [11; Remark 1, p. 395]). (We shall be brief not bothering about the domain of T; some limiting arguments are necessary for $T=\mathcal{F}^{-1}(m\mathcal{F}\cdot)$.) Now, the property (D) is obvious since $\mathcal{A}(1/\rho, N)\subset\mathcal{A}(1/\rho_0, N)$ if $1\leqq\rho\leqq\rho_0$. This completes the proof.

PROOF OF THEOREM 1.4. Same as that of Theorem 1.2; use $(1+|\xi|^2)^{\pm\lambda/2}$ instead of $|\xi|^{\pm\lambda}$.

PROOF OF THEOREM 1.5. Only the property (E) for the case $\rho_0>0$ needs a proof; we shall write $\rho_0=1/p$ and $\sigma_0=\tau$. Suppose that $m\in\mathcal{M}(h^p, X_\tau)$, $1/p>\tau$ and that $G=\mathcal{F}^{-1}m$ has a compact support, say that support$(G)\subset\{|x|\leqq1\}$. Take any element $f\in h^p$ and decompose it as follows: $f=\sum_{j=1}^\infty f_j$, support$(f_j)\subset\{|x-x_j|\leqq2\}$, and

$$\sum_{j=1}^\infty\|f_j\|_{h^p}^p\leqq C\|f\|_{h^p}^p,$$

where x_j's are points of $\mathbf{R}^n$ such that $|x_i-x_j|\geqq\delta$ if $i\neq j$ with a constant $\delta>0$ depending only on n. Such a decomposition is easily obtained if $p>1$ since then

$h^p=L^p$; if $p\leqq 1$, the atomic decomposition of $f\in h^p$ ([6; Lemma 5]) can be arranged to provide such a decomposition. Now, since support$(G*f_j)\subset\{|x-x_j|\leqq 3\}$ and the balls $\{|x-x_j|\leqq 3\}$ have bounded overlaps, we have

$$(1.3)\qquad \|G*f\|_{L^p}^p\leqq C\sum_{j=1}^{\infty}\|G*f_j\|_{L^p}^p\leqq C\sum_{j=1}^{\infty}\|G*f_j\|_{X_\tau}^p\leqq CA^p\sum_{j=1}^{\infty}\|f_j\|_{h^p}^p\leqq CA^p\|f\|_{h^p}^p,$$

where A is the norm (quasi-norm) of m in $\mathcal{M}(h^p, X_\tau)$. Since $\mathcal{M}(h^p, h^p)=\mathcal{M}(h^p, L^p)$ (cf. [13; Theorem 3.4]), (1.3) implies that $m=\mathcal{F}G\in\mathcal{M}(h^p, h^p)$. Hence, by interpolation, $m\in\mathcal{M}(h^p, X_\sigma)$ for $\tau\leqq\sigma\leqq 1/p$. (Some limiting arguments are necessary in order to make the above reasoning rigorous.) This completes the proof.

§ 2. Mapping properties of some special multipliers.

In this section and in Section 4, we use the following notation:

$$m(\xi;\, A,\, a,\, b)=A(|\xi|)|\xi|^b\exp(i|\xi|^a),\qquad \xi\in\mathbf{R}^n,$$

where a and b are real numbers and A is a function on $(0, \infty)$.

The multipliers studied in this section are the following ones:

$$m(\xi;\, \phi A,\, a,\, -b)=\phi(|\xi|)A(|\xi|)|\xi|^{-b}\exp(i|\xi|^a),\quad a\geqq 0,\quad a\neq 1,\quad b\in\mathbf{R},$$

$$m(\xi;\, \phi A,\, -d,\, c)=\phi(|\xi|)A(|\xi|)|\xi|^{c}\exp(i|\xi|^{-d}),\quad d\geqq 0,\quad c\in\mathbf{R}.$$

We assume that the function A is smooth on $(0, \infty)$ and satisfies the following two conditions:

$$(2.1)\qquad |(d/dx)^k A(x)|\leqq C_k x^{-k},\quad k=0, 1, 2, \cdots;$$

$$(2.2)\qquad \{x\mid 2^j<x<2^{j+1},\ |A(x)|\geqq 1\}\neq\emptyset\quad \text{for each integer } j.$$

When we consider $m(\xi;\, \phi A,\, -d,\, c)$ as a distribution, we define it by

$$m(\xi;\, \phi A,\, -d,\, c)=\lim_{\varepsilon\downarrow 0}\psi(|\xi|/\varepsilon)\phi(|\xi|)A(|\xi|)|\xi|^c\exp(i|\xi|^{-d});$$

this limit exists in the space of distributions (cf. Lemma 2.6 given below).

Mapping properties of the above multipliers are given in the following theorems.

THEOREM 2.1. *If the function A satisfies the conditions* (2.1) *and* (2.2) *and if $a\geqq 0$, $a\neq 1$ and $b\in\mathbf{R}$, then*

$$\dot{D}(m(\cdot;\, \phi A,\, a,\, -b))\cap\{(\rho, \sigma)\mid \rho\geqq\sigma,\ \rho+\sigma\geqq 1\}$$
$$=\left\{(\rho, \sigma)\ \middle|\ \begin{array}{l}\rho\geqq\sigma,\ \rho+\sigma\geqq 1,\ \rho-(1-a)\sigma\leqq b/n+a/2,\\ \rho-\sigma\leqq b/n+a/2\end{array}\right\}.$$

THEOREM 2.2. *If the function A satisfies* (2.1) *and* (2.2) *and if $d\geqq 0$ and $c\in\mathbf{R}$, then*

$$\dot{D}(m(\cdot\,;\phi A,\,-d,\,c))\cap\{(\rho,\,\sigma)\mid\,\rho\geqq\sigma,\;\rho+\sigma\geqq 1\}$$
$$=\left\{(\rho,\,\sigma)\,\middle|\,\begin{array}{l}\rho\geqq\sigma,\;\rho+\sigma\geqq 1,\;-\rho+(1+d)\sigma\leqq d/2+c/n,\\ -\rho+\sigma\leqq d/2+c/n\end{array}\right\}.$$

THEOREM 2.3. *Let A, a and b be as in Theorem 2.1. Then* $D(m(\cdot\,;\phi A,\,a,\,-b))=\dot{D}(m(\cdot\,;\phi A,\,a,\,-b))$.

THEOREM 2.4. *Suppose that the function A satisfies* (2.1) *and* (2.2) *and that $d>0$. If $c>-n-nd/2$, then*

$$D(m(\cdot\,;\phi A,\,-d,\,c))\cap\{(\rho,\,\sigma)\mid\rho\geqq\sigma,\;\rho+\sigma\geqq 1\}$$
$$=\left\{(\rho,\,\sigma)\,\middle|\,\begin{array}{l}\rho\geqq\sigma,\;\rho+\sigma\geqq 1,\;-\rho+(1+d)\sigma\leqq d/2+c/n,\\ (1+d)\sigma<1+d/2+c/n\end{array}\right\}. \tag{2.3}$$

If $c=-n-nd/2$, then $D(m(\cdot\,;\phi A,-d,c))=\{(\rho,\sigma)\mid\rho\geqq 1,\sigma\leqq 0\}$. If $c<-n-nd/2$, then $D(m(\cdot\,;\phi A,\,-d,\,c))$ is empty.

We shall begin with the proof of Theorem 2.2. This theorem is derived from the lemmas given below with the aid of Theorem 1.2. In the following lemmas, we assume that A is a function satisfying (2.1) and (2.2) and simply denote $m(\cdot\,;\phi A,\,-d,\,c)$ by $m_{d,c}$.

LEMMA 2.1. *If $d>0$, $c\in\mathbf{R}$, $-1/p+(1+d)/q=d/2+c/n$, $0<p\leqq q<2$ and $p<1$, then $m_{d,c}\in\mathcal{M}(H^p,\,H^q)$. If $d=0$, $c\in\mathbf{R}$, $-1/p+1/q=\min\{0,\,c/n\}$, $p<1$ and $q<2$, then $m_{d,c}\in\mathcal{M}(H^p,\,H^q)$.*

LEMMA 2.2. *If $d>0$, $c\geqq 0$ and $1/p=1/2+c/nd$, then $m_{d,c}\in\mathcal{M}(H^p,\,H^p)$.*

LEMMA 2.3. *If $d>0$ and $c=-n-nd/2$, then $m_{d,c}\in\mathcal{M}(L^1,\,L^\infty)$. If $d>0$ and $c<-n-nd/2$, then $m_{d,c}\in\mathcal{M}(H^p,\,L^\infty)$, where $-1/p=d/2+c/n$.*

LEMMA 2.4. *If $d>0$, $-n-nd/2<c<0$, $1/p=1/2-c/n(2+d)$ and $1/q=1-1/p$, then $m_{d,c}\in\mathcal{M}(H^p,\,H^q)$.*

LEMMA 2.5. *If $d\geqq 0$, $c\in\mathbf{R}$, $0<p\leqq q<\infty$ and $m_{d,c}\in\mathcal{M}(H^p,\,H^q)$, then $-1/p+(1+d)/q\leqq d/2+c/n$.*

PROOF OF LEMMA 2.1. We shall derive the fact $m_{d,c}\in\mathcal{M}(H^p,\,H^q)$ with p and q as indicated in the lemma from the following estimates for $m_{d,c}$: $|(\partial/\partial\xi)^\alpha m_{d,c}(\xi)|\leqq C_\alpha|\xi|^{c-(1+d)|\alpha|}$ and $m_{d,c}(\xi)=0$ outside a compact set.

By virtue of the characterization of H^p in terms of atoms (see [10]) and that of H^q in terms of Riesz transforms, it is sufficient to show the estimate

$$\|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)\|_{L^q}\leqq C,\quad f\in\mathcal{A}_r(p),\quad 0<r<\infty,$$

where $\mathcal{A}_r(p)$ is the set of all functions f such that support$(f)\subset\{|x|\leqq r\}$, $\|f\|_{L^\infty}\leqq r^{-n/p}$ and $\int f(x)x^\alpha dx=0$ for $|\alpha|\leqq[n/p-n]$ (cf. [12; §2]). First, we can obtain the estimate

$$\|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)\|_{L^q(|x|\geqq 2r)}\leqq C,\quad f\in\mathcal{A}_r(p),\quad 0<r<\infty,$$

by slightly modifying the calculations given in [12; §2]; we shall omit the detailed calculations. Secondly, if $0<r<1$, then, for $f\in\mathcal{A}_r(p)$ and $|x|\leqq 2r$,

$$|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)(x)|=\left|\int_{|y|<r}\left\{K(x-y)-\sum_{|\alpha|\leqq N}D^\alpha K(x)(-y)^\alpha/\alpha!\right\}f(y)dy\right|$$
$$\leqq C\,r^{N+1-n/p+n}\leqq C,$$

where $K=\mathcal{F}^{-1}m_{d,c}$ and $N=[n/p-n]$; thus

$$\|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)\|_{L^q(|x|\leqq 2r)}\leqq C,\quad f\in\mathcal{A}_r(p),\quad 0<r<1.$$

Thus there remains only the estimate

$$\|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)\|_{L^q(|x|\leqq 2r)}\leqq C,\quad f\in\mathcal{A}_r(p),\quad r\geqq 1;$$

this is obtained in the following way. Let $f\in\mathcal{A}_r(p)$ and $r\geqq 1$. Hölder's inequality gives

$$\|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)\|_{L^q(|x|\leqq 2r)}\leqq Cr^{n(1/q-1/2)}\|\mathcal{F}^{-1}(m_{d,c}\mathcal{F}f)\|_{L^2}. \tag{2.4}$$

If $c\geqq 0$, then the right hand side of (2.4) is dominated by

$$Cr^{n(1/q-1/2)}\|f\|_{L^2}\leqq Cr^{n(1/q-1/2)}r^{-n/p+n/2}\leqq C;$$

if $c<0$, then, since $m_{d,c}\in\mathcal{M}(H^s, L^2)$ with $1/s=1/2-c/n$ (fractional integration), the right hand side of (2.4) is dominated by

$$Cr^{n(1/q-1/2)}\|f\|_{H^s}\leqq Cr^{n(1/q-1/2)}r^{-n/p+n/s}=Cr^{nd(-1/q+1/2)}\leqq C.$$

This completes the proof of Lemma 2.1.

PROOF OF LEMMA 2.2. Lemma 2.2 can be derived from Lemma 2.1 (the case $p=q$) and the fact $m_{d,0}\in\mathcal{M}(L^2, L^2)$ by means of the analytic interpolation; cf. [12; §2].

PROOF OF LEMMA 2.3. The latter half of the lemma is derived from the former half with the aid of the fact that

$$|\xi|^{c+n+nd/2}\in\mathcal{M}(H^p, L^1),\quad 1/p-1=(-c-n-nd/2)/n>0,$$

(fractional integration). The former half is a direct consequence of the following lemma, which will show that $\mathcal{F}^{-1}(m_{d,c})\in L^\infty$ for $d>0$ and $c=-n-nd/2$.

LEMMA 2.6. *Let $\rho(x)=x^c$, $c\in\mathbf{R}$, $\sigma(x)=x^{-d}$, $d>0$ and A be a smooth function on $(0,\infty)$ satisfying (2.1). Then the limit*

$$m=\lim_{\varepsilon\downarrow 0}\phi(|x|/\varepsilon)\phi(|x|)A(|x|)\rho(|x|)\exp(i\sigma(|x|)),\quad x\in\mathbf{R}^n,$$

exists in $\mathcal{E}'(\mathbf{R}^n)$ (*the space of distributions with compact supports*) *and its Fourier transform* $H=\mathcal{F}m$ *has the following estimate*:

$$H(\xi)=C_1(r)(x_r/r)^{(n-1)/2}(\sigma''(x_r))^{-1/2}\rho(x_r)(A(x_r)+o(1)),\tag{2.5}$$

where $r=|\xi|$, $o(1)\to 0$ *as* $r\to\infty$,

$$C_1(r)=\exp[i(-\pi n/4+\pi/2+\sigma(x_r)+rx_r)]$$

and x_r *is given by* $\sigma'(x_r)+r=0$.

We shall put off the proof of this lemma to the next section and proceed to the

PROOF OF LEMMA 2.4. This lemma is derived from the facts $m_{d,0}\in\mathcal{M}(L^2, L^2)$ and $m_{d,-n-nd/2}\in\mathcal{M}(L^1, L^\infty)$ (Lemma 2.3) by means of the analytic interpolation; cf. [13; Proof of Proposition 5.3, pp. 295-296].

PROOF OF LEMMA 2.5. First, we shall treat the case $d=0$. We can take numbers $t_1, \cdots, t_N$ such that $1/2\leqq t_j\leqq 2$ and $\sum_{j=1}^N|A(t_js)|^2\geqq 1/2$ for all $s>0$. By the Hörmander-Mihlin criterion, the functions

$$A_k(\xi)=\Big(\sum_{j=1}^N|A(t_j|\xi|)|^2\Big)^{-1}\phi(4|\xi|)\overline{A(t_k|\xi|)},\quad k=1,\cdots,N,$$

all belong to $\mathcal{M}(H^p, H^p)$. Hence, $m_{0,c}\in\mathcal{M}(H^p, H^q)$ implies that

$$e^{-i}\sum_{k=1}^N A_k(\xi)t_k^{-c}m_{0,c}(t_k\xi)=\phi(4|\xi|)|\xi|^c\in\mathcal{M}(H^p, H^q).\tag{2.6}$$

Now, since $\mathcal{F}^{-1}(\phi(|\xi|)|\xi|^{n/p-n+\delta})\in H^p$ for any $\delta>0$, (2.6) in turn implies that $\mathcal{F}^{-1}(\phi(4|\xi|)|\xi|^{c+n/p-n+\delta})\in H^q$ for any $\delta>0$, which is possible only when $c+n/p-n+\delta>n/q-n$ for any $\delta>0$, or $-1/p+1/q\leqq c/n$. This settles the case $d=0$.

Next, consider the case $d>0$. If $m_{d,c}\in\mathcal{M}(H^p, H^q)$, then, for any $\delta>0$, the limit

$$\lim_{\varepsilon\downarrow 0}\mathcal{F}^{-1}(\phi(|\xi|/\varepsilon)\phi(|\xi|)A(|\xi|)|\xi|^c\exp(i|\xi|^{-d})|\xi|^{n/p-n+\delta})$$

must exist in $H^q(\mathbf{R}^n)$ and hence in particular the function

$$|x|^{(-c-n/p-nd/2-\delta)/(1+d)}(A((|x|/d)^{-1/(1+d)})+o(1)),\quad |x|\to\infty,$$

(cf. Lemma 2.6) belongs to $L^q(\mathbf{R}^n)$, which is possible only when $-c-n/p-nd/2-\delta<-n(1+d)/q$ (here the condition (2.2) is used). Since $\delta>0$ is arbitrary, we have $-1/p+(1+d)/q\leqq d/2+c/n$. This completes the proof of Lemma 2.5.

Now we have completed the proof of Theorem 2.2 except for the proof of Lemma 2.6.

PROOF OF THEOREM 2.1. Similar to that of Theorem 2.2 (cf. [13]); use, instead of Lemma 2.6, the following

LEMMA 2.7. *Let* $\rho(x)=x^{-b}$, $b\in\mathbf{R}$, $\sigma(x)=x^{a}$ *and* A *be a smooth function on* $(0, \infty)$ *satisfying* (2.1). *If* $0<a<1$, *then*:

$$H(\xi)=\mathcal{F}[\phi(|x|)\rho(|x|)A(|x|)\exp(i\sigma(|x|))](\xi), \quad x, \xi\in\mathbf{R}^{n},$$

is a smooth function in $\mathbf{R}^{n}\setminus\{0\}$; $(\partial/\partial\xi)^{\alpha}H(\xi)=O(|\xi|^{-N})$ *as* $|\xi|\to\infty$ *for all* α *and all* $N>0$; *in a neighborhood of the origin,* $H(\xi)$ *can be written as*

$$H(\xi)=C_{2}(r)(x_{r}/r)^{(n-1)/2}(-\sigma''(x_{r}))^{-1/2}\rho(x_{r})(A(x_{r})+o(1))$$
$$+(a \text{ smooth function of } r^{2}),$$

where $r=|\xi|$, $o(1)\to 0$ *as* $r\to 0$, x_{r} *is given by* $\sigma'(x_{r})-r=0$ *and* $C_{2}(r)$ *is given by*

$$C_{2}(r)=\exp[i(\pi n/4-\pi/2+\sigma(x_{r})-rx_{r})].$$

If $a>1$, *then* $H(\xi)$ *is a smooth function throughout* $\mathbf{R}^{n}$ *and has the following asymptotic behavior as* $r=|\xi|\to\infty$:

$$H(\xi)=C_{3}(r)(x_{r}/r)^{(n-1)/2}(\sigma''(x_{r}))^{-1/2}\rho(x_{r})(A(x_{r})+o(1)),$$

where x_{r} *is given by* $\sigma'(x_{r})-r=0$ *and* $C_{3}(r)$ *is given by*

$$C_{3}(r)=\exp[i(\pi n/4+\sigma(x_{r})-rx_{r})].$$

The proof of this lemma is similar to that of Lemma 2.6, which will be given in the next section.

PROOF OF THEOREM 2.3. From the identification of H^{p}, h^{p}, $\tilde{\Lambda}_{s}$ and Λ_{s} as the spaces of Triebel-Lizorkin type (Remark 1.1), we see that, for $f\in\mathcal{S}'(\mathbf{R}^{n})$ with support$(\mathcal{F}f)\subset\{|\xi|\geqq 1/2\}$,

$$\|f\|_{H^{p}}\approx\|f\|_{h^{p}}, \quad 0<p<\infty, \tag{2.7}$$

and $\|f\|_{\tilde{\Lambda}_{s}}\approx\|f\|_{\Lambda_{s}}$, $s>0$. It also holds that $\|f\|_{\mathrm{BMO}}\approx\|f\|_{\mathrm{bmo}}$ for $f\in\mathcal{S}'(\mathbf{R}^{n})$ with support$(\mathcal{F}f)\subset\{|\xi|\geqq 1/2\}$; this can be derived from (2.7) with $p=1$ by means of duality. Using these inequalities, we see that $\dot{D}(m)=D(m)$ whenever support(m) $\subset\{|\xi|\geqq 1\}$. This completes the proof.

PROOF OF THEOREM 2.4. If f is a smooth function such that support$(f)\subset$ $\{|x|\leqq 1\}$ and $\int f(x)dx=1$ (such a function belongs to h^{p}, $p\leqq 1$, while not to H^{p}, $p\leqq 1$), then, from Lemma 2.6, we see that $\mathcal{F}^{-1}(m(\cdot\,;\phi A, -d, c)\mathcal{F}f)$ belongs to L^{q} if and only if $(1+d)/q<1+d/2+c/n$ (observe that the function $C_{1}(r)$ oscillates "slowly"). This fact causes the restriction $(1+d)\sigma<1+d/2+c/n$ in (2.3). We shall omit the proof of the other facts; they are similar to that of Theorem 2.2, or rather reduced to Theorem 2.2 by means of Theorem 1.4.

§ 3. Proofs of Lemmas 2.6 and 2.7.

The first half of Lemma 2.7 for $A(x)\equiv 1$ can be found in [18; Part II]. The proof of [18] might be extended to obtain the second half of Lemma 2.7 for $A(x)\equiv 1$ and Lemma 2.6 for $A(x)\equiv 1$, but it cannot be applied to the cases $A(x)\not\equiv 1$ since it is based on the integration of holomorphic functions (the method of steepest descent). Here we shall give a proof which is based on the calculus of real functions and hence is available for the cases $A(x)\not\equiv 1$. We shall present the proof of Lemma 2.6; that of Lemma 2.7 is similar to it.

PROOF OF LEMMA 2.6. Throughout this proof, we shall simply denote $\phi(s)\rho(s)A(s)$ by $K(s)$.

If $f\in C^{\infty}(\mathbf{R}^n)$, then integration by parts gives

$$\begin{aligned}(3.1)\qquad &\int_{\mathbf{R}^n}\phi(|x|/\varepsilon)K(|x|)e^{i\sigma(|x|)}f(x)dx\\ &=\int_0^2 e^{i\sigma(s)}\Big(\frac{\partial}{\partial s}\cdot\frac{i}{\sigma'(s)}\Big)^J[\phi(s/\varepsilon)K(s)\langle f\rangle(s)s^{n-1}]ds,\end{aligned}$$

where

$$\langle f\rangle(s)=\int_{|\omega|=1}f(s\omega)\mu(d\omega),$$

μ being the $(n-1)$-dimensional area measure on $\{\omega\in\mathbf{R}^n\,|\,|\omega|=1\}$. The integrand on the right hand side of (3.1) is, in absolute value, dominated by $C\rho(s)s^{n-1}\sigma(s)^{-J}$ with a constant C independent of $s>0$ and $\varepsilon>0$. Hence, if J is sufficiently large, we easily see that the right hand side of (3.1) has a limit as ε tends to zero. Thus $\lim_{\varepsilon\downarrow 0}\phi(|x|/\varepsilon)K(|x|)\exp(i\sigma(|x|))$ exists in $\mathcal{E}'(\mathbf{R}^n)$.

Applying (3.1) to $f(x)=\phi(|\xi||x|)e^{-i\xi\cdot x}$ and letting $\varepsilon\downarrow 0$, we obtain, if J is sufficiently large,

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}^n}\phi(|x|/\varepsilon)K(|x|)e^{i\sigma(|x|)}\phi(|\xi||x|)e^{-i\xi\cdot x}dx\\ &=\int_0^{2/|\xi|}e^{i\sigma(s)}\Big(\frac{\partial}{\partial s}\cdot\frac{i}{\sigma'(s)}\Big)^J[K(s)\phi(|\xi|s)\langle e^{-i\xi\cdot x}\rangle(s)s^{n-1}]ds.\end{aligned}$$

The integrand on the right hand side of this equality is, in absolute value, dominated $C_J\rho(s)\sigma(s)^{-J}s^{n-1}$ with a constant C_J independent of $s>0$ and $\xi\in\mathbf{R}^n$. Hence, for sufficiently large J,

$$\begin{aligned}&\Big|\lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}^n}\phi(|x|/\varepsilon)K(|x|)e^{i\sigma(|x|)}\phi(|\xi||x|)e^{-i\xi\cdot x}dx\Big|\\ &\le C_J\rho(1/r)\sigma(1/r)^{-J}r^{-n},\quad r=|\xi|;\end{aligned}$$

J can be taken arbitrarily large. Thus, in order to prove (2.5), it is sufficient to prove that the function

(3.2) $$\tilde{H}(r)=(2\pi)^{-n/2}\int_{\mathbf{R}^n} K(|x|)e^{i\sigma(|x|)}\phi(r|x|)e^{-i\xi\cdot x}dx\,, \quad r=|\xi|,$$

has the asymptotic behavior indicated in the right hand side of (2.5).

First, consider the case $n=1$. We can write

$$\tilde{H}(r)=I_+(r)+I_-(r)\,,$$

where

$$I_\pm(r)=(2\pi)^{-1/2}\int_0^\infty K(x)e^{i\sigma(x)}\phi(rx)e^{\pm irx}dx\,.$$

$I_-(r)$ can be estimated in the following way. Integration by parts gives

$$I_-(r)=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{i(\sigma(x)-rx)}\Big(\frac{\partial}{\partial x}\cdot\frac{i}{\sigma'(x)-r}\Big)^J[K(x)\phi(rx)]dx\,;$$

the integrand in the right hand side of this equation can be expressed as a finite linear combination of the following terms:

(3.3) $$e^{i(\sigma(x)-rx)}(\sigma'(x)-r)^{-J-j}\sigma^{(n_1)}(x)\cdots\sigma^{(n_j)}(x)(\partial/\partial x)^m[K(x)\phi(rx)]\,,$$

where $m+n_1+\cdots+n_j=J+j$, $0\leqq m\leqq J$, $0\leqq j\leqq J$, $1\leqq m+j\leqq J$, and $n_1, \cdots, n_j\geqq 2$. The term (3.3) is majorized in absolute value by $C_J\rho(x)|\sigma'(x)-r|^{-J-j}\sigma(x)^j x^{-J-j}$ with a constant C_J independent of x and r. Thus

$$\begin{aligned}|I_-(r)|&\leqq C_J\sum_{j=0}^{J}\int_{1/r}^{2}\rho(x)|\sigma'(x)-r|^{-J-j}\sigma(x)^j x^{-J-j}dx\\&\leqq C_J\sum_{j=0}^{J}\Big\{\int_{1/r}^{x_r}\rho(x)\sigma(x)^{-J}dx+r^{-J-j}\int_{x_r}^{2}\rho(x)\sigma(x)^j x^{-J-j}dx\Big\}\\&\leqq C_J\rho(x_r)\sigma(x_r)^{-J}x_r\end{aligned}$$

for all sufficiently large J. Hence we can neglect $I_-(r)$ in estimating $\tilde{H}(r)$.

We decompose $I_+(r)$ as follows:

$$I_+(r)=I_{+,1}(r)+I_{+,2}(r)\,,$$

$$I_{+,1}(r)=(2\pi)^{-1/2}\int e^{i(\sigma(x)+rx)}K(x)\phi(rx)\psi(|x-x_r|/\varepsilon x_r)dx\,,$$

$$I_{+,2}(r)=(2\pi)^{-1/2}\int e^{i(\sigma(x)+rx)}K(x)\phi(|x-x_r|/\varepsilon x_r)dx\,,$$

where $\varepsilon>0$ is a small number independent of r. $I_{+,1}(r)$ can be treated just in the same way as $I_-(r)$; we have $|I_{+,1}(r)|\leqq C_J\rho(x_r)\sigma(x_r)^{-J}x_r$ for all large J. Hence we can neglect the contribution of $I_{+,1}(r)$.

In order to obtain the asymptotic behavior of $I_{+,2}(r)$ as $r\to\infty$, we shall employ the *method of stationary phase*. First, we make a change of variables $x\to u=u(r, x)$ so that we have

$$\sigma(x)+rx=\sigma(x_r)+rx_r+\sigma''(x_r)u^2/2 \quad\text{and}\quad (du/dx)(x_r)=1\,;$$

if $\varepsilon>0$ is sufficiently small, then, for large $r>0$, we have a smooth function $[x_r-2\varepsilon x_r, x_r+2\varepsilon x_r]\ni x\mapsto u\in[u_1, u_2]$ with a smooth inverse $[u_1, u_2]\ni u\mapsto x\in[x_r-2\varepsilon x_r, x_r+2\varepsilon x_r]$, where $|u_1|, |u_2|\leqq 3\varepsilon x_r$. (Observe that $\varepsilon>0$ can be chosen independent of r so long as r is large.) Then $I_{+,2}(r)$ can be rewritten as

$$I_{+,2}(r)=\exp[i(\sigma(x_r)+rx_r)]\int f(u)\exp(i\sigma''(x_r)u^2/2)du\,, \tag{3.4}$$

where

$$f(u)=(2\pi)^{-1/2}K(x)\phi(|x-x_r|/\varepsilon x_r)(dx/du)\,.$$

Next, we rewrite (3.4) as follows:

$$\begin{aligned}I_{+,2}(r)=&\exp[i(\sigma(x_r)+rx_r)]f(0)\int\phi(|u|/3\varepsilon x_r)\exp(i\sigma''(x_r)u^2/2)du\\&+\exp[i(\sigma(x_r)+rx_r)]\int\frac{\partial}{\partial u}\left[\frac{i(f(u)-f(0))\phi(|u|/3\varepsilon x_r)}{\sigma''(x_r)u}\right]\\&\cdot\exp(i\sigma''(x_r)u^2/2)du\,.\end{aligned} \tag{3.5}$$

The first term in the right hand side of (3.5) reduces to

$$\exp[i(\sigma(x_r)+rx_r)](\sigma''(x_r))^{-1/2}K(x_r)\{e^{i\pi/4}-R(3\varepsilon x_r\sqrt{\sigma''(x_r)})\}\,,$$

where the function R is defined by

$$R(t)=\lim_{a\to\infty}(2\pi)^{-1/2}\int\phi(|v|/t)\phi(|v|/a)\exp(iv^2/2)dv\,.$$

It holds that $R(t)=O(t^{-N})$ for every $N>0$ as $t\to\infty$. In order to estimate the second term in the right hand side of (3.5), observe that, for large r,

$$|(d/du)^k x|\leqq C_k x_r^{1-k},\quad k=1, 2, \cdots,$$

and

$$|(d/du)^k f(u)|\leqq C_k\rho(x_r)x_r^{-k},\quad k=0, 1, 2, \cdots.$$

From these estimates, we easily see that the second term in the right hand side of (3.5) is of order $o(\rho(x_r)/\sqrt{\sigma''(x_r)})$ as $r\to\infty$. Thus we see that $I_{+,2}(r)$ and hence $\tilde{H}(r)$ have the asymptotic behavior as indicated in (2.5) in the case $n=1$.

Next, consider the case $n\geqq 2$. We can rewrite (3.2) as

$$\tilde{H}(r)=r^{-n/2+1}\int_0^\infty K(s)\phi(rs)s^{n/2}e^{i\sigma(s)}J_{(n-2)/2}(rs)ds\,, \tag{3.6}$$

where $J_{(n-2)/2}$ is the Bessel function. By Hankel's asymptotic expansion of the Bessel function, we can write

$$J_{(n-2)/2}(rs)=\sum_{m=0}^{M-1}b_m^- e^{-irs}(rs)^{-1/2-m}+\sum_{m=0}^{M-1}b_m^+ e^{irs}(rs)^{-1/2-m}+e_M(rs)\,,\quad M=1, 2, \cdots, \tag{3.7}$$

where $b_0^\pm=(2\pi)^{-1/2}\exp[\pm i(-\pi n/4+\pi/4)]$ and

$$|e_M(rs)|\leqq C_M(rs)^{-1/2-M} \quad \text{for} \quad rs\geqq 1.$$

Replace the $J_{(n-2)/2}(rs)$ in (3.6) by the right hand side of (3.7); then $\tilde{H}(r)$ can be rewritten as a sum of $(2M+1)$-integrals corresponding to the $(2M+1)$-terms in the right hand side of (3.7). The integrals involving the terms $b_m^{\pm}$ can be estimated by the 1-dimensional result; those $2M$-integrals together give a function whose asymptotic behavior is given by the right hand side of (2.5). As for the integral involving $e_M(rs)$, we have, if M is sufficiently large,

$$\left|r^{-n/2+1}\int_0^\infty K(s)\psi(rs)s^{n/2}e^{i\sigma(s)}e_M(rs)ds\right|\leqq C'_M\rho(1/r)r^{-n},$$

which is of order $o((x_r/r)^{(n-1)/2}(\sigma''(x_r))^{-1/2}\rho(x_r))$ as $r\to\infty$ so long as $c>-n/2$ (recall that $\rho(s)=s^c$). Hence we have completely proved (2.5) in the case $c>-n/2$.

We shall show that (2.5) is valid also in the case $c\leqq -n/2$. Here, instead of $\tilde{H}(r)$, we shall directly deal with the function

$$H_n(r\,;K)=\lim_{\varepsilon\downarrow 0} r^{-n/2+1}\int_0^\infty \psi(s/\varepsilon)K(s)s^{n/2}e^{i\sigma(s)}J_{(n-2)/2}(rs)ds\,.$$

By the reasoning given at the beginning of this proof, the following integration by parts is easily legitimated:

$$H_n(r\,;K)=\lim_{\varepsilon\downarrow 0} r^{-n/2+1}\int_0^\infty \psi(s/\varepsilon)e^{i\sigma(s)}\frac{\partial}{\partial s}\Big(\frac{K(s)s^{n/2}J_{(n-2)/2}(rs)}{-i\sigma'(s)}\Big)ds\,.$$

From this, using the formula $(\partial/\partial s)(s^{-\nu}J_\nu(rs))=-rs^{-\nu}J_{\nu+1}(rs)$, we obtain

$$H_n(r\,;K)=H_n(r\,;K_0)+r^2H_{n+2}(r\,;K_1)\,,$$

where $K_0(s)=s^{-n+1}(d/ds)(iK(s)s^{n-1}/\sigma'(s))$ and $K_1(s)=K(s)/i\sigma'(s)s$. Repeated application of this process yields

$$H_n(r\,;K)=\sum_{j=0}^{M} r^{2j}H_{n+2j}(r\,;G_j)\,,$$

where G_j, $j=0, 1, \cdots, M$, are smooth functions on $(0, \infty)$ vanishing on $[2, \infty)$ with the estimates

$$|(d/ds)^kG_j(s)|\leqq C_{j,k}s^{c+Md-k}, \quad k=0, 1, 2, \cdots,$$

and, in particular, $G_M(s)=K(s)(i\sigma'(s)s)^{-M}$. If we take M so large that $c+Md>-n/2$, then, as we have already shown, we can apply the formula (2.5) to $H_{n+2j}(r\,;G_j)$, $j=0, 1, \cdots, M$. In this way, we find again the formula (2.5) for $H_n(r\,;K)$. This completes the proof of Lemma 2.6.

§4. Converse to Theorems 1.1~1.5.

In the following theorems, (A)~(E) refer to the properties mentioned in Theorems 1.1~1.5.

THEOREM 4.1. *If a set $K \subset \mathbf{R}^2$ has the properties* (A) *and* (B) *and if K is closed, then there exists an $m \in \mathcal{D}'(\mathbf{R}^n \setminus \{0\})$ such that $\dot{D}(m)=K$.*

THEOREM 4.2. *If a set $K \subset \mathbf{R}^2$ has the properties* (A), (B) *and* (C) *and if K is closed, then there exists an $m \in \mathcal{D}'(\mathbf{R}^n \setminus \{0\})$ with bounded support such that $\dot{D}(m)=K$.*

THEOREM 4.3. *If a set $K \subset \mathbf{R}^2$ has the properties* (A), (B) *and* (D) *and if K is closed, then there exists an $m \in \mathcal{S}'(\mathbf{R}^n)$ such that $D(m)=K$.*

THEOREM 4.4. *If a set $K \subset \mathbf{R}^2$ has the properties* (A), (B), (C) *and* (D) *and if K is closed, then there exists an $m \in \mathcal{S}'(\mathbf{R}^n)$ with compact support such that $D(m)=K$.*

THEOREM 4.5. *If a set $K \subset \mathbf{R}^2$ has the properties* (A), (B), (D) *and* (E) *and if K is closed, then there exists an $m \in \mathcal{S}'(\mathbf{R}^n)$ such that $\mathcal{F}^{-1}m$ has a compact support and $D(m)=K$.*

We shall prove these theorems by using the multipliers studied in § 2. We begin with the

PROOF OF THEOREM 4.2. Let θ_j, $j=1, 2, \cdots$, be smooth functions on $(0, \infty)$ with the following properties: support$(\theta_j) \subset (1, 2)$, supports of θ_j's are disjoint, and $\{s \mid \theta_j(s)=1\} \neq \emptyset$ for every j. Set

$$A_j(s)=\sum_{k=-\infty}^{\infty} \theta_j(2^k s)\,;$$

then each A_j has the properties (2.1) and (2.2). Now, let K be as mentioned in the theorem. Then, by virtue of the separation theorem for convex sets and Theorem 2.2, it is possible to take countably many numbers $c_j \in \mathbf{R}$ and $d_j \geqq 0$ such that

$$K=\bigcap_{j=1}^{\infty} \dot{D}(m(\cdot\,;\phi A_j,\,-d_j,\,c_j))=\bigcap_{j=1}^{\infty} \dot{D}(m(\cdot\,;\phi A_j^2,\,-d_j,\,c_j))\,.$$

Take $\varepsilon_j>0$ so small that $\|\varepsilon_j m(\cdot\,;\phi A_j,\,-d_j,\,c_j)\|_{\rho,\sigma} \leqq 2^{-j}$ for $(\rho, \sigma) \in K \cap \{\rho \leqq j, \sigma \geqq -j\}$, where $\|\ \|_{\rho,\sigma}$ denotes the quasi-norm in $\mathcal{M}(\tilde{X}_\rho, \tilde{X}_\sigma)$. Consider the following distribution:

$$m=\sum_{j=1}^{\infty} \varepsilon_j m(\cdot\,;\phi A_j,\,-d_j,\,c_j) \in \mathcal{D}'(\mathbf{R}^n \setminus \{0\})\,.$$

This m has certainly a bounded support; we shall show that $\dot{D}(m)=K$. The inclusion $\dot{D}(m) \supset K$ (*i.e.*, $m \in \mathcal{M}(\tilde{X}_\rho, \tilde{X}_\sigma)$ for all $(\rho, \sigma) \in K$) is obvious. Suppose that $(\rho, \sigma) \in \dot{D}(m)$, *i.e.*, $m \in \mathcal{M}(\tilde{X}_\rho, \tilde{X}_\sigma)$. Then, for each j, $A_j(|\xi|)m(\xi)$ also belongs to $\mathcal{M}(\tilde{X}_\rho, \tilde{X}_\sigma)$ and hence, since A_j's have disjoint supports, $m(\cdot\,;\phi A_j^2,\,-d_j,\,c_j)$ belongs to $\mathcal{M}(\tilde{X}_\rho, \tilde{X}_\sigma)$. Thus

$$\dot{D}(m) \subset \bigcap_{j=1}^{\infty} \dot{D}(m(\cdot\,; \phi A_j^2, -d_j, c_j)) = K.$$

Thus we have $\dot{D}(m)=K$. This completes the proof.

PROOF OF THEOREM 4.1. Similar to that of Theorem 4.2.

PROOF OF THEOREM 4.4. Only the following five cases are possible to occur. Case 1, K is the empty set; Case 2, $K=\{\rho \geqq 1, \sigma \leqq 0\}$; Case 3, $K=\{\rho \geqq \sigma, \rho \geqq 0\}$; Case 4, $K=\{\rho \geqq \sigma, \rho \geqq 0, \sigma \leqq \sigma_0\}$ with some $\sigma_0 \geqq 1$; Case 5, $K \subset \{\sigma<1\}$ and $K \cap \{\sigma>0\}$ is not empty. Cases 1 and 2 are settled by Theorem 2.4. In Case 3, $K=D(\phi(|\xi|))$. In Case 4,

$$K=D(\phi(4|\xi|)|\xi|^c(\log(1/|\xi|))^{c'}\exp(i|\xi|^{-d})),$$

where $d>0$, $1+d/2+c/n=\sigma_0(1+d)$ and $c'/\sigma_0<-1$; this we can show by slightly modifying the calculations given in § 2. Consider Case 5. In this case, it is sufficient to construct an m which has a compact support and satisfies

$$D(m) \cap \{0<\sigma<1\} = K \cap \{0<\sigma<1\} \tag{4.1}$$

(observe that such an m necessarily satisfies $D(m)=K$). Let A_j be as in the proof of Theorem 4.2. Since each $A_j(|\xi|)$ belongs to $\mathcal{M}(X_\sigma, X_\sigma)$ for $0<\sigma<1$ (note that $X_\sigma=L^{1/\sigma}$ for $0<\sigma<1$), by the reasoning given in the proof of Theorem 4.2, it is possible to take $\varepsilon_j>0$, $d_j>0$ and $c_j \in \mathbf{R}$ so that the distribution

$$m=\sum_{j=1}^{\infty} \varepsilon_j m(\cdot\,; \phi A_j, -d_j, c_j),$$

which has certainly a compact support, satisfies (4.1). This completes the proof.

PROOF OF THEOREM 4.3. Using the reasoning given in the proofs of Theorems 4.2 and 4.4, we can construct m_1 and $m_2 \in \mathcal{S}'(\mathbf{R}^n)$ such that support$(m_1) \subset \{|\xi| \leqq 3/2\}$, support$(m_2) \subset \{|\xi| \geqq 2\}$ and $K=D(m_1) \cap D(m_2)$. If we set $m=m_1+m_2$, then we have $D(m)=K$.

PROOF OF THEOREM 4.5. If K satisfies the conditions of the theorem, then, in the same way as in the proof of Theorem 4.2, we can construct an $m \in \mathcal{S}'(\mathbf{R}^n)$ which satisfies $D(m)=K$ by setting

$$m=\sum_{j=1}^{\infty} \varepsilon_j m(\cdot\,; \phi A_j, a_j, -b_j),$$

where $\varepsilon_j>0$, $0 \leqq a_j<1$, $b_j \in \mathbf{R}$, and A_j's are functions given in the proof of Theorem 4.2. Since the functions $\phi(|x|)\mathcal{F}^{-1}(m(\cdot\,; \phi A_j, a_j, -b_j))$ belong to $\mathcal{S}(\mathbf{R}^n)$, we can choose $\varepsilon_j>0$ so small that $\phi(|x|)\mathcal{F}^{-1}m$ also belongs to $\mathcal{S}(\mathbf{R}^n)$. If $\{\varepsilon_j\}$ is so chosen, then $\phi(|x|)\mathcal{F}^{-1}m$ has a compact support and $D(\mathcal{F}(\phi(|x|)\mathcal{F}^{-1}m))=D(m)=K$ since $m-\mathcal{F}(\phi(|x|)\mathcal{F}^{-1}m)=\mathcal{F}(\psi(|x|)\mathcal{F}^{-1}m) \in \mathcal{S} \subset \mathcal{M}(X_\rho, X_\sigma)$ for all (ρ, σ) with $\rho \geqq \sigma$

and $\rho\geqq 0$. This completes the proof.

§ 5. H^1-boundedness and weak (1, 1)-boundedness.

Many important convolution operators $f\mapsto K*f$ appearing in Fourier analysis are not bounded in L^1 but bounded in H^1 and of weak type (1, 1), *i.e.*, have the estimate

$$|\{x\,|\,|(K*f)(x)|>\lambda\}|\leqq C\lambda^{-1}\|f\|_{L^1},\quad \lambda>0,\quad f\in L^1. \tag{5.1}$$

It is natural to raise the question whether either one of the estimates (the H^1-estimate or the weak (1, 1)-estimate) implies the other. In this section, we shall show that there are no such implications.

1°) It is easy to give an example of m such that the operator $T_m: f\mapsto \mathcal{F}^{-1}(m\mathcal{F}f)$ is of weak type (1, 1) but not bounded in H^1. $m(\xi)=(\xi-1)/|\xi-1|$, $\xi\in\mathbf{R}$, is an example; for this m, T_m is composed of the Hilbert transform and multiplications by $e^{\pm ix}$ and hence of weak type (1, 1), but it is not bounded in H^1 since, for $f\in\mathcal{S}$,

$$(T_m f)(x)=(i/\pi)\Big(\int f(y)e^{-iy}dy\Big)x^{-1}e^{ix}+O(x^{-2})\quad\text{as}\quad |x|\to\infty,$$

and thus $T_m f$ is not integrable for generic $f\in\mathcal{S}\cap H^1$. The compactly supported multiplier $m(\xi)=\phi(|\xi|/2)(\xi-1)/|\xi-1|$, $\xi\in\mathbf{R}$, is also such an example.

2°) *There is an m such that T_m is of weak type* (1, 1), *not bounded in H^1 and $\mathcal{F}^{-1}m$ has a compact support.*

PROOF. Let X be the set of all $K\in\mathcal{S}'$ such that support$(K)\subset\{|x|\leqq 1\}$ and the operator $f\mapsto K*f$ is of weak type (1, 1). Define a quasi-norm $\|\ \|_X$ by $\|K\|_X=\|K\|_{\text{weak}(1,1)}$, where

$$\|K\|_{\text{weak}(1,1)}=\sup\{\lambda|\{|K*f|>\lambda\}|/\|f\|_{L^1}\mid \lambda>0,\ f\in\mathcal{S}\}.$$

By using the inequality $\|\mathcal{F}K\|_{L^\infty}\leqq\|K\|_{\text{weak}(1,1)}$ (cf. Knopf [9]), we easily see that $(X, \|\ \|_X)$ is a complete quasi-normed space. Now, assume that our assertion is false, *i.e.*, that $\mathcal{F}K\in\mathcal{M}(H^1, H^1)$ for all $K\in X$. Then, by the closed graph theorem, there is a constant A such that

$$\|\mathcal{F}K\|_{\mathcal{M}(H^1,H^1)}\leqq A\|K\|_{\text{weak}(1,1)} \tag{5.2}$$

for all $K\in X$. Since dilation $K\mapsto r^{-n}K(r^{-1}\cdot)$ does not change both the quasi-norms $\|\mathcal{F}\cdot\|_{\mathcal{M}(H^1,H^1)}$ and $\|\cdot\|_{\text{weak}(1,1)}$, (5.2) holds for all K with compact supports. Consider the following distribution:

$$K_{\varepsilon,N}(x)=(i/\pi)\psi(|x|/\varepsilon)\phi(|x|/N)x^{-1}e^{ix},\quad x\in\mathbf{R},$$

(for simplicity, suppose that we are considering the 1-dimensional case). Since $\|K_{\varepsilon,N}\|_{\text{weak}(1,1)}$ is bounded for $0<\varepsilon<N<\infty$, (5.2) shows that

$$(\xi-1)/|\xi-1|=\lim_{\varepsilon\downarrow 0,\, N\to\infty}\mathcal{F}K_{\varepsilon,N}\in\mathcal{M}(H^1, H^1),$$

which is a contradiction. This completes the proof.

3°) *An example of* m *such that the operator* $T_m: f\mapsto\mathcal{F}^{-1}(m\mathcal{F}f)$ *is bounded in* H^1 *but not of weak type* (1, 1) *is given by*

$$m(\xi)=\phi(|\xi|/2)|\xi|^{-n/2}(\log|\xi|)^c\exp[i|\xi|(\log|\xi|)^d],\quad \xi\in\mathbf{R}^n,$$

where $d\neq 0$ *and* $-1/2<c+nd/2\leqq 0$.

PROOF. First, observe that m vanishes in $\{|\xi|\leqq 2\}$ and that it has the estimates

$$|(\partial/\partial\xi)^\alpha m(\xi)|\leqq C_\alpha f(|\xi|)g(|\xi|)^{|\alpha|}$$

with $f(x)=x^{-n/2}(\log x)^c$ and $g(x)=(\log x)^d$, from which we can conclude that $m\in\mathcal{M}(H^1, H^1)$ so long as $f(x)(xg(x))^{n/2}=O(1)$ as $x\to\infty$ or $c+nd/2\leqq 0$ (see [12; Theorem 1″]). Secondly, we can calculate the asymptotic form of $\mathcal{F}^{-1}m$: if $d>0$, then

$$(\mathcal{F}^{-1}m)(x)=d^{-1/2}C_d(r)r^{-n/2+c/d+1/2d}(1+o(1))\quad\text{as}\quad r=|x|\to\infty, \tag{5.3}$$

where

$$C_d(r)=\exp[i(\pi n/4+t_r(\log t_r)^d-rt_r)]$$

with t_r determined by

$$\begin{cases}(\log t_r)^d+d(\log t_r)^{d-1}=r,\\ t_r\to\infty\quad\text{as}\quad r^{1/d}\to\infty;\end{cases} \tag{5.4}$$

if $d<0$, then

$$\begin{aligned}(\mathcal{F}^{-1}m)(x)=&(-d)^{-1/2}C_d(r)r^{-n/2+c/d+1/2d}(1+o(1))\\ &+(\textit{a smooth function of } r^2)\quad\text{as}\quad r=|x|\to 0,\end{aligned} \tag{5.5}$$

where

$$C_d(r)=\exp[i(\pi n/4-\pi/2+t_r(\log t_r)^d-rt_r)]$$

with t_r determined by (5.4). (The same formulas as those in Lemma 2.7 can be applied to these cases; proof indicated in § 3 needs only slight modifications, the main one of which is to use the estimate

$$|\sigma'(t_r\pm\varepsilon t_r)-r|\approx|\sigma''(t_r)|t_r\quad\text{as}\quad t_r\to\infty$$

for $\sigma(x)=x(\log x)^d$, $d\neq 0$.) If $c+nd/2>-1/2$, then, from (5.3) or (5.5), we see that $\mathcal{F}^{-1}m\notin L(1,\infty)$ (the Lorentz space), *i.e.*, the estimate

$$|\{x\,|\,|(\mathcal{F}^{-1}m)(x)|>\lambda\}|\leqq C\lambda^{-1},\quad 0<\lambda<\infty,$$

does not hold, and hence *a fortiori* T_m is not of weak type (1, 1). This completes the proof.

4°) Let m be as mentioned in 3°) with $d<0$. Then, by using integration by parts, we see that $(\partial/\partial x)^{\alpha}(\mathcal{F}^{-1}m)(x)=O(|x|^{-N})$ as $|x|\to\infty$ for all α and all $N>0$, and hence $\phi(|x|)\mathcal{F}^{-1}m\in\mathcal{S}$. Thus the distribution

$$K=\phi(|x|)\mathcal{F}^{-1}m=\mathcal{F}^{-1}m-(a\ function\ in\ \mathcal{S})$$

is an example of K such that K has a compact support and the operator $f\mapsto K*f$ is bounded in H^1 but not of weak type (1, 1).

5°) *There is an $m\in\mathcal{S}'$ with compact support such that the operator $T_m: f\mapsto\mathcal{F}^{-1}(m\mathcal{F}f)$ is bounded in H^1 but not of weak type* (1, 1). In fact it can be shown that there is a continuous function m with compact support such that T_m is bounded in H^1 but $\mathcal{F}^{-1}m\notin L(1, \infty)$. We shall omit the proof, which is similar to that in 2°).

References

[1] Benedek, A. and R. Panzone, The spaces L^P, with mixed norm, Duke Math. J. **28** (1961), 301-324.

[2] Bergh, J. and J. Löfström, Interpolation Spaces, Springer Verl., Berlin-Heidelberg-New York, 1976.

[3] Calderón. A.P. and A. Torchinsky, Parabolic maximal functions associated with a distribution, Adv. in Math. **16** (1975), 1-64; II, *ibid.* **24** (1977), 101-171.

[4] Coifman, R.R. and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. Math. Soc. **83** (1977), 569-645.

[5] Fefferman, C. and E.M. Stein, H^p spaces of several variables, Acta Math. **129** (1972), 137-193.

[6] Goldberg, D., A local version of real Hardy spaces, Duke Math. J. **46** (1979), 27-42.

[7] Hörmander, L., Estimates for translation invariant operators in L^p spaces, Acta Math. **104** (1960), 93-140.

[8] Johnson, R., Multipliers of H^p spaces, Ark. Mat. **16** (1978), 235-249.

[9] Knopf, P.M., Weak-type multipliers, Studia Math. **67** (1980), 73-84.

[10] Latter, R.H., A characterization of $H^p(\boldsymbol{R}^n)$ in terms of atoms, Studia Math. **62** (1978), 93-101.

[11] Latter, R.H. and A. Uchiyama, The atomic decomposition for parabolic H^p spaces, Trans. Amer. Math. Soc. **253** (1979), 391-398.

[12] Miyachi, A., On some Fourier multipliers for $H^p(\boldsymbol{R}^n)$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), 157-179.

[13] Miyachi, A., On some singular Fourier multipliers, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 267-315.

[14] Peetre, J., On spaces of Triebel-Lizorkin type, Ark. Mat. **13** (1975), 123-130.

[15] Taibleson, M.H., On the theory of Lipschitz spaces of distributions on Euclidean n-space, I, J. Math. Mech. **13** (1964), 407-479; II, *ibid.* **14** (1965), 821-839; III, *ibid.* **15** (1966), 973-981.

[16] Triebel, H., Fourier Analysis and Function Spaces, Teubner-Texte zur Math., Teubner Verl., Leipzig, 1977.

[17] Triebel, H., Spaces of Besov-Hardy-Sobolev Type, Teubner-Texte zur Math., Teubner Verl., Leipzig, 1978.

[18] Wainger, S., Special Trigonometric Series in k-dimensions, Mem. Amer. Math. Soc. No. 59, Amer. Math. Soc., Providence, Rhode Island, 1965.

(Received November 9, 1981)

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