Some properties of mild hyperfunctions and an application to propagation of micro-analyticity in boundary value problems

By Kiyômi KATAOKA

Abstract

The notion of mildness for hyperfunctions ([5]) was successfully applied in [6] to the microlocal analysis of boundary value problems. In the present article, we study some further properties of mild hyperfunctions, particularly under Holmgren-type coordinate transformations. As corollaries, we obtain a general theorem on the propagation of micro-analyticity up to the boundary of solutions of microdifferential equations, and a theorem on the compatibility between topological boundary values and "Trace" of mild distributions. The former result has already been announced in [7]. The latter one is the perfect form of the similar result in [5].

Introduction.

In [5] we introduced the notion of "mildness" for hyperfunctions on a real analytic boundary (say, $x_1=0$); that is, a subclass of hyperfunctions defined in one side of the boundary which have boundary values for any normal derivative of finite or infinite order. In particular this notion covers the theory of non-characteristic boundary values of hyperfunction solutions of partial differential equations by Komatsu-Kawai and Schapira.

We review here some fundamental properties of mild hyperfunctions. Set $M=\boldsymbol{R}^n \ni (x_1, x')$, $M_+=\{x\in M;\ x_1\geqq 0\}$ and $N=\{x\in M;\ x_1=0\}$. Let $f(x)$ be a hyperfunction defined in $\{x\in M;\ 0<x_1<\delta,\ |x'-x'_0|<\delta\}$. Then *$f(x)$ is said to be mild at $(0, x'_0)\in N$ from the positive side of N* if $f(x)$ is written as a sum of boundary values of holomorphic functions defined in such domains as

$$(1)\qquad D(x'_0, \Gamma; \varepsilon)=\{z\in \boldsymbol{C}^n;\ |z_1|+|z'-x'_0|<\varepsilon,\ \mathrm{Im}\, z'\in\Gamma,\ |\mathrm{Im}\, z_1|+(-\mathrm{Re}\, z_1)_+<\varepsilon|\mathrm{Im}\, z'|\}.$$

Here ε is a small positive number, Γ is an open proper convex cone in $\boldsymbol{R}^{n-1}$ and $(x)_+=x$ if $x\geqq 0$, $=0$ if $x<0$. More exactly speaking, there exist some small $\varepsilon>0$ and some holomorphic functions $F_1(z), \cdots, F_k(z)$ defined on $D(x'_0, \Gamma_1; \varepsilon), \cdots, D(x'_0, \Gamma_k; \varepsilon)$ such that

$$(2)\qquad f(x)=\sum_{j=1}^{k} F_j(x_1,\ x'+i0\Gamma_j)$$

holds on $\{x_1>0,\ |x_1|+|x'-x'_0|<\varepsilon\}$. As for the uniqueness of such an expression, we have the edge of the wedge theorem of Martineau's type. This is more delicate than the ordinary edge of the wedge theorem. In fact, when $\dim M=2$ (hence $\Gamma=(0, +\infty)$, or $(-\infty, 0)$), it is stated as follows: Let $F_\pm(z)$ be any holomorphic functions defined on $D(0, (0, +\infty); \varepsilon)$, $D(0, (-\infty, 0); \varepsilon)$ respectively. Suppose that

$$f(x)=F_+(x_1,\ x_2+i0)-F_-(x_1,\ x_2-i0)=0$$

on $\{x_1>0,\ |x_1|+|x_2|<\varepsilon\}$ as a hyperfunction. Then F_+ and F_- are both analytically extensible to a neighborhood of

$$\{z\in \boldsymbol{C}^n;\ \mathrm{Im}\, z=0,\ \mathrm{Re}\, z_1\geqq 0,\ |\mathrm{Re}\, z_1|+|\mathrm{Re}\, z_2|<\varepsilon\}$$

and coincide there with each other. In particular, they can be prolonged analytically through the real boundary N. Hence we can justify several operations on mild hyperfunctions by using the expression (2) and this theorem. For example, the boundary value of a mild hyperfunction is defined by

$$\text{(3)}\qquad \mathrm{Trace}(f)=f(+0,\ x')\equiv\sum_{j=1}^{k} F_j(0,\ x'+i0\cdot\Gamma_j)$$

and the extension with support in M_+ is defined by

$$\text{(4)}\qquad \mathrm{ext}(f)=f(x)Y(x_1)\equiv\sum_{j=1}^{k}(F_j(x_1,\ z')Y(x_1))_{\mathrm{Im}\, z'\to 0\cdot\Gamma_j},$$

where $Y(t)$ is the Heaviside function.

We denote by $\mathring{\mathcal{B}}_{N|M_+}$ the sheaf on N of mild hyperfunctions from the positive side of N (in [5], [6] we used the notation $\hat{\mathcal{B}}_{N|M_+}$, $\hat{C}_{N|M_+}$ instead of $\mathring{\mathcal{B}}_{N|M_+}$, $\mathring{C}_{N|M_+}$ respectively). Following the usual procedure we microlocalize $\mathring{\mathcal{B}}_{N|M_+}$ over the cotangential spherical bundle iS^*N of N. That is, a germ $f(x)$ of $\mathring{\mathcal{B}}_{N|M_+}$ at $(0, x'_0)\in N$ is said to be ρ-microanalytic at $(0, x'_0)$ in the direction $i\eta'_0dx'$ $(\eta'_0\in \boldsymbol{R}^{n-1}\setminus\{0\})$ if $f(x)$ is written in the form (2) such that for every $j=1, \cdots, k$

$$\Gamma_j\cap\{v\in \boldsymbol{R}^{n-1};\ \langle v,\ \eta'_0\rangle<0\}\neq\emptyset\,.$$

Here ρ means the projection:

$$\text{(5)}\qquad iS^*M\underset{M}{\times}N\setminus iS^*_NM\ni(0,\ x';\ i\eta dx)\longmapsto(x';\ i\eta' dx')\in iS^*N\,.$$

Then the stalk of $\mathring{C}_{N|M_+}$ at $(x'_0;\ i\eta'_0)\in iS^*N$ is defined as an equivalence class:

$$\text{(6)}\qquad \mathring{\mathcal{B}}_{N|M_+}|_{x'_0}/\{f(x)\in \mathring{\mathcal{B}}_{N|M_+}|_{x'_0};\ f(x) \text{ is } \rho\text{-microanalytic at } x'_0 \text{ in the direction } i\eta'_0dx'\}.$$

Let π_N be the projection: $iS^*N\to N$ and $\mathcal{A}_M$ be the sheaf of germs of real analytic functions on M. Then these sheaves satisfy the following exact sequence on N:

$$\text{(7)}\qquad 0\longrightarrow \mathcal{A}_M|_N\longrightarrow \mathring{\mathcal{B}}_{N|M_+}\longrightarrow \pi_{N*}\mathring{C}_{N|M_+}\longrightarrow 0\,.$$

The first main result (Theorem 1.2) is connected with the injectivity of the restriction:

$$\hat{C}_{N|M_+} \longrightarrow \hat{C}_{N'|M'_+}$$

at $(x'_0; i\eta'_0 dx')$. Here $M_+=\{x_1\geqq 0\}$, $M'_+=\{x_1-\varphi(x')\geqq 0\}$ with real non-negative valued analytic function $\varphi(x')$ satisfying $\varphi(x'_0)=0$. As a direct corollary, we obtain Theorem 1.4 concerning the propagation of micro-analyticity of solutions up to the boundary for micro-differential equations; for example, $P(x, D)=D_1^2-(x_1-x_2^2\pm x_3^2)(D_2^2+D_3^2)$ at $x=0$ (see Theorem 1.5). The other main result (Theorem 2.5) is connected with coincidence of 'Trace' and the topological boundary values for those mild distributions that are extensible through the boundary as distributions. This is a direct corollary of the expression theorem for such a mild distribution (Theorem 2.4). A similar result has already been obtained in [5] where we suppose the properness of the support along the boundary. This assumption is, however, too strong and unnatural for the case of extensible mild distributions. In fact, in order to get our present result, a very different method (a kind of Holmgren's method) is needed.

§ 1. Injectivity of the restriction homomorphism.

We set M, M_+, N as before. Further we denote by $X=\boldsymbol{C}^n$ the complexification of $M=\boldsymbol{R}^n$, and by S^*X the cotangential spherical bundle

$$\{(z;\zeta dz)\in \boldsymbol{C}^n\times((\boldsymbol{C}^n\setminus\{0\})/\boldsymbol{R}^+)\}$$

of X. Let N' be another real analytic submanifold of M given by $\{x\in M=\boldsymbol{R}^n;\ x_1-\varphi(x')=0\}$ with real non-negative analytic function $\varphi(x')$. Set $M'_+=\{x\in M;\ x_1-\varphi(x')\geqq 0\}$. Then, as seen in [5], micro-differential operators defined on a neighborhood of

$$(8) \qquad \{(z;\zeta dz)\in S^*X;\ z=(0, x'_0),\ \zeta_1\in \boldsymbol{C},\ \zeta'=i\eta'_0\}$$

$$(\text{or } \{(z;\tau_1 d(z_1-\varphi(z'))+\tau' dz')\in S^*X;\ z=(\varphi(x'_0), x'_0),\ \tau_1\in\boldsymbol{C},\ \tau'=i\eta'_0\})$$

operate on the stalk of $\hat{C}_{N|M_+}$ at $(0, x'_0; i\eta'_0 dx')\in iS^*N$ (resp. $\hat{C}_{N'|M'_+}$ at $(\varphi(x'_0), x'_0; i\eta'_0 dx')\in iS^*N'$).

PROPOSITION 1.1. *Suppose that* $\varphi(x'_0)=0$ (*hence* $\operatorname{grad}\varphi(x'_0)=0$). *Then the restriction homomorphism*

$$\mathfrak{R}: \hat{C}_{N|M_+} \longrightarrow \hat{C}_{N'|M'_+}$$

is well-defined at $(0, x'_0; i\eta'_0 dx')\in iS^*N$ (*and also* $\in iS^*N'$). *Further* $\mathfrak{R}$ *commutes with the operation of any micro-differential operator defined on a neighborhood of* (8).

PROOF. To prove the first statement, by Theorem 2.1.23 of [5] we have only to see that the restriction $\mathcal{R}: \tilde{A}_{M_+} \to \tilde{A}_{M'_+}$ is well-defined as a sheaf homomorphism on $\tau_N^{-1}(x'_0) \cong \tau_{N'}^{-1}(x'_0)$. Here $\tilde{A}_{M_+}$ is the sheaf on the tangential spherical bundle SN of N, which is the sheaf of defining functions for mild hyperfunctions, and $\tau_N: SN \to N$ is the projection. Hereafter we assume $x'_0=0$. Recall that the stalk of $\tilde{A}_{M_+}$ at $(0;\theta'_0)\in SN$ with $|\theta'_0|=1$ is written as follows:

$$\lim_{\overrightarrow{r\to+0}} \Gamma(\{|z|<r,\ (-x_1)_+ + \sqrt{|y|^2-\langle y',\theta'_0\rangle^2} < r\langle y',\theta'_0\rangle\},\ \mathcal{O}_{C^n})$$

where $z=x+iy$ (see (2.8) in [5]). Put $w=u+iv=(z_1-\varphi(z'), z_2, \cdots, z_n)$. Then

$$\begin{aligned} D_r &= \{|z|<r,\ (-x_1)_+ + \sqrt{|y|^2-\langle y',\theta'_0\rangle^2} < r\langle y',\theta'_0\rangle\} \\ &\supset \{|w|+|\varphi(w')|<r,\ (-u_1-\mathrm{Re}\,\varphi(w'))_+ \\ &\qquad + \sqrt{(v_1+\mathrm{Im}\,\varphi(w'))^2+|v'|^2-\langle v',\theta'_0\rangle^2} < r\langle v',\theta'_0\rangle\} \\ &\supset \{(-\mathrm{Re}\,\varphi(w'))_+ + |\mathrm{Im}\,\varphi(w')| + (-u_1)_+ \\ &\qquad + \sqrt{|v|^2-\langle v',\theta'_0\rangle^2} < r\langle v',\theta'_0\rangle,\ |w|+|\varphi(w')|<r\}. \end{aligned}$$

Since $\varphi(0)=\mathrm{grad}\,\varphi(0)=0$, we can take a small $\varepsilon>0$ such that $\sup_{|w|\leqq\varepsilon}(|\varphi(w')|+|w|) < r$, $\inf_{|w'|\leqq\varepsilon}\left(\frac{r}{4}|v'|+\mathrm{Re}\,\varphi(w')\right)\geqq 0$ and $\inf_{|w'|\leqq\varepsilon}\left(\frac{r}{4}|v'|-|\mathrm{Im}\,\varphi(w')|\right)\geqq 0$. Hence we have

$$D_r \supset \left\{|w|<\varepsilon,\ (-u_1)_+ + \sqrt{|v|^2-\langle v',\theta'_0\rangle^2} + \frac{r}{2}|v'| < r\langle v',\theta'_0\rangle\right\}$$

$$\supset \left\{|w|<\varepsilon,\ (-u_1)_+ + \left(1+\frac{r}{2}\right)\sqrt{|v|^2-\langle v',\theta'_0\rangle^2} < \frac{r}{2}\langle v',\theta'_0\rangle\right\}.$$

Therefore any germ of $\tilde{A}_{M_+}$ at $(0;\theta'_0)\in SN$ defines a germ of $\tilde{A}_{M'_+}$ at $(0;\theta'_0)\in SN'$.

Let $P(x, D)$ be a micro-differential operator defined on a neighborhood of (8) and let $f(x)$ be a germ of $\hat{C}_{N|M_+}$ at $(x'_0; i\eta'_0)$. We may take $x'_0=0$ and $\eta'_0 dx' = dx_n$. Then there exist a small positive number r and a section $F(z)$ of $\tilde{A}_{M_+}$ such that $F(z)$ is a holomorphic function on

$$\Omega_r = \left\{z\in \boldsymbol{C}^n;\ |z|<r,\ \mathrm{Im}\,z_n > r(|\mathrm{Im}\,z_2|+\cdots+|\mathrm{Im}\,z_{n-1}|)+\frac{1}{r}((-\mathrm{Re}\,z_1)_+ + |\mathrm{Im}\,z_1|)\right\}$$

satisfying $F(x_1, \cdots, x_{n-1}, x_n+i0)=f(x)$ as a germ of $\hat{C}_{N|M_+}$ at $(x'_0; i\eta'_0)$. Further we may assume that $P(z, D_z)$ is defined on

$$\left\{(z;\zeta)\in S^*X;\ |z|\leqq 2r,\ |\zeta_1|\leqq \frac{2n}{r}|\zeta_n|,\ |\zeta_2|\leqq 2nr|\zeta_n|,\ \cdots,\ |\zeta_{n-1}|\leqq 2nr|\zeta_n|\right\}.$$

In particular, $P(z, D_z)$ has the following expansion:

$$P(z, D_z) = \sum_{J=(j_1,\cdots,j_n)\in Z^n} a_J(z) D_z^J.$$

Here J moves over all the multi-indices such that $|J|=j_1+\cdots+j_n\leqq m$ (*the order of* P) and $j_1\geqq 0, \cdots, j_{n-1}\geqq 0$, and $\{a_J(z)\}_J$ are holomorphic functions on $\{|z|\leqq 2r\}$ satisfying

$$\sup_{|z|\leq 2r}|a_J(z)|\leq C(m-|J|)!\ A^{m-|J|}\left(\frac{r}{2n}\right)^{j_1}\left(\frac{1}{2nr}\right)^{j_2+\cdots+j_{n-1}}$$

for some positive constants C and A. Consequently, as seen in the proof of Theorem 1.8 of [6], $P(x, D)f(x)$ is written as the boundary value of

$$P^{\varepsilon}(z, D_z)F(z)=\sum_{j_n\geq 0}a_J(z)D_z^J F(z)+\sum_{j_n<0}a_J(z)\cdot\int_{i\varepsilon}^{z_n}\frac{(z_n-s)^{|j_n|-1}}{(|j_n|-1)!}$$
$$\times D_{z_1}^{j_1}, \cdots, D_{z_{n-1}}^{j_{n-1}}F(z_1, \cdots, z_{n-1}, s)ds$$

for sufficiently small $\varepsilon>0$. Note that $P^{\varepsilon}(z, D_z)F(z)$ is holomorphic in a domain with the same edge as Ω_r. Therefore $\mathscr{R}(P(x, D)f)=\mathscr{R}([P^{\varepsilon}(z, D_z)F(z)])=[P^{\varepsilon}(z, D_z)\mathscr{R}(F(z))]=P(x, D)\mathscr{R}(f)$.

Now we will show the injectivity of $\mathscr{R}$.

THEOREM 1.2. *We inherit the notation from Proposition* 1.1. *Then the restriction homomorphism*

$$\mathscr{R}:\hat{C}_{N|M_+}\longrightarrow\hat{C}_{N'|M'_+}$$

defined at $(x_0'; i\eta_0'dx')$ *is injective.*

PROOF. The theorem directly follows from the edge of the wedge theorem for $\tilde{A}_{M_+}$ in [5] when $n=\dim M=2$. So we assume $n>2$. Take a coordinate system of M as in the preceding proposition. Let $f(x)$ be a germ of $\hat{C}_{N|M_+}$ at $(0; idx_n)\in iS^*N$ such that $\mathscr{R}(f)=0$ at $(0; idx_n)\in iS^*N'$. Hence there exist some holomorphic functions $F(z)$, $G_1(z), \cdots, G_k(z)$ defined on D and $D_1, \cdots, D_k$ respectively such that $f(x)=[F(x_1, \cdots, x_{n-1}, x_n+i0)]$ as a germ of $\hat{C}_{N|M_+}$ at $(0; idx_n)$ and that $F(z)=G_1(z)+\cdots+G_k(z)$ on $D\cap D_1\cap\cdots\cap D_k$ $(\neq\emptyset)$. Here,

$$D=\{z\in\boldsymbol{C}^n;\ |z|<\delta,\ |\mathrm{Im}\,z_1|+(-\mathrm{Re}\,z_1)_+<r\langle\mathrm{Im}\,z', \xi'\rangle$$
$$\text{for every } \xi' \text{ satisfying } |\xi'|=1 \text{ and } |\xi'-(0, \cdots, 0, 1)|<2r\},$$

and for $j=1, \cdots, k$

$$D_j=\{|z|<\delta,\ |\mathrm{Im}(z_1-\varphi(z'))|+(-\mathrm{Re}(z_1-\varphi(z')))_+<r\langle\mathrm{Im}\,z', \xi'\rangle$$
$$\text{for every } \xi'\in K_j\}$$

with some positive constants δ, r $(r<1/3)$ and compact subsets $K_1, \cdots, K_k$ of $S^{n-2}=\{\xi'\in\boldsymbol{R}^{n-1};\ |\xi'|=1\}$ satisfying $K_j\subset\{r<|\xi'-(0, \cdots, 0, 1)|<3r\}$ for every j. By softness of $\hat{C}_{N|M_+}$, we can choose $\{(G_j(z), K_j)\}_j$ as follows: There are some vectors $\eta^j=(\eta_2^j, \cdots, \eta_{n-1}^j, 0)\in S^{n-2}$ for $j=1, \cdots, k$ and a small positive number ε such that

$$K_j\subset L_j=\{\xi';\ \langle\xi', \eta^j\rangle>\varepsilon,\ r<|\xi'-(0, \cdots, 0, 1)|<3r\}.$$

Now we use the curvilinear wave expansion of $F(z)$ (see Theorem 1.1.8 in [4]); that is, setting

$$K_\alpha(z', \xi') = \frac{(n-2)!}{(2\pi)^{n-1}} \cdot \frac{(1-i\alpha\langle z', \xi'\rangle)^{n-3}\left\{1-i\alpha\langle z', \xi'\rangle - \alpha^2\left(\sum_{j=2}^{n} z_j^2 - \langle z', \xi'\rangle^2\right)\right\}}{\left\{-i\langle z', \xi'\rangle + \alpha\left(\sum_{j=2}^{n} z_j^2 - \langle z', \xi'\rangle^2\right)\right\}^{n-1}},$$

under suitable conditions on y'_0, α and R,

$$R_{\alpha, y'_0} F(z_1, z', \xi') = \int_{\{\mathrm{Im}\, w' = y'_0,\, |\mathrm{Re}\, w'| \leq R\}} K_\alpha(z'-w', \xi') F(z_1, w') dw'$$

is analytically extended to a non-void fixed domain when ξ' moves over S^{n-2} and there the inversion formula (with parameter z_1)

$$F(z_1, z') = \int_{|\xi'|=1} \mathrm{R}_{\alpha, y'_0} F(z_1, z', \xi') d\sigma(\xi')$$

holds. Set $\alpha = R^{-1/2}$, $R_1 = R/2$ and $y'_0 = (0, \cdots, 0, R^{3/2}/16)$ in Theorem 1.1.8 of [4]. Then for the Radon transform $\mathrm{R}_{\alpha, y'_0} F(x_1, z', \xi')$ of $F(x_1, z')$ with real analytic parameter $x_1 \in [0, \delta/2]$ all the assumptions are satisfied if

$$(9) \qquad 0 < R < \min\{\delta/4, 1/16, (2\delta)^{2/3}\}.$$

Note that $\mathrm{R}_{\alpha, y'_0} F(z_1, z', \xi')$ represents a section of $\tilde{A}_{M+}$ on

$$\{(x'; v') \in SN;\ |x'| < R/2,\ \langle v', \xi'\rangle > 0\}$$

for every ξ'. In fact this is holomorphic in a domain containing

$$\{C(|\mathrm{Im}\, z_1| + (-\mathrm{Re}\, z_1)_+) < \langle \mathrm{Im}\, z', \xi'\rangle - 2\alpha(|\mathrm{Im}\, z'|^2 - \langle \mathrm{Im}\, z', \xi'\rangle^2),$$
$$|\mathrm{Im}\, z'| + |z_1| < 1/C,\ |\mathrm{Re}\, z'| < R/2\}$$

with some $C > 0$. Hence, in order to prove this theorem, we have only to show that $\mathrm{R}_{\alpha, y'_0} F(z_1, z', \xi')$ is holomorphic in a neighborhood of $\{z_1 = 0, z' = 0, \xi' = (0, \cdots, 0, 1)\}$. To simplify the argument, we consider only $\mathrm{R}_{\alpha, y'_0} F(0, z', (0, \cdots, 0, 1))$. Since $\varphi(0) = \mathrm{grad}\, \varphi(0) = 0$, there exists a positive constant λ such that $|\mathrm{Im}\, \varphi(w')| + |\mathrm{Re}\, \varphi(w')| \leqq \lambda |w'|^2$ on $\{|w'| \leqq \delta\}$. Therefore the inclusion

$$(10) \qquad D_j \supset \left\{z \in \boldsymbol{C}^n;\ z_1 = 0,\ |z'| < \delta,\ \langle \mathrm{Im}\, z', \xi'\rangle > \frac{\lambda}{r}|z'|^2 \text{ for every } \xi' \in L_j\right\}$$

holds for every j. In particular, $D \cap D_1 \cap \cdots \cap D_k \supset \{z_1 = 0,\ \mathrm{Im}\, z' = y'_0,\ |\mathrm{Re}\, z'| \leqq R\}$ holds if $\min\{R^{3/2} \cos\theta/16;\ 0 \leqq \theta \leqq 2\,\mathrm{Sin}^{-1}(1/2)\}$ is greater than $2\lambda R^2/r$, that is,

$$(11) \qquad R < (r/64\lambda)^2.$$

Hence from the identity $F(z) = G_1(z) + \cdots + G_k(z)$ we obtain

$$\mathrm{R}_{\alpha, y'_0} F(0, z', \xi') = \sum_{j=1}^{k} \mathrm{R}_{\alpha, y'_0} G_j(0, z', \xi').$$

On the other hand, by Theorem 1.1.8 or Lemma 1.1.6 in [4] we know that $\mathrm{R}_{\alpha, y'_0} G_j(0, z', (0, \cdots, 0, 1))$ is holomorphic on

$$\bigcup_{v'\in E_j}\left\{z'\in \boldsymbol{C}^{n-1};\ |x'|<R/2,\ |y'|<R^{3/2}/8,\ (y_n-v_n)-2\sqrt{R}^{-1}\cdot\sum_{j=2}^{n-1}(y_j-v_j)^2>0\right\}$$

for every j. Here (recall (10)),

$$E_j=\{|v'|<R^{3/2}/8,\ \langle v',\xi'\rangle>2\lambda R^2/r \text{ for every } \xi'\in L_j\}$$

$$\supset\left\{v'=t_1\eta^j+(0,\cdots,0,t_2)\in \boldsymbol{R}^{n-1};\ t_1^2+t_2^2<R^3/64,\ t_1\geqq 0,\ t_2\leqq 0 \right.$$

$$\left. \text{and } t_1\varepsilon+\left(1-\frac{r^2}{2}\right)t_2>2\lambda R^2/r\right\}.$$

Therefore $\mathrm{R}_{\alpha,y'_0}G_j(0,z',(0,\cdots,0,1))$ is holomorphic at $z'=0$ if

$$\left\{(t_1,t_2)\in \boldsymbol{R}^2;\ t_1^2+t_2^2<R^3/64,\ t_1\geqq 0,\ t_2\leqq 0,\ t_1\varepsilon+\left(1-\frac{r^2}{2}\right)t_2>2\lambda R^2/r\right\}$$

$$\cap\{-t_2>2\sqrt{R}^{-1}t_1^2\}\neq\emptyset.$$

That is,

$$(12)\quad \begin{cases} R<(r\varepsilon\tan\theta/16\lambda)^{2/3} \quad\text{and} \\ 4\left(1+\sqrt{1-\dfrac{16\lambda R^{3/2}}{r\varepsilon}\cot\theta}\right)^{-2}+\tan^2\theta\left\{2\left(1+\sqrt{1-\dfrac{16\lambda R^{3/2}}{r\varepsilon}\cot\theta}\right)^{-1}-1\right\}^2 \\ <r^2\varepsilon^2/256\lambda^2R, \end{cases}$$

where $\theta=\mathrm{Tan}^{-1}\left(\varepsilon\Big/\left(1-\frac{r^2}{2}\right)\right)$. Hence conditions (9), (11) and (12) are all satisfied if R is taken small enough. So we conclude that $\mathrm{R}_{\alpha,y'_0}F(0,z',(0,\cdots,0,1))$ is holomorphic at $z'=0$. Easily to see, this argument allows that $\mathrm{R}_{\alpha,y'_0}F(z_1,z',\xi')$ is holomorphic in a neighborhood of $\{z_1=0,\ z'=0,\ \xi'=(0,\cdots,0,1)\}$ if R is taken small enough. This completes the proof.

Now we recall the notion of "N_+-regularity" for micro-differential operators defined in [6] (cf. [11]). We inherit the notation M, N, M_+ from Introduction and denote by X the complexification of M.

DEFINITION 1.3. Let $p_0=(0,x'_0;\ i\eta_0dx)$ be a point of $iS^*M|_N$ and $P(x,D)$ be a micro-differential operator defined at p_0. Then $P(x,D)$ is said to be N_+-regular at p_0 if the following condition is satisfied: For any germ $u(x)$ of $\mathcal{C}_{M_+|X}$ at p_0 satisfying $P(x,D)u(x)\in\mathcal{C}_{N|X}$ at p_0, $u(x)$ belongs to $\mathcal{C}_{N|X}$ at p_0 if $\mathrm{Supp}(u)\subset\{(x;\ \zeta_1dx_1+i\eta'dx')\in S^*_{M_+}X;\ x_1=0\}$.

Here $\mathcal{C}_{M_+|X}$ and $\mathcal{C}_{N|X}$ are the sheaves on $S^*_{M_+}X$, S^*_NX of relative microfunctions with respect to the couples (M_+,X), (N,X) respectively (see § 4 of [4]; $S^*_{M_+}X$ and S^*_NX are closed subsets of S^*X). Furthermore, $\mathcal{E}_X$ (the sheaf on S^*X of micro-differential operators) operates on them as sheaf homomorphisms.

The meaning of N_+-regularity is explained as follows: Suppose that $\sigma(P)(0,x'_0,\zeta_1,i\eta'_0)\not\equiv 0$ ($\sigma(P)$ is the principal symbol of P). Then by the preparation theorem for micro-differential operators, $P(x,D)$ is written in the form

$$P(x, D)=R(x, D)P'(x, D),\quad P'(x, D)=D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D').$$

Here $R(x, D)$ is elliptic at p_0, order$(P')=m$ and the equation

$$\zeta_1^m+\sigma_1(P_1)(0, x_0', i\eta_0')\zeta_1^{m-1}+\cdots+\sigma_m(P_m)(0, x_0', i\eta_0')=0$$

has a zero with multiplicity m at $\zeta_1=i\eta_{0,1}$. Then, the N_+-regularity of P at p_0 is equivalent to the following statement. For any $\mathring{C}_{N|M_+}$-solution $u(x)$ of $P'(x, D)u(x)=0$ at $p_0'=(x_0'; i\eta_0'dx')\in iS^*N$, $u(x)$ vanishes as a germ of $\mathring{C}_{N|M_+}$ at p_0' (or equivalently, all the boundary values $\{D_1^ju(+0, x')\}_{j=0,\dots,m-1}$ vanish as microfunctions at p_0') if $u(x)$ vanishes as a section of microfunction over $\{(x; i\eta)\in iS^*M;\ 0<x_1<\varepsilon,\ |x'-x_0'|<\varepsilon,\ |\eta-\eta_0|<\varepsilon\}$ for some small $\varepsilon>0$.

THEOREM 1.4. *Let* $N=\{x\in M;\ x_1=0\}$, $N'=\{x\in M;\ x_1=\varphi(x')\}$ *be real analytic submanifolds of* M *with real non-negative valued analytic function* $\varphi(x')$ *satisfying* $\varphi(0)=0$, $\nabla\varphi(0)=0$. *Let* $P(x, D)$ *be a micro-differential operator defined at* $p_0=(0; i\eta_0dx)\in iS^*M$ *satisfying* $\sigma(P)(0, \zeta_1, i\eta_0')\not\equiv 0$. *Then* $P(x, D)$ *is* N_+*-regular at* p_0 *if* $P(x, D)$ *is* N_+'*-regular at* p_0.

REMARK. Three wide classes of N_+-regular operators are known; the non-microcharacteristic operators by P. Schapira (see [11]), the operators semi-hyperbolic in the positive side of N and the diffractive operators ([2], [6]). The first and the third classes of N_+-regular operators are, however, invariable under the perturbation of the boundary N as above. Hence we obtain a new class of N_+-regular operators only from the second class (cf. Theorem 1.5).

PROOF. Without loss of generality we may assume that $p_0=(0; idx_n)$ and $P(x, D)$ has the form

$$D_1^m+P_1(x, D')D_1^{m-1}+\cdots+P_m(x, D'),$$

where m is the order of P and $\{\zeta_1\in C;\ \sigma(P)(0, \zeta_1, 0, \cdots, 0, i)=0\}=\{0\}$. Let $u(x)$ be any germ of $\mathring{C}_{N|M_+}$ at $(0; idx_n)\in iS^*N$ such that $Pu=0$ and that $u(x)$ vanishes as a section of microfunction over

$$\{(x; i\eta)\in iS^*M;\ 0<x_1<\varepsilon,\ |x'|<\varepsilon,\ |\eta-(0, \cdots, 0, 1)|<\varepsilon\}$$

for some $\varepsilon>0$. Then, by Proposition 1.1, the restriction $u'(x)=\mathcal{R}(u)\in\mathring{C}_{N'|M_+'}$ also satisfies $Pu'=0$. Therefore u' vanishes as a germ of $\mathring{C}_{N'|M_+'}$ at $(0; idx_n)\in iS^*N'$ because P is N_+'-regular at p_0 and $\sigma(P)(0, \zeta_1, 0, \cdots, 0, i)$ does not vanish in $C\setminus\{0\}$. Hence $u=0$ as a germ of $\mathring{C}_{N|M_+}$ at $(0; idx_n)\in iS^*N$ by Theorem 1.2. This completes the proof.

THEOREM 1.5. *Let* $P(x, D)$ *be a second-order micro-differential operator of the form*

$$P(x, D)=D_1^2-A(x, D')$$

defined at $(0; idx_n)$. *Suppose that* $\sigma_2(A)(0, e_n)=0$ *with* $e_n=(0, \cdots, 0, 1)\in \boldsymbol{R}^{n-1}$ *and that* $\sigma_2(A)(x, \xi')$ *is real and non-negative in* $\{(x; \xi')\in \boldsymbol{R}^n\times \boldsymbol{S}^{n-2}; C|x'|^2\leqq x_1, |x|<1/C$ *and* $|\xi'-e_n|<1/C\}$ *for sufficiently large* $C>0$. *Then for any* $\hat{C}_{N|M_+}$-*solution* $u(x)$ *of* $P(x, D)u(x)=0$ *at* $(0; idx_n)\in iS^*N$, *the micro-analyticity of* u *propagates up to the boundary* $\{x_1=0\}$ *near* $(0; idx_n)$: *More exactly,* $SS(u(+0, x'))\cup SS(D_1u(+0, x'))\not\ni(0; idx_n)\in iS^*N$ *if* $u(x)$ *vanishes as a section of microfunction over*

$$\{(x; i\eta)\in iS^*M;\ 0<x_1<\varepsilon,\ |x'|<\varepsilon,\ |\eta-(0, e_n)|<\varepsilon\}$$

for some $\varepsilon>0$.

PROOF. Take $N'=\{x\in M; x_1=C|x'|^2\}$. Then we have only to see the N'_+-regularity of $P(x, D)$ at $(0; idx_n)$. Set the coordinate system $y=(y_1, y')=(x_1-C|x'|^2, x')$ and rewrite $\sigma(P)(x, \xi)$ by the new coordinates $(y; \theta dy)$:

$$\sigma(P)=\theta_1^2-\sigma_2(A)(y_1+C|y'|^2, y', \theta'-2C\theta_1y').$$

Now we claim that P is hyperbolic in $\{y_1>0\}$ in the dy_1-direction near $(0; idy_n)\in iS^*M$; that is, for sufficiently small $\varepsilon>0$,

$$F(\tau, y, \theta')=\tau^2-\sigma_2(A)(y_1+C|y'|^2, y', \theta'-2C\tau y')$$

never vanishes in

$$\{(\tau, y, \theta')\in \boldsymbol{C}\times\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ y_1>0,\ \tau\notin\boldsymbol{R},\ |y|<\varepsilon,\ |\theta'-e_n|<\varepsilon,\ |\tau|<1\}.$$

Write $F(\tau, y, \theta')$ in the form

$$(1+g(\tau, y, \theta'))\tau^2+2h(y, \theta')\tau-\alpha(y, \theta')$$

and

$$\alpha(y, \theta')=\sigma_2(A)(y_1+C|y'|^2, y', \theta'),$$

where $g(\tau, y, \theta')$ and $h(y, \theta')$ are analytic in a neighborhood of $\{(\tau, y, \theta')\in \boldsymbol{C}\times\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}; y=0, \theta'=e_n\}$, they are real valued functions for real τ, and satisfy $g(\tau, 0, \theta')=h(0, \theta')\equiv 0$. Choose a small number $\varepsilon>0$ such that g, h and α are analytic in

$$V=\{(\tau, y, \theta')\in\boldsymbol{C}\times\boldsymbol{R}^n\times\boldsymbol{R}^{n-1};\ |\tau|\leqq 1,\ |y|\leqq\varepsilon,\ |\theta'-e_n|\leqq\varepsilon\}$$

and that

$$\sup_V \max\{|g(\tau, y, \theta')|,\ |h(y, \theta')|,\ |\alpha(y, \theta')|\}<1/5. \tag{13}$$

Further we can take ε small enough such that, for any $(y, \theta')\in\{(y, \theta')\in\boldsymbol{R}^n\times\boldsymbol{R}^{n-1}; y_1\geqq 0, |y|\leqq\varepsilon, |\theta'-e_n|\leqq\varepsilon\}$, the equation $F(\tau, y, \theta')=0$ in τ has just two zeros in $\{\tau\in\boldsymbol{C}; |\tau|<1\}$ counting multiplicities (consider $F(\tau, 0, e_n)\equiv\tau^2$ and use Rouché's theorem). In fact these zeros are real for such (y, θ') because $F(\tau, y, \theta')=0$ has two zeros in the real interval $(-1, 1)$ (remark that $F(0, y, \theta')\leqq 0$, $F(\pm 1, y, \theta')>0$). Thus our claim has been justified. Therefore, by Theorem 1.12 of [6], P is N'_+-regular at $(0; idy_n)$.

§ 2. Extensible mild distributions.

Let $f(x)$ be a distribution defined on $\{x\in \boldsymbol{R}^n;\ |x|<R,\ x_1>0\}$. We call $f(x)$ *an extensible mild distribution at* $x=0$ if $f(x)$ is mild at $x=0$ from $x_1>0$ and $f(x)$ is extensible to $x_1\leqq 0$ near $x=0$ as a distribution. Then we have a theorem on "good expressions" of extensible mild distributions.

We prepare three lemmas. Set $M=\boldsymbol{R}^n\ni(x_1,\ x')$, $N=\{x\in M;\ x_1=0\}$, $M_+=\{x\in M;\ x_1\geqq 0\}$ and $X=M^C=\boldsymbol{C}^n$ as before, and recall the definitions of sheaves $\mathcal{C}_{M_+|X}$ and $\mathcal{C}_{N|X}$ (cf. Definition 1.3). Then another definition of $\hat{\mathcal{C}}_{N|M_+}$ given in § 2 of [5] is as follows:

$$\hat{\mathcal{C}}_{N|M_+}=\iota_*^+\mathcal{C}_{M_+|X}\cap j_*\mathcal{C}_{N|X}|_{iS^*N\times\{\infty\}}/\iota_*\mathcal{C}_{N|X}\,,$$

where $\bar{G}_+=\{(0,\ x';\ \zeta_1,\ i\eta')\in\boldsymbol{R}^{n-1}\times(\boldsymbol{C}\times i\boldsymbol{R}^{n-1}\setminus\{0\}/\boldsymbol{R}^+);\ \mathrm{Re}\,\zeta_1\geqq 0\}$ and

$$\iota^+:\ \bar{G}_+\setminus S_Y^*X\ni(0,\ x';\ \zeta_1 dx_1+i\eta' dx')\longmapsto(x';\ i\eta' dx')\in iS^*N\,,$$

$$\iota\ :\ S_N^*X\setminus S_Y^*X\ni(0,\ x';\ \zeta_1 dx_1+i\eta' dx')\longmapsto(x';\ i\eta' dx')\in iS^*N\,,$$

$$j\ :\ S_N^*X\setminus S_Y^*X\hookrightarrow(S_N^*X\setminus S_Y^*X)\cup iS^*N\times\{\infty\}$$
$$=\{(0,\ x';\ \zeta_1,\ i\eta');\ \zeta_1\in\boldsymbol{P}^1=\boldsymbol{C}\cup\{\infty\}\}.$$

LEMMA 2.1. *Setting* $K_\delta=\{(0,\ x';\ \zeta_1,\ i\eta')\in S_N^*X;\ \delta|\zeta_1|\geqq|\eta'|\}$ *for every* $\delta>0$, *we define the sheaf* $\hat{\mathcal{C}}_{N|M_+}(\delta)$ *on* iS^*N *by*

$$\hat{\mathcal{C}}_{N|M_+}(\delta)=\iota_*^+\mathcal{C}_{M_+|X}\cap(\iota|_{K_\delta})_*\mathcal{C}_{N|X}/\iota_*\mathcal{C}_{N|X}\,.$$

Then $\hat{\mathcal{C}}_{N|M_+}(\delta)$ *is a soft subsheaf of* $\hat{\mathcal{C}}_{N|M_+}$ *and*

$$\varinjlim_{\delta\to+0}\hat{\mathcal{C}}_{N|M_+}(\delta)=\hat{\mathcal{C}}_{N|M_+}\,. \tag{14}$$

PROOF. The softness follows directly from the proof of the softness of $\hat{\mathcal{C}}_{N|M_+}$ (Theorem 2.1.12 in [5]). The other statements are trivial.

LEMMA 2.2. *Let V be a proper convex open cone in* $\boldsymbol{R}^{n-1}$ *and* $f(x)$ *be a germ of* $\hat{\mathcal{B}}_{N|M_+}$ *at* $(0,\ x_0')\in N$. *Suppose that* $f(x)$ *represents a section of* $\hat{\mathcal{C}}_{N|M_+}(\delta)$ *on* $\{x'=x_0'\}\times i\boldsymbol{S}^{n-2}$ *with support in* $\{(x';\ i\eta')\in iS^*N;\ \eta'\in V\}$. *Then there exists the unique holomorphic function* $F(z)$ *defined in*

$$\{z\in\boldsymbol{C}^n;\ |z_1|+|z'-x_0'|<\varepsilon,\ \mathrm{Im}\,z'\in\mathrm{int}(V^\circ),$$
$$\sqrt{|\mathrm{Im}\,z_1|^2+(-\mathrm{Re}\,z_1)_+^2}<\delta\cdot\inf\{\langle\mathrm{Im}\,z',\ \eta'\rangle;\ |\eta'|=1,\ \eta'\in V\}\} \tag{15}$$

for some $\varepsilon>0$ *such that* $f(x)=F(x_1,\ x'+i0\cdot\mathrm{int}(V^\circ))$ *holds near* $(0,\ x_0')$.

PROOF. Note that the dual cone Γ° of $\Gamma=\{(\zeta_1,\ i\eta')\in\boldsymbol{C}\times i\boldsymbol{R}^{n-1};\ \eta'\in V,\ \delta|\zeta_1|<|\eta'|\}$ is equal to $\{(w_1,\ iv')\in\boldsymbol{C}\times i\boldsymbol{R}^{n-1};\ -\mathrm{Re}(w_1\zeta_1)+\langle v',\ \eta'\rangle\geqq 0$ for every $(\zeta_1,\ i\eta')\in\Gamma\}=\{(w_1,\ iv')\in\boldsymbol{C}\times i\boldsymbol{R}^{n-1};\ v'\in V^\circ,\ |w_1|\leqq\delta\cdot\inf_{|\eta'|=1,\ \eta'\in V}\langle v',\ \eta'\rangle\}$. Then the lemma follows from the arguments similar to the proof of Lemma 2.2 in [6].

The following lemma is a special case of Theorem 2.4 (cf. [8]).

LEMMA 2.3. *Let $F(z)$ be a holomorphic function defined in $D(0, \Gamma; R)$, where Γ is a proper convex open cone in $\mathbf{R}^{n-1}$ and $R>0$* (see (1)). *Let $g(x)$ be a continuous function defined in $\{x_1 \geqq 0, |x_1|+|x'| \leqq R\}$ such that g is of C^{n+1}-class in $\{x_1>0, |x_1|+|x'|<R\}$. Suppose that the boundary value $F(x_1, x'+i0)$ coincides with $g(x)$ on $\{x_1>0, |x_1|+|x'|<R\}$ as a hyperfunction. Define the subset $E(V; C, \mu)$ of $\mathbf{C}^n$ by*

$$(16) \qquad E(V; C, \mu)=\{z \in \mathbf{C}^n;\ |z_1|+|\operatorname{Im} z'| \leqq \mu,\ \operatorname{Im} z' \in V,\ \operatorname{Re} z_1 \geqq -C|\operatorname{Im} z'|^2,$$
$$|\operatorname{Im} z_1| \leqq C(\operatorname{Re} z_1+C|\operatorname{Im} z'|^2)\cdot|\operatorname{Im} z'|\},$$

where V is a cone in $\mathbf{R}^{n-1}$ and C, μ are positive numbers. Then, for every $C>0$, every proper convex compact subcone V of Γ and every compact subset K of $\{x' \in \mathbf{R}^{n-1}; |x'|<R\}$, there exists a small positive number μ such that

$$\sup\{|F(z)|;\ z \in E(V; C, \mu),\ \operatorname{Re} z' \in K\}<+\infty .$$

PROOF. Take C, V, K as above. Then there exists a small positive number μ $(<1/2C)$ such that the set

$$\{((1+Csw)t+Cw^2,\ x'+wy') \in \mathbf{C} \times \mathbf{C}^{n-1};\ x' \in K,\ y' \in V \cap \mathbf{S}^{n-2},\ 0 \leqq t \leqq \mu,$$
$$-1 \leqq s \leqq 1, \text{ and } w \in \mathbf{C} \text{ such that } |\operatorname{Re} w| \leqq 2\mu,\ 0<\operatorname{Im} w \leqq \mu\}$$

is contained in the convex subset

$$D'=\{z \in \mathbf{C}^n;\ |z_1|+|z'|<R,\ \operatorname{Im} z' \in \Gamma,$$
$$|\operatorname{Im} z_1|+(-\operatorname{Re} z_1)_+<R \cdot \inf_{\eta' \in \Gamma^\circ \cap S^{n-2}} \langle \operatorname{Im} z', \eta' \rangle\}$$

of $D(0, \Gamma; R)$. Therefore, for every $(x', y') \in K \times (V \cap \mathbf{S}^{n-2})$ and every $(t, s) \in (0, \mu] \times [-1, 1]$, $G(w)=F((1+Csw)t+Cw^2, x'+wy')$ is holomorphic in $\{w \in \mathbf{C}; |\operatorname{Re} w| \leqq 2\mu, 0<\operatorname{Im} w \leqq \mu\}$. Clearly, the boundary value $G(u+i0)$ coincides with $g((1+Csu)t+Cu^2, x'+uy')$ as a hyperfunction of u on $\{u \in \mathbf{R}; |u|<2\mu\}$ because $t>0$. Since $g((1+Csu)t+Cu^2, x'+uy')$ is a continuous function, $G(w)$ is continuous in $\{w \in \mathbf{C}; |\operatorname{Re} w|<2\mu, 0 \leqq \operatorname{Im} w \leqq \mu\}$. In particular, by the maximum principle we have

$$(17) \qquad \sup\{|G(w)|;\ 0<\operatorname{Im} w \leqq \mu,\ |\operatorname{Re} w| \leqq \mu\}$$
$$\leqq \max\{\sup\{|g(x)|;\ x_1 \geqq 0,\ |x_1|+|x'| \leqq R\},$$
$$\sup\{|G(w)|;\ |\operatorname{Re} w| \leqq \mu \text{ and } \operatorname{Im} w=\mu,$$
$$\text{or } |\operatorname{Re} w|=\mu \text{ and } 0<\operatorname{Im} w \leqq \mu\}\}.$$

Note that, for a sufficiently small number $\delta>0$, the inclusions

$$H=\{((1+Csw)t+Cw^2,\ x'+wy')\in \boldsymbol{C}\times\boldsymbol{C}^{n-1};\ (x',\ y')\in K\times(V\cap \boldsymbol{S}^{n-2}),\ 0\leqq t\leqq \mu,$$
$$-1\leqq s\leqq 1 \text{ and } w\in\boldsymbol{C} \text{ such that } |\mathrm{Re}\, w|=\mu,\ 0<\mathrm{Im}\, w\leqq\delta\}$$
$$\subset\{z\in\boldsymbol{C}^n;\ \mathrm{Re}\, z\in U,\ \mathrm{Im}\, z\in W,\ 0<|\mathrm{Im}\, z|\leqq\lambda\}\subset D'$$

hold. Here λ is some positive number, U is some relatively compact open subset of $\{x_1>0,\ |x_1|+|x'|<R\}$ and W is some compact proper convex cone in $\boldsymbol{R}^n$. Now we claim that $F(z)$ is continuous on $\{\mathrm{Re}\, z\in U,\ \mathrm{Im}\, z\in \overline{W}$ and $0\leqq|\mathrm{Im}\, z|\leqq\lambda\}$. In fact, since the boundary value $g(x)$ of $F(z)$ is a C^{n+1}-function in a neighborhood of $\overline{U}$, this claim follows directly from the formula concerning the curvilinear wave decomposition of δ-function (cf. § 1 of [4]). Hence $F(z)$ is bounded on H because $\overline{H}\cap\boldsymbol{R}^n$ is compactly contained in U. Further, since the set

$$\{((1+Csw)t+Cw^2,\ x'+wy');\ (x',\ y')\in K\times(V\cap\boldsymbol{S}^{n-2}),\ 0\leqq t\leqq\mu,\ -1\leqq s\leqq 1, \text{ and}$$
$$w\in\boldsymbol{C} \text{ such that } |\mathrm{Re}\, w|=\mu \text{ and } \delta\leqq\mathrm{Im}\, w\leqq\mu, \text{ or } |\mathrm{Re}\, w|\leqq\mu \text{ and } \mathrm{Im}\, w=\mu\}$$

is compactly contained in $D(0,\ \Gamma;\ R)$, we know by (17) that

$$\sup\{|F((1+Csw)t+Cw^2,\ x'+wy')|;\ (x',\ y')\in K\times(V\cap\boldsymbol{S}^{n-2}),\ 0\leqq t\leqq\mu,\ -1\leqq s\leqq 1$$
$$\text{and } w\in\boldsymbol{C} \text{ such that } |\mathrm{Re}\, w|\leqq\mu,\ 0<\mathrm{Im}\, w\leqq\mu\}<+\infty.$$

In particular, putting $w=iv\in i\boldsymbol{R}$, we have

$$\sup\{|F((1+iCsv)t-Cv^2,\ x'+ivy')|;\ (x',\ y')\in K\times(V\cap\boldsymbol{S}^{n-2}),\ 0\leqq t\leqq\mu,$$
$$-1\leqq s\leqq 1 \text{ and } 0<v\leqq\mu\}<+\infty.$$

This completes the proof.

THEOREM 2.4. *Every extensible mild distribution $f(x)$ at $x=0$ has the following expression:*

$$f(x)=\sum_{j=1}^{k}F_j(x_1,\ x'+i0\Gamma_j)$$

on $\{x_1>0,\ |x_1|+|x'|<\varepsilon\}$ for some $\varepsilon>0$. Here each $F_j(z)$ $(1\leqq j\leqq k)$ is holomorphic in $D(0,\ \Gamma_j;\ \varepsilon)$ for some proper convex open cone $\Gamma_j\subset\boldsymbol{R}^{n-1}$, and it satisfies the following estimate: There exists a positive constant s such that, for every $C>0$,

$$\sup\{|F_j(z)|\cdot|\mathrm{Im}\, z'|^s;\ z\in E(\Gamma_j;\ C,\ \mu),\ |\mathrm{Re}\, z'|<\varepsilon/2\}<+\infty \tag{18}$$

holds with some small $\mu>0$ (as for $E(\Gamma;\ C,\ \mu)$, see (16)).

PROOF. Let $\varphi(x)$ be an extensible mild distribution at $x=0$. Then we can find a $C^{l(n)+2n+1}$-function $f(x)$ defined in $\Omega=\{x\in\boldsymbol{R}^n;\ |x_1|<2R,\ |x'|<2R\}$ satisfying

$$\varphi(x)=P(D)f(x) \quad \text{on} \quad \Omega,$$

where $P(D)$ is an elliptic differential operator with constant coefficients; $l(n)$ is the integer depending only on n which appears later. Since $\varphi(x)$ is mild at $x=0$

from $x_1>0$ and $\{x_1=0\}$ is non-characteristic with respect to $P(D)$, $f(x)$ is also mild at $x=0$ from $x_1>0$ (cf. Definition 2.1.1 in [5]). Considering (14), we may suppose that, for some small number $\delta>0$, $f(x)$ represents a section of $\mathring{C}_{N|M_+}(\delta)$ on $\{(x'; i\eta')\in iS^*N;\ |x'|<2R\}$ $(N=\{x_1=0\})$ and that

$$\mathrm{SS}(f)\cap iS^*\Omega\cap\{x_1>0\}\subset\{\delta|\eta_1|<|\eta'|\}$$

(recall the definition of mildness). Set the constant

$$\varepsilon=\min\{\delta/16\sqrt{16R^2+1},\ 1/2R\}. \tag{19}$$

By the arguments in the proof of the flabbiness of $\mathcal{C}$ (see Ch. III Corollary 2.1.5 of [10]. Use the softness of $\mathcal{D}'$ in place of the flabbiness of $\mathcal{B}$. Then, $l(n)$ is the total derivative loss in this process), we can modify $f(x)$ to $\tilde{f}(x)$ in the following way: $\tilde{f}(x)$ is a C^{2n+1}-function defined in $U=\{|x_1|<R,\ |x'|<7R/4\}$ satisfying $f=\tilde{f}+g_0$ on $U\cap\{0\leqq x_1<\varepsilon R^2/2\}$ with some analytic function g_0 on $\{0\leqq x_1<\varepsilon R^2/2,\ |x'|<7R/4\}$, and

$$\mathrm{SS}(\tilde{f})\cap iS^*U\cap\{x_1>0\}\subset\{\delta|\eta_1|<|\eta'|\}\setminus\{x_1=\varepsilon R^2,\ |x'|=R\}.$$

Let $S^{n-2}=A_1\cup\cdots\cup A_k$ be a decomposition of S^{n-2} into so small measurable subsets $A_1, \cdots, A_k$ that

$$\sup_{v'\in A_j^\circ}\big(\inf_{\xi'\in A_j}\langle v',\xi'\rangle/|v'|\cdot|\xi'|\big)>1/2. \tag{20}$$

for every $j=1, \cdots, k$. Then, consider the following integral:

$$g_j(x)=\int_{A_j}d\sigma(\xi')\int_{\{u'\in R^{n-1};|u'|\leq R\}}K_\alpha(x'-u',\xi')\tilde{f}(x_1+\varepsilon|x'-u'|^2,u')du' \tag{21}$$

for $j=1, \cdots, k$, where $\alpha>0$ and

$$K_\alpha(x',\xi')=\frac{(n-2)!}{(-2\pi i)^{n-1}}\cdot\frac{(1-i\alpha\langle x',\xi'\rangle)^{n-3}\cdot\{1-i\alpha\langle x',\xi'\rangle-\alpha^2(|x'|^2-\langle x',\xi'\rangle^2)\}}{\{\langle x',\xi'\rangle+i\alpha(|x'|^2-\langle x',\xi'\rangle^2)+i0\}^{n-1}}$$

is a curvilinear wave decomposition of $\delta(x')$ (cf. Ch. III Corollary 2.1.5 of [10], and §1 of [4]). The g_j's become C^{n+1}-functions defined in $\{(x_1, x')\in \boldsymbol{R}\times\boldsymbol{R}^{n-1};\ |x_1|<\mu,\ |x'|<\mu\}$ for some $\mu>0$ because $\tilde{f}(x)$ is real analytic in a neighborhood of $\{x_1=\varepsilon R^2,\ |x'|=R\}$. Further they satisfy

$$f(x)=\sum_{j=1}^{k}g_j(x)+g_0(x) \tag{22}$$

on $\{0\leqq x_1<\mu,\ |x'|<\mu\}$ for some smaller $\mu>0$.

Next, we will show that each $g_j(x)$ can be written as the boundary value of a holomorphic function $G_j(z)$. To do so, we must express the integral in the definition of g_j by using suitable defining functions of $\tilde{f}(x)$. Choose an open covering $\{V_q\}_{q=1,\dots,p}$ of S^{n-2} such that each V_q is a proper convex open cone in $\boldsymbol{R}^{n-1}$ and that V_q° contains a proper convex open cone $\Gamma_q\subset\boldsymbol{R}^{n-1}$ satisfying

$$\inf\{\langle\eta',v'\rangle;\ \eta'\in V_q,\ v'\in\Gamma_q,\ |\eta'|=|v'|=1\}>1/2. \tag{23}$$

Since $\tilde{f}(x)$ represents a section of $\mathring{C}_{N|M_+}(\delta)$ on $\{(x'; i\eta'); |x'|<7R/4\}$ by the softness of $\mathring{C}_{N|M_+}(\delta)$ we can decompose $\tilde{f}(x)$ into the sum $f_1(x)+\cdots+f_p(x)$ of sections $f_1(x), \cdots, f_p(x)$ of $\mathring{\mathcal{B}}_{N|M_+}$ on $\{|x'|<3R/2\}$ such that each f_q $(1\leqq q\leqq p)$ represents a section of $\mathring{C}_{N|M_+}(\delta)$ on $\{(x'; i\eta'); |x'|<3R/2\}$ with support in $\{\eta'\in V_q\}$. Let $h_q(x)$ be the section of $\mathcal{C}_M$ on $\{(x; i\eta)\in iS^*M; x_1\leqq\nu, |x'|\leqq 4R/3\}$ induced by $\mathrm{ext}(f_q)=f_q(x)Y(x_1)$ for every $q=1, \cdots, p$ with some small number $\nu>0$ $(\nu<\varepsilon R^2/2)$. Then, since $\mathrm{Supp}(h_q)\cap\{0<x_1\leqq\nu\}\subset\{\eta'\in V_q, \delta|\eta_1|<|\eta'|\}$ and $[\tilde{f}(x)Y(x_1)]=h_1+\cdots+h_p$ holds as a section of $\mathcal{C}_M$ on $\{x_1\leqq\nu, |x'|\leqq 4R/3\}$ for some smaller $\nu>0$, we can extend each $h_q(x)$ to a section of $\mathcal{C}_M$ on $\{(x; i\eta); x_1<R, |x'|<4R/3\}$ for $q=1, \cdots, p$ as follows: $[\tilde{f}(x)Y(x_1)]=h_1+\cdots+h_p$ on $\{(x; i\eta); x_1<R, |x'|<4R/3\}$ and

$$\mathrm{Supp}(h_q)\cap\{0<x_1<R, |x'|<4R/3\}\subset\{\eta'\in V_q, \delta|\eta_1|<|\eta'|\}\setminus\{x_1=\varepsilon R^2, |x'|=R\}$$

for every $q=1, \cdots, p$. Take a hyperfunction $H_q(x)$ on $\{x\in M; x_1<R, |x'|<4R/3\}$ satisfying $h_q(x)=[H_q(x)]$ on $\{x_1<R, |x'|<4R/3\}$ for every $q=1, \cdots, p$. Then we know that each $H_q(x)$ defines a section of $\mathring{C}_{N|M_+}(\delta)$ on $\{(x'; i\eta'); |x'|<4R/3\}$ with support in $\{\eta'\in V_q\}$, that

$$\mathrm{SS}(H_q)\cap\{0<x_1<R, |x'|<4R/3\}\subset\{\eta'\in V_q, \delta|\eta_1|<|\eta'|\}\setminus\{x_1=\varepsilon R^2, |x'|=R\},$$

and that $H_0(x)=\tilde{f}(x)Y(x_1)-\sum_{q=1}^{p}H_q(x)$ is analytic in $\{x_1<R, |x'|<4R/3\}$. After modifying H_1, we may assume $H_0\equiv 0$. Therefore $\tilde{f}(x)=\sum_{q=1}^{p}H_q(x)$ holds in $\{0<x_1<R, |x'|<4R/3\}$. Further, by Lemma 2.2 and (23), $H_q(x)$ can be expressed in $\{0<x_1<2R/3, |x'|<5R/4\}$ as the boundary value of the holomorphic function $H_q(z)$ defined on

$$\text{(24)}\qquad D_q=\Big\{z\in \boldsymbol{C}^n; |\mathrm{Im}\, z'|<\lambda, \mathrm{Re}\, z_1<2R/3, |\mathrm{Re}\, z'|<5R/4, \mathrm{Im}\, z'\in\Gamma_q, \sqrt{|\mathrm{Im}\, z_1|^2+(-\mathrm{Re}\, z_1)_+^2}<\frac{1}{2}\delta|\mathrm{Im}\, z'|\Big\}$$

$$\cup\{|\mathrm{Im}\, z'|+|z_1-\varepsilon R^2|<\lambda, R-\lambda<|\mathrm{Re}\, z'|<R+\lambda\}$$

for every $q=1, \cdots, p$ with some small $\lambda>0$. Hence, $g_j(x)$ is written as the boundary value of

$$\text{(25)}\quad G_j(z)=\sum_{q=1}^{p}\int_{A_j}d\sigma(\xi')\Big\{\int_{\{|\mathrm{Re}\, w'|\leqq R,\ \mathrm{Im}\, w'=v_q'\}}K_\alpha(z'-w', \xi')H_q(z_1+\varepsilon(z'-w')^2, w')dw'$$
$$+\int_{\{|\mathrm{Re}\, w'|=R,\ \mathrm{Im}\, w'\in[0,1]\cdot v_q'\}}K_\alpha(z'-w', \xi')H_q(z_1+\varepsilon(z'-w')^2, w')dw'\Big\}$$

for every $j=1, \cdots, k$ (see (21), $(z'-w')^2=\sum_{s=2}^{n}(z_s-w_s)^2$), where v_q' is a small vector in Γ_q. In fact, the second term of (25) is holomorphic at $(z_1, z')=0$ if v_q' is sufficiently small, and the first term of (25) is holomorphic in the interior of

$$\Omega_j=\bigcap_{|u'|\leq R}\{(z_1,z')\in \boldsymbol{C}\times\boldsymbol{C}^{n-1};\ \mathrm{Re}\,z_1+\varepsilon|\mathrm{Re}\,z'-u'|^2-\varepsilon|\mathrm{Im}\,z'-v_q'|^2<2R/3,$$
$$\sqrt{(\mathrm{Im}\,z_1+2\varepsilon\langle\mathrm{Re}\,z'-u',\ \mathrm{Im}\,z'-v_q'\rangle)^2+(-\mathrm{Re}\,z_1-\varepsilon|\mathrm{Re}\,z'-u'|^2+\varepsilon|\mathrm{Im}\,z'-v_q'|^2)_+^2}$$
$$<\frac{1}{2}\delta|v_q'|,\ \langle\mathrm{Im}\,z'-v_q',\ \xi'\rangle-\alpha(|\mathrm{Im}\,z'-v_q'|^2-\langle\mathrm{Im}\,z'-v_q',\ \xi'\rangle^2)>0$$
$$\text{for every } q=1,\cdots,p \text{ and every } \xi'\in A_j\cap\boldsymbol{S}^{n-2}\}.$$

Note that

$$\sqrt{(\mathrm{Im}\,z_1+2\varepsilon\langle\mathrm{Re}\,z'-u',\ \mathrm{Im}\,z'-v_q'\rangle)^2+(-\mathrm{Re}\,z_1-\varepsilon|\mathrm{Re}\,z'-u'|^2+\varepsilon|\mathrm{Im}\,z'-v_q'|^2)_+^2}$$
$$\leqq\sqrt{|\mathrm{Im}\,z_1|^2+(-\mathrm{Re}\,z_1)_+^2}+\varepsilon\sqrt{4(R+|\mathrm{Re}\,z'|)^2+|\mathrm{Im}\,z'-v_q'|^2}\cdot|\mathrm{Im}\,z'-v_q'|$$
$$\leqq\sqrt{|\mathrm{Im}\,z_1|^2+(-\mathrm{Re}\,z_1)_+^2}+\frac{\delta}{16}\cdot(|\mathrm{Im}\,z'|+|v_q'|)$$

if $|\mathrm{Re}\,z'|\leqq R$ and $|\mathrm{Im}\,z'-v_q'|\leqq 1$ (recall (19)). Therefore, taking $|v_q'|=|\mathrm{Im}\,z'|/3>0$, we see that $G_j(z)$ is holomorphic in

$$S_j=\{|z|<r,\ \sqrt{|\mathrm{Im}\,z_1|^2+(-\mathrm{Re}\,z_1)_+^2}<\delta|\mathrm{Im}\,z'|/12,\ \mathrm{Im}\,z'\in W_j\} \tag{26}$$

for every $j=1,\cdots,k$ with some small $r>0$. Here,

$$W_j=\{v'\in\boldsymbol{R}^{n-1};\ \langle v',\xi'\rangle>|v'|/2 \text{ for every } \xi'\in A_j\cap\boldsymbol{S}^{n-2}\}$$

is a non-void proper convex open cone (recall (20)). On the other hand, since the boundary value $g_j(x)$ on $\{x_1>0,\ |x'|<r\}$ of $G_j(z)$ is of C^{n+1}-class in a neighborhood of $\{x_1\geqq 0,\ |x'|<r\}$, we can apply Lemma 2.3 to the $G_j(z)$'s $(1\leqq j\leqq k)$. Recall that $\varphi(x)=P(D)f(x)=\sum_{j=1}^{k}P(D)g_j(x)+P(D)g_0(x)$ on $\{0<x_1<\mu,\ |x'|<\mu\}$ and that g_0 is analytic on $\{0\leqq x_1<\mu,\ |x'|<\mu\}$ (see (22)). Hence we know by the Cauchy estimate that $P(D)G_j(z)$ satisfies the estimate of type (18) for every $j=1,\cdots,k$. This completes the proof.

As a direct corollary, we obtain the following theorem concerning coincidence of the two definitions of boundary values for extensible mild distributions. Kaneko has obtained a similar result as for hyperfunction solutions of differential equations ([1]). The present author generalized his result to the case of mild hyperfunctions ([5]). These results are, however, essentially connected with the properness of the support along the boundary.

THEOREM 2.5. *Let $f(x)$ be a distribution defined in $\{x\in\boldsymbol{R}^n;\ |x_1|<R,\ |x'|<R\}$ with some $R>0$. Suppose that $f(x)$ is mild on $\{x\in\boldsymbol{R}^n;\ x_1=0,\ |x'|<R\}$ from $x_1>0$. Then, for every $s=0,1,2,\cdots$, $D_1^sf(\varepsilon,x')$ converges in $\mathcal{D}'(\{|x'|<R\})$ to $\mathrm{Trace}(D_1^sf)=D_1^sf(+0,x')$ as $\varepsilon\to+0$, and $f(x)\mathrm{Y}(x_1-\varepsilon)$ converges in $\mathcal{D}'(\{x\in\boldsymbol{R}^n;\ x_1<R,\ |x'|<R\})$ to $\mathrm{ext}(f)=f(x)\mathrm{Y}(x_1)$ as $\varepsilon\to+0$.*

PROOF. The question being local with respect to x', we easily know the theorem from the expression of $f(x)$ obtained in Theorem 2.4.

At the last of this section, we give an example of extensible mild distributions. The defining function of this example blows up in $\{\mathrm{Re}\, z_1<0\}$.

EXAMPLE 2.6. We construct the defining holomorphic function $F(z_1, z_2)$ of an extensible mild distribution. Recall the following entire function studied by Morimoto and Yoshino in [9]:

$$T(z)=\frac{-1}{2\pi i}\int_{\Gamma_a}\frac{\exp(\exp(w^2))}{z-w}dw,$$

where $\Gamma_a=\{w=u+iv\in \boldsymbol{C};\ u=a$ and $|v|\leqq\pi/2a$, or $u\geqq a$ and $|uv|=\pi/2\}$ with a large positive number a ($>\mathrm{Re}\, z$, see Figure 1).

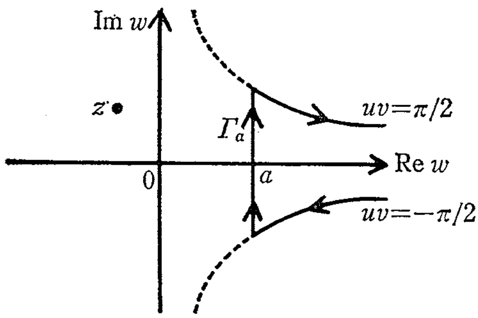

Figure 1. Some properties of mild hyperfunctions and an application to propagation of microanalyticity in boundary value problems.

Because $|\exp(\exp(w^2))|=\exp(\exp(u^2-v^2)\cdot\cos 2uv)$, this integral is well defined for every $z\in \boldsymbol{C}$. Further, we know that, for every $\lambda>0$,

(27) $$|T(z)|=O(|z|^{-1})\quad \textit{as}\ |z|\to\infty\ \textit{satisfying}\ |\arg z|\geqq\lambda,$$

and that

(28) $$T(x)=\exp(\exp(x^2))+O(1)\quad \textit{as real}\ x\to+\infty.$$

Firstly, we define a holomorphic function $G(\zeta, z_2)$ by

(29) $$G(\zeta, z_2)=T(-e^{\pi i/4}\cdot\zeta^{-1}z_2^{-1/2})\ \text{on}\ \{\zeta\in \boldsymbol{P}^1\setminus\{0\},\ \mathrm{Im}\, z_2>0\}$$

($0<\arg z_2<\pi$). Then, $F(z_1, z_2)$ is introduced as the inverse quantized Legendre transform of G:

(30) $$F(z_1, z_2)=\oint_{|\zeta|=r}G(\zeta, z_2-i\zeta z_1)d\zeta,$$

where r is a small positive number (cf. p. 370 in [5]). $F(z_1, z_2)$ is holomorphic in $\{(z_1, z_2)\in \boldsymbol{C}^2;\ \mathrm{Im}\, z_2>0\}$, and satisfies

(31) $$\sup\{|F(x_1, z_2)|\cdot|\mathrm{Im}\, z_2|;\ 0<x_1<A,\ \mathrm{Im}\, z_2>0,\ |z_2|<A\}<\infty$$

for every $A>0$. Hence $F(x_1, x_2+i0)$ is an extensible mild distribution. On the other hand, we have

$$|F(-Ct^{3/2-\varepsilon}, it)| \geqq C' \exp(\exp(13C^2/2t^{2\varepsilon})) \quad as \quad t \to +0 \tag{32}$$

for every $C, \varepsilon>0$ with some positive constant C' depending only on C and ε (compare with the estimate in Lemma 2.3).

PROOF. Fix a point $(x_1, z_2) \in \{(x_1, z_2) \in \boldsymbol{R} \times \boldsymbol{C};\ 0<x_1<A,\ \operatorname{Im} z_2>0,\ |z_2|<A\}$. Putting $S(w)=T(w)-T(0)-T'(0)w$, we know from (27) that

$$|S(w)| \leqq B|w|^2/(1+|w|) \text{ for every } w \in \boldsymbol{C} \text{ such that } |\arg w| \geqq \frac{\pi}{4} \tag{33}$$

with some positive constant B. Then, we have

$$F(x_1, z_2)+2\pi i e^{\pi i/4} T'(0) z_2^{-1/2} = \oint_{|\zeta|=r} S(-e^{\pi i/4}\zeta^{-1}(z_2-i\zeta x_1)^{-1/2})d\zeta$$

$$=\Big(\int_\delta^\tau+\int_{-\tau}^{-\delta}\Big)S(-e^{\pi i/4}(it)^{-1}(z_2+tx_1)^{-1/2})i\,dt$$

$$+\int_0^\pi S(-i\delta^{-1}e^{(\pi/4-\theta)i}(z_2-\delta x_1 e^{i\theta})^{-1/2})\delta e^{i\theta}d\theta$$

$$+\int_0^\pi S(i\tau^{-1}e^{(\pi/4-\theta)i}(z_2+\tau x_1 e^{i\theta})^{-1/2})(-\tau)e^{i\theta}d\theta\,,$$

where δ, τ are arbitrary positive numbers satisfying $0<\delta<\tau$, $\tau>Ax_1^{-1}$ and $\delta<x_1^{-1}\operatorname{Im} z_2$. Take $\tau=2Ax_1^{-1}$ and $\delta=\operatorname{Im} z_2/2A$. Then, since $|z_2+\tau x_1 e^{i\theta}| \geqq A$ and $S(0)=0$, the third term is estimated by a constant depending only on A. Further, by using (33), we easily deduce

$$|\textit{the first term}+\textit{the second term}| \leqq C|\operatorname{Im} z_2|^{-1},$$

where C is a positive constant depending only on A. Thus we get (31). In the next place, to show (32), we transform (30) as follows:

$$F(-\alpha t, it)=\oint_{|\zeta|=r} T(-\zeta^{-1}t^{-1/2}(1+\alpha\zeta)^{-1/2})d\zeta$$

$$=\frac{1}{2\pi i}\oint_{|\zeta|=r} d\zeta \int_{-i\infty+R}^{i\infty+R} \frac{T(w)}{w+\zeta^{-1}t^{-1/2}(1+\alpha\zeta)^{-1/2}}dw\,,$$

where α, $t>0$, $0<r<\alpha^{-1}$ and $R>r^{-1}t^{-1/2}(1-\alpha r)^{-1/2}$ (use (27)). Hence,

$$F(-\alpha t, it)=\frac{1}{2\pi i}\int_{-i\infty+R}^{i\infty+R}\frac{T(w)}{w}dw \oint_{|\zeta|=r}\frac{\zeta(1+\alpha\zeta)^{1/2}}{\zeta(1+\alpha\zeta)^{1/2}+(\sqrt{t}\,w)^{-1}}d\zeta\,.$$

By Rouché's theorem we know that $\zeta(1+\alpha\zeta)^{1/2}+(\sqrt{t}\,w)^{-1}$ has only one zero in $\{|\zeta|<r\}$ if $|w| \geqq R > r^{-1}t^{-1/2}(1-\alpha r)^{-1/2}$. Denoting by $\zeta(w)$ this zero, we get

$$F(-\alpha t, it)=\int_{-i\infty+R}^{i\infty+R}\frac{T(w)}{w}dw\cdot\frac{\zeta(w)(1+\alpha\cdot\zeta(w))}{1+3\alpha\cdot\zeta(w)/2}\,.$$

Easily to see, $\zeta(w)$ is holomorphic in $C\backslash(-\infty, \sqrt{27\alpha^2/4t}]$ and satisfies $\overline{\zeta(w)}=\zeta(\bar{w})$. Therefore, if $27\alpha^2/4t>1$,

$$F(-\alpha t, it)=\Big(\int_{1+i0}^{1+i\infty}+\int_{1-i\infty}^{1-i0}\Big)\frac{\zeta(w)(1+\alpha\cdot\zeta(w))}{1+3\alpha\cdot\zeta(w)/2}\cdot\frac{T(w)}{w}dw$$

$$+\int_1^{\sqrt{27\alpha^2/4t}}\frac{2}{3}(\zeta(u-i0)-\zeta(u+i0))\Big(1+\frac{2}{9\alpha^2}\Big|\zeta(u+i0)+\frac{2}{3}\alpha^{-1}\Big|^{-2}\Big)\frac{T(u)}{u}du\,.$$

Note that $\frac{1}{i}(\zeta(u+i0)-\zeta(u-i0))>0$ on $u\in(1, \sqrt{27\alpha^2/4t})$. Then, from (27) and (28), we easily obtain estimate (32) (remark that $\zeta(w)$ is written as $\alpha^{-1}\tilde{\zeta}(\alpha^{-1}\sqrt{t}\,w)$, where $\tilde{\zeta}(w)$ does not depend on α and t).

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan

Present address
Department of Mathematics
Faculty of Science
Tokyo Metropolitan University
Fukazawa, Meguro-ku, Tokyo
158 Japan