Microlocal analysis of partial differential operators with irregular singularities

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Summary. We denote the variables in $\boldsymbol{R}^{n+1}$ by $x=(x_0, x')$, where $x_0 \in \boldsymbol{R}$ and $x' \in \boldsymbol{R}^n$. We consider partial differential operators of the form

$$P(x, \partial/\partial x)=\sum_{|\alpha|\leq m} a_\alpha(x) x_0^{\kappa(|\alpha|)}(\partial/\partial x)^\alpha,$$

where $\kappa(j)$ is some integer $\geqq 0$, $a_\alpha(x)$ is real analytic in a neighborhood of $x=0$, and $a_{(m,0,\cdots,0)}=1$. We define the irregularity $\iota \in [1, \infty)$ and the characteristic exponents $\lambda_1, \cdots, \lambda_{\kappa(m)} \in \boldsymbol{C}$ of the operator P at the point $\mathring{x}^*=(0; \sqrt{-1}, 0, \cdots, 0) \in \sqrt{-1}\, T^* \boldsymbol{R}^{n+1}$.

It will be proved that if $\iota > 1$ and all the characteristic exponents of P are distinct, then P is equivalent microlocally to the operator

$$\begin{array}{rccc} x_0^{\kappa(m)}: & \mathcal{C}_{\boldsymbol{R}^{n+1}} & \longrightarrow & \mathcal{C}_{\boldsymbol{R}^{n+1}} \\ & \Downarrow & & \Downarrow \\ & u & \longmapsto & x_0^{\kappa(m)} u \end{array}$$

in a neighborhood of $\mathring{x}^*$.

§ 0. Introduction.

An ordinary differential operator of the form

$$t^m (d/dt)^m + \sum_{j=0}^{m-1} a_j(t) t^j (d/dt)^j,$$

where $a_j(t)$ is real analytic in a neighborhood of $t=0$, is said to have a singular point of the first kind at $t=0$. It is well-known that such an operator has a regular singular point at $t=0$.

Kashiwara and Oshima [5] considered partial differential operators of analogous type. We denote the variables in $\boldsymbol{R}^{n+1}$ by $x=(x_0, x')$, where $x_0 \in \boldsymbol{R}$ and $x'=(x_1, \cdots, x_n) \in \boldsymbol{R}^n$. They called a partial differential operator written in the form

$$P=\sum_{|\alpha|\leq m} a_\alpha(x) x_0^{|\alpha|}(\partial/\partial x)^\alpha,$$

a partial differential operator with regular singularities along the hypersurface $N=\{x_0=0\}$, where $a_\alpha(x)$ is real analytic in a neighborhood of $x=0$ and $a_{(m,0,\cdots,0)}$

$=1$. They proved that in a neighborhood of $\mathring{x}^*=(0;\sqrt{-1},0,\cdots,0)\in\sqrt{-1}\,T^*\boldsymbol{R}^{n+1}$, such an operator is equivalent to the very simple operator x_0^m microlocally.

In this paper we consider partial differential operators of a more general type. Let $P(x,\partial/\partial x)$ be of the form

$$P(x,\partial/\partial x)=\sum_{|\alpha|\leq m}a_\alpha(x)x_0^{\kappa(|\alpha|)}(\partial/\partial x)^\alpha,$$

where $a_\alpha(x)$ is real analytic at $x=0$, $a_{(m,0,\dots,0)}=1$ and $\kappa(j)$ is some integer $\geqq 0$. After Aoki [3], we define the irregularity ι of $P(x,\partial/\partial x)$ by

$$\iota=\max\left\{\left(\max_{0\leq j\leq m-1}\frac{\kappa(m)-\kappa(j)}{m-j}\right),1\right\}.$$

If $\iota=1$, the method of Kashiwara and Oshima [5] is applicable and we can prove that P is equivalent to $x_0^{\kappa(m)}$ at $\mathring{x}^*$. Thus we consider only the case $\iota>1$. Such an investigation has been done only when $n=0$, i.e., when P is an ordinary differential operator with an irregular singular point at the origin. In fact Aoki [1] and Kashiwara [4] proved independently that if $n=0$, P is equivalent to the operator $x_0^{\kappa(m)}$ at $(0;\sqrt{-1})\in\sqrt{-1}\,T^*\boldsymbol{R}$. The purpose of this paper is to generalize this result to partial differential operators.

Assume that $\iota>1$ and $n\geqq 0$. In this case we say that the partial differential operator $P(x,\partial/\partial x)$ has *irregular singularities along the hypersurface* $\{x_0=0\}$. In the classical theory of an irregular singular point of an ordinary differential operator, situations become rather complicated unless all the characteristic exponents are distinct. Analogous difficulty arises in our microlocal analysis of irregular singularities of a partial differential operator. To avoid this difficulty, we consider the simplest case, i.e., the case where all the "characteristic exponents" are distinct.

From now on, we always assume that $\iota>1$. Now we define the characteristic exponents of $P(x,\partial/\partial x)$ to be the roots $\lambda_1,\cdots,\lambda_{\kappa(m)}$ of the algebraic equation

$$\lambda^{\kappa(m)}+\sum_{\pi(P)}a_{(j,0,\dots,0)}(0)\lambda^{\kappa(j)}=0,$$

where

$$\pi(P)=\left\{0\leqq j\leqq m-1;\ \frac{\kappa(m)-\kappa(j)}{m-j}=\iota\right\}.$$

We remark that, since we assume $\iota>1$, we have $\kappa(m)>\kappa(j)$ for $j\in\pi(P)$.

Now we have the following

THEOREM 1. *Assume that $\iota>1$ and that $\lambda_i\neq\lambda_j$, if $i\neq j$. Then we can construct holomorphic microlocal operators $Q_1,\cdots,Q_{\kappa(m)}$ defined at $\mathring{x}^*=(0;\sqrt{-1},0,\cdots,0)\in\sqrt{-1}\,T^*\boldsymbol{R}^{n+1}$ such that the following sequence*

$$0\longrightarrow\bigoplus_{\kappa(m)}\delta(x_0)\otimes\mathscr{B}_N\xrightarrow{(Q_1,\cdots,Q_{\kappa(m)})}\mathcal{C}_M\xrightarrow{P}\mathcal{C}_M\longrightarrow 0$$

is exact in the sense of sheaf theory at $\mathring{x}^*$.

To prove the above theorem, we need to consider a $\kappa(m)\times\kappa(m)$ matrix $x_0 I_{\kappa(m)}+A(x, D)$ of microdifferential operators of fractional order. Here the total symbol $\sigma(A)(x, \xi)$ of $A(x, D)$ satisfies

$$\frac{\partial\sigma(A)}{\partial x_0}(x, \xi)=0,$$

and, if $1\leqq\mu,\ \nu\leqq\kappa(m)$, the (μ, ν)-element $\sigma(A)_{(\mu,\nu)}(x, \xi)$ of $\sigma(A)(x, \xi)$ satisfies

$$|\sigma(A)(x, \xi)|<c|\xi_0|^{-1/t} \quad \text{if } |\xi'|<\varepsilon|\xi_0|$$

for some constants c and ε. Furthermore, we assume that $\sigma(A)(x, \xi)$ admits an asymptotic expansion

$$\sigma(A)(x, \xi)\sim\sum_{-p/q\geqq j\in(1/q)\mathbf{Z}} A_j(x, \xi),$$

the precise meaning of which will be explained in § 1, and here p and q are integers satisfying $1\leqq p<q$. Now we have the following.

THEOREM 2. *If the eigenvalues of $A_{-p/q}(x, \xi)$ are all distinct, there exist $\kappa(m)\times\kappa(m)$ matrices $E(x, D)$ and $F(x, D)$ of holomorphic microlocal operators defined at $\mathring{x}^*$ such that $E(x, D)F(x, D)=F(x, D)E(x, D)=I_{\kappa(m)}$ and that*

$$E(x, D)\{x_0 I_{\kappa(m)}+A(x, D)\}F(x, D)=x_0 I_{\kappa(m)}.$$

Theorem 2 says that $x_0 I_{\kappa(m)}+A(x, D)$ is equivalent to $x_0 I_{\kappa(m)}$, and from this fact we can prove Theorem 1 which says that $P(x, \partial/\partial x)$ is equivalent to $x_0^{\kappa(m)}$. (Compare the exact sequence in Theorem 1 with the following one:

$$0\longrightarrow\bigoplus_{j=0}^{\kappa(m)-1}\delta^{(j)}(x_0)\otimes\mathcal{B}_N\hookrightarrow\mathcal{C}_M\xrightarrow{x_0^{\kappa(m)}}\mathcal{C}_M\longrightarrow 0.)$$

Such a type of partial differential operators was also investigated in Nourrigat [6] in the category of distribution theory. He proved that under certain conditions such an operator is C^∞-hypoelliptic, i.e.,

$$u\in\mathcal{D}',\ Pu\in C^\infty\Rightarrow u\in C^\infty.$$

However we stress the fact that such an operator behaves completely differently in hyperfunction theory.

Now we explain the plan of this paper.

In § 1, we shall review some important facts concerning microdifferential operators of fractional order and the irregularity of a partial differential operator, which are useful for us.

In § 2, we shall introduce a matrix of microdifferential operators of fractional order, which will be transformed into the canonical form in § 3 and § 4.

In § 5, we shall return to the original scalar operator.

In § 3 and § 4, we need several estimates, the proof of which will be given in Appendix 1 and Appendix 2.

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§ 1. Preliminaries.

Let M be an $(n+1)$-dimensional real analytic manifold, X a complex neighborhood of M, and T^*X the cotangent vector bundle of X. We denote by $\mathcal{C}_M$ the sheaf of microfunctions on $T^*_M X \cong \sqrt{-1}\,T^*M$, and by $\mathcal{E}^R_X$ the sheaf of holomorphic microlocal operators on T^*X. The sheaf $\mathcal{C}_M$ (resp. $\mathcal{E}^R_X$) is often abbreviated to $\mathcal{C}$ (resp. $\mathcal{E}^R$). The sheaf $\mathcal{C}_M$ is canonically endowed with the structure of a left $\mathcal{E}^R_X|_{\sqrt{-1}T^*M}$-Module. Not only a usual microdifferential operator but also a microdifferential operator of fractional order is a section of $\mathcal{E}^R_X$. (See Sato-Kawai-Kashiwara [7] and Aoki [2].)

In our approach to irregular singularities of a partial differential operator, microdifferential operators of fractional order are very useful. Thus we recall some results concerning $\mathcal{E}^R_X$ proved in Aoki [2].

1. We denote the local coordinates of X by $x=(x_0, x')=(x_0, x_1, \cdots, x_n)$, and the dual variables of x by $\xi=(\xi_0, \xi')=(\xi_0, \xi_1, \cdots, \xi_n)$. Let $\mathring{x}^*=(0; \sqrt{-1}, 0, \cdots, 0)$ be a point in T^*X, and $A(x, D)$ be an element of $\mathcal{E}^R{}_{\mathring{x}^*}$. Aoki defined the symbol $\sigma(A)$ of A, which is a holomorphic function on

$$\Gamma_\varepsilon=\{(x, \xi)\in T^*X;\ |x|<\varepsilon,\ |\xi'|<\varepsilon|\xi_0|,\ |\mathrm{Re}\,\xi_0|<\varepsilon\,\mathrm{Im}\,\xi_0 \text{ and } \varepsilon|\xi|>1\}$$

for some constant $\varepsilon>0$. Furthermore $\sigma(A)$ satisfies

$$(1)\qquad \sup_{\Gamma_\varepsilon}|\sigma(A)(x, \xi)e^{-\delta|\xi|}|<\infty \qquad \text{for any } \delta>0.$$

Conversely, let $a(x, \xi)$ be a holomorphic function on Γ_ε for some constant $\varepsilon>0$, which satisfies

$$(2)\qquad \sup_{\Gamma_\varepsilon}|a(x, \xi)e^{-\delta|\xi|}|<\infty \qquad \text{for any } \delta>0.$$

Then there exists an element A of $\mathcal{E}^R{}_{\mathring{x}^*}$ such that the symbol $\sigma(A)$ of A is holomorphic on Γ_ε, and satisfies (1) and

$$(3)\qquad \sup_{\Gamma_\varepsilon}|a(x, \xi)-\sigma(A)(x, \xi)|\,e^{\delta|\xi|}<\infty \qquad \text{for some } \delta>0.$$

We write $\sigma(A)\equiv a$ if (3) is valid.

2. Let $i>0$ be an integer. For some $\varepsilon>0$ fixed, assume that there exists a sequence

$$\{a_j(x, \xi);\ j=\cdots, -2/i, -1/i, 0, 1/i, 2/i, \cdots\}$$

of holomorphic functions on Γ_ε. We say that $\sum_j a_j(x, \xi)$ is a formal symbol of fractional order if for any $\delta>0$ there exists some $C_\delta>0$ such that

(4) $$\sup_{\Gamma_\varepsilon}|a_j(x, \xi)|\leqq C_\delta \delta^j|\xi|^j/[j]! \qquad \text{for } j=1/i,\ 2/i,\ \cdots$$

and if for some $C>0$

(5) $$\sup_{\Gamma_\varepsilon}|a_j(x, \xi)|\leqq C^{j+1}|\xi|^j[-j]! \qquad \text{for } j=0,\ -1/i,\ \cdots,$$

where $[|j|]$ denotes the largest integer smaller than or equal to $|j|$. Let $\{a_j(x, \xi);\ j\in(1/i)\boldsymbol{Z}\}$ be a formal symbol of fractional order. We can find a function $a(x, \xi)$ holomorphic on Γ_ε such that for any $\delta>0$

$$\sup_{\Gamma_\varepsilon}|a(x, \xi)e^{-\delta|\xi|}|<\infty$$

and for some δ, $C>0$,

(6) $$\sup_{\Gamma_\varepsilon}|a(x, \xi)-\sum_{j\geqq j_0} a_j(x, \xi)|\leqq C\delta^{j_0}|\xi|^{j_0}[-j_0]!$$

for $j_0=0,\ -1/i,\ -2/i,\ \cdots$. We write $a(x, \xi)\sim\Sigma a_j(x, \xi)$ if (6) holds. For this function $a(x, \xi)$, there exists an operator $A\in\mathcal{E}^R_{\dot{x}*}$ such that $\sigma(A)\equiv a$, as stated in **1.** Then we have $\sigma(A)\sim\Sigma a_j$. Such an operator is called a microdifferential operator of fractional order.

If $A=(A_{(\mu,\nu)})$ is a matrix of microdifferential operators of fractional order, and if the symbol $\sigma(A_{(\mu,\nu)})(x, \xi)$ of each element $A_{(\mu,\nu)}$ of A admits an asymptotic expansion $\sigma(A_{(\mu,\nu)})\sim\Sigma a_{j,(\mu,\nu)}$, then we write $\sigma(A)\sim\Sigma A_j(x, \xi)$ where A_j is the matrix $A_j=(a_{j,(\mu,\nu)})$.

3. Fix an integer $i>0$. Let $A(x, D)$ be a microdifferential operator of fractional order such that

$$\sigma(A)\sim\sum_{j\in(1/i)\boldsymbol{Z}} a_j(x, \xi)$$

in the sense of **2.** Then the symbol $\sigma(A^*)$ of the formal adjoint A^* of A admits an asymptotic expansion

$$\sigma(A^*)\sim\sum_{j\in(1/i)\boldsymbol{Z}}(A^*)_j(x, \xi)$$

where

$$(A^*)_j(x, \xi)=\sum_{\substack{k\in(1/i)\boldsymbol{Z}\\ \delta\in(\boldsymbol{Z}_+)^{n+1}\\ k-|\delta|=j}}\frac{(-1)^{|\delta|}}{\delta!}\partial_\xi^\delta\partial_x^\delta a_k(x, -\xi)$$

with $\boldsymbol{Z}_+=\{0,1,2,\cdots\}$.

Let B be another microdifferential operator of fractional order with

$$\sigma(B)\sim\sum_{j\in(1/i)\boldsymbol{Z}} b_j(x, \xi).$$

The symbol $\sigma(BA)$ of the composite operator BA admits an asymptotic expansion

$$\sigma(BA)\sim\sum_{j\in(1/i)\boldsymbol{Z}}(ba)_j(x, \xi)$$

where

$$(ba)_j = \sum_{\substack{k, l \in (1/i)Z \\ \delta \in (Z_+)^{n+1} \\ k+l-|\delta|=j}} \frac{1}{\delta !} \partial_\xi^\delta b_k(x, \xi) \partial_x^\delta a_l(x, \xi).$$

§ 2. Reduction to a matrix equation.

As before, we denote the variables of $\boldsymbol{R}^{n+1}$ by $x=(x_0, x')$, where $x_0 \in \boldsymbol{R}$ and $x'=(x_1, \cdots, x_n) \in \boldsymbol{R}^n$. We consider a partial differential operator $P(x, \partial/\partial x)$ of the form

$$P(x, \partial/\partial x) = \sum_{|\alpha| \leq m} a_\alpha(x) x_0^{\kappa(|\alpha|)} (\partial/\partial x)^\alpha,$$

where $\kappa(j)$ is some integer $\geqq 0$, $a_\alpha(x)$ is real analytic in a neighborhood of $x=0$, and $a_{(m,0,\cdots,0)}=1$. If the irregularity ι is larger than 1, we say that $P(x, \partial/\partial x)$ has irregular singularities along the hypersurface $\{x_0=0\}$. In this paper we assume that $\iota > 1$.

At $\mathring{x}^*$ we define a microdifferential operator $Q(x, D)$ by

$$(7) \qquad \sigma(Q)(x, \xi) = \{\sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha\}^{-1}.$$

Here $\sigma(Q)(x, \xi)$ denotes the total symbol of $Q(x, D)$, and not the principal symbol. Define $P'(x, D)$ by

$$(8) \qquad P'(x, D) = Q(x, D) P(x, D).$$

It is easy to see that there exist integers $\bar{\kappa}(0), \cdots, \bar{\kappa}(m-1) \geqq 0$ such that $\frac{\kappa(m)-\bar{\kappa}(j)}{m-j} \leqq \iota$, $\frac{\kappa(m)-\bar{\kappa}(j)}{m-j} = \iota$ if and only if $\frac{\kappa(m)-\kappa(j)}{m-j} = \iota$, and that

$$(9) \qquad P'(x, D) = x_0^{\kappa(m)} + \sum_{k=1}^m P'_{-k}(x, D) x_0^{\bar{\kappa}(m-k)},$$

where $P'_{-k}(x, D)$ is a microdifferential operator of order $-k$. Furthermore, if we define

$$\pi(P) = \left\{0 \leqq j \leqq m-1;\ \frac{\kappa(m)-\kappa(j)}{m-j} = \iota\right\} = \left\{j;\ \frac{\kappa(m)-\bar{\kappa}(j)}{m-j} = \iota\right\},$$

the principal symbol $\sigma_{-k}(P'_{-k})(x, \xi)$ of $P'_{-k}(x, D)$ satisfies

$$(10) \qquad \sigma_{-k}(P'_{-k})(0, x', \xi) = \sigma(Q)(0, x', \xi) \sum_{|\alpha|=m-k} a_\alpha(0, x') \xi^\alpha,$$

provided $m-k \in \pi(P)$.

If u and f are elements of $C_{\mathring{x}^*}$, the equation $P'u=f$ is equivalent to

$$(x_0 I_{\kappa(m)} + A'(x, D)) \begin{pmatrix} u \\ x_0 u \\ \vdots \\ x_0^{\kappa(m)-1} u \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f \end{pmatrix},$$

where the $\kappa(m)\times\kappa(m)$ matrix $A'(x, D)$ is given by

$$A'(x, D)=\begin{pmatrix} 0, & & -1, & & \\ & \ddots & & \ddots & \\ & & 0, & & -1 \\ A'_{(\kappa(m),1)}, & & \cdots\cdots, & & A'_{(\kappa(m),\kappa(m))} \end{pmatrix}, \tag{11}$$

$$A'_{(\kappa(m),j)}(x, D)=\sum_{\substack{1\leq k\leq m \\ j=\kappa(m-k)+1}} P'_{-k}(x, D). \tag{12}$$

We transform this matrix $x_0 I_{\kappa(m)}+A'$ into another matrix $x_0 I_{\kappa(m)}+A''$ as follows:

$$x_0 I_{\kappa(m)}+A''=\begin{pmatrix} 1 & & & \\ & D_0^{1/\iota} & & \\ & & \ddots & \\ & & & D_0^{(\kappa(m)-1)/\iota} \end{pmatrix}(x_0 I_{\kappa(m)}+A')\begin{pmatrix} 1 & & & \\ & D_0^{-1/\iota} & & \\ & & \ddots & \\ & & & D_0^{-(\kappa(m)-1)/\iota} \end{pmatrix}.$$

$A''(x, D)$ is a $\kappa(m)\times\kappa(m)$ matrix of microdifferential operators of fractional order. The symbol $\sigma(A'')(x, \xi)$ of $A''(x, D)$ admits an asymptotic expansion

$$\sigma(A'')(x, \xi)\sim\sum_{-p/q\geq j\in(1/q)\mathbf{Z}} A''_j(x, \xi)$$

in the sense of § 1. Here p and q are two integers determined as follows: From the definition, the irregularity ι of P is a rational number q/p, where p and q are two integers relatively prime, and $1\leqq p<q$. Furthermore, we have

$$A''_{-p/q}(x, \xi)=\begin{pmatrix} 0, & & -\xi_0^{-p/q} & & \\ & \ddots & & \ddots & \\ & & 0, & & -\xi_0^{-p/q} \\ A''_{-p/q,(\kappa(m),1)}, & & \cdots\cdots, & & A''_{-p/q,(\kappa(m),\kappa(m))} \end{pmatrix}, \tag{13}$$

where

$$A''_{-p/q,(\kappa(m),j)}(x, \xi)=\sum_{m-k\in\pi(P)} \sigma_{-k}(A'_{(\kappa(m),j)})(x, \xi)\xi_0^{k-1/\iota}. \tag{14}$$

Now we need the Weierstrass preparation theorem for a matrix of microdifferential operators of fractional order.

LEMMA 1. *There exists a $\kappa(m)\times\kappa(m)$ matrix of microdifferential operators of fractional order $W(x, D)$ such that*

$$\sigma(W)(x, \xi)\sim I_{\kappa(m)}+\sum_{0>j\in(1/q)\mathbf{Z}} W_j(x, \xi) \tag{15}$$

and

$$(x_0 I_{\kappa(m)}+A''(x, D))W(x, D)=x_0 I_{\kappa(m)}+A(x, D). \tag{16}$$

Furthermore, the symbol $\sigma(A)(x, \xi)$ of A satisfies

$$\frac{\partial\sigma(A)}{\partial x_0}(x, \xi)=0, \tag{17}$$

(18) $$\sigma(A)(x', \xi) \sim \sum_{-p/q \geqq j \in (1/q)\boldsymbol{Z}} A_j(x', \xi),$$

and

(19) $$A_{-p/q}(x', \xi) = A''_{-p/q}(0, x', \xi).$$

The proof of this lemma is just the same as Theorem 2.2.2 in Sato-Kawai-Kashiwara [7].

REMARK. If an operator $H(x, D)$ satisfies $\partial\sigma(H)/\partial x_0 = 0$, we denote this operator by $H(x', D)$ as well as by $H(x, D)$.

LEMMA 2. *There exists a $\kappa(m)\times\kappa(m)$ matrix $W'(x, D)$ of microdifferential operators of fractional order such that $W(x, D)W'(x, D) = W'(x, D)W(x, D) = I_{\kappa(m)}$.*

PROOF. Since the principal symbol of $W(x, D)$ is $I_{\kappa(m)}$, we can construct the parametrix $W'(x, D)$ of $W(x, D)$ just as in Theorem 2.1.1. in Sato-Kawai-Kashiwara [7]. Q. E. D.

Therefore we just have to consider $x_0 I_{\kappa(m)} + A(x', D)$ instead of $x_0 I_{\kappa(m)} + A''(x, D)$.

In this paper, we assume the following

Hypothesis. *All the characteristic exponents of $P(x, \partial/\partial x)$ are distinct.*

This is equivalent to the following

Hypothesis′. *All the eigenvalues $\lambda'_1(x', \xi), \cdots, \lambda'_{\kappa(m)}(x', \xi)$ of $A_{-p/q}(x', \xi)$ are distinct.*

In fact we have

$$\lambda_j = -\lambda'_j(0, \xi_0, 0)\xi_0^{p/q} \qquad j = 1, \cdots, \kappa(m).$$

This can be proved using (7), (10), (12), (13), (14) and (19).

Under this Hypothesis′, we shall transform $x_0 I_\kappa + A(x', D)$ into a diagonal matrix $x_0 I_{\kappa(m)} + B'(x', D)$ of microdifferential operators of fractional order, in § 3. And then we shall transform this diagonal matrix into $x_0 I_{\kappa(m)}$, in § 4

§ 3. Diagonalization of a matrix of microdifferential operators.

In the rest of this paper, we denote by κ the constant $\kappa(m)$. For a $\kappa\times\kappa$ matrix $H = (H_{(\mu,\nu)})_{\mu,\nu}$, we define $|H| = \kappa \max_{\mu,\nu} |H_{(\mu,\nu)}|$. We remind the reader that the irregularity ι of $P(x, \partial/\partial x)$ is a rational number p/q, where $1 \leqq p < q$.

LEMMA 3. *Let $A(x', D)$ be a $\kappa\times\kappa$ matrix of microdifferential operators of fractional order satisfying the conditions of* Theorem 2. *Then we can construct* *an invertible* $\kappa\times\kappa$ *matrix* $T(x', D)$ *of microdifferential operators of fractional order defined at* $\mathring{x}^*$ *such that*

$$|\sigma(T)(x', \xi)|,\ |\sigma(T^{-1})(x', \xi)|\leqq C \quad \textit{for some } C>0, \tag{20}$$

$$T(x', D)(x_0 I_\kappa + A(x', D)) = (x_0 I_\kappa + B(x', D))T(x', D). \tag{21}$$

Here $B(x', D)$ *satisfies the following conditions*:

$$\sigma(B)(x', \xi)\sim \sum_{-p/q\geqq j\in(1/q)Z} B_j(x', \xi), \tag{22}$$

$$B_{j,(\mu,\nu)}(x', \xi)=0 \quad \textit{if } j\geqq -1 \textit{ and } \mu\neq\nu, \tag{23}$$

and

$$B_{-p/q}(x', \xi)=M(x', \xi), \tag{24}$$

where

$$M(x', \xi)=\begin{pmatrix} \lambda_1'(x', \xi) & & \\ & \ddots & \\ & & \lambda_\kappa'(x', \xi) \end{pmatrix}. \tag{25}$$

PROOF. It is easy to find an invertible $\kappa\times\kappa$ matrix $T'(x', \xi)$ such that

$$|T'(x', \xi)|,\ |T'^{-1}(x', \xi)|\leqq C'$$

with some constant C', and that

$$T'(x', \xi)A_{-p/q}(x', \xi)T'^{-1}(x', \xi)=M(x', \xi).$$

We define $T'(x', D)$ by $\sigma(T')(x', \xi)=T'(x', \xi)$. Then we have

$$T'(x', D)(x_0 I_\kappa + A(x', D))\{T'(x', D)\}^{-1}=x_0 I_\kappa + M(x', D)+M'(x', D)$$

where $\sigma(M)(x', \xi)=M(x', \xi)$ and $\sigma(M')(x', \xi)\sim \sum_{-p/q\geqq j\in(1/q)Z} M_j(x', \xi)$. Here $M_j(x', \xi)$ is a $\kappa\times\kappa$ matrix homogeneous of order j with respect to ξ.

Next, we define $T''(x', D)$ by $\sigma(T'')(x', \xi)=I_\kappa+\sum_{0\geqq j\geqq -(q-p)/q} T_j''(x', \xi)$, where $T_j''(x', \xi)$ is homogeneous of order j with respect to ξ, which will be determined later. Furthermore, we define $C(x', D)$ by $\sigma(C)(x', \xi)=\sum_{-(p+1)/q\geqq j\geqq -1} C_j(x', \xi)$, where $C_j(x', \xi)$ is a diagonal matrix homogeneous of order j with respect to ξ, which will also be determined later. Then we have

$$\begin{aligned} &\sigma(T''(x', D)(x_0 I_\kappa + M(x', D)+M'(x', D)))(x', \xi) \\ &-\sigma((x_0 I_\kappa + M(x', D)+C(x', D))T''(x', D))(x', \xi) \\ &=T''(x', \xi)(M(x', \xi)+M_{-(p+1)/q}(x', \xi)+\cdots+M_{-1}(x', \xi)) \\ &-(M(x', \xi)+C_{-(p+1)/q}(x', \xi)+\cdots+C_{-1}(x', \xi))T''(x', \xi) \\ &+S(x', \xi), \end{aligned} \tag{26}$$

where $S(x', \xi)\sim \sum_{-1\gtrless j\in(1/q)Z} S_j(x', \xi)$, $S_j(x', \xi)$ being homogeneous of order j with respect to ξ.

We can determine the matrices

$$T''_{-j/q}(x', \xi) \text{ and } C_{-(p+j)/q}(x', \xi) \qquad 1\leqq j\leqq q-p$$

by induction on j, in such a manner that in the right-hand side of (26) the term homogeneous of order j vanishes if $j\geqq -1$. For instance, the term homogeneous of order $-\frac{p+1}{q}$ is $M_{-(p+1)/q}-C_{-(p+1)/q}+T_{-1/q}M-MT_{-1/q}$. Now we define

$$C_{-(p+1)/q,(\mu,\nu)}=\begin{cases} 0 & \text{if } \mu\neq\nu, \\ M_{-(p+1)/q,(\mu,\mu)} & \text{if } \mu=\nu, \end{cases}$$

$$T''_{-1/q,(\mu,\nu)}=\begin{cases} (\lambda'_\mu-\lambda'_\nu)^{-1}M_{-(p+1)/q,(\mu,\nu)} & \text{if } \mu\neq\nu, \\ 0 & \text{if } \mu=\nu. \end{cases}$$

Then this term is equal to 0. The term homogeneous of order j, $-\frac{p+2}{q}\leqq j\leqq -1$, can be treated in a similar way. This is a well-known method in the theory of ordinary differential operators. Q. E. D.

Instead of $x_0I_\kappa+A(x', D)$ we may consider $x_0I_\kappa+B(x', D)$, which satisfies (22)-(25). We define $B'(x', D)$ by

$$\sigma(B')(x', \xi)=\sum_{-p/q\geqq j\geqq -1} B_j(x', \xi).$$

By (23), this matrix satisfies $B'_{(\mu,\nu)}(x', D)=0$ if $\mu\neq\nu$.

The purpose of this section is to transform the matrix $x_0I_\kappa+B(x', D)$ into $x_0I_\kappa+B'(x', D)$. For this purpose we must construct a matrix $U(x', D)$ of micro-differential operators of fractional order which satisfies

$$(x_0I_\kappa+B(x', D))D_0^{(q-p)/q}U(x', D)=D_0^{(q-p)/q}U(x', D)(x_0I_\kappa+B'(x', D)).$$

This equation can be rewritten as follows:

$$\text{(27)} \qquad \frac{\partial}{\partial\xi_0}\sigma(U)(x', \xi)+\frac{q-p}{q}\xi_0^{-1}\sigma(U)(x', \xi)-\sigma(BU-UB')(x', \xi)\sim 0.$$

We solve this equation by successive approximation: At first, we shall find a function $U^{(0)}(x', \xi)$ which satisfies

$$\text{(28)}_0 \qquad \frac{\partial}{\partial\xi_0}U^{(0)}(x', \xi)+\frac{q-p}{q}\xi_0^{-1}U^{(0)}(x', \xi) -\sigma(B')(x', \xi)U^{(0)}(x', \xi)+U^{(0)}(x', \xi)\sigma(B')(x', \xi)=0.$$

And then, for $i=1, 2, 3, \cdots$, we shall find $U^{(i)}(x', \xi)$ which satisfies

$$(28)_i \qquad \frac{\partial}{\partial\xi_0}U^{(i)}(x',\xi)+\frac{q-p}{q}\xi_0^{-1}U^{(i)}(x',\xi)$$
$$-\sigma(B')(x',\xi)U^{(i)}(x',\xi)+U^{(i)}(x',\xi)\sigma(B')(x',\xi)$$
$$=\sum_{(h,\delta,i')\in\pi_1(i)}\frac{1}{\delta!}\partial_{\xi'}^{\delta}B_h(x',\xi)\partial_{x'}^{\delta}U^{(i')}(x',\xi)$$
$$-\sum_{(h,\delta,i')\in\pi_2(i)}\frac{1}{\delta!}\partial_{\xi'}^{\delta}U^{(i')}(x',\xi)\partial_{x'}^{\delta}B_h(x',\xi),$$

where

$$(29) \qquad \pi_1(i)=\{(h,\delta,i')\in(1/q)\boldsymbol{Z}\times(\boldsymbol{Z}_+)^n\times\boldsymbol{Z}_+;$$
$$h-|\delta|-(i'/q)=-(i/q)-1,\ |h|+|\delta|\geqq 1\}$$

and

$$(30) \qquad \pi_2(i)=\{(h,\delta,i')\in(1/q)\boldsymbol{Z}\times(\boldsymbol{Z}_+)^n\times\boldsymbol{Z}_+;$$
$$h-|\delta|-(i'/q)=-(i/q)-1,\ h\geqq -1,\ \delta\neq 0\}.$$

If $(h,\delta,i')\in\pi_1(i)$ or $\pi_2(i)$, we have $i'\leqq i$. In fact, since

$$|h|+|\delta|\geqq 1,\quad \text{if } (h,\delta,i')\in\pi_1(i) \text{ or } \pi_2(i)$$

we have

$$i'=i+q-q(|h|+|\delta|)\leqq i.$$

This means that if we have already found $U^{(0)},\cdots,U^{(i-1)}$, then the right-hand side of $(28)_i$ is a known function.

The equation $(28)_i$ is an ordinary differential equation, where the unknown function is $U^{(i)}(x',\xi)$, the variable is ξ_0, and the parameters are (x',ξ'). The difficulty in this equation arises from the fact that it has an irregular singular point at $\xi_0=\infty$. To overcome this difficulty, we employ the technique developed by Turrittin in [8] in a modified form.

From now on, we write $\tau=\xi_0^{1/q}$. If $f(\xi_0)$ is some function, we denote also by $f(\tau)$ the function acquired by substituting $\tau=\xi_0^{1/q}$ in $f(\xi_0)$, for the sake of simplicity. Then $(28)_i$, $i=0,1,2,\cdots$, is rewritten as

$$(31)_i \qquad \frac{\partial}{\partial\tau}U^{(i)}(x',\tau,\xi')+(q-p)\tau^{-1}U^{(i)}(x',\tau,\xi')$$
$$-q\tau^{q-1}\{\sigma(B')(x',\tau,\xi')U^{(i)}(x',\tau,\xi')-U^{(i)}(x',\tau,\xi')\sigma(B')(x',\tau,\xi')\}$$
$$=F^{(i)}(x',\tau,\xi'),$$

where

$$(32)_0 \qquad F^{(0)}(x',\tau,\xi')=0$$

and

$$(32)_i \qquad F^{(i)}(x', \tau, \xi')=q\tau^{q-1}\sum_{\pi_1(i)}\frac{1}{\delta!}\partial_{\xi'}^{\delta}B_h(x', \tau, \xi')\partial_{x'}^{\delta}U^{(i')}(x', \tau, \xi')$$
$$-q\tau^{q-1}\sum_{\pi_2(i)}\frac{1}{\delta!}\partial_{\xi'}^{\delta}U^{(i')}(x', \tau, \xi')\partial_{x'}^{\delta}B_h(x', \tau, \xi')$$

for $i=1, 2, 3, \cdots$.

It is easy to see that $U^{(0)}=\tau^{-q+p}$ is a solution of $(31)_0$. For $i\geqq 0$, we have the following

PROPOSITION 1. *Let $\varepsilon>0$ and $r>0$ be small enough. Then there exists a formal power series*

$$(33)_i \qquad U^{(i)}=\sum_{j/q+|\alpha|\leq-(i+q-p)/q}U_{j\alpha}^{(i)}(x')\tau^j\xi'^{\alpha},$$

where $j\in Z_-=\{0, -1, -2, \cdots\}$ and $\alpha\in(Z_+)^n$, which satisfies $(28)_i$ formally. Here $U_{j\alpha}^{(i)}(x')$ is holomorphic on $\{x'\in C^n; |x'|<\varepsilon\}$ and there exists some constant $C>0$ such that

$$(34)_i \qquad |\partial_{x'}^{\Delta}U_{j,\alpha,(\mu,\nu)}^{(i)}(x')|\leqq C\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-j+(q-p)|\Delta|\right]!}{((q^2+1)i+q|\alpha|)!}\right)^{1/(q-p)}r^{(j-|\Delta|)/2-|\alpha|-4qi}$$

for any $1\leqq\mu, \nu\leqq\kappa$, $i\in Z_+$, $j\in Z_-$, $\alpha\in(Z_+)^n$ and $\Delta\in(Z_+)^n$, provided $|x'|<\varepsilon$.

PROOF. If $i=0$, the assertion is trivial since we may take $U^{(0)}=\tau^{-(q-p)}$. Assume that $i\gneqq 0$ and that the assertion is valid for $i'=0, 1, \cdots, i-1$. In the asymptotic expansion

$$\sigma(B)(x', \xi)\sim\sum_{-p/q\geq h\in(1/q)Z}B_h(x', \xi),$$

we can choose each $B_h(x', \xi)$ to be homogeneous in ξ of order h. Thus from now on we assume that each $B_h(x', \xi)$ has been chosen to be homogeneous in ξ of order h. Considering the Taylor expansion of $B_h(x', \xi)$ with respect to ξ' at $\xi'=0$, we have

$$(35) \qquad B_h(x', \tau, \xi')=\sum_{\substack{j\in Z_-, \alpha\in(Z_+)^n\\ j/q+|\alpha|=h}}B_{j\alpha}(x')\tau^j\xi'^{\alpha}.$$

Here each $B_{j\alpha}(x')$ is holomorphic on $\{x'\in C; |x'|<\varepsilon\}$, and there exist some $a>0$ and some $R>0$ such that

$$(36) \qquad |\partial_{x'}^{\Delta}B_{j\alpha}(x')|\leqq aR^{h-|\alpha|-|\Delta|}[|h|+|\Delta|]! \quad (h=j/q+|\alpha|)$$

for any $j\in Z_-$, any $\alpha\in(Z_+)^n$, and any $\Delta\in(Z_+)^n$, if $|x'|<\varepsilon$.

Substituting $(33)_{i'}$, $i'=0, \cdots, i-1$, and (35) into $(32)_i$, we see that $F^{(i)}$ is a formal power series of the form

$$(37)_i \qquad F^{(i)}=\sum_{j\in Z_-, \alpha\in(Z_+)^n}F_{j\alpha}^{(i)}(x')\tau^j\xi'^{\alpha}.$$

Since $j/q+|\alpha|\leqq-(i'+q-p)/q$ in $(33)_{i'}$, we have $j/q+|\alpha|\leqq-(i+q-p+1)/q$ in $(37)_i$. Here $F_{j\alpha}^{(i)}(x')$ is given by

$$F_{j\alpha}^{(i)}(x')=q\sum{}^*\frac{1}{\delta!}\cdot\frac{\beta!}{(\beta-\delta)!}B_{k\beta}(x')\partial_{x'}^{\delta}U_{l\gamma}^{(i')}(x')$$
$$-q\sum{}^{**}\frac{1}{\delta!}\cdot\frac{\gamma!}{(\gamma-\delta)!}U_{l\gamma}^{(i')}(x')\partial_{x'}^{\delta}B_{k\beta}(x'),$$

where the summation $\sum^*$ is taken for

$$\Big\{(h,\beta,\gamma,\delta,i',k,l)\in\Big(\frac{1}{q}\boldsymbol{Z}\Big)\times(\boldsymbol{Z}_+)^n\times(\boldsymbol{Z}_+)^n\times(\boldsymbol{Z}_+)^n\times\boldsymbol{Z}_+\times\boldsymbol{Z}\times\boldsymbol{Z};$$
$$(h,\delta,i')\in\pi_1(i),\ \beta+\gamma-\delta=\alpha,\ k+l+q-1=j,\ \beta\geqq\delta\Big\}$$

while the summation $\sum^{**}$ for

$$\Big\{(h,\beta,\gamma,\delta,i',k,l)\in\Big(\frac{1}{q}\boldsymbol{Z}\Big)\times(\boldsymbol{Z}_+)^n\times(\boldsymbol{Z}_+)^n\times(\boldsymbol{Z}_+)^n\times\boldsymbol{Z}_+\times\boldsymbol{Z}\times\boldsymbol{Z};$$
$$(h,\delta,i')\in\pi_2(i),\ \beta+\gamma-\delta=\alpha,\ k+l+q-1=j,\ \beta\geqq\delta\Big\}.$$

In Appendix 1 we shall prove that

$$(38)_i\quad |\partial_{x'}^{\Delta}F_{j\alpha}^{(i)}(x')|\leqq C\left(\frac{\Big[\Big(q^2+\frac{q-p}{q}\Big)i-(j+1)+(q-p)|\Delta|\Big]!}{((q^2+1)i+q|\alpha|-(q-p))!}\right)^{1/(q-p)}r^{(j+1-|\Delta|)/2-|\alpha|-4qi+1}$$

for any $i\in\boldsymbol{Z}_+$, $j\in\boldsymbol{Z}_-$, $\alpha\in(\boldsymbol{Z}_+)^n$ and $\Delta\in(\boldsymbol{Z}_+)^n$, if $|x'|<\varepsilon$.

Admitting this for the moment we proceed as follows: Equating the coefficient of $\tau_{\xi}^{j,\alpha}$ in both sides of $(31)_i$, we obtain

$$(j+1+q-p)U_{j+1,\alpha}^{(i)}(x')-q\sum{}^{***}(B_{k\beta}(x')U_{l\gamma}^{(i)}(x')-U_{l\gamma}^{(i)}(x')B_{k\beta}(x'))=F_{j\alpha}^{(i)}(x'),$$

where the summation $\sum^{***}$ is taken for

$$\{(k,l,\beta,\gamma)\in\boldsymbol{Z}\times\boldsymbol{Z}\times(\boldsymbol{Z}_+)^n\times(\boldsymbol{Z}_+)^n;$$
$$(k/q)+|\beta|=-p/q,\ -(p+1)/q,\ \cdots,\ -1,\ k+l+q-1=j,\ \beta+\gamma=\alpha\}.$$

It follows that $l\geqq j+1-q+p$ in each term.

Calculating the (μ,ν)-element of the above equation, we have

$$(j+1+q-p)U_{j+1,\alpha,(\mu,\mu)}^{(i)}(x')=F_{j\alpha,(\mu,\mu)}^{(i)}(x')$$

and

$$-q(B_{-p,0,(\mu,\mu)}(x')-B_{-p,0,(\nu,\nu)}(x'))U_{j+1-q+p,\alpha,(\mu,\nu)}^{(i)}(x')$$
$$=-(j+1+q-p)U_{j+1,\alpha,(\mu,\nu)}^{(i)}(x')$$
$$+q\sum_{k\leqq-p}{}^{***}(B_{k\beta,(\mu,\mu)}(x')-B_{k\beta,(\nu,\nu)}(x'))U_{l\gamma,(\mu,\nu)}^{(i)}(x')$$
$$+F_{j\alpha,(\mu,\nu)}^{(i)}(x')\qquad\text{if } \mu\neq\nu.$$

Thus we can solve this equation inductively by setting

$$U^{(i)}_{j+1,\alpha,(\mu,\mu)}(x')=(j+1+q-p)^{-1}F^{(i)}_{j\alpha(\mu,\mu)}(x')$$

and

$$\begin{aligned}U^{(i)}_{j-q+p+1,\alpha,(\mu,\nu)}(x')=&-\{q(B_{-p,0,(\mu,\mu)}(x')-B_{-p,0,(\nu,\nu)}(x'))\}^{-1}\\&\times\{-(j+1+q-p)U^{(i)}_{j+1,\alpha,(\mu,\nu)}(x')\\&+q\sum_{k\lneqq -p}^{***}(B_{k\beta,(\mu,\mu)}(x')-B_{k\beta,(\nu,\nu)}(x'))U^{(i)}_{l\gamma,(\mu,\nu)}(x')\\&+F^{(i)}_{j\alpha,(\mu,\nu)}(x')\} \qquad \text{if } \mu\neq\nu.\end{aligned}$$

This is a well-known procedure in the theory of ordinary differential equations.

Now we have constructed a formal solution of the equation $(31)_i$. The fact that $\frac{j}{q}+|\alpha|\leqq-\frac{i+q-p}{q}$ in $(34)_i$ follows from the fact that $\frac{j}{q}+|\alpha|\leqq-\frac{i+q-p+1}{q}$ in $(37)_i$. The estimate $(34)_i$ can be proved as follows: Let $j\leqq-\frac{i+q-p+1}{q}$. Assume that if $l\gneqq j-q+p+1$ and $\mu\neq\nu$, then $\partial^{\Delta}_{x'}U^{(i)}_{l\gamma,(\mu,\nu)}(x')$ satisfies the estimate $(34)_i$ for any Δ, $\gamma\in(Z_+)^n$. (There is nothing to assume if $j\gneqq -i-2q+2p$.) Then we have

$$|U^{(i)}_{j+1,\alpha,(\mu,\mu)}|\leqq\frac{Cr}{|j+1+q-p|}\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j+1)\right]!}{((q^2+1)i+q|\alpha|-(q-p))!}\right)^{1/(q-p)}r^{(j+1)/2-|\alpha|-4qi}.$$

Since $j+1+q-p\leqq-i-q|\alpha|$, we have

$$\frac{r}{|j+1+q-p|}\leqq\prod_{k=1}^{q-p}((q^2+1)i+q|\alpha|-(q-p)+k)^{-1/(q-p)},$$

provided $r\leqq 1/(q^2+1)$. Thus $U^{(i)}_{j+1,\alpha,(\mu,\mu)}$ satisfies $(34)_i$ with $\Delta=0$. The case $\Delta\gneqq 0$ can be proved similarly.

On the other hand, from the above assumption about $U^{(i)}_{l\gamma,(\mu,\nu)}$ with $l\gneqq j-q+p+1$ and $\mu\neq\nu$, we have

$$\begin{aligned}&|\{q(B_{-p,0,(\mu,\mu)}-B_{-p,0,(\nu,\nu)})\}^{-1}(j+1+q-p)U^{(i)}_{j+1,\alpha,(\mu,\nu)}|\\&\leqq r^{q-p}aC\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-j+q-p-1\right]!}{((q^2+1)i+q|\alpha|)!}\right)^{1/(q-p)}r^{(j-q+p+1)/2-|\alpha|-4qi},\end{aligned}$$

and

$$\begin{aligned}&|\sum_{k\lneqq -p}^{***}(B_{k\beta,(\mu,\mu)}-B_{k\beta,(\nu,\nu)})U^{(i)}_{l\gamma,(\mu,\nu)}|\\&\leqq 2aCR^{-1}\sum_{k\lneqq -p}^{***}R^{-|\beta|}\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-l\right]!}{((q^2+1)i+q|\gamma|)!}\right)^{1/(q-p)}r^{(l/2)-|\gamma|-4qi}.\end{aligned}$$

Since $(l/q)+|\gamma|\leqq-(i+q-p)/q$, we have

$$\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-l\right]!}{((q^2+1)i+q|\gamma|)!}\leqq\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j-q+p+1)\right]!}{((q^2+1)i+q|\alpha|)!}$$

and we have

$$\sum_{k\leqq -p}^{***} R^{-|\beta|} r^{(l/2)-|\gamma|-4qi} \leqq \frac{2^n(q-p+1)r^{1/2}}{1-(2r/R)} r^{(j-q+p+1)/2-|\alpha|-4qi},$$

provided $r\leqq 4^{-1}R$. Thus we have

$$\Big|\sum_{k\leqq -p}^{***}(B_{k\beta,(\mu,\mu)}-B_{k\beta,(\nu,\nu)})U_{l\gamma,(\mu,\nu)}^{(i)}\Big|$$
$$\leqq r^{1/4}C\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j-q+p+1)\right]!}{((q^2+1)+q|\alpha|)!}\right)^{1/(q-p)} r^{(j-q+p+1)/2-|\alpha|-4qi},$$

provided $r\leqq \dfrac{R^4}{2^{4n+4}a^4(q-p+1)^4}$. Finally we have

$$|F_{j\alpha}^{(i)}(x')|\leqq r^{1/4}C\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j-q+p+1)\right]!}{((q^2+1)i+q|\alpha|)!}\right)^{1/(q-p)} r^{(j-q+p+1)/2-|\alpha|-4qi}.$$

From these estimates we obtain the estimate $(34)_i$ with $\Delta=0$, for $U_{j-q+p+1,\alpha,(\mu,\nu)}^{(i)}(x')$, $\mu\neq\nu$. The case $\Delta\gneqq 0$ is similar. Q. E. D.

In the rest of this paper, we shall denote by the same letters such as C, r, ε and ρ, several constants which do not depend on the indices i, j, α, δ and Δ. From Proposition 1, we easily obtain the following

COROLLARY. *There exist some constants C, $r>0$ such that*

$$|\partial_{x'}^{\Delta}U_{j\alpha}^{(i)}(x')|\leqq C\frac{\left[\frac{-j}{q-p}\right]!\left[\frac{i}{q}+|\Delta|\right]!}{\left[\frac{i}{q-p}\right]!\left[\frac{q}{q-p}|\alpha|\right]!} r^{j-|\alpha|-i-|\Delta|} \tag{39$_i$}$$

for any $i\in \mathbf{Z}_+$, $j\in \mathbf{Z}$, $\alpha\in(\mathbf{Z}_+)^n$ and $\Delta\in(\mathbf{Z}_+)^n$, provided $|x'|<\varepsilon$.

REMARK. Unfortunately, the formal power series $(33)_i$ does not define a symbol of a microdifferential operator of fractional order, because the estimate $(34)_i$ or $(39)_i$ grows too rapidly as $\dfrac{j}{q}+|\alpha|\to-\infty$. To overcome this defect, we shall consider the true solution of $(28)_i$ corresponding to the formal solution $(33)_i$. For this purpose, we employ the method developed by Turrittin [8], using the Laplace transformation, or rather the Leroy transformation.

Let $\sigma=\tau^{q-p}$. We define the functions $B_{h,(J)}(x',\sigma,\xi')$, $J=1, 2, \cdots, q-p$, by

$$B_{h,(J)}(x',\sigma,\xi')=\sum_{\substack{(j/q)+|\alpha|=h\\ p+j-J\in(q-p)\mathbf{Z}}} B_{j\alpha}(x')\sigma^{(p+j-J)/(q-p)}\xi'^{\alpha}, \tag{40}$$

as in Turrittin [8].

We have

$$B_h(x',\tau,\xi')=\sum_{J=1}^{q-p}\tau^{J-p}B_{h,(J)}(x',\sigma,\xi').$$

We write $B'_{(J)}(x', \sigma, \xi')=\sum_{h\geqq -1} B_{h,(J)}(x', \sigma, \xi')$.

In the same way, we define the formal power series $U^{(i)}_{(J)}=U^{(i)}_{(J)}(x', \sigma, \xi')$, $J=1, 2, \cdots, q-p$, by

$$U^{(i)}_{(J)}=\sum_{\substack{j-J\in(q-p)Z\\ \alpha\in(Z_+)^n}} U^{(i)}_{j,\alpha}(x')\sigma^{(j-J)/(q-p)}\xi'^{\alpha}.$$

Then formally we have

$$(41)_{i,J}\qquad (q-p)\frac{\partial}{\partial\sigma}U^{(i)}_{(J)}+(J+q-p)\sigma^{-1}U^{(i)}_{(J)}$$
$$-q\sum_{K+L=J}\{B'_{(K)}U^{(i)}_{(L)}-U^{(i)}_{(L)}B'_{(K)}\}$$
$$-q\sigma\sum_{K+L=J+q-p}\{B'_{(K)}U^{(i)}_{(L)}-U^{(i)}_{(L)}B'_{(K)}\}=F^{(i)}_{(J)}\qquad J=1, 2, \cdots, q-p$$

where

$$(42)_{i,J}\quad F^{(i)}_{(J)}=q\sum_{K+L=J}\Big\{\sum_{\pi_1(i)}\frac{1}{\delta!}\partial^{\delta}_{\xi'}B_{h,(K)}\partial^{\delta}_{x'}U^{(i')}_{(L)}-\sum_{\pi_2(i)}\frac{1}{\delta!}\partial^{\delta}_{\xi'}U^{(i')}_{(L)}\partial^{\delta}_{x'}B_{h,(K)}\Big\}$$
$$+q\sigma\sum_{K+L=J+q-p}\Big\{\sum_{\pi_1(i)}\frac{1}{\delta!}\partial^{\delta}_{\xi'}B_{h,(K)}\partial^{\delta}_{x'}U^{(i')}_{(L)}-\sum_{\pi_2(i)}\frac{1}{\delta!}\partial^{\delta}_{\xi'}U^{(i')}_{(L)}\partial^{\delta}_{x'}B_{h,(K)}\Big\}.$$

We define

$$C_{h,(J)}(x', \sigma, \xi')=\sigma\sum_{\substack{(j/q)+|\alpha|=h\\ p+j-J\in(q-p)Z\\ (p+j-J)/(q-p)\leqq -2}} B_{j,\alpha}(x')\sigma^{(p+j-J)/(q-p)}\xi'^{\alpha}.$$

To proceed, we must consider the Fourier transform $\hat{B}_{h,(J)}(x', s, \xi')$ (resp. $\hat{C}_{h,(J)}(x', s, \xi')$) of $B_{h,(J)}(x', \sigma, \xi')$ (resp. $C_{h,(J)}(x', \sigma, \xi')$) with respect to the σ-variable. If $|\sigma|\geqq\Big\{\frac{1}{2R}(1+|\xi'|)\Big\}^{(q-p)/q}$, we have

$$|B_{h,(J)}(x', \sigma, \xi')|,\quad |C_{h,(J)}(x', \sigma, \xi')|\leqq a_h|\sigma|$$

by (40) and (36). Here a_h is some constant depending on h. Fixing $\xi'\in \boldsymbol{C}^n$ arbitrarily we define

$$\gamma(\xi')=\Big\{\sigma\in\boldsymbol{C};\ -\infty<\operatorname{Re}\sigma<\infty,\ \operatorname{Im}\sigma=\Big(\frac{1}{2R}(1+|\xi'|)\Big)^{(q-p)/q}\Big\}.$$

Then if $s\in\boldsymbol{R}$, the integral

$$\hat{B}_{h,(J)}(x', s, \xi')=\int_{\gamma(\xi')}e^{-\sqrt{-1}s\sigma}B_{h,(J)}(x', \sigma, \xi')d\sigma$$
$$=e^{s\operatorname{Im}\sigma}\int_{\boldsymbol{R}}e^{-\sqrt{-1}s\operatorname{Re}\sigma}B_{h,(J)}(x', \operatorname{Re}\sigma+\sqrt{-1}\operatorname{Im}\sigma, \xi')d\operatorname{Re}\sigma$$

is well-defined as the Fourier transform: $\mathcal{S}'_{\operatorname{Re}\sigma}\to\mathcal{S}'_s$. We can prove

$$(43)\qquad \hat{B}_{h,(J)}(x', s, \xi')=-2\pi\sqrt{-1}\tilde{B}_{h,(J)}(x', s, \xi')Y(s).$$

where $Y(s)$ denotes the Heaviside function and

$$\tilde{B}_{h,(J)}(x', s, \xi') = \sum_{\substack{(j/q)+|\alpha|=h \\ p+j-J\in(q-p)\boldsymbol{Z}}} B_{j,\alpha}(x') \frac{(-\sqrt{-1}\, s)^{|p+j-J|/(q-p)-1}}{\left(\frac{|p+j-J|}{q-p}-1\right)!} \xi'^{\alpha}. \tag{44}$$

Defining in the same way $\hat{C}_{h,(J)}(x', s, \xi') = \int_{\gamma(\xi')} e^{-\sqrt{-1}s\sigma} C_{h,(J)}(x', \sigma, \xi') d\sigma$, we have

$$\hat{C}_{h,(J)}(x', s, \xi') = -2\pi\sqrt{-1}\, \tilde{C}_{h,(J)}(x', s, \xi') Y(s), \tag{45}$$

where

$$\tilde{C}_{h,(J)}(x', s, \xi') = \sum_{\substack{(j/q)+|\alpha|=h \\ p+j-J\in(q-p)\boldsymbol{Z} \\ (p+j-J)/(q-p)\leqq -2}} B_{j,\alpha}(x') \frac{(-\sqrt{-1}\, s)^{|p+j-J|/(q-p)-2}}{\left(\frac{|p+j-J|}{q-p}-2\right)!} \xi'^{\alpha}. \tag{46}$$

Concerning these families of functions, we have the following

LEMMA 4. *$\tilde{B}_{h,(J)}(x', s, \xi')$ and $\tilde{C}_{h,(J)}(x', s, \xi')$ are holomorphic on*

$$\{(x', s, \xi') \in \boldsymbol{C}^n \times \boldsymbol{C} \times \boldsymbol{C}^n ;\ |x'| < \varepsilon\},$$

if $\varepsilon > 0$ is small enough. Furthermore, there exist some constants a_1, $R_1 > 0$ such that in this domain

$$|\partial_{x'}^{\Delta} \partial_{\xi'}^{\delta} \tilde{B}_{h,(J)}(x', s, \xi')| \leqq a_1 \exp\{R_1^{-1}(1+|\xi'|)^{(q-p)/q}|s|\} \times [|h|+|\Delta|+|\delta|]!\, R_1^{h-|\Delta|-|\delta|} \times \frac{|s|^{((q|h|-p+q|\delta|+J)/(q-p)-1)}}{\Gamma\left(\frac{q|h|-p+q|\delta|+J}{q-p}\right)} \tag{47}$$

and

$$|\partial_{x'}^{\Delta} \partial_{\xi'}^{\delta} \tilde{C}_{h,(J)}(x', s, \xi')| \leqq a_1 \exp\{R_1^{-1}(1+|\xi'|)^{(q-p)/q}|s|\} \times [|h|+|\Delta|+|\delta|]!\, R_1^{h-|\Delta|-|\delta|} \times \frac{|s|^{((q|h|-p+q|\delta|+J)/(q-p)-2)}}{\Gamma\left(\frac{q|h|-p+q|\delta|+J}{q-p}-1\right)} \tag{48}$$

for any $\Delta \in (\boldsymbol{Z}_+)^n$, $\delta \in (\boldsymbol{Z}_+)^n$, $h \in \frac{1}{q}\boldsymbol{Z}$ and $J \in \{1, 2, \cdots, q-p\}$. If $h \geqq -1$, we have

$$|\partial_{x'}^{\Delta} \tilde{C}_{h,(J)}(x', s, \xi')| \leqq a_1 |\Delta|!\, R_1^{-|\Delta|}(1+|\xi'|)^{(2q-2p-J)/q} \exp\{R_1^{-1}(1+|\xi'|)^{(q-p)/q}|s|\} \tag{49}$$

for any $\Delta \in (\boldsymbol{Z}_+)^n$ and $J \in \{1, 2, \cdots, q-p\}$.

PROOF.

$$|\partial_{\xi'}^{\delta} \tilde{B}_{h,(J)}(x', s, \xi')|$$

$$\leqq \sum_{(j/q)+|\alpha|=h} |B_{j,\alpha}(x')| \frac{|s|^{(|p+j-J|/(q-p)-1)}}{\left(\frac{|p+j-J|}{q-p}-1\right)!} \cdot \frac{\alpha!}{(\alpha-\delta)!} |\xi'|^{|\alpha-\delta|} \qquad \text{(by (44))}$$

$$\leqq a \sum R^{h-|\alpha|} [|h|]! \frac{|s|^{((q|h|-p+q|\alpha|+J)/(q-p)-1)}}{\left[\frac{q|h|-p+q|\alpha|+J}{q-p}-1\right]!} \cdot \frac{\alpha!}{(\alpha-\delta)!} |\xi'|^{|\alpha-\delta|} \qquad \text{(by (36))}$$

$$\leqq a\sum R^{h-|\alpha|}[|h|]!\frac{|s|^{((q|h|-p+q|\alpha|+J)/(q-p)-1)}}{\left[\frac{q|h|-p+q|\alpha|+J}{q-p}-1\right]!}\delta!\,2^{|\alpha|}|\xi'|^{|\alpha-\delta|}$$

$$\leqq a[|h|+|\delta|]!\left(\frac{R}{2}\right)^{h-|\delta|}\frac{|s|^{((q|h|-p+q|\delta|+J)/(q-p)-1)}}{\left[\frac{q|h|-p+q|\delta|+J}{q-p}-1\right]!}$$

$$\times\sum_{\alpha\geqq\delta}\left(\frac{2|\xi'|}{R}\right)^{|\alpha-\delta|}\frac{|s|^{(q|\alpha-\delta|)/(q-p)}}{\left[\frac{q|\alpha-\delta|}{q-p}\right]!}$$

which proves (47) with $\Delta=0$. The case $\Delta\geqq 0$ is a consequence of the Cauchy integral formula for the derivatives of a holomorphic function. (48) and (49) are proved in a similar way. Q.E.D.

Let $\theta\in[\pi,-\pi]$ be a real number. We denote $S_\theta=e^{\sqrt{-1}\theta}\boldsymbol{R}$. We define

$$\hat{B}_{h,(J),\theta}(x', s, \xi')=-2\pi\sqrt{-1}\,\tilde{B}_{h,(J)}(x', s, \xi')Y(e^{-\sqrt{-1}\theta}s)$$

and

$$\hat{C}_{h,(J),\theta}(x', s, \xi')=-2\pi\sqrt{-1}\,\tilde{C}_{h,(J)}(x', s, \xi')Y(e^{-\sqrt{-1}\theta}s)$$

for $x'\in\boldsymbol{C}^n$, $|x'|\leqq\varepsilon$, $s\in S_\theta$ and $\xi'\in\boldsymbol{C}^n$. The suffix θ will be omitted if confusion is not likely.

We write

$$\hat{B}'_{(J),\theta}(x', s, \xi')=\sum_{h\geqq -1}\hat{B}_{h,(J),\theta}(x', s, \xi')$$

and

$$\hat{C}'_{(J),\theta}(x', s, \xi')=\sum_{h\geqq -1}\hat{C}_{h,(J),\theta}(x', s, \xi').$$

Now we define

$$(50)_i\qquad \tilde{U}^{(i)}_{(J)}(x', s, \xi')=\sum_{\substack{j-J\in(q-p)\boldsymbol{Z}\\ \alpha\in(\boldsymbol{Z}_+)^n}}U^{(i)}_{j,\alpha}(x')\frac{(-\sqrt{-1}\,s)^{(|j-J|/(q-p)-1)}}{\left(\frac{|j-J|}{q-p}-1\right)!}\xi'^{\alpha}.$$

Because of (39), this power series converges if $x'\in\boldsymbol{C}^n$, $|x'|<\varepsilon$, $\xi'\in\boldsymbol{C}^n$, $s\in\boldsymbol{C}$ and $|s|\leqq r/2$.

Note that

$$\frac{|j-J|}{q-p}-1=\frac{|j|+J}{q-p}-1\geqq\frac{i+q-p+J}{q-p}-1=\frac{i+J}{q-p}.$$

From this fact we obtain by direct calculus

$$(51)_i\qquad |\tilde{U}^{(i)}_{(J)}(x', s, \xi')|\leqq C\frac{[i/q]!\,r^{-i}}{\left[\frac{i+J}{q-p}\right]!}|s|^{(i+J)/(q-p)}\exp\{r^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$

with some constants $C, r>0$, if $|x'|<\varepsilon$, $|s|<r/2$. Let Δ, $\delta\in(\boldsymbol{Z}_+)^n$. Differentiate both sides of $(50)_i$ Δ_1 times by $x_1, \cdots, \Delta_n$ times by x_n, δ_1 times by $\xi_1, \cdots, \delta_n$ times by ξ_n. Using (39), we can easily prove that there exist some constants

C, $r>0$ such that

$$(52)_i \qquad |\partial_{x'}^{\Delta}\partial_{\xi'}^{\delta}\tilde{U}_{(J)}^{(i)}(x', s, \xi')| \leqq C\frac{\left[\frac{i}{q}+|\Delta+\delta|\right]!\, r^{-i-|\Delta+\delta|}}{\left[\frac{i+J+q|\delta|}{q-p}\right]!}|s|^{(i+J+q|\delta|)/(q-p)}\exp\{r^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$

for any Δ, $\delta(Z_+)^n$, provided $|s|<r/2$.

If $\theta\in[\pi, -\pi]$, we define

$$\hat{U}_{(J),\theta}^{(i)}(x', s, \xi')=-2\pi\sqrt{-1}\tilde{U}_{(J)}^{(i)}(x', s, \xi')Y(e^{-\sqrt{-1}\theta}s)$$

for $x'\in C^n$, $|x'|<\varepsilon$, $s\in S_\theta$, $-\infty<e^{-\sqrt{-1}\theta}s<r/2$, and $\xi'\in C^n$.

Now fix an arbitrary $\theta\in[-\pi, \pi]$. The fact that $U_{(J),\theta}^{(i)}(x', s, \xi')$ satisfies $(40)_{i,J}$ formally means that $\hat{U}_{(J),\theta}^{(i)}(x', s, \xi')$ satisfies the following equation $(53)_{i,J}$ rigidly, if $|x'|<\varepsilon$, $s\in S_\theta$, and $-\infty<e^{-\sqrt{-1}\theta}s<r/2$:

$$\begin{aligned}(53)_{i,J}\quad &(q-p)\sqrt{-1}\,s\hat{U}_{(J),\theta}^{(i)}(x', s, \xi')+\frac{J+q-p}{\sqrt{-1}}\int_0^s\hat{U}_{(J),\theta}^{(i)}(x', t, \xi')dt\\
&-\frac{q}{2\pi}\sum_{K+L=J}\int_0^s\{\hat{B}'_{(K),\theta}(x', t, \xi')\hat{U}_{(L),\theta}^{(i)}(x', s-t, \xi')\\
&\qquad-\hat{U}_{(L),\theta}^{(i)}(x', s-t, \xi')\hat{B}'_{(K),\theta}(x', t, \xi')\}\,dt\\
&-q\sum_{K+L=J+q-p}\{B_{K-q,0}(x')\hat{U}_{(L),\theta}^{(i)}(x', s, \xi')-\hat{U}_{(L),\theta}^{(i)}(x', s, \xi')B_{K-q,0}(x')\}\\
&-\frac{q}{2\pi}\sum_{K+L=J+q-p}\int_0^s\{\hat{C}'_{(K),\theta}(x', t, \xi')\hat{U}_{(L),\theta}^{(i)}(x', s-t, \xi')\\
&\qquad-\hat{U}_{(L),\theta}^{(i)}(x', s-t, \xi')\hat{C}'_{(K),\theta}(x', t, \xi')\}\,dt\\
&=\hat{F}_{(J),\theta}^{(i)}(x', s, \xi')\end{aligned}$$

where

$$\begin{aligned}(54)_{i,J}\quad &\hat{F}_{(J),\theta}^{(i)}(x', s, \xi')\\
&=\frac{q}{2\pi}\sum_{K+L=J}\Big\{\sum_{\pi_1(i)}\frac{1}{\delta!}\int_0^s\partial_{\xi'}^{\delta}\hat{B}_{h,(K),\theta}(x', t, \xi')\partial_{x'}^{\delta}\hat{U}_{(L),\theta}^{(i')}(x', s-t, \xi')dt\\
&\qquad-\sum_{\pi_2(i)}\frac{1}{\delta!}\int_0^s\partial_{\xi'}^{\delta}\hat{U}_{(L),\theta}^{(i')}(x', s-t, \xi')\partial_{x'}^{\delta}\hat{B}_{h,(K),\theta}(x', t, \xi')dt\Big\}\\
&+q\sum_{K+L=J+q-p}\sum_{\pi_3(i)}\frac{1}{\delta!}\partial_{\xi'}^{\delta}\hat{U}_{(L),\theta}^{(i')}(x', s, \xi')\partial_{x'}^{\delta}B_{K-q,0}(x')\\
&+\frac{q}{2\pi}\sum_{K+L=J+q-p}\Big\{\sum_{\pi_1(i)}\frac{1}{\delta!}\int_0^s\partial_{\xi'}^{\delta}\hat{C}_{h,(K),\theta}(x', t, \xi')\partial_{x'}^{\delta}\hat{U}_{(L),\theta}^{(i')}(x', s-t, \xi')dt\\
&\qquad-\sum_{\pi_2(i)}\frac{1}{\delta!}\int_0^s\partial_{\xi'}^{\delta}\hat{U}_{(L),\theta}^{(i')}(x', s-t, \xi')\partial_{x'}^{\delta}\hat{C}_{h,(K),\theta}(x', t, \xi')dt\Big\}.\end{aligned}$$

We have written

$$\pi_3(i, K)=\{(i', \delta)\in Z_+\times(Z_+)^n\,;\ i'+q|\delta|-K=i,\ \delta\neq 0\}.$$

Now we choose an arbitrary θ such that

$$\theta+\frac{\pi}{2}\notin\{\arg(\lambda_\mu(x')-\lambda_\nu(x'))\,;\ 1\leqq\mu,\ \nu\leqq\kappa,\ \mu\neq\nu,\ |x'|<\varepsilon\}. \tag{55}$$

If we assume that the solution $\hat{U}^{(i')}_{(J),\theta}$ of the equation $(53)_{i',J}$ can be continued to a solution on the whole of S_θ for $i'=0, 1, \cdots, i-1$, and $J=1, 2, \cdots, q-p$, then $\hat{F}^{(i)}_{(J),\theta}$ is a function which is already known on the whole of S_θ. The system $\{(53)_{i,J}\,;\ J=1, 2, \cdots, q-p\}$ is a system of Volterra's integral equation of the third kind, and the condition (55) means that we can continue $\{\hat{U}^{(i)}_{(J),\theta}\,;\ J=1, 2, \cdots, q-p\}$ to a solution on the whole of S_θ. Thus we can extend each $\hat{U}^{(i)}_{(J),\theta}$ on the whole of S_θ, inductively. We shall estimate this solution $\hat{U}^{(i)}_{(J)}=\hat{U}^{(i)}_{(J),\theta}$.

PROPOSITION 2. *Let $\theta\in[\pi, -\pi]$ satisfy* (55). *If ε, $\rho>0$ are small enough, there exists some constant $C>0$ such that $\hat{U}^{(i)}_{(J)}(x', s, \xi')$ satisfies*

$$(56)_i\qquad |\partial^\Delta_{x'}\partial^\delta_{\xi'}\hat{U}^{(i)}_{(J)}(x', s, \xi')|$$
$$\leqq C\rho^{-(6i+|\Delta+\delta|+J/(q-p))}\exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$
$$\times\sum_{k=0}^{qi}\sum_{l=0}^{\infty}\frac{\left[\dfrac{i+k}{q}+|\Delta+\delta|\right]!}{(k\,!)^{1/q}}\cdot\frac{|s|^{(i+J+l+q|\delta|)/(q-p)}(1+|\xi'|)^{l/q}}{\Gamma\left(\dfrac{i+J+l+q|\delta|}{q-p}+1\right)}$$

for any Δ, $\delta\in(Z_+)^n$ and $i\in Z_+$, provided x', $\xi'\in C^n$, $|x'|<\varepsilon$, and $s\in S_\theta$.

PROOF. It is trivial that $(56)_0$ is valid. In fact, since $U^{(0)}(x', \tau, \xi')=\tau^{-(q-p)}$, we have

$$\hat{U}^{(0)}_{(J)}(x', s, \xi')=\begin{cases} -2\pi sY(e^{-\sqrt{-1}\theta}s) & J=q-p,\\ 0 & J\neq q-p.\end{cases}$$

Now let $i\geqq 0$ be an integer. Assume that $(56)_0, \cdots, (56)_{i-1}$ is true. We shall prove in Appendix 2 that we then have

$$(57)_i\qquad |\partial^\Delta_{x'}\partial^\delta_{\xi'}\hat{F}^{(i)}_{(J)}(x', s, \xi')|$$
$$\leqq(1+|s|)C\rho^{-(6i+|\Delta+\delta|+J/(q-p))+1}\exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$
$$\times\sum_{k=0}^{qi}\sum_{l=0}^{\infty}\frac{\left(\left[\dfrac{i+k}{q}\right]+|\Delta+\delta|\right)!}{(k\,!)^{1/q}}\cdot\frac{|s|^{(i+J+l+q|\delta|)/(q-p)}(1+|\xi'|)^{l/q}}{\Gamma\left(\dfrac{i+J+l+q|\delta|}{q-p}+1\right)}$$

provided $|x'|<\varepsilon$.

Admitting this for the moment, we prove $(56)_i$ with $\Delta=\delta=0$ as follows: Assume that there exists a point $(x', s, \xi')\in C^n\in S\times C^n$ with $|x'|<\varepsilon$, at which the estimate $(56)_i$ does not hold with $\Delta=\delta=0$. Choosing such a point (x', s, ξ') arbitrarily, we fix these points $x'\in C^n$ and $\xi'\in C^n$. Define $\hat{s}\in[0, \infty]$ by

$$\hat{s}=\inf\{|s|\,;\ s\in S_\theta,\ \text{the estimate } (56)_i \text{ is not valid with } \Delta=\delta=0\}.$$

$\hat{s}\geqq r/2$ because of $(52)_i$. On the other hand, $\hat{s}\lneqq\infty$ because of the above assumption. In this way we are led to a contradiction just in the same manner as in Turrittin [8]. Thus we obtain $(56)_i$ with $\Delta=\delta=0$. The case $(\Delta, \delta)\gneqq 0$ can be proved analogously. Q. E. D.

COROLLARY. *Let $\theta\in[\pi, -\pi]$ satisfy (55). If $r>0$ is small enough, there exists a constant $C>0$ such that*

$$(58)\qquad |\hat{U}^{(i)}_{(J)}(x', s, \xi')|\leqq Cr^{-i}\{r^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}\frac{[i/q]!\,|s|^{[i/(q-p)]}}{[i/(q-p)]!}.$$

If $\mathrm{Im}(e^{\sqrt{-1}\,\theta}\sigma)>2r^{-1}(1+|\xi'|)^{(q-p)/q}$, then the function

$$(59)\qquad U^{(i)}_{(J)}(x', \sigma, \xi')=\frac{1}{2\pi}\int_{S_\theta} e^{\sqrt{-1}\,\sigma s}\hat{U}^{(i)}_{(J)}(x', s, \xi')ds$$

is well-defined. Furthermore, $U^{(i)}_{(J)}(x', \sigma, \xi')$ is holomorphic on

$$\Omega_\sigma=\{(x', \sigma, \xi')\in \boldsymbol{C}^n\times\boldsymbol{C}\times\boldsymbol{C}^n\,;\ |x'|<\varepsilon,\ \mathrm{Im}(e^{\sqrt{-1}\,\theta}\sigma)>2r^{-1}(1+|\xi'|)^{(q-p)/q}\}.$$

Since $\hat{U}^{(i)}_{(J)}(x', s, \xi')$ satisfies $(52)_{i,J}$, it follows that $U^{(i)}_{(J)}(x', \sigma, \xi')$ satisfies $(41)_{i,J}$ rigidly on Ω_σ.

In the same way, the function

$$U'^{(i)}_{(J)}(x', \sigma, \xi')=\frac{1}{2\pi}\int_{S_\theta} e^{\sqrt{-1}\,\sigma s}\hat{U}^{(i)}_{(J)}(x', s, \xi')(\sqrt{-1}\,s)^{-[i/(q-p)]}ds$$

is also well-defined and is holomorphic on Ω_σ. From (58) it follows that

$$(60)\qquad |U'^{(i)}_{(J)}(x', \sigma, \xi')|\leqq(q-1)Cr^{-i}\frac{\left[\frac{i}{q}\right]!}{\left[\frac{i}{q-p}\right]!}$$

on Ω_σ. Since

$$U^{(i)}_{(J)}(x', \sigma, \xi')=\left(\frac{\partial}{\partial\sigma}\right)^{[i/(q-p)]}U'^{(i)}_{(J)}(x', \sigma, \xi'),$$

it follows that

$$|U^{(i)}_{(J)}(x', \sigma, \xi')|\leqq Cr^{-i}\varepsilon^{-[i/(q-p)]}[i/q]!\,|\sigma|^{-[i/(q-p)]}$$

on

$$\Omega'_\sigma=\{(x', \sigma, \xi')\in \boldsymbol{C}^n\times\boldsymbol{C}\times\boldsymbol{C}^n\,;\ |x'|<\varepsilon,$$
$$\mathrm{Im}(e^{\sqrt{-1}\,\theta}\sigma)>\varepsilon|\sigma|+2r^{-1}(1+|\xi'|)^{(q-p)/q}\}.$$

We define

$$(61)\qquad U^{(i)}(x', \xi)=\sum_{J=1}^{q-p}\xi_0^{J/q}[U^{(i)}_{(J)}(x', \sigma, \xi')]_{\sigma=\xi_0(q-p)/q}\,.$$

Then $U^{(i)}(x', \xi)$ satisfies $(31)_i$ rigidly on

$$\Omega_{\xi_0}=\{(x',\xi)\in \boldsymbol{C}^n\times \boldsymbol{C}^{n+1};\ |x'|<\varepsilon,$$
$$\operatorname{Im}((e^{\sqrt{-1}q\theta/(q-p)}\xi_0)^{(q-p)/q})>2r^{-1}(1+|\xi'|)^{(q-p)/q}\},$$

and there exist some constants C, $r>0$ such that

$$|U^{(i)}(x',\xi)|\leqq Cr^{-i}[i/q]!|\xi_0|^{-((q-p)/p)[i/(q-p)]+(q-p)/q} \tag{62}$$

on

$$\Omega'_{\xi_0}=\{(x',\xi)\in \boldsymbol{C}^n\times \boldsymbol{C}^{n+1};\ |x'|<\varepsilon,$$
$$\operatorname{Im}((e^{\sqrt{-1}q\theta/(q-p)}\xi_0)^{(q-p)/q})>\varepsilon|\xi_0|^{(q-p)/q}+2r^{-1}(1+|\xi'|)^{(q-p)/q}\}.$$

Let us choose a θ which satisfies (55) and $\frac{\pi}{6}<\theta<\frac{\pi}{3}$. Such θ always exists if we take $\varepsilon>0$ small enough. It is easy to see that

$$\Omega'_{\xi_0}\supset\Omega''_{\xi_0}=\Big\{(x',\xi)\in \boldsymbol{C}^n\times \boldsymbol{C}^{n+1};\ |x'|<\varepsilon,\ 0<\arg\xi_0<q\pi/2(q-p),$$
$$|\xi_0|>\Big(\frac{2r^{-1}}{2^{-1}-\varepsilon}\Big)^{q/(q-p)}(1+|\xi'|)\Big\}.$$

Then (62) means that there exists a $\kappa\times\kappa$ matrix $U(x',D)$ of microdifferential operators of fractional order defined at $\mathring{x}^*=(0;\sqrt{-1},0,\cdots,0)$, such that

$$\sigma(U)(x',\xi)\sim\sum_i U^{(i)}(x',\xi) \tag{63}$$

in the sense of § 1. Then from $(28)_i$, $i=0,1,2,\cdots$, and (63) we can conclude

$$(x_0I_\kappa+B(x',D))D_0^{(q-p)/q}U(x',D)=D_0^{(q-p)/q}U(x',D)(x_0I_\kappa+B'(x',D)).$$

Furthermore we can construct the parametrix of $U(x',D)$ as follows. From (62) and (63) it follows that

$$|\sigma(U)(x',\xi)-\sum_{i\leqq 3q-3p}U^{(i)}(x',\xi)|\leqq C'|\xi_0|^{-(2q-2p)/q} \tag{64}$$

with some $C'>0$. For $i\leqq 3q-3p$, we define

$$U''^{(i)}_{(J)}(x',\sigma,\xi')$$
$$=\int_{S_\theta}e^{\sqrt{-1}\sigma s}\Big\{\hat{U}^{(i)}_{(J)}(x',s,\xi')-\sum_{\substack{j-J\in(q-p)Z\\ (j-J)/(q-p)\geqq-3\\ \alpha\in(Z_+)^n}}U^{(i)}_{j,\alpha}(x')\frac{(-\sqrt{-1}s)^{|j-J|/(q-p)-1}}{\Big(\frac{|j-J|}{q-p}-1\Big)!}\xi'^\alpha\Big\}$$
$$\times(\sqrt{-1}s)^{-3}ds.$$

From (62) and (63) we have

$$|U''^{(i)}_{(J)}(x',\sigma,\xi')|\leqq C''$$

with some $C''>0$. We have

$$U^{(i)}(x',\xi)=\sum_J\Big(\sum_{(j-J)/(q-p)\geqq-3,\ \alpha\in(Z_+)^n}U^{(i)}_{j,\alpha}(x')\xi_0^{j/q}\xi'^\alpha\Big)$$
$$+\sum_J\xi_0^{J/q}\Big[\Big(\frac{\partial}{\partial\sigma}\Big)^3U''^{(i)}_{(J)}(x',\sigma,\xi')\Big]_{\sigma=\xi_0}(q-p)/q.$$

The first term in the right-hand side is of order $|\xi_0|^{-(i+q-p)/q}$ because $\frac{j}{q}+|\alpha| \leqq -(i+q-p)/q$ in the summation, and the second term is of order $|\xi_0|^{-(2q-2p)/q}$. Thus

$$|U^{(i)}(x', \xi)| \leqq C''' |\xi_0|^{-(q-p+1)/q} \tag{65}$$

with some $C'''>0$, provided $1 \leqq i \leqq 3q-3p$. Since $U^{(0)}=\xi_0^{-(q-p)/q}$ we can construct the parametrix of $U(x', D)$ using (62), (64) and (65), just in the same way as in Sato-Kawai-Kashiwara [8].

Thus we have transformed the operator

$$x_0 I_\kappa + B(x', D)$$

into

$$x_0 I_\kappa + B'(x', D).$$

REMARK. Note that $U(x', D)$ is a matrix of 0th order microdifferential operators of fractional order. This means that we can transform the original matrix $x_0 I_\kappa + A(x', D)$ into $x_0 I_\kappa + B'(x', D)$ using only matrices of 0th order microdifferential operators of fractional order. Now we remind the reader the fact that $B'(x', D)$ is a diagonal matrix. We shall transform $x_0 I_\kappa + B'(x', D)$ furthermore into the canonical form $x_0 I_\kappa$ in the next section. For this purpose we shall need infinite order holomorphic microlocal operators.

§ 4. A diagonal matrix of microdifferential operators.

The purpose of this section is to find a matrix $V(x', D)$ of holomorphic microlocal operators such that

$$(x_0 I_\kappa + B'(x', D))V(x', D) = V(x', D) x_0 I_\kappa$$

where $B'(x', D)$ is the diagonal matrix of microdifferential operators of fractional order introduced in § 3. The symbol of $B'(x', D)$ was given by

$$\sigma(B')(x', \xi) = \sum_{-(p/q) \geqq j \geqq -1} B_j(x', \xi),$$

where $B_j(x', \xi)$ is homogeneous in ξ of order j.

We denote by $\mathring{x}^*$ the point $(0; \sqrt{-1}, 0, \cdots, 0) \in \sqrt{-1}\, T^* \boldsymbol{R}^{n+1}$, as before. Define $X(x', D_0) \in \mathcal{E}^R_{\mathring{x}^*}$ by

$$\sigma(X)(x', \xi_0) = \exp(\mathcal{X}(x', \xi_0)).$$

Here $\mathcal{X}(x', \xi_0)$ is a function of the form

$$\mathcal{X}(x', \xi_0) = \sum_{\substack{j \in (1/q)\boldsymbol{Z} \\ 1-(p/q) \geqq j \geqq (1/q)}} \mathcal{X}_j(x') \xi_0^j + \mathcal{X}_0(x') \log \xi_0,$$

where $\mathcal{X}_j(x')$ is a diagonal matrix of holomorphic functions defined on $|x'|<\varepsilon$, which will be determined later.

It is easy to see that the parametrix $X^{-1}(x', D_0)$ of $X(x', D_0)$ is given by

(66) $$\sigma(X^{-1})(x', \xi_0)=\exp(-\mathcal{X}(x', \xi_0)).$$

In this paper we always denote by $\sigma(H)$ the total symbol of a holomorphic microlocal operator H, and not the principal symbol. (66) is true because

(67) $$\frac{\partial}{\partial x_0}\sigma(X)=\frac{\partial}{\partial \xi_1}\sigma(X)=\cdots=\frac{\partial}{\partial \xi_n}\sigma(X)=0$$

and $\sigma(X)$ is a diagonal matrix.

Now we have

$$\begin{aligned}&\sigma(X^{-1}(x', D_0)B'(x', D)X(x', D_0))(x', \xi)\\ &\sim\sum_{i=0}^{\infty}\sum_{|\delta|=i}\frac{1}{\delta!}\partial_{\xi'}^{\delta}\sigma(X^{-1}(x', D_0)B'(x', D))\partial_{x'}^{\delta}\sigma(X(x', D_0))\\ &=\sum_{i=0}^{\infty}\sum_{|\delta|=i}\frac{1}{\delta!}\partial_{\xi'}^{\delta}\sigma(B')(x', \xi)\partial_{x'}^{\delta}\exp(\mathcal{X}(x', \xi_0))\times\exp(-\mathcal{X}(x', \xi_0)).\end{aligned}$$

The last equality is valid because of our special situations that all the matrices are diagonal and that (67) holds.

To investigate this asymptotic expansion more precisely, it is convenient to prepare the following

LEMMA 5. *Let $\delta\in(\boldsymbol{Z}_+)^n$ and $j\in\boldsymbol{Z}_+$ satisfy $0\leqq j\leqq|\delta|$. Then there exist holomorphic functions $C_{\delta j}(x', \xi_0)$ which are defined on $\{|x'|<\varepsilon,\ |\xi_0|>1/\varepsilon\}$ ($\varepsilon>0$ is a small constant), such that*

(68) $$\partial_{x'}^{\delta}\exp(\mathcal{X}(x', \xi_0))=\sum_{j=0}^{|\delta|}C_{\delta j}(x', \xi_0)\exp(\mathcal{X}(x', \xi_0))$$

and that with some constants $C, r>0$ we have

(69) $$|\partial_{x'}^{\Delta}C_{\delta j}(x', \xi_0)|\leqq C(|\Delta|+j)!\,r^{-|\Delta|-2|\delta|}|\xi_0|^{((q-p)/q)(|\delta|-j)}$$

for any δ, $\Delta\in(\boldsymbol{Z}_+)^n$ and $j\in\boldsymbol{Z}_+$.

PROOF. If $\delta=0$, it is enough to define $C_{00}=1$. Let $\delta\gneqq 0$. Assume that for any $\alpha\in(\boldsymbol{Z}_+)^n$ satisfying $\alpha\lneqq\delta$, we have constructed the desired function $C_{\alpha j}(x', \xi_0)$, $0\leqq j\leqq|\alpha|$. Without loss of generality, we may assume that $\delta_n\neq 0$. Define $\alpha\in(\boldsymbol{Z}_+)^n$ by

$$\alpha=(\delta_1, \cdots, \delta_{n-1}, \delta_n-1).$$

Then we have

$$\begin{aligned}\partial_{x'}^{\delta}\exp(\mathcal{X}(x', \xi_0))&=\partial_{x_n}\partial_{x'}^{\alpha}\exp(\mathcal{X}(x', \xi_0))\\ &=\partial_{x_n}\Big\{\sum_{j=0}^{|\delta|-1}C_{\alpha j}(x', \xi_0)\exp(\mathcal{X}(x', \xi_0))\Big\}\\ &=\sum_{j=0}^{|\delta|-1}\{\partial_{x_n}C_{\alpha j}(x', \xi_0)+C_{\alpha j}(x', \xi_0)\partial_{x_n}\mathcal{X}(x', \xi_0)\}\\ &\quad\times\exp(\mathcal{X}(x', \xi_0)).\end{aligned}$$

We define $C_{\delta j}(x', \xi_0)$, $0 \leqq j \leqq |\delta|$, by

$$C_{\delta j} = \begin{cases} C_{\alpha 0}\partial_{x_n}\mathscr{X} & \text{if } j=0, \\ \partial_{x_n}C_{\alpha, j-1} + C_{\alpha j}\partial_{x_n}\mathscr{X} & \text{if } 1 \leqq j \leqq |\delta|-1, \\ \partial_{x_n}C_{\delta, j-1} & \text{if } j=|\delta|. \end{cases}$$

The estimate (69) can be proved easily. Q.E.D.

COROLLARY.

$$C_{\delta 0} = (\partial_{x_1}\mathscr{X})^{\delta_1}(\partial_{x_2}\mathscr{X})^{\delta_2}\cdots(\partial_{x_n}\mathscr{X})^{\delta_n}. \tag{70}$$

Thus we have

$$\sigma(X^{-1}B'X)(x', \xi) \sim \sum_{i=0}^{\infty}\Big(\sum_{|\delta|=i}\sum_{j=0}^{|\delta|} X_{\delta j}(x', \xi)\Big) \tag{71}$$

where we have defined

$$X_{\delta j}(x', \xi) = \frac{1}{\delta!}\partial_{\xi'}^{\delta}\sigma(B')(x', \xi)C_{\delta j}(x', \xi_0).$$

From (69), it follows that

$$|X_{\delta j}(x', \xi)| \leqq Cr^{-|\delta|}j!\,|\xi_0|^{-(p/q)-|\delta|+((q-p)/q)(|\delta|-j)} \tag{72}$$

with some constants $C>0$ and $r>0$.

Defining $\Phi_i(x', \xi)$, $i \in \boldsymbol{Z}_+$, by

$$\Phi_i(x', \xi) = \sum_{|\delta|=i}\sum_{j=0}^{|\delta|} X_{\delta j}(x', \xi)$$

(71) can be rewritten as

$$\sigma(X^{-1}B'X)(x', \xi) \sim \sum_{i=0}^{\infty}\Phi_i(x', \xi). \tag{73}$$

From (72) it follows easily that for any $i \in \boldsymbol{Z}_+$, we have

$$|\Phi_i(x', \xi)| \leqq Cr^{-i}i!\,|\xi_0|^{-(p/q)-i}\exp(r^{-1}|\xi_0|^{(q-p)/q}).$$

On the other hand, we define $\Psi_j(x', \xi)$, $j=0, 1, 2, \cdots$, by

$$\Psi_j(x', \xi) = \sum_{|\delta| \geqq j} X_{\delta j}(x', \xi).$$

From (72) it follows that for any $j \in \boldsymbol{Z}_+$, we have

$$|\Psi_j(x', \xi)| \leqq Cr^{-j}j!\,|\xi_0|^{-(p/q)-j}. \tag{74}$$

Thus $\sum_{i=0}^{\infty}\Psi_j(x', \xi)$ defines some holomorphic microlocal operator. Furthermore, it also follows from (72) that for any $i \in \boldsymbol{Z}_+$, we have

$$\Big|\sum_{j=0}^{i}\Phi_j(x', \xi) - \sum_{j=0}^{i}\Psi_j(x', \xi)\Big| \leqq Cr^{-i}i!\,|\xi_0|^{-i}\exp(r^{-1}|\xi_0|^{(q-p)/q}). \tag{75}$$

This means that the two operators defined by $\sum_{i=0}^{\infty}\Phi_i(x', \xi)$ and by $\sum_{j=0}^{\infty}\Psi_j(x', \xi)$ respectively, coincide. (This fact is proved by Aoki [2].) Now from (74) we conclude that $X^{-1}(x', D_0)B'(x', D)X(x', D_0)$ is a finite-order microdifferential operator of fractional order, despite of the fact that $X(x', D_0)$ and $X^{-1}(x', D_0)$ are holomorphic microlocal operators of infinite order.

From (74) and (75) we have

$$\begin{aligned}&\sigma(X^{-1}(x', D_0)\{x_0 I_\kappa + B'(x', D)\}X(x', D_0))(x', \xi)\\&\equiv x_0 I_\kappa - \frac{\partial}{\partial\xi_0}\mathfrak{X}(x', \xi_0) + \Psi_0(x', \xi) \qquad (\text{mod } 0(|\xi_0|^{-(p/q)-1})).\end{aligned}$$

We define $\tilde{\mathfrak{X}}(x', \xi_0)$ by

$$\tilde{\mathfrak{X}}(x', \xi_0) = \sum_{k=1/q}^{(q-p)/q}\mathfrak{X}_k(x')\xi_0^k.$$

Then from (70) we have

$$X_{\delta 0} \equiv \frac{1}{\delta!}(\partial_{x_1}\tilde{\mathfrak{X}})^{\delta_1}\cdots(\partial_{x_n}\tilde{\mathfrak{X}})^{\delta_n}\partial_{\xi'}^{\delta}\sigma(B') \qquad (\text{mod } 0(|\xi_0|^{-1-(1/2q)})).$$

Thus we have

$$\Psi_0(x', \xi) \equiv \sum_{j=-(p/q), -(p+1)/q, \cdots, -1}\Psi_0^{(j)}(x', \xi) \qquad (\text{mod } 0(|\xi_0|^{-1-(1/2q)}))$$

where $\Psi_0^{(j)}(x', \xi)$, $-\frac{p}{q} \geqq j \geqq -1$, are some functions homogeneous of order j with respect to ξ, and which are composed of

$$\mathfrak{X}_{(q-p)/q}(x'), \cdots, \mathfrak{X}_{j+1+(1/q)}(x')$$

and $\sigma(B')(x', \xi)$.

Now we define $\mathfrak{X}_{(q-p)/q}(x'), \cdots, \mathfrak{X}_0(x')$ inductively by

$$\begin{cases}\mathfrak{X}_j(x') = \dfrac{1}{j}\Psi_0^{(j-1)}(x', \xi_0, 0)\xi_0^{-j+1} & \dfrac{q-p}{q} \geqq j \geqq \dfrac{1}{q},\\ \mathfrak{X}_0(x') = \Psi_0^{(-1)}(x', \xi_0, 0)\xi_0 .\end{cases}$$

It follows that

$$\begin{aligned}&\sigma(X^{-1}(x', D_0)\{x_0 I_\kappa + B'(x', D)\}X(x', D_0))(x', \xi)\\&= x_0 I_\kappa + \sum_{j=-(p/q), -(p+1)/q, \cdots, -1}B_j''(x', \xi) + B'''(x', \xi),\end{aligned}$$

where $B_j''(x', \xi)$, $-\frac{p}{q} \geqq j \geqq -1$, are homogeneous of order j with respect to ξ vanishing at $\xi'=0$, and $B'''(x', \xi)$ satisfies

$$|B'''(x', \xi)| \leqq \text{const.}\, |\xi_0|^{-1-(1/2q)}.$$

Let $B''(x', D)$ be defined by

$$\sigma(B'')(x', \xi)=\sum_{j=-(p/q),-(p+1)/q,\cdots,-1} B''_j(x', \xi)+B'''(x', \xi).$$

It follows that

$$|\sigma(B'')(x', \xi)|\leqq \text{const.}\,|\xi_0|^{-1-(1/2q)}(1+|\xi'|).$$

Thus we have proved the

PROPOSITION 3. *Let $\varepsilon>0$ be small enough. Then we have*

$$X^{-1}(x', D_0)\{x_0 I_\kappa+B'(x', D)\}X(x', D_0)=x_0 I_\kappa+B''(x', D), \tag{76}$$

where there exist some constants $a>0$ and $R>0$ such that

$$|\partial_{\xi'}^{\Delta}\sigma(B'')(x', \xi)|<aR^{-|\Delta|}\Delta!\,|\xi_0|^{-1-(1/2q)-|\Delta|}(1+|\xi'|) \tag{77}$$

for any $\Delta\in(\boldsymbol{Z}_+)^n$, on Γ_ε.

In this proposition, $B''(x', D)$ is a diagonal matrix of holomorphic microlocal operators. But this fact is not important. In fact the remainder of this paper is valid for any $\kappa\times\kappa$ matrix of holomorphic microlocal operators whose symbol satisfies the inequality (77).

Now we shall construct a matrix $Z(x', D)$ of holomorphic microlocal operators such that

$$\{x_0 I_\kappa+B''(x', D)\}Z(x', D)=Z(x', D)x_0 I_\kappa$$

by successive approximation. We define $Z^{(i)}(x', \xi)$, $i=0, 1, \cdots$, inductively by

$$\begin{cases} Z^{(0)}(x', \xi)=I_\kappa \\ Z^{(i)}(x', \xi)=\sum \dfrac{1}{\delta!}\displaystyle\int_\infty^{\xi_0}\partial_{\xi'}^{\delta}\sigma(B'')(x', \xi)\partial_{x'}^{\delta}Z^{(i')}(x', \xi)d\xi_0, \end{cases} \tag{78}$$

where the summation is taken over $\{(i', \delta)\in\boldsymbol{Z}_+\times(\boldsymbol{Z}_+)^n;\, i'+|\delta|+1=i\}$. We have the following

PROPOSITION 4. *Let $\varepsilon>0$ be small enough. Then there exist constants $C>0$ and $r>0$ such that $Z^{(i)}(x', \xi)$, $i=0, 1, \cdots$, are holomorphic on Γ_ε and satisfy*

$$|\partial_{x'}^{\Delta}Z^{(i)}(x', \xi)|\leqq C\sum_{j=0}^{i}\frac{|\xi_0|^{-j/2q}(1+|\xi'|)^j}{j!}|\xi_0|^{-i+j}(i-j+|\Delta|)!\,r^{-2i-|\Delta|} \tag{79}$$

on Γ_ε for any $i\in Z_+$ and $\Delta\in(\boldsymbol{Z}_+)^n$.

PROOF. If $i=0$, the assertion is trivial. Assume that $i\geqq 1$ and that $Z^{(0)}, \cdots, Z^{(i-1)}$ satisfy the statement. Let $(x', \xi)\in\Gamma_\varepsilon$ and $\tilde{\xi}_0=t\xi_0$, $t\in[1, \infty]$. Then $(x', \tilde{\xi}_0, \xi')\in\Gamma_\varepsilon$. This means that $Z^{(i)}(x', \xi)$ is well-defined on Γ_ε. From (77), (78) and (79) it follows that

$$
\begin{aligned}
&|Z^{(i)}(x', \xi)| \\
&\leqq \sum_{i'+|\Delta|+1=i} \frac{1}{\Delta!}\left|\int_{\infty}^{\xi_0} \partial_{\xi'}^{\Delta} \sigma(B'')(x', \tilde{\xi}_0, \xi') \partial_{x'}^{\Delta} Z^{(i')}(x', \tilde{\xi}_0, \xi') d\tilde{\xi}_0\right| \\
&\leqq \sum_{i'+|\Delta|+1=i} \frac{1}{\Delta!}\int_{|\xi_0|}^{\infty} a\Delta!\, R^{-|\Delta|}|\tilde{\xi}_0|^{-(1/2q)-1-|\Delta|}(1+|\xi'|) \\
&\qquad\qquad \times C\sum_{j=0}^{i'} \frac{|\tilde{\xi}_0|^{-(j/2q)}(1+|\xi'|)^j}{j!}|\tilde{\xi}_0|^{-i'+j}(i'-j+|\Delta|)!\, r^{-2i'-|\Delta|} d|\tilde{\xi}_0| \\
&\leqq aC \sum_{\substack{i'+|\Delta|+1=i \\ 0\leqq j\leqq i'}} R^{-|\Delta|} r^{-2i'-|\Delta|}(i'-j+|\Delta|)!\, |\xi_0|^{-|\Delta|-i'+j}(1+|\xi'|)^{j+1} \\
&\qquad\qquad \times \frac{1}{j!}\int_{|\xi_0|}^{\infty} |\tilde{\xi}_0|^{-(j+1)/2q-1} d|\tilde{\xi}_0| \\
&\leqq 2qaC \sum_{i'+|\delta|+1=i}\left(\sum_{j=0}^{i-1} \frac{|\xi_0|^{-(j+1)/2q}(1+|\xi'|)^{j+1}}{(j+1)!}|\xi_0|^{-i+j+1}(i-j-1)!\right)(r/R)^{|\delta|} r^{-2i+2} \\
&\leqq C\sum_{j=1}^{i} \frac{|\xi_0|^{-j/2q}(1+|\xi'|)^j}{j!}|\xi_0|^{-i+j}(i-j)!\, r^{-2i},
\end{aligned}
$$

if $r\leqq\min(R/2, 4qa)$. Thus we have proved (79) with $\Delta=0$. The case $\Delta\gneqq 0$ can be proved in a similar way. Q.E.D.

The inequality (79) means that

$$|Z^{(i)}(x', \xi)|\leqq Cr^{-i}i!\,|\xi_0|^{-i}\exp\{r^{-1}|\xi_0|^{1-(1/4q)}\}$$

on Γ_ε with some constant $C>0$ and $r>0$. Thus there exists a matrix $Z(x', D)$ of holomorphic microlocal operators satisfying

$$\{x_0 I_\kappa + B''(x', D)\}Z(x', D) = Z(x', D)x_0 I_\kappa$$

and

$$\sigma(Z)(x', \xi)\sim\sum_{i=0}^{\infty} Z^{(i)}(x', \xi).$$

Now we construct the parametrix $\tilde{Z}(x', D)$ of $Z(x', D)$ as follows: We define $\tilde{Z}^{(i)}(x', \xi)$, $i=0, 1, \cdots$, by

$$
(80)\qquad
\begin{cases}
\tilde{Z}^{(0)}(x', \xi) = I_\kappa \\
\tilde{Z}^{(i)}(x', \xi) = -\displaystyle\sum_{i'+|\delta|+1=i} \frac{1}{\delta!}\int_{\infty}^{\xi_0} \partial_{\xi'}^{\delta} \tilde{Z}^{(i')}(x', \xi) \partial_{x'}^{\delta} \sigma(B'')(x', \xi) d\xi_0 .
\end{cases}
$$

Just like $Z^{(i)}(x', \xi)$, we can prove that $\tilde{Z}^{(i)}(x', \xi)$, $i=0, 1, \cdots$, are holomorphic on Γ_ε and satisfy

$$
(81)\qquad |\tilde{Z}^{(i)}(x', \xi)|\leqq C\sum_{j=0}^{i} \frac{|\xi_0|^{-j/2q}(1+|\xi'|)^j}{j!}|\xi_0|^{-i+j}(i-j)!\, r^{-i}.
$$

Thus we can conclude that there exists a matrix $\tilde{Z}(x', D)$ of holomorphic microlocal operators satisfying

$$\tilde{Z}(x', D)\{x_0 I_\kappa + B''(x', D)\} = x_0 I_\kappa \tilde{Z}(x', D)$$

and

$$\sigma(\tilde{Z})(x', \xi) \sim \sum_{i=0}^{\infty} \tilde{Z}^{(i)}(x', \xi).$$

We define $\tilde{W}(x', D)$ by

$$\tilde{W}(x', D) = Z(x', D)\tilde{Z}(x', D).$$

Then we have

$$\sigma(\tilde{W})(x', \xi) \sim \sum_{i=0}^{\infty} \tilde{W}^{(i)}(x', \xi),$$

where

$$\tilde{W}^{(i)}(x', \xi) = \sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} Z^{(j)}(x', \xi) \partial_{x'}^{\delta} \tilde{Z}^{(k)}(x', \xi) \tag{82}$$

Now we have

$$\frac{\partial}{\partial \xi_0} \tilde{W}^{(i)} = \sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi_0} \partial_{\xi'}^{\delta} Z^{(j)} \partial_{x'}^{\delta} \tilde{Z}^{(k)} + \sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} Z^{(j)} \partial_{\xi_0} \partial_{x'}^{\delta} \tilde{Z}^{(k)}.$$

From (78), it follows that

$$\begin{aligned} &\sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} \partial_{\xi_0} Z^{(j)} \partial_{x'}^{\delta} \tilde{Z}^{(k)} \\ &= \sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} \Big\{ \sum_{j'+|\delta'|+1=j} \frac{1}{\delta'!} \partial_{\xi'}^{\delta'} \sigma(B'') \partial_{x'}^{\delta'} Z^{(j')} \Big\} \partial_{x'}^{\delta} \tilde{Z}^{(k)}. \end{aligned}$$

From (82), we can prove that the right-hand side of this equation is equal to

$$\sum_{i'+|\delta|+1=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} \sigma(B'') \partial_{x'}^{\delta} \tilde{W}^{(i')}.$$

Similarly we have

$$\sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} Z^{(j)} \partial_{x'}^{\delta} \partial_{\xi_0} \tilde{Z}^{(k)} = - \sum_{i'+|\delta|+1=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} \tilde{W}^{(i')} \partial_{x'}^{\delta} \sigma(B'').$$

Thus we have

$$\frac{\partial}{\partial \xi_0} \tilde{W}^{(i)} = \sum_{i'+|\delta|+1=i} \frac{1}{\delta!} \{ \partial_{\xi'}^{\delta} \sigma(B'') \partial_{x'}^{\delta} \tilde{W}^{(i')} - \partial_{\xi'}^{\delta} \tilde{W}^{(i')} \partial_{x'}^{\delta} \sigma(B'') \}. \tag{83}$$

It is trivial that $\tilde{W}^{(0)} = 1$. Assume that $i \geqq 1$ and that

$$\tilde{W}^{(1)} = \cdots = \tilde{W}^{(i-1)} = 0.$$

Then (83) means that $\dfrac{\partial}{\partial \xi_0} \tilde{W}^{(i)} = 0$. On the other hand, from (79) and (81) we have

$$\begin{aligned} |\tilde{W}^{(i)}| &= \Big| \sum_{j+k+|\delta|=i} \frac{1}{\delta!} \partial_{\xi'}^{\delta} Z^{(j)} \partial_{x'}^{\delta} \tilde{Z}^{(k)} \Big| \\ &\leqq \text{const.} \sum_{j+k+|\delta|=i} \{ |\xi_0|^{-(j/2q)-|\delta|} (1+|\xi'|)^j |\xi_0|^{-(k/2q)-|\delta|} (1+|\xi'|)^k \} \\ &\leqq \text{const.} |\xi_0|^{-(i/2q)} (1+|\xi'|)^i. \end{aligned}$$

Thus we have $\tilde{W}^{(i)} = 0$. Now we have proved that

$$Z(x', D)\tilde{Z}(x', D) = \tilde{W}(x', D) = I_{\kappa}.$$

Similarly we can prove that

$$\tilde{Z}(x', D)Z(x', D)=I_\kappa .$$

Summing up, we have proved that $V(x', D)=X(x', D)Z(x', D)$ satisfies

$$\{x_0 I+B'(x', D)\}V(x', D)=V(x', D)x_0 I_\kappa$$

and that $V(x', D)$ is an invertible matrix of holomorphic microlocal operators. This proves Theorem 2.

§ 5. Proof of Theorem 1.

Let $u, f\in \mathcal{C}_{\dot{x}^*}$. Using the notation of § 2, we define

$$\vec{u}=W^{-1}(x, D)\begin{pmatrix}1 & & & \\ & D_0^{1/\iota} & & \\ & & \ddots & \\ & & & D_0^{(\kappa-1)/\iota}\end{pmatrix}\begin{pmatrix}u \\ x_0 u \\ \vdots \\ x_0^{\kappa-1}u\end{pmatrix}$$

and

$$\vec{f}=\begin{pmatrix}1 & & & \\ & D_0^{1/\iota} & & \\ & & \ddots & \\ & & & D_0^{(\kappa-1)/\iota}\end{pmatrix}\begin{pmatrix}0 \\ 0 \\ \vdots \\ f\end{pmatrix}.$$

Then the equation $Pu=f$ is equivalent to $(x_0 I_\kappa+A(x', D))\vec{u}=\vec{f}$. Applying Theorem 2 to this equation, there exists an invertible matrix of holomorphic microlocal operators $E(x', D)$ such that $x_0 E(x', D)u=E(x', D)f$.

It is easy to see that

$$\mathrm{Ker}_{\oplus_\kappa \mathcal{C}_{\dot{x}^*}}(x_0 I_\kappa)=\bigoplus_\kappa(\delta(x_0)\otimes \mathcal{B}_N)$$

and

$$E^{-1}(x', D):\mathrm{Ker}_{\oplus_\kappa \mathcal{C}_{\dot{x}^*}}(x_0 I_\kappa)\xrightarrow{\sim}\mathrm{Ker}_{\oplus_\kappa \mathcal{C}_{\dot{x}^*}}(x_0 I_\kappa+A(x', D)).$$

Define $\vec{v}=W(x, D)\vec{u}$. Then it is easy to see that

$$\begin{array}{ccc}\mathrm{Ker}_{\oplus_\kappa \mathcal{C}_{\dot{x}^*}}(x_0 I_\kappa+A(x', D)) & \xrightarrow{\sim} & \mathrm{Ker}_{\mathcal{C}_{\dot{x}^*}}P, \\ \Cup & & \Cup \\ \vec{u} & \mapsto & v_1\end{array}$$

where we have denoted by v_1 the first element of the κ vector $\vec{v}$.

On the other hand, $\mathrm{Cok}_{\mathcal{C}_{\dot{x}^*}}P=0$ because $\mathrm{Cok}_{\oplus_\kappa \mathcal{C}_{\dot{x}^*}}(x_0 I_\kappa)=0$.

Appendix 1. Proof of $(38)_i$.

Let $i\geqq 1$. Assume that the estimates $(34)_0, \cdots, (34)_{i-1}$ are valid. Under this assumption, we prove the estimate $(38)_i$.

$F_{j\alpha}^{(i)}(x')$ was given by

$$F_{j\alpha}^{(i)}(x')=q\sum{}^{*}\frac{1}{\delta!}\cdot\frac{\beta!}{(\beta-\delta)!}B_{k\beta}(x')\partial_{x'}^{\delta}U_{l\gamma}^{(i')}(x')$$
$$-q\sum{}^{**}\frac{1}{\delta!}\cdot\frac{\gamma!}{(\gamma-\delta)!}U_{l\gamma}^{(i')}(x')\partial_{x'}^{\delta}B_{k\beta}(x')\,.$$

Here we have retained the notation used in §3. We denote by I_j, $j=1, 2$, the jth term of the right-hand side in this equation. Using $(34)_0, \cdots, (34)_{i-1}$ and (36) we have

$$|I_1|\leqq qaC\sum{}^{*}\binom{\beta}{\delta}R^{h-|\beta|}[|h|]!\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i'-l+(q-p)|\delta|\right]!}{((q^2+1)i'+q|\gamma|)!}\right)^{1/(q-p)}\times r^{(l-|\delta|)/2-|\gamma|-4qi'}.$$

We may assume that $R\leqq 1$. Now we have

$$[|h|]!\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i'-l+(q-p)|\delta|\right]!}{((q^2+1)i'+q|\gamma|)!}\right)^{1/(q-p)}$$
$$\leqq\left(\frac{\left[q^2+(q-p)\left(\frac{i'}{q}+|\delta|+|h|\right)-l\right]!}{((q^2+1)i'+q|\gamma|)!}\right)^{1/(q-p)}$$
$$\leqq\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j+1)\right]!}{((q^2+1)i+q|\alpha|-(q-p))!}\right)^{1/(q-p)}.$$

On the other hand, if $r\leqq\min\left(\frac{R}{8}, \frac{R}{2^q}, \frac{R}{24\cdot 4^n qa}\right)$, we have

$$qaC\sum{}^{*}2^{|\beta|}R^{h-|\beta|}r^{(l-|\delta|)/2-|\gamma|-4qi'}\leqq qaCr^{(j+1)/2-|\alpha|-4qi+1}\sum{}^{*}\left(\frac{2r}{R}\right)^{|\beta|}\left(\frac{r}{R}\right)^{|h|}r^{|\delta|}$$
$$\leqq\frac{1}{2}Cr^{(j+1)/2-|\alpha|-4qi+1}.$$

Thus we obtain

$$|I_1|\leqq\frac{1}{2}C\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j+1)\right]!}{((q^2+1)i+q|\alpha|-(q-p))!}\right)^{1/(q-p)}r^{(j+1)/2-|\alpha|-4qi+1}.$$

Similarly, we can prove

$$|I_2|\leqq\frac{1}{2}C\left(\frac{\left[\left(q^2+\frac{q-p}{q}\right)i-(j+1)\right]!}{((q^2+1)i+q|\alpha|-(q-p))!}\right)^{1/(q-p)}r^{(j+1)/2-|\alpha|-4qi+1}.$$

Thus we have proved $(38)_i$ with $\Delta=0$. The case with $\Delta\gneqq 0$ can be proved in a similar manner. Q.E.D.

Appendix 2. Proof of $(57)_i$.

Let $i\geqq 1$. Assume that $(56)_0, \cdots, (56)_{i-1}$ are valid. Under this assumption, we give the proof of the estimate $(57)_i$. We denote by Π_j, $j=1, \cdots, 5$, the jth term of the right-hand side of $(54)_{i,J}$. Furthermore we divide Π_5 into two parts: $\Pi_5=\Pi_{51}+\Pi_{52}$ where

$$\Pi_{51}=-\frac{1}{2\pi}\sum_{K+L=J+q-p}\ \sum_{\substack{\pi_2(i)\\ h\geqq -1}}$$

and

$$\Pi_{52}=-\frac{1}{2\pi}\sum_{K+L=J+q-p}\ \sum_{\substack{\pi_2(i)\\ h\leqq -1}}.$$

Using (47) and $(56)_0, \cdots, (56)_{i-1}$, we obtain

$$|\Pi_1|\leqq\frac{qa_1C}{2\pi}\sum_{K+L=J}\ \sum_{\pi_1(i)}\exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$
$$\times\frac{[|h|+|\delta|]!}{\delta!}R_1^{h-|\delta|}\rho^{-(6i'+|\delta|+L/(q-p))}\sum_{k=0}^{qi'}\sum_{l=0}^{\infty}\frac{\left[\frac{i'+k}{q}+|\delta|\right]!}{(k!)^{1/q}}(1+|\xi'|)^{l/q}$$
$$\times\int_0^{|s|}\frac{|t|^{((q|h|-p+q|\delta|+K)/(q-p)-1)}}{\Gamma\left(\frac{q|h|-p+q|\delta|+K}{q-p}\right)}\cdot\frac{(|s|-|t|)^{((i'+L+l)/(q-p))}}{\Gamma\left(\frac{i'+L+l}{q-p}+1\right)}d|t|.$$

We may assume that $R_1\leqq 1$. Now we have $\frac{[|h|+|\delta|]!}{\delta!}\leqq 2^{|h|+|\delta|}n^{|\delta|}[|h|]!$ and

$$[|h|]!\frac{\left[\frac{i'+k}{q}+|\delta|\right]!}{(k!)^{1/q}}\leqq\frac{\left[\frac{i'+q|h|+q|\delta|+k}{q}\right]!}{(k!)^{1/q}}=\frac{\left[\frac{i+k+q}{q}\right]!}{(k!)^{1/q}}.$$

Note that

$$\int_0^{|s|}\frac{|t|^{((q|h|-p+q|\delta|+K)/(q-p)-1)}}{\Gamma\left(\frac{q|h|-p+q|\delta|+K}{q-p}\right)}\cdot\frac{(|s|-|t|)^{((i'+L+l)/(q-p))}}{\Gamma\left(\frac{i'+L+l}{q-p}+1\right)}d|t|$$
$$=\frac{|s|^{((i+J+l)/(q-p)+1)}}{\Gamma\left(\frac{i+J+l}{q-p}+2\right)}\leqq(1+|s|)\frac{|s|^{((i+J+l)/(q-p))}}{\Gamma\left(\frac{i+J+l}{q-p}+1\right)}\cdot\frac{q-p}{i}.$$

Assume that $\rho\leqq\min\left(\frac{R_1}{8n},\frac{R_1}{2^{q+1}},\frac{\pi}{12\times 2^n q^2(q-p)^2a_1}\right)$. Then we have

$$|\Pi_1|\leqq(1+|s|)\frac{q(q-p)^2a_1}{2\pi}C\sum_{\pi_1(i)}\left(\frac{R_1}{2}\right)^h\left(\frac{R_1}{2n}\right)^{-|\delta|}\rho^{-(6i'+|\delta|+J/(q-p))}$$
$$\times\exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$
$$\times\sum_{k=0}^{qi-q}\sum_{l=0}^{\infty}\frac{\left[\frac{i+k+q}{q}\right]!}{(k!)^{1/q}}\cdot\frac{|s|^{((i+J+l)/(q-p))}(1+|\xi'|)^{l/q}}{\Gamma\left(\frac{i+J+l}{q-p}+1\right)\times i}$$
$$\leqq\frac{C}{6}(1+|s|)\rho^{-(6i+J/(q-p))+1}\exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\}$$
$$\times\sum_{k=0}^{qi-q}\sum_{l=0}^{\infty}\frac{\left[\frac{i+k+q}{q}\right]!}{((k+q)!)^{1/q}}\cdot\frac{|s|^{((i+J+l)/(q-p))}(1+|\xi'|)^{l/q}}{\Gamma\left(\frac{i+J+l}{q}+1\right)}.$$

We can estimate $\Pi_2, \cdots, \Pi_4$ and Π_{52} in the same way using $(56)_0, \cdots, (56)_{i-1}$ and (47) or (48). Using (49), we have

$$
\begin{aligned}
|\Pi_{51}| \leqq & \frac{qa_1C}{2\pi} \sum_{K+L=J+q-p} \sum_{\substack{\pi_2(i) \\ h\geqq -1}} \frac{|\delta|!}{\delta!} R_1^{-|\delta|}(1+|\xi'|)^{(2q-2p-K)/q} \rho^{-(6i'+|\delta|+L/(q-p))} \\
& \times \sum_{k=0}^{qi'} \sum_{l=0}^{\infty} \frac{\left[\frac{i'+k}{q}+|\delta|\right]}{(k!)^{1/q}} \cdot \frac{|s|^{((i'+L+l+q|\delta|)/(q-p))}(1+|\xi'|)^{l/q}}{\Gamma\left(\frac{i'+L+l+q|\delta|}{q-p}+1\right)} \\
& \times \int_0^{|s|} \exp\{R_1^{-1}(1+|\xi'|)^{(q-p)/q}|t|+\rho^{-1}(1+|\xi'|)^{(q-p)/q}(|s|-|t|)\}\, d|t| \\
\leqq & \frac{q^2(q-p)^2a_1C}{2\pi(\rho^{-1}-R_1^{-1})} \sum_{K+L=J+q-p} \sum_{\pi_2(i)} \left(\frac{R_1}{n}\right)^{-|\delta|} \rho^{-(6i'+|\delta|+L/(q-p))} \\
& \times \sum_{k=0}^{qi-q} \sum_{l=0}^{\infty} \frac{\left[\frac{i+k+q}{q}\right]!}{((k+q)!)^{1/q}} \cdot \frac{|s|^{((i+J+(q-p-K+l))/(q-p))}(1+|\xi'|)^{((q-p-K+l)/q)}}{\Gamma\left(\frac{i+J+(q-p-K+l)}{q-p}+1\right)} \\
& \times \exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\} \\
\leqq & \frac{C}{6}(1+|s|)\rho^{-(6i+J/(q-p))+1} \exp\{\rho^{-1}(1+|\xi'|)^{(q-p)/q}|s|\} \\
& \times \sum_{k=0}^{qi-q} \sum_{l=0}^{\infty} \frac{\left[\frac{i+k+q}{q}\right]!}{((k+q)!)^{1/q}} \cdot \frac{|s|^{((i+J+l)/(q-p))}(1+|\xi'|)^{l/q}}{\Gamma\left(\frac{i+J+l}{q-p}+1\right)}.
\end{aligned}
$$

Thus we have proved $(57)_i$ with $\Delta=0$. The case with $\Delta\gneqq 0$ can be proved in a similar manner. Q. E. D.

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