Some remarks on Dehn surgery along graph knots

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§ 1. Introduction.

A knot $K \subset S^3$ is said to be a *graph knot* when its complement $M = S^3 - \operatorname{Int} U(K)$ is a graph manifold, i.e., there is a family of disjoint tori $T = \cup T_i$ in $\operatorname{Int} M$, such that each component M_j of $M - \operatorname{Int} U(T)$ is an S^1-bundle over a surface, where $U(K)$, $U(T)$ denote regular neighborhoods of K, T respectively. In this case each M_j is an orientable S^1-bundle over an orientable punctured surface of genus 0, since M is contained in S^3 (cf. [6]). In this paper we give the theorems below (Theorems 2 and 3) concerning Dehn surgery on graph knots. In § 2 we examine some properties of 3-manifolds represented by plumbing graphs and characterize the reduced forms of the graphs which represent Seifert manifolds. In § 3, we determine the types of graph knots by using plumbing graphs and give the reduced forms of the graphs which represent the given graph knots (Theorem 1). Finally in § 4 we give proofs of the theorems stated below.

For a given knot K let $C(p, q: K)$ be the cable of K with linking number p and winding number q (we assume that $q \geqq 2$ and $(p, q)=1$) and let $\chi(K: r)$ be a manifold obtained by r-surgery on K where r is a rational number.

THEOREM 2. *For a nontrivial graph knot K and a rational number r, $\chi(K: r)$ is a Seifert manifold with an orientable orbit surface if and only if (K, r) is in the following list.*

(1) *$K=C(p, q)$ the (p, q) torus knot, r=any rational number other than pq.*
(2) *$K=C(p_1, q_1: C(p_2, q_2))$, $r=p_1q_1+1/s$ for any $s \in \boldsymbol{Z}$.*
(3) *$K=C(2p_1q_1+\delta, 2: C(p_1, q_1: C(p_2, q_2)))$, $r=4p_1q_1+\delta$ where $\delta=\pm 1$.*
(4) *$K=C(p, q) \# C(p', q')$, $r=pq+p'q'$.*

In cases (1) and (2), $\chi(K, r)$ is either a lens space or a Seifert manifold with orbit surface S^2 and 3 exceptional fibers. In case (3), $\chi(K, r)$ is a Seifert manifold with orbit surface S^2 and 3 exceptional fibers and in case (4), with 4 exceptional fibers.

REMARK. Cases (1), (2), and (3) are compatible with Corollary 7.4, Theorem 7.5 in Gordon [3], which have been obtained by different methods from ours. In (1) and (2), the cases when the resulting manifolds are lens spaces are exactly the same as the cases stated in Moser [7], Theorem 1 in Fintushel-Stern [1], and

Theorem 7.5 in [3]. See [1], [3], and [7] for the explicit description of the resulting lens spaces in these cases.

THEOREM 3. *Suppose that* $\chi(K, r)$ *is a non-prime manifold* $M=M_1\#\cdots\#M_k$ *for a graph knot* K *and* $r\in\mathbf{Q}$, *where* M_i *is a prime manifold not homeomorphic to* S^3 *for each* i. *Then* $k=2$ *and one of* M_i*'s is a lens space.*

In this paper we will assume that all the manifolds and maps are piecewise-linear and all the 3-manifolds are oriented without otherwise stated.

§ 2. Reduced plumbing graphs[1)]

Let Γ be an integrally weighted graph for plumbing which we call a plumbing graph. A vertex v of Γ of weight $a\in\mathbf{Z}$ corresponds to a D^2-bundle over S^2 of euler number a and an edge e of weight $\varepsilon=\pm1$ represents a plumbing whose intersection number is ε. We denote by P_Γ the 4-manifold associated with Γ and put $M_\Gamma=\partial P_\Gamma$. Then M_Γ has a graph structure as follows. Consider the vertex v of weight a whose valency (i.e., the number of edges which contain v) is r. Let Γ_v be the part of Γ in Figure 1. Then Γ_v represents a 3-manifold M_v which is constructed as follows. Let B_r be an r-punctured 2-sphere, D_0 be a 2-disk in $\operatorname{Int} B_r$, q_i for $i\geqq1$ (resp. q_0) be a curve $\partial_iB_r\times *$ in $\partial_iB_r\times S^1$ where ∂_iB_r denotes the i-th boundary component of B_r (resp. a curve $\partial D_0\times *\subset\partial D_0\times S^1$), and h be an S^1-fiber $*\times S^1\subset B_r\times S^1$. We assume that the orientation of q_i's and h are induced by the natural one of $B_r\times S^1$, so $\sum_{i=0}^{r} q_i$ is null-homologous in $(B_r-\operatorname{Int} D_0)\times *$. We fix the canonical coordinate of $\partial_iB_r\times S^1$ (resp. $\partial D_0\times S^1$) determined by the peripheral system $\{q_i, h\}$ (resp. $\{q_0, h\}$) and the coordinate of $\partial D^2\times S^1\subset D^2\times S^1$ determined by $\{\partial D^2\times *, *\times S^1\}$. Then M_v with the induced orientation from M_Γ is homeomorphic (with an orientation-preserving homeomorphism) to $(B_r-\operatorname{Int} D_0)\times S^1\bigcup_f D^2\times S^1$, where we assume that the orientation of $\partial D_0\times S^1$ (resp. $\partial D^2\times S^1$) is induced by the natural one of $D_0\times S^1$ (resp. $D^2\times S^1$) and $f:\partial D_0\times S^1\to\partial D^2\times S^1$ is represented by $\begin{pmatrix}1 & 0\\ -a & 1\end{pmatrix}$. Note that the orientation of $\partial D_0\times S^1$ is determined by $\{-q_0, h\}$ here. Then M_v is homeomorphic to $B_r\times S^1$. The edge e of weight ε corresponds to $M_e\cong T^2$ and if the vertices v and v' are joined by e, then M_e is a component of ∂M_v, say $\partial_1M_v=\partial_1B_r\times S^1$ (and a component of $\partial M_{v'}$, say $\partial_1M_{v'}$) and M_v and $M_{v'}$ are glued by a homeomorphism $g:\partial_1M_v\to\partial_1M_{v'}$ represented by $\begin{pmatrix}0 & \varepsilon\\ \varepsilon & 0\end{pmatrix}$. Note that if we replace the coordinate $\{q_1, h\}$ of ∂_1M_v by $\{-q_1, -h\}$,

1) W. D. Neumann [10] has studied the properties of plumbing graphs in more general context independently.

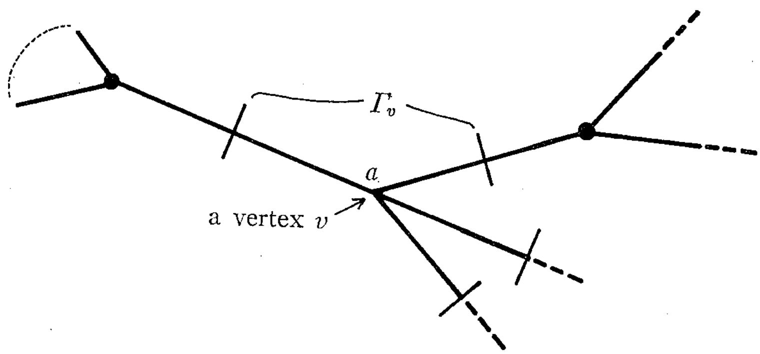
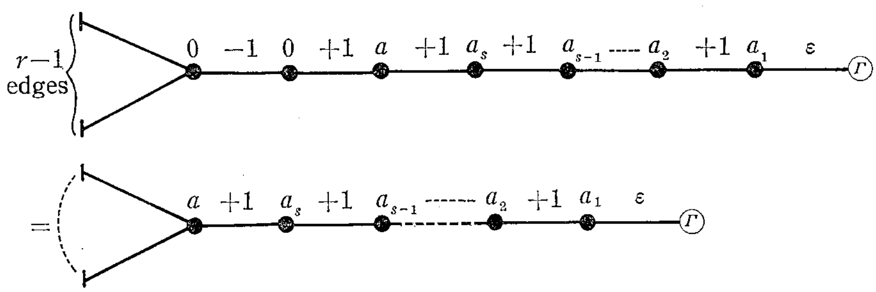


Figure 1.

then all the weights of the edges containing v reverse their signs. We also consider M_Γ with non-empty boundary, in which case Γ is represented as $\vdash$—Ⓐ, for example, where $\vdash$— corresponds to a boundary component of M_Γ. The following proposition shows that any oriented graph manifold whose any component is homeomorphic to (a punctured S^2) $\times S^1$ is represented by a plumbing graph. (The converse is also true.)

PROPOSITION 1. (i) *Let M_Γ be a 3-manifold corresponding to the graph $\vdash$—Ⓣ, where $\vdash$— corresponds to the boundary component $\partial' M_\Gamma \cong T^2$. Put $M = M_\Gamma \bigcup_g B_r \times S^1$ for a given homeomorphism $g: \partial_1 B_r \times S^1 \to \partial' M_\Gamma$ represented by $\begin{pmatrix} s & u \\ t & v \end{pmatrix}$ by the canonical coordinates, where s, t, u and v are integers such that $sv - tu = -1$. Then M is represented by the following graph.*



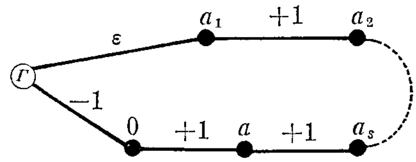
such that

$$\begin{pmatrix} s & u \\ t & v \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -a_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -a_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -a_s & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s-au & u \\ t-av & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

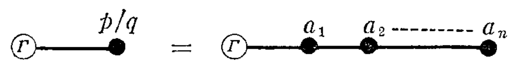
where $u/v = a_1 - 1/a_2 - 1/a_3 - \cdots - 1/a_s = [a_1, a_2, \cdots, a_s]$, $\varepsilon = +1$ or -1, and all *the other numbers are integers.*

(ii) *Let M be a manifold obtained from M_Γ by identifying the two boundary components by a homeomorphism $g: \partial_1 M_\Gamma \to \partial_2 M_\Gamma$ represented by $\begin{pmatrix} s & u \\ t & v \end{pmatrix}$ where M_Γ corresponds to the graph* ⊢—Ⓘ—⊣ *and the edges* ⊢— *represent $\partial_1 M_\Gamma$ and $\partial_2 M_\Gamma$. Then M is represented by the following graph.*

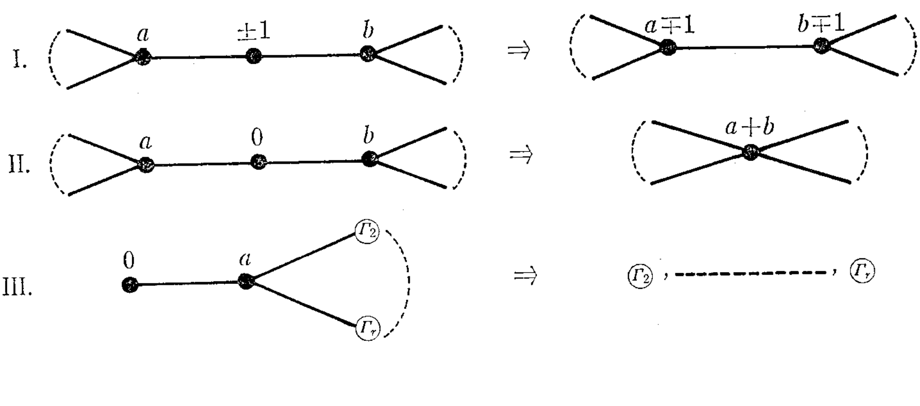


where a, a_i, and ε are as in (i).

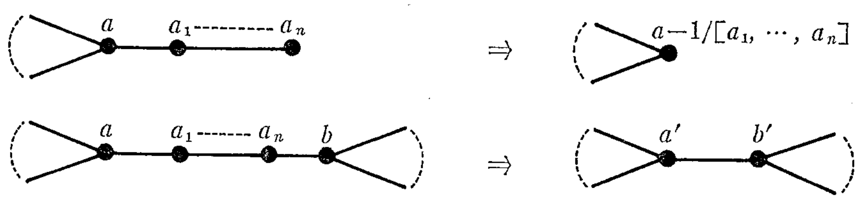
Proposition 1 is easily seen by the construction stated above and so we omit the proof. (cf. [2], [4] and [11].) In the presentation in (i) and (ii), the part of the vertex of weight a is inserted only for convenience's sake. The weight of the edge is irrelevant unless the edge lies on the cycle in the graph. Therefore we omit them if the graph is a tree. Note that we can represent M_Γ for any graph Γ by a framed link (see [11]). In the remainder of this paper we assume that all the graphs are trees unless otherwise stated. A graph manifold M_Γ is defined also when the weights of Γ are rational numbers. The following two graphs represent the same 3-manifold.



where $p, q, a_1, \cdots, a_n \in \boldsymbol{Z}$ and $p/q = [a_1, \cdots, a_n]$ (See [1], [11], [13]). We assume that any rational number is represented as p/q so that $p, q \in \boldsymbol{Z}$, $(p, q) = 1$, and $p \geqq 0$ from now on. Furthermore there are well-known reduction processes of Γ as an integrally weighted graph which do not alter the homeomorphism type of M_Γ as follows (See [11], [13]).



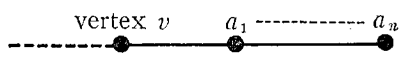
The following two processes are induced by I and II.



for some $a', b' \in \boldsymbol{Z}$, where $[a_1, \cdots, a_n]^{-1} \in \boldsymbol{Z}$ in both of the processes.

DEFINITION ([11], [13]). An integrally weighted plumbing graph Γ is said to be *reduced* if no more moves arising from I~III can be performed on it.

DEFINITION 2. A vertex of valency $\geqq 3$ is said to be a *multiple vertex*. A *linear branch* Γ_1 for a multiple vertex v is a linear subgraph of Γ which is attached to v as follows,



where all the vertices of Γ_1 are of valency 2 except for the one of the right hand side which is of valency 1.

DEFINITION 3. A *weight of a linear branch* Γ_1 for a multiple vertex v is a rational number $[a_1, \cdots, a_n]$. If this number is an inverse of an integer or equal to 0, then Γ_1 can be reduced away. Otherwise we call Γ_1 an *exceptional branch.*

In the remainder of this paper we assume that any graph is integrally weighted except for linear branches which are represented as $\vdash\!\!\!\overset{p/q}{\bullet}$ where p/q denotes the weight. We examine some basic properties of M_Γ for a plumbing graph Γ.

PROPOSITION. 2 *A graph manifold M_Γ with $\partial M_\Gamma \neq \emptyset$ which corresponds to a reduced connected graph Γ is irreducible.*

PROOF. We divide Γ into the subgraphs $\bigcup_{i=1}^{t} \Gamma_i$ by cutting along some edges so that Γ_i has at most 1 multiple vertex for each i. Then M_{Γ_i} is a Seifert manifold with boundary since Γ_i has no linear branch of weight 0 (cf. [14]). If $M_{\Gamma_i} \cap M_{\Gamma_j} \neq \emptyset$, then it is a torus and we denote it by T_{ij}. If $t \leqq 1$, then Proposition 2 is true by [5], II Lemma 2.3. Suppose that $t \geqq 2$. If T_{ij} is incompressible both in M_{Γ_i} and M_{Γ_j} for any i and j, then $M_\Gamma = \cup M_{\Gamma_i}$ is irreducible, and contains an incompressible torus. If T_{ij} is not incompressible in M_{Γ_i} for some i and j, then M_{Γ_i} is a solid torus. Therefore Γ_i has no multiple vertex since any Seifert fibration of a solid torus contains at most 1 exceptional fiber ([5], VI Lemma 3.3). Then the number of the component t is reduced by 1. Therefore Proposition 2 is proved by induction on t.

COROLLARY TO PROPOSITION 2. *Let M_Γ be as in Proposition 2. Then either each component of ∂M_Γ is incompressible or M_Γ is a solid torus.*

The proof is straightforward.

PROPOSITION 3. *Suppose that M_Γ is a solid torus for a connected reduced graph Γ. Then Γ is linear.*

PROOF. Proposition 3 is clear by the proof of Proposition 2.

The following two lemmas are necessary for the proofs of Theorems 2 and 3.

LEMMA 1. *Let Γ be a connected graph (not necessarily reduced). If M_Γ is not irreducible, then Γ is either reduced to $\overset{0}{\bullet}$ and hence $M_\Gamma \cong S^2\times S^1$, or Γ is reduced to a disconnected graph and hence M_Γ is non-prime.*

PROOF. Suppose that there is no linear branch of weight 0 in Γ. Decompose M_Γ as in the proof of Proposition 2. Since M_Γ is not irreducible, t must be reduced by 1 by the previous argument. If $t\leqq 1$, M_Γ is a Seifert manifold of type o_1 (we use the terminology in [12]). Then M_Γ is irreducible unless $M_\Gamma\cong S^2\times S^1$ ([12], [15]). But if $M_\Gamma\cong S^2\times S^1$, then Γ can be reduced to $\overset{0}{\bullet}$ (cf. Lemma 5.1 and Satz 5.13 in [11]). Thus Lemma 1 is proved by the reduction process III and induction.

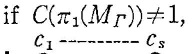
LEMMA 2. *Suppose that a reduced connected graph Γ contains more than 1 multiple vertices. Then M_Γ contains an incompressible torus and $\pi_1(M_\Gamma)$ has a trivial center.*

PROOF. We assume that $\partial M_\Gamma=\emptyset$, but the proof below can be modified so that it is also applicable to the case when $\partial M_\Gamma\neq\emptyset$. We take a multiple vertex v such that it has exactly one non-linear branch. Such a vertex exists since Γ is a tree. We decompose Γ as $\Gamma=\Gamma_1\cup\Sigma\cup\Gamma_2$ such that Γ_1 contains exactly one multiple vertex v, Γ_2 contains all the other multiple vertices, and Σ is a linear subgraph which connects Γ_1 and Γ_2. Then M_Γ is represented as a union of a Seifert manifold M_{Γ_1}, $M_\Sigma\cong T^2\times I$, and a graph manifold $M_{\Gamma_2}=M_{\Lambda_0}\cup(\bigcup_{j=1}^{k} M_{\Lambda_j})$ where Λ_0 contains exactly one multiple vertex v' which is the nearest one to v, and each Λ_j for $j\geqq 1$ is a non-linear branch of v'. The map $\pi_1(\partial M_{\Gamma_i})\rightarrow\pi_1(M_{\Gamma_i})$ induced by the inclusion is injective for both $i=1$ and 2 by Corollary to Proposition 2 and Proposition 3. We consider the presentations of $\pi_1(M_{\Gamma_i})$ and $\pi_1(M_{\Lambda_j})$. M_{Γ_1} is a Seifert manifold with orbit surface D^2, with m-exceptional fibers of types, say, $(\alpha_1, \beta_1), \cdots, (\alpha_m, \beta_m)$ for some $m \geqq 2$. Then we have the following presentation of $\pi_1(M_{\Gamma_1})$.

$$\pi_1(M_{\Gamma_1}) = \{q_1, \cdots, q_m, h \mid q_i^{\alpha_i} h^{\beta_i} = 1, [q_i, h] = 1, \text{ for } i \geqq 1\}$$

where we use the notation given in the first paragraph in § 2 both for the curves and for their homotopy classes. Then $\pi_1(M_{\Gamma_1})/\langle h \rangle = Z_{\alpha_1} * \cdots * Z_{\alpha_m}$ is centerless since $\alpha_i \geqq 2$ and $m \geqq 2$, where $\langle h \rangle$ denotes the normal closure of h. Therefore $C(\pi_1(M_{\Gamma_1}))$=the center of $\pi_1(M_{\Gamma_1})$ is $\langle h \rangle$ which is infinite cyclic. Similarly we get

$$\pi_1(M_{\Lambda_0}) = \{q_1', \cdots, q_n', h' \mid q_i'^{\alpha_i'} h'^{\beta_i'} = 1 \text{ for } k+1 \leqq i \leqq n, [q_i', h'] = 1 \text{ for } i \geqq 1\},$$

where q_i' (resp. h') corresponds to q_i (resp. h) and q_i' for $i \geqq k+1$ (resp. $1 \leqq i \leqq k$) is in the boundary of the tubular neighbourhood of the exceptional fiber (resp. in $M_{\Lambda_0} \cap M_{\Lambda_i}$). Therefore $C(\pi_1(M_{\Lambda_0})) = \langle h' \rangle$ since $\pi_1(M_{\Lambda_0})$=(the free group of rank k) $* Z_{\alpha'_{k+1}} * \cdots * Z_{\alpha'_n}$ $(n \geqq 2)$. Then $\pi_1(\partial M_{\Gamma_1}) \lneqq \pi_1(M_{\Gamma_1})$ and $\pi_1(\partial_n M_{\Lambda_0}) \lneqq \pi_1(M_{\Lambda_0})$ for any n, (where $H \lneqq G$ means that H is a proper subgroup of G). Furthermore it is easily verified by induction on the number of multiple vertices that $\pi_1(\partial M_{\Lambda_j}) \lneqq \pi_1(M_{\Lambda_j})$ for each $j \geqq 1$. Next consider $\pi_1(M_{\Gamma_2})$. We apply Corollary 4.5 in [8] successively to $\pi_1(M_{\Gamma_2}) = \pi_1(M_{\Lambda_0}) \underset{Z^2}{*} \pi_1(M_{\Lambda_1}) \underset{Z^2}{*} \cdots \underset{Z^2}{*} \pi_1(M_{\Lambda_k})$. If $k \geqq 1$, we can see that $C(\pi_1(M_{\Gamma_2}))$ is contained in an infinite cyclic subgroup of $\pi_1(\partial M_{\Gamma_2}) \lneqq \pi_1(M_{\Gamma_2})$ generated by h'. If $k=0$, $C(\pi_1(M_{\Gamma_2})) = \langle h' \rangle$ since $M_{\Gamma_2} \cong M_{\Lambda_0}$ is itself a Seifert manifold. Finally we consider $\pi_1(M_\Gamma) = \pi_1(M_{\Gamma_1}) \underset{Z^2}{*} \pi_1(M_{\Gamma_2})$ where $\boldsymbol{Z}^2$ is contained in $\pi_1(M_{\Gamma_i})$ via the inclusion $T^2 \times \{i-1\} \cong \partial M_{\Gamma_i} \subset M_{\Gamma_i}$ for $i=1, 2$, (where $T^2 \times \{i-1\} \subset T^2 \times I \cong M_\Sigma$). Again by Corollary 4.5 in [8], we get $C(\pi_1(M_\Gamma)) = \boldsymbol{Z}^2 \cap C(\pi_1(M_{\Gamma_1})) \cap C(\pi_1(M_{\Gamma_2}))$. On the other hand M_{Γ_i}'s are glued by a homeomorphism $\phi: \partial M_{\Gamma_1} \to \partial M_{\Gamma_2}$ such that $\phi(h) = h'^{\gamma} q'^{\delta}$ for some $\gamma, \delta \in \boldsymbol{Z}$, where h, q (resp. h', q') are the canonical curves in ∂M_{Γ_1} (resp. in ∂M_{Γ_2}). Therefore if $C(\pi_1(M_\Gamma)) \neq 1$, then $\phi(h) = h'^{\pm 1}$. This implies that $[c_1, \cdots, c_s] = 0$ where $\Sigma =$ 

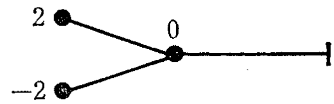
. (In particular $M_\Sigma \neq \emptyset$.) But in this case $|c_i| \leqq 1$ for some i, which contradicts the assumption. By the proof above it is easily verified that M_Γ contains an incompressible torus. This proves Lemma 2.

COROLLARY TO LEMMA 2. *Suppose that Γ is reduced and connected. Then M_Γ is a Seifert manifold of type o_1 if and only if Γ is star-shaped.*

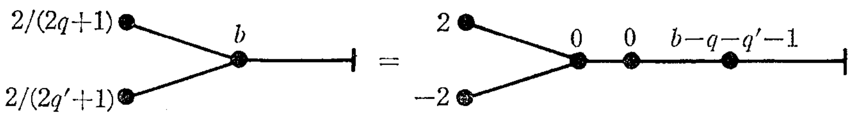
The proof is straightforward. See also [13].

REMARK 1. We can also obtain Lemmas 1, 2 and this corollary by comparing Waldhausen's reduction processes with the reductions of plumbing graphs. But the proof given above is of some interest.

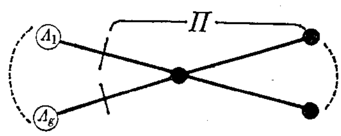
Remark 2. We can also characterize the reduced graphs which represent Seifert manifolds of type n_2 by the method stated in Remark 1. To do so, we examine the reduction processes of Waldhausen [16], 6.2.1~6.2.10 for the graph structure corresponding to the graph Γ which represents a Seifert manifold of type n_2, then apply Satz 8.1 in [16]. The most essential process is the exchange of the twisted S^1-bundle over the Moebius band N for the Seifert manifold Q with orbit space D^2 and two exceptional fibers of weight 2 which is homeomorphic to N. Q is represented as



in which the boundary curve $\partial D^2 \times *$ is homologous to the S^1-fiber of N. Note that



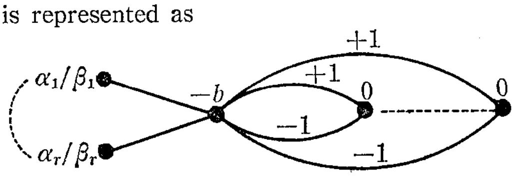
These graphs also represent Q. (The case when q, $q'=0$ or -1 is essential.) We can see that Γ has the following reduced form if $M_\Gamma = \{b, (n_2, g); (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$

(♯) 

r-exceptional linear branches

where A_i—⊣ represents Q for $i=1, \cdots, g$ and the remaining part Π of Γ is star-shaped. The weights are chosen so that if A_i—⊣'s are replaced by N's then their S^1-fibers are homologous to the one of M_Π which is a Seifert manifold of type o_1. There are several exceptions in [16], but these are also represented by the graphs of the form (♯) if the graphs are trees except for the case when $M_\Gamma = RP^3 \sharp RP^3$ or a lens space (cf. [13]).

Remark 3. If we consider graphs with cycles, there are several other examples which represent Seifert manifolds. For example, $\{b, (o_1, g): (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)\}$ is represented as



with g cycles.

§ 3. Representations for graph knots.

Let $K\subset S^3$ be an oriented graph knot (We fix the natural orientation of S^3.) and $U(K)$ be a tubular neighborhood of K. By Proposition 1, $S^3-\mathrm{Int}\,U(K)$ is represented by the graph of the form

We denote the vertex which lies on the right hand side of Γ by v'. Let L_Γ be a framed link associated with Γ, $L_{v'}$ be its component associated with v', and $m_{v'}$ be a meridian of $L_{v'}$ which is linked with no other component of L_Γ. Then a tubular neighborhood $U(m_{v'})$ of $m_{v'}$ is naturally contained in M_Γ, and (S^3, K) $\cong((M_\Gamma-\mathrm{Int}\,U(m_{v'}))\bigcup_h U(K), K)$ for some homeomorphism $h:\partial(M_\Gamma-\mathrm{Int}\,U(m_{v'}))$ $\to\partial U(K)$. By Proposition 1, Γ can be arranged so that h maps a meridian of $\partial U(m_{v'})$ to a meridian of $\partial U(K)$. It follows that the trivial surgery on K is represented by $(\Gamma)\!-\!\overset{\infty}{\bullet}=\Gamma$ and hence Γ represents S^3. Under this condition a framed link presentation of K corresponding to Γ (in which K is represented by $m_{v'}$) is uniquely determined up to framing and orientation. (See Figure 2, in which the circle denoted by K represents $m_{v'}$.) If $U(m_{v'})$ is framed by an oriented pair of a meridian M' and a preferred longitude L' in the framed link picture, then $U(m_{v'})$ is identified with $U(K)$ so that M' and L' are represented by an oriented pair of a meridian M and a preferred longitude L on $\partial U(K)$ as follows,

$$M'=M,\ L'=L+\phi_\Gamma M \text{ for some integer } \phi_\Gamma.$$

(Note. ϕ_Γ depends on Γ, but it will be denoted by ϕ_K once the graph Γ is fixed for K.) Then we represent K by the graph of the form $(\Gamma)\!-\!\overset{v_K}{\circ}$, such that the vertex v_K corresponds to $m_{v'}$ (and hence corresponds to K) and $(\Gamma)\!-\!\overset{u/v}{\bullet}$ represents u/v-surgery on $m_{v'}$ in the framed link picture. (The correspondence between $(\Gamma)\!-\!\overset{v_K}{\circ}$ and a framed link is suggested in Figure 2.) It is easy to see that $(\Gamma)\!-\!\overset{u/v}{\bullet}=\chi(K, u/v+\phi_\Gamma)$. Γ has the weights according to the orientation of $S^3-\mathrm{Int}\,U(K)$ induced by the one of S^3, and the orientation of K is determined according to the weight of the edge containing v_K. (Consider the associated framed link.) Furthermore we may assume that the following two graphs represent the same knot by the link calculus in Figure 2. (We omit the weights on the edges.)

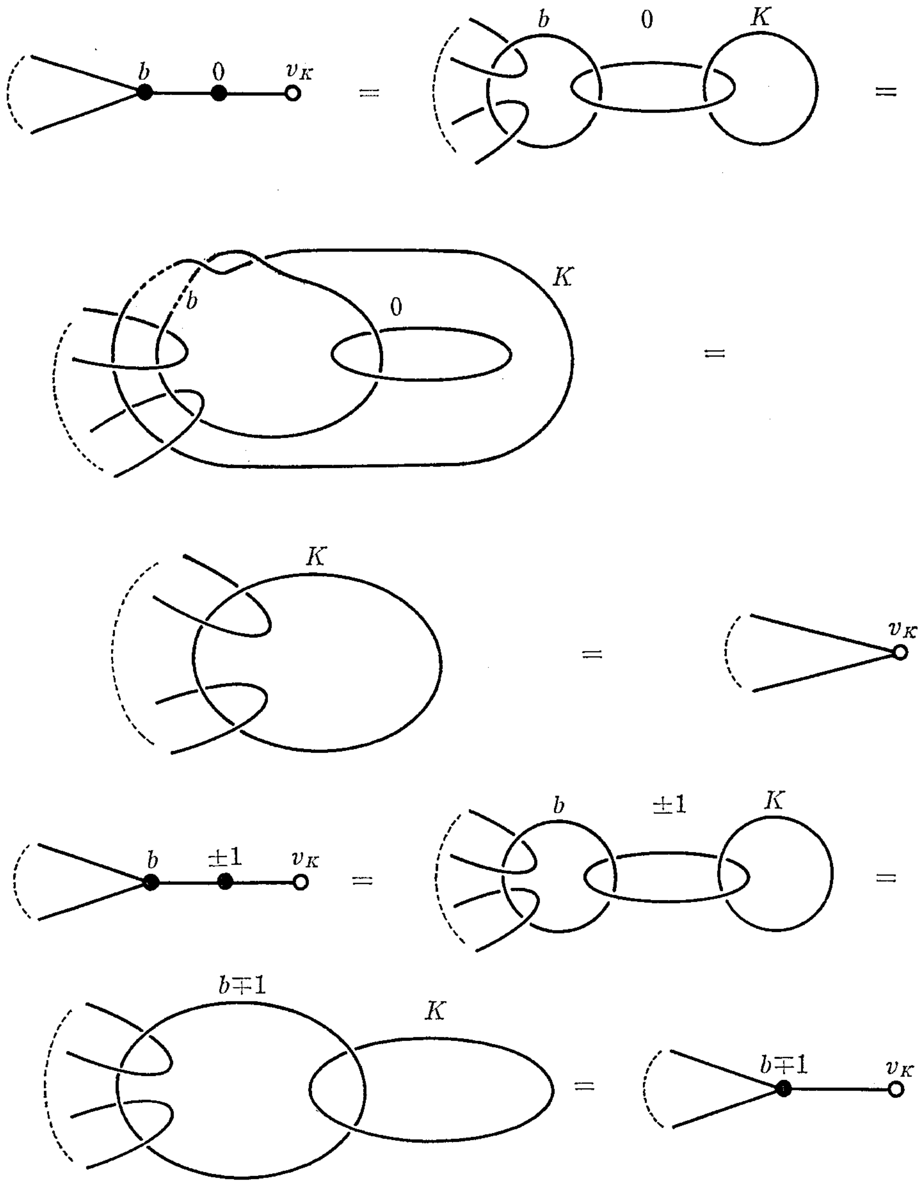
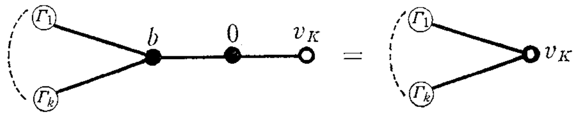
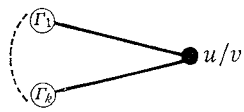


Figure 2.



The graph on the right side is assumed to be determined by the framed link picture as in Figure 2 and to be framed so that the graph below represents u/v-surgery on the circle denoted by K in the corresponding framed link picture.



If $u/v=\infty$, then we ignore it, i.e., we ignore v_K in the original graph. Then Γ_i satisfies the following conditions (cf. [6]),

(i) Γ_i is a tree. Γ_i and Γ_j are mutually disjoint for $i \neq j$.

(ii) $M_{\Gamma_i} \cong S^3$ if we ignore v_K.

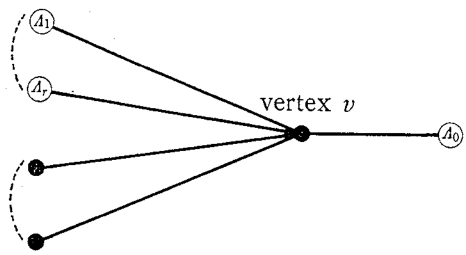
In this section we obtain a reduced graph for the graph knot $K = (\Gamma)\!-\!\overset{v_K}{\circ}$ by applying the reduction processes and their inverses to it. The weights of the edges in Γ are irrelevant since Γ is a tree. Finally we will see that the weight of the edge containing v_K is also irrelevant.

PROPOSITION 4. *Let Γ be a connected graph such that $M_\Gamma \cong S^3$. Then for any multiple vertex v, there are at most 2 exceptional linear branches for v. This conclusion is valid also when $M_\Gamma \cong B^2 \times S^1$.*

PROOF. We can reduce Γ by the reduction process III so that Γ has no linear branch of weight 0 without changing the condition about the exceptional branches. If the number of multiple vertices of Γ, which we denote by t_Γ, is at most 1, then Proposition 4 is true ([5] VI. 3.2). If $t_\Gamma \geqq 2$, decompose Γ as $\Gamma_0 \cup \Gamma_1$ so that $M_{\Gamma_0} \cap M_{\Gamma_1} = \partial M_{\Gamma_0} = \partial M_{\Gamma_1} \cong T^2$ and each Γ_i is non-linear. Either one of M_{Γ_i}, say M_{Γ_0}, is a solid torus. Then Γ is reduced to the graph Γ' which contains Γ_1 and $t_{\Gamma'} < t_\Gamma$. On the other hand there is a graph Γ'' of the form $(\Gamma_0)\!-\!\overset{p/q}{\bullet}$ for some $p/q \in \boldsymbol{Q}$ such that $M_{\Gamma''} \cong S^3$ and $t_{\Gamma''} < t_\Gamma$. Thus Proposition 4 is proved by induction on t_Γ. The case when $M_\Gamma \cong B^2 \times S^1$ is similar.

Consider the graph knot $(\Gamma)\!-\!\overset{v_K}{\circ}$. We omit the weight of the edge containing v_K. We assume that $(\Gamma)\!\!-\!\!\!\dashv$ is reduced. Let v be the multiple vertex of Γ nearest to v_K. Then K is represented as follows,

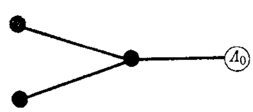
(##)

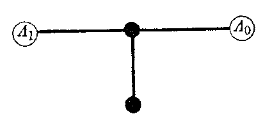


where (Λ_0)—| denotes the linear branch containing v_K, each (Λ_i)—| is a non-linear branch for $i=1, \cdots, r$, and all the others are exceptional linear branches of v. Put s=the number of exceptional linear branches of v. Then $s \leqq 2$ by Proposition 4. Note that K is unknotted if Γ is linear (cf. Figure 2). Therefore we may assume that Γ is non-linear hereafter.

Case I. Λ_0 is not reduced to $\overset{0}{\bullet}$ if v_K is ignored.

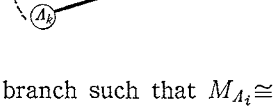
Let Λ' be the remaining part for $\bigcup_{i \geqq 1} (\Lambda_i)$—| where v_K in $\Lambda_0 \subset \Lambda'$ is ignored. Then $M_{\Lambda'}$ must be a solid torus since (Λ_i)—| is non-linear for $i \geqq 1$ and $M_\Gamma = S^3$ if v_K is ignored. Therefore it turns out that $r \leqq 1$ and $s \leqq 1$ if $r=1$ since (Λ_i)—| corresponds to a manifold which is ∂-irreducible. Thus K is represented by one of the following graphs.



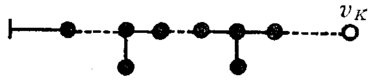


Case II. Λ_0 is reduced to $\overset{0}{\bullet}$ if v_K is ignored.

In this case K is represented as follows.

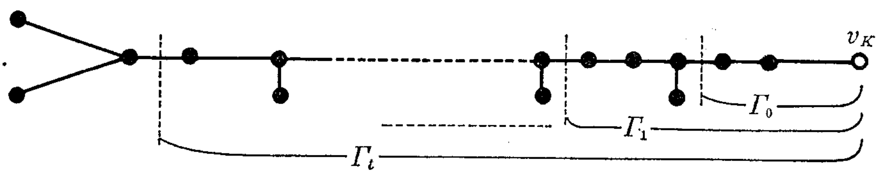


where (Λ_i)—| is a non-linear branch such that $M_{\Lambda_i} \cong S^3$ for each $i \geqq 1$. Furthermore $k \geqq 2$ and there is no exceptional linear branch attached to the vertex v.

We can apply the previous arguments to the graph of the form (##) for another multiple vertex v (in this case |—(Λ_0) is assumed to be a graph of the form 

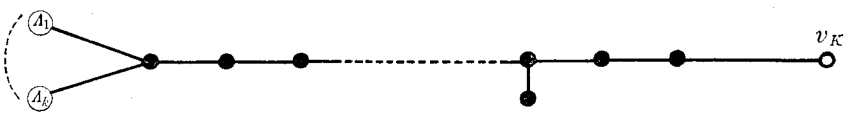
which is reduced to a linear graph if v_K is ignored). Then by induction we can reduce the graph to one of the following types.

A.



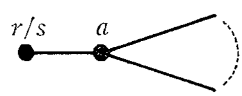
where Γ_i is a subgraph of Γ as above such that Γ_i is contained in Γ_{i+1} for $0 \leqq i \leqq t-1$.

B.

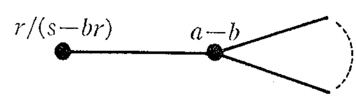


where A_j is a non-linear graph such that $M_{A_j} \cong S^3$ for $j=1, \cdots, k$. Let Γ_i be a subgraph as in type A for $i=0, \cdots, t$.

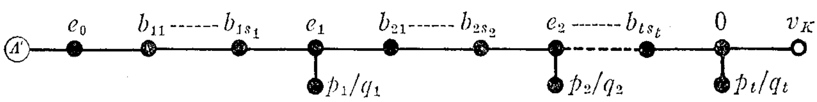
In any case, if we ignore v_K, Γ_i must be reduced to a linear graph of weight $1/e_i$ for some $e_i \in \boldsymbol{Z}$ except for Γ_t in case B. The last one must be reduced to $\overset{0}{\vdash\!\!-\!\!\bullet}$ since $M_\Gamma \cong S^3$. In particular Γ_0 is reduced away. On the other hand the graph of the form



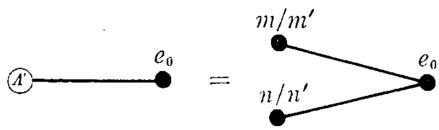
can be replaced by

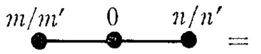


by the composition of reduction processes and their inverses. Thus we can choose the following representation of K.

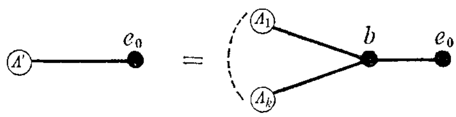


where $[b_{n1}, \cdots, b_{ns_n}, 0, p_n/q_n]=1/e_{n-1}$ for $n=1, \cdots, t$, and $\overset{0}{A'\!-\!\bullet}$ represents S^3. (Note that $[b_{n1}, \cdots, b_{ns_n}] \neq 0$ since $p_n \geqq 2$ for $n=1, \cdots, t$.) Furthermore if K is of type A, then

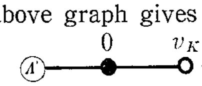


where m, m', n and n' are integers such that $mn'+nm'=\varepsilon=\pm 1$, since 

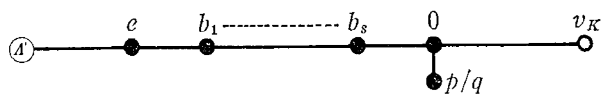
$= S^3$. If K is of type B, then



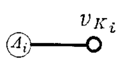
for $k\geqq 2$.

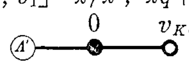
In any case the above graph gives the presentation of t-fold nontrivial iterated cable of the knot 
. This follows from

PROPOSITION 2 in [1]. *The knot K represented by the graph*



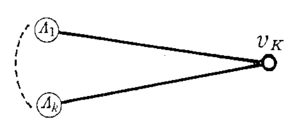
where $[b_1, \cdots, b_s, 0, p/q]=1/e$, is the $(\delta x+p\phi_K, p)$-cable of the knot

$K_i =$ 

where $[b_s, \cdots, b_1]=x/x'$, $xq+px'=\delta=\pm 1$, and $\phi_{K'}$ is an integer defined for the presentation 

of K' as in the first paragraph of § 3. (Conversely any cable is represented as above.)

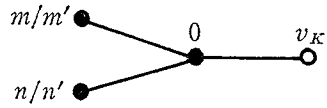
The knot of type B for $t=0$ is reduced to the graph of the form



where (A_i)—○ for each i gives the presentation of a graph knot K_i which has the form as above as is seen by induction. Therefore the only possible types of graph knots are knots in the class generated by the trivial knot under the operations of connected sum and cabling, all of which are invertible as is seen by induction. Thus the representation makes sense without the weight on the edge which contains v_K. In particular the above graph represents $K_1 \# K_2 \# \cdots \# K_k$. It does not depend on the orientations of knots. Thus we get

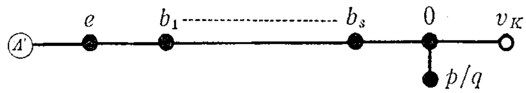
THEOREM 1. (i) *K is a graph knot if and only if it is in the class generated by the trivial knot under the operations of connected sum and cabling. In particular any graph knot is invertible.*

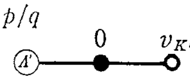
(ii) *For any graph knot K, one of its reduced presentation is given as follows,*

(1) 

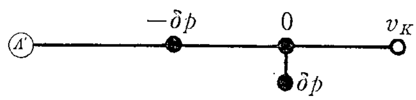
where $mn'+m'n=\varepsilon=\pm 1$

for $K=C(m, \varepsilon n)$

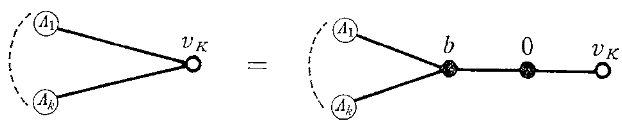
(2) 

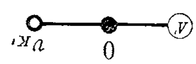
for $K=C(\delta x+p\phi_{K'}, p; K')$ *where* $K'=$ 

. *All the numbers are as in Proposition 2 in [1]. In particular*

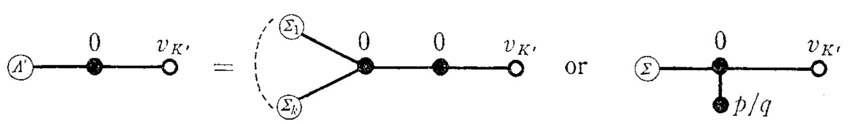


for $K=C(\delta+p\phi_{K'}, p; K')$

(3) 

for $K=K_1 \# \cdots \# K_k$ *where* 

REMARK. In (2), we can assume that



(iii) The integer ϕ_K defined for the standard representation of K given in (ii) (See the first paragraph of this section for the definition of ϕ_K.) is determined inductively as follows.

(1) $\phi_K=\varepsilon mn$ for $K=C(m, \varepsilon n)$

(2) $\phi_K=p(\delta x+p\phi_{K'})$ for $K=C(\delta x+p\phi_{K'}, p; K')$

(3) $\phi_K=\phi_{K_1}+\cdots+\phi_{K_k}$ for $K=K_1 \# \cdots \# K_k$ for the first presentation in (ii), (3). ϕ_K for the second presentation depends on b but if we take $b=0$, it is well-defined and identical to the first case.

REMARK. Theorem 1 (i) has been obtained by [3] by different methods.

§ 4. Proofs of Theorems 2 and 3.

In § 4 we assume that any graph knot $K \subset S^3$ is represented by the reduced graph given in Theorem 1 and we fix it. Then $\chi(K, u/v+\phi_K) = \textcircled{\Gamma}\!-\!\overset{u/v}{\bullet}$ which we denote by $\Gamma(u/v)$ (See § 3.), where ϕ_K is defined as in Theorem 1 (iii). We may assume that $v \neq 0$.

PROOF OF THEOREM 2. Suppose that $\Gamma(u/v)$ is a Seifert manifold of type o_1 for the given knot $K = \textcircled{\Gamma}\!-\!\overset{v_K}{\circ}$ and for some $u/v \in \boldsymbol{Q}$. First suppose that $K = K_1 \# \cdots \# K_k$ for $k \geqq 2$. Since $\Gamma(u/v)$ must be reduced to a star-shaped graph (§ 2), it turns out that $k=2$, $u/v=0$, and both of Λ_1 and Λ_2 has at most one multiple vertex, i.e., K is a connected-sum of 2 torus knots. Thus we obtain Theorem 2 (4). From now on we assume that K is of type A or of type B for $t \geqq 1$. We use the following simple lemma.

LEMMA 3. *We assume that* $p' \geqq 2$, $q' \neq 0$.

(i) *Suppose that* $\overset{p/q}{\bullet}\!-\!\!-\!\overset{b}{\bullet}\!-\!\!-\!\overset{p'/q'}{\bullet} = \overset{p/q}{\bullet}\!-\!\!-\!\overset{b'}{\bullet}\!-\!\!-\!\overset{p'/q'}{\bullet} = S^3$ *for* $b \neq b'$. *Then* $p=1$, $b-b'=\pm 1$, *and* $p'=2$.

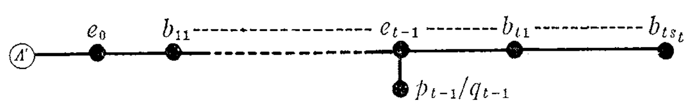
(ii) *Suppose that* $\overset{p/q}{\bullet}\!-\!\!-\!\overset{b}{\bullet}\!-\!\!-\!\overset{p'/q'}{\bullet} = \overset{p/q}{\bullet}\!-\!\!-\!\overset{b'}{\bullet}\!-\!\!-\!\overset{p'/q'}{\bullet} = S^2 \times S^1$. *Then* $b=b'$.

(iii) *There are no integers such that* $\overset{p/q}{\bullet}\!-\!\!-\!\overset{b}{\bullet}\!-\!\!-\!\overset{p'/q'}{\bullet} = S^3$ *and* $\overset{p/q}{\bullet}\!-\!\!-\!\overset{b'}{\bullet}\!-\!\!-\!\overset{p'/q'}{\bullet} = S^2 \times S^1$.

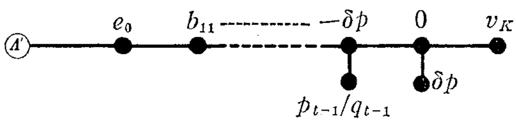
The proof is straightforward. (i) is identical to Lemma 5 in [1].

Claim 1. $u/v \neq 0$.

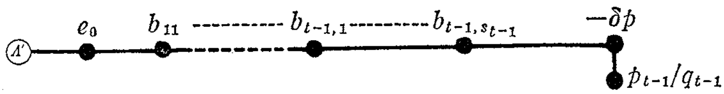
PROOF. Suppose that $\Gamma(0)$ is a Seifert manifold (of type o_1). If $t \geqq 2$, then the subgraph



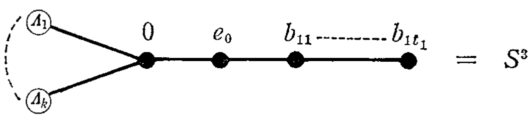
must represent S^3 since $\Gamma(0)$ is a sum of the above graph and $\overset{p_t/q_t}{\bullet}$. Therefore $\overset{b_{t1}}{\bullet}\text{-------}\overset{b_{ts_t}}{\bullet}$ must be reduced away (§ 2) and so it must represent S^3. (Note that $[b_{t1}, \cdots, b_{ts_t}] \neq 0$ by the construction in § 3, Theorem 1 (ii) since $p_t \geqq 2$.) Then it follows that Γ is of the form



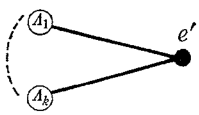
for $\delta=\pm 1$, $p\geqq 2$. The following graph must represent S^3 by the assumption.



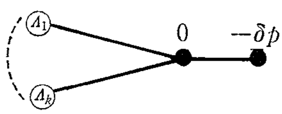
Hence $[b_{t-1,1}, \cdots, b_{t-1,s_{t-1}}, -\delta p, p_{t-1}/q_{t-1}]$ (or $[e_0, b_{11}, \cdots, -\delta p, p_1/q_1]$ for type B, $t=2$)$=0$, or $1/w$ which contradicts Lemma 3 since $[b_{t-1,1}, \cdots, b_{t-1,s_{t-1}}, 0, p_{t-1}/q_{t-1}]=1/e_{t-2}$ (§3 Theorem 1). If K is of type B for $t=1$, then



and so $[e_0, b_{11}, \cdots, b_{1t_1}]=1/w$, or 0 (i.e., $[b_{11}, \cdots, b_{1t_1}]=1/e_0$. Therefore

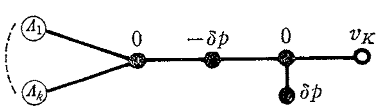


for some $e'\in \boldsymbol{Z}$ (the first case) or



for $\delta=\pm 1$, $p\geqq 2$ (the second case)

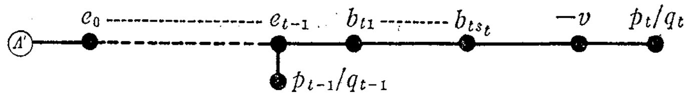
must represent S^3 (Theorem 1). Note that in the second case $[b_{11}, \cdots, b_{1t_1}]=1/e_0$ and hence K is of the following form. (Recall the first assumption in §4.)



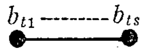
But either one of the graphs does not represent S^3 (cf. constructions in §3). The case when K is of type A for $t\leqq 1$ is also excluded.

Claim 2. $t\leqq 2$.

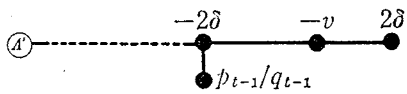
PROOF. The proof is similar to the one of Theorem 1 in [1]. Suppose that $t\geqq 3$. Then $u=1$ since $p_t\geqq 2$ and $[b_{t1}, \cdots, b_{ts_t}]\neq 0$.

$\Gamma(1/v) =$ 

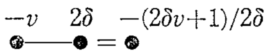
Then 

$= S^3$. On the other hand, $[b_{t1}, \cdots, b_{ts_t}, 0, p_t/q_t]=1/e_{t-1}$ and $v \neq 0$. Hence 

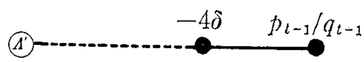
$= S^3$, $v=\pm 1$, and $p_t=2$ by Lemma 3. Therefore

$\Gamma(1/v) =$ 

(by Theorem 1 (ii).)

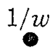
Since 

must represent S^3, $v=-\delta$ and

$\Gamma(1/v) =$ 

Then it follows that 

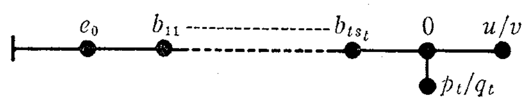
$=$ 

or 

, which contradicts Lemma 3.

Claim 3. K is not of type B for $t \geqq 1$.

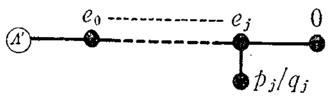
PROOF. If K is of type B for $t \geqq 1$, then the subgraph of $\Gamma(u/v)$ of the form



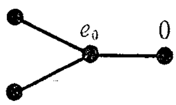
must be reduced to a linear graph of weight 0 or $1/w$, where $t \leqq 2$ by Claim 2. If $t=2$, then the subgraph is reduced to a linear graph of weight $[e_0, b_{11}, \cdots, b_{1s_1}, -4\delta, p_1/q_1]$ for $\delta=\pm 1$ by the proof of Claim 2. But this case also contradicts Lemma 3 since $[e_0, b_{11}, \cdots, b_{1s_1}, 0, p_1/q_1]=0$ by the construction. If $t=1$, then $u=1$ ($u \neq 0$ by Claim 1) and $[e_0, b_{11}, \cdots, b_{1s_1}, -v, p_1/q_1]=0$ or $1/w$, which is also a contradiction since $v \neq 0$.

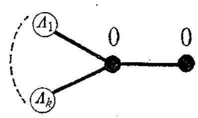
The preceding arguments show that if K is of type A for $t=2$, then $p_2=2$ and $u/v=1/v=\pm 1$, i.e., $K=C(\delta+2\phi_{K'}, 2; K')$ where K' is any nontrivial cable of a nontrivial torus knot. In this case $\Gamma(u/v)$ is ϕ_K+1/v-surgery on K and it turns out that $\delta=-v$ by the proof of Claim 2. Furthermore in this case the resulting manifold has exactly 3 exceptional fibers by Lemma 3. If K is of type A and $t=1$, then $u=1$, i.e., $K=C(\delta x+p\phi_{K'}, p; K')$ where $K'=C(m, n)$ and in this case $\Gamma(u/v)$ is a ϕ_K+1/v-surgery on K. The case when K is of type A for $t=0$ reduces to [7]. Thus if we use Theorem 1 and rewrite symbols, we get Theorem 2 (1), (2) and (3).

PROOF OF THEOREM 3. Suppose that $\Gamma(u/v)$ represents a non-prime manifold. Then K cannot be of type B for $t=0$. By Lemma 1 we see that either

$\Gamma(u/v) =$ 

for some j or

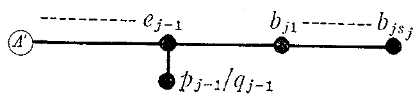
$\Gamma(u/v) =$ 

for type A or 

for type B.

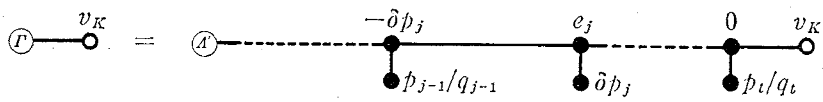
In the second case $\Gamma(u/v)$ is either S^3 (type B) or a connected sum of 2 lens spaces (type A). Then we only consider the first case.

Case 1. $j \geqq 2$.

$\Gamma(u/v) =$ 

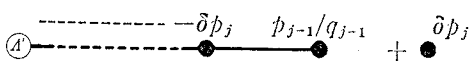
$+$ $\bullet\, p_j/q_j$.

Put the first graph on the right hand side as Λ. Suppose that M_Λ is not prime. Then by Lemma 1, $[b_{j1}, \cdots, b_{js_j}]=1/w$ for some $w=\boldsymbol{Z}$. Therefore we may assume by the construction in §3 that



$(\delta=\pm 1)$

and

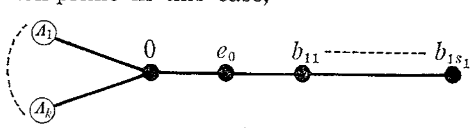
$\Gamma(u/v) =$ 

$+$ $\bullet\, \delta p_j$.

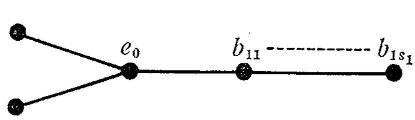
But in this case $[b_{j-1,1}, \cdots, b_{j-1,s_{j-1}}, -\delta p_j, p_{j-1}/q_{j-1}]$ (or $[e_0, b_{11}, \cdots, -\delta p_2, p_1/q_1]$ for type B, $j=2)=0$ or $1/w$. This also contradicts Lemma 3. Thus M_Λ is prime.

Case 2. $j=1$.

If M_Λ is non-prime in this case,

$\Lambda =$ 

(if K is of type B), or

$\Lambda =$ 

(if K is of type A)

would represent a non-prime manifold. But it is impossible in either case since $[b_{11}, \cdots, b_{1s_1}] \neq 0$. Thus Theorem 3 is proved.

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