Semipositive line bundles

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Introduction.

In this paper we want to study semipositive line bundles. As we explained in [F2], there are several different ways to define the notion of semipositivity. The strongest one is the semiampleness, i.e., a line bundle L is semiample if $\mathrm{Bs}|mL|=\emptyset$ for some positive integer m. The weakest one is the numerical semipositivity, i.e., a line bundle L on a space S is numerically semipositive if $LC\geqq 0$ for any curve C in S.

Semiample line bundles have many nice properties. For example, the graded algebra $G(S, L)=\bigoplus_{t\geqq 0} H^0(S, tL)$ is finitely generated (see (1.3)). So, it would be important to find a good sufficient condition for a line bundle to be semiample. Our theorem (1.10) improves upon Zariski's famous result in [Z]. Although our criterion is still too strong, it seems difficult to obtain a better one.

On complex analytic manifolds there is another notion (called the geometrical semipositivity) based on the real differential (1, 1)-form representing the Chern class $c_1(L)$. The significance of this notion lies in the vanishing theorem (4.9), which is slightly stronger than Kodaira's original one.

In [F2] we introduced a couple of other notions, but now the cohomological semipositivity turned out to be equivalent to the numerical semipositivity. This is a consequence of a strengthened version (5.1) of Serre's vanishing theorem. As applications we obtain the results (6.2), (6.5), (6.9), (6.10), (6.12) etc., which were known in [F2] only in case of characteristic zero.

In the final section we give a generalization of Ramanujam's vanishing theorem in positive characteristic cases (see (7.5) and (7.8)).

Notation, Convention ond Terminology.

Basically we employ the notation as those in [EGA], [Ha 2], [F1] and [F3]. We work either in the category of algebraic spaces which are *proper* over a fixed algebraically closed field $\mathfrak{K}$, or in the category of *compact* complex analytic spaces. An object in these categories will be called a "*space*". When we say simply "*space*" in a statement, then it is valid in both categories. A *variety* means an irreducible reduced space. A non-singular variety is called a *manifold*. Vector bundles are identified with the locally free sheaves of their sections.

Tensor products of line bundles are denoted additively, while multiplicative notations are used for intersection products of Chow rings. The pull-back of a line bundle L to a space T by a given morphism $T\to S$ will be sometimes denoted simply by L when there is little fear of confusion. Otherwise we usually write L_T.

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§1. Semiampleness.

(1.1) DEFINITION. A line bundle L on a space S is said to be *semiample* if there exists a positive integer m such that $\mathrm{Bs}|mL|=\emptyset$, i.e., $\mathcal{O}_S[mL]$ is generated by global sections.

(1.2) PROPOSITION. *If L is semiample, then f^*L is semiample for any morphism $f: T\to S$.*

This is obvious.

(1.3) THEOREM. *For any semiample line bundle L on a space S, the graded algebra $G(S, L)=\bigoplus_{t\geq 0} H^0(S, tL)$ is finitely generated.*

To prove this fact, we introduce several terminology.

(1.4) Given any linear system Λ on a space S, we set $L=[\Lambda]\in\mathrm{Pic}(S)$ and let V_Λ be the linear subspace of $H^0(S, L)$ corresponding to Λ. By a *Λ-module system* we mean a system consisting of a coherent sheaf $\mathcal{F}$ on S, an integer q and linear subspaces M_t of $H^q(\mathcal{F}[tL])$ for each integer t such that the image of $M_t\otimes V_\Lambda$ under the natural mapping $H^q(\mathcal{F}[tL])\otimes V_\Lambda\to H^q(\mathcal{F}[(t+1)L])$ is contained in M_{t+1} for every integer t. $\mathcal{F}$ is called the associated sheaf of this system. q is called the degree.

Let R_Λ be the graded subalgebra of $G(S, L)$ generated by V_Λ. Then $M=\bigoplus_{t\in Z} M_t$ is a graded R_Λ-module in a natural way. Sometimes M is called a Λ-module system. M is said to be *bounded* if $M_t=0$ for any sufficiently small integer t.

(1.5) LEMMA. *Let M be a bounded Λ-module system as above. Then M is a finitely generated R_Λ-module if and only if the natural mapping $M_t\otimes V_\Lambda\to M_{t+1}$ is surjective for any sufficiently large integer t.*

The proof is easy. In this case M is called a finitely generated Λ-module system.

(1.6) LEMMA. *Let M be a bounded Λ-module system as in* (1.4) *and let* $\varphi: \mathcal{F}[-L]\to\mathcal{F}$ *be a homomorphism induced by an element δ of V_Λ. Set* $\mathcal{K}=\mathrm{Ker}(\varphi)$, $\mathcal{C}=\mathrm{Coker}(\varphi)$ *and* $N_t=\mathrm{Im}(M_t\to H^q(\mathcal{C}[tL]))$. *Suppose that* $M_t=H^q(\mathcal{F}[tL])$ *and* $H^{q+1}(\mathcal{K}[tL])=0$ *for any sufficiently large integer t and that* $N=\bigoplus_t N_t$ *is a finitely generated Λ-module system. Then M is finitely generated too.*

PROOF. Set $\mathcal{J}=\mathrm{Im}(\varphi)$ and $I_t=H^q(\mathcal{J}[tL])$. Then, for any sufficiently large integer t, $M_{t-1}\to I_t$ is surjective since $H^{q+1}(\mathcal{K}[tL])=0$. Hence $K_t=\mathrm{Ker}(M_t\to N_t)$ comes from M_{t-1}. This implies $K_t\subset\mathrm{Im}(M_{t-1}\otimes V_\Lambda\to M_t)$ because $\delta\in V_\Lambda$. On the other hand, $N_{t-1}\otimes V_\Lambda\to N_t$ is surjective by (1.5). Hence $M_{t-1}\otimes V_\Lambda\to M_t\to N_t$ is surjective. Combining these observations we infer that $M_{t-1}\otimes V_\Lambda\to M_t$ is surjective. So (1.5) applies.

(1.7) THEOREM. *Let Λ be a linear system on a space S such that* $\mathrm{Bs}\,\Lambda=\emptyset$. *Then any bounded Λ-module system* $M=\bigoplus M_t$ *is finitely generated.*

PROOF. We use the induction on $n=\dim(\mathrm{Supp}(\mathcal{F}))$, where $\mathcal{F}$ is the associated sheaf of the system M. For any general element δ of V_Λ, the induced homomorphism $\varphi: \mathcal{F}[-L]\to\mathcal{F}$ is injective by [F3; (1.2)]. Moreover, $\dim(\mathrm{Supp}(\mathcal{C}))<n$ for $\mathcal{C}=\mathrm{Coker}(\varphi)$. Take a bounded Λ-module system F such that $M\subset F$ and $F_t=H^q(\mathcal{F}[tL])$ for $t\gg 0$. Using (1.6) and applying the induction hypothesis we infer that F is finitely generated. So F is Noetherian module since R_Λ is a Noetherian algebra. Hence its submodule M is finitely generated.

(1.8) PROOF OF (1.3). Take $m>0$ such that $\mathrm{Bs}|mL|=\emptyset$. Then, for each $r=0, 1, \cdots, m-1$, $\bigoplus_{t\geq 0} H^0(S, (tm+r)L)$ is a finitely generated $|mL|$-module system by (1.7). So the mapping $H^0(S, tL)\otimes H^0(S, mL)\to H^0(S, (t+m)L)$ is surjective for any sufficiently large integer t by (1.5). This implies that $G(S, L)$ is finitely generated as an algebra.

(1.9) *Problem.* Find a good sufficient condition for a line bundle to be semiample.

Here we prove the following

(1.10) THEOREM. *Let Λ be a linear system on a space S. Suppose that the restriction of* $L=[\Lambda]$ *to* $B=\mathrm{Bs}\,\Lambda$ *is ample. Then there exists an integer k such that* $\mathrm{Bs}|tL|=\emptyset$ *for any* $t\geqq k$. *In particular L is semiample.*

(1.11) LEMMA. *Let things be as in* (1.10). *Then any bounded Λ-module system* $M=\bigoplus M_t$ *with* $q>0$ *is finitely generated.*

PROOF. We use the induction on $d=\dim V_\Lambda$. If $d=0$, then $B=S$ and L is ample. So $M_t=0$ for $t\gg 0$ and the assertion is obvious.

When $d>0$, take a general element δ of V_Λ and let $\varphi: \mathcal{F}[-L]\to\mathcal{F}$ be the induced homomorphism, where $\mathcal{F}$ is the associated sheaf of M. Then $\mathcal{F}_D=\mathrm{Coker}(\varphi)$ is supported in the zero-subscheme D of δ and we have $\dim \Lambda_D<\dim \Lambda$. Moreover, $\mathcal{K}=\mathrm{Ker}(\varphi)$ is supported on a subset of B by [F3; (1.2)], on which L is ample. Take a bounded Λ-module system $F=\oplus F_t$ such that $M\subset F$ and $F_t=H^q(\mathcal{F}[tL])$ for $t\gg 0$. Then, applying (1.6), we infer that F is finitely generated. So F is a Noetherian R_Λ-module and hence M is finitely generated.

REMARK. If $q>\dim B$, we get the same conclusion even if L_B is not ample.

(1.12) LEMMA. *Let R be the symmetric algebra of a vector space V of finite dimension. So, R is a polynomial algebra with a natural grading. Let $M=\bigoplus_{t\geq 0} M_t$ be a finitely generated graded R-module. Then, for any general element δ of V, the multiplication mapping $\varphi_t: M_t\to M_{t+1}$ by δ is injective for every sufficiently large integer t.*

PROOF. Let $P=\mathrm{Proj}(R)=\boldsymbol{P}(V)$. Then there is a coherent sheaf $\mathcal{F}$ on P such that $M_t\cong H^0(P, \mathcal{F}(t))$ for every $t\gg 0$. The homomorphism $\varphi: \mathcal{F}\to\mathcal{F}(1)$ induced by $\delta\in V\cong H^0(P, \mathcal{O}(1))$ is injective by [F3; (1.2)]. Hence φ_t is injective for any large t.

(1.13) Now we prove (1.10) by induction on $d=\dim V_\Lambda$. Let $\varphi: \mathcal{O}_S[-L]\to\mathcal{O}_S$ be the homomorphism induced by a general element δ of V_Λ. Then $\mathrm{Coker}(\varphi)=\mathcal{O}_D$ for the zero-subscheme D of δ and $\mathcal{K}=\mathrm{Ker}(\varphi)$ is supported on a subset in B. Using the exact sequence $0\to\mathcal{K}\to\mathcal{O}_S[-L]\to\mathcal{J}\to 0$ where $\mathcal{J}=\mathrm{Im}(\varphi)$, we infer that $H^1(S, (t-1)L)\cong H^1(\mathcal{J}[tL])$ for any $t\gg 0$, because L is ample on $\mathrm{Supp}(\mathcal{K})$. Combined with (1.11) and (1.12), this implies that $H^1(\mathcal{J}[tL])\to H^1(S, tL)$ is injective for $t\gg 0$. Hence $H^0(S, tL)\to H^0(D, tL_D)$ is surjective and $\mathrm{Bs}|tL|=\mathrm{Bs}|tL_D|$ for $t\gg 0$. So, applying the induction hypothesis to Λ_D, we obtain the desired conclusion.

(1.14) COROLLARY. *Let L be a line bundle on a space S. Suppose that* $\mathrm{Bs}|L|$ *is a finite set. Then* $\mathrm{Bs}|tL|=\emptyset$ *for $t\gg 0$ and hence L is semiample.*

REMARK. This was proved by Zariski [Z; §6] when S is a normal projective variety. Our result (1.10) is valid even if S is a non-algebraic analytic space. However, as a matter of fact, the existence of such a linear system Λ implies that any irreducible component X of S is algebraic if $X\cap B\neq\emptyset$.

(1.15) *Example.* It is possible that $\mathrm{Bs}|tL|\neq\emptyset$ for infinitely many positive integers t even if L is a semiample line bundle on a variety V such that $L^n>0$, $n=\dim V$.

Let L_1 be a line bundle on a non-singular elliptic curve C such that $\deg L_1 \geqq 2$. Let L_{-1} be an m-torsion in Pic(C). Let P be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(L_1 \oplus L_{-1})$ over C, let D_1 and D_{-1} be the sections of $\pi: P \to C$ corresponding to the quotient bundles L_1 and L_{-1}, and let H be the tautological line bundle on P. Then the restriction of H to D_j ($\cong C$) is L_j and $D_j \in |H - \pi^* L_{-j}|$ for $j = \pm 1$. Given any positive integer t, we write $t = mq + r$ for some non-negative integers q, r with $r < m$. Then $|tH| = rD_{-1} + \Lambda_t$ for some linear system Λ_t with Bs $\Lambda_t = \emptyset$. Hence, although H is semiample and $H^2 > 0$, we have $\mathrm{Bs}|tH| \neq \emptyset$ for any integer t with $t \not\equiv 0 \pmod m$.

(1.16) It should be possible to improve (1.10). One might ask: *Is L semiample if the restriction of L to* $\mathrm{Bs}|L|$ *is semiample?*

Unfortunately, the answer is No.

Example. Let C be a non-singular elliptic curve and let E_r be an indecomposable vector bundle on C with rank r, $\deg(E_r) = 0$ and $h^0(C, E_r) = 1$. Atiyah [At] shows that such a vector bundle E_r exists uniquely for each $r \geqq 1$ and that there is a non-splitting exact sequence $0 \to \mathcal{O}_C \to E_r \to E_{r-1} \to 0$ for each r. Let $P = \boldsymbol{P}(E_2)$ and let H be the tautological line bundle on P. Then H contains a unique member D, which is a section of $P \to C$. Moreover $H_D \cong \mathcal{O}_D$. However, H is not semiample. In fact, [At] shows that $E_r \cong S^{r-1}(E_2)$ for every $r \geqq 2$. Hence $h^0(P, tH) = h^0(C, S^t E_2) = h^0(C, E_{t-1}) = 1$. So $|tH| = tD$ and $\mathrm{Bs}|tH| = D$.

(1.17) DEFINITION. Given a line bundle L on a space S, by SBs(L) we denote the intersection $\bigcap_{t>0} \mathrm{Bs}|tL|$, which is a Zariski closed subset of S. This will be called the stable base locus of L.

Using this notion, we will prove a variant (1.19) of (1.10).

(1.18) PROPOSITION. *For any line bundle L on a space S, there is a positive integer m such that* $\mathrm{Bs}|mL| = \mathrm{SBs}(L)$. *In particular, L is semiample if and only if* $\mathrm{SBs}(L) = \emptyset$.

PROOF. Assuming the contrary, we will derive a contradiction. Let $B = \mathrm{SBs}(L)$ and $B_t = \mathrm{Bs}|tL|$ for each positive integer t. Set $n_t = \dim(B_t - B)$ and let r_t be the number of irreducible components of B_t of dimension n_t not contained in B. Take a general point x on one of such components. Then $x \notin B$ and hence $x \notin B_u$ for some $u > 0$. Clearly $B_{tu} \subset B_t \cap B_u$. Hence $n_{tu} \leqq n_t$. Moreover, if $n_{tu} = n_t$, then $r_{tu} < r_t$. So $(n_{tu}, r_{tu}) < (n_t, r_t)$ with respect to the lexicographical order. Repeating similarly we find a sequence $(n_t, r_t) > (n_{tu}, r_{tu}) > (n_{tuv}, r_{tuv}) > \cdots$ of infinite length, which is impossible.

(1.19) THEOREM. *Let L be a line bundle on a normal variety V. For any* *connected component* X *of* SBs(L), *the restriction of* L *to* X *is not ample. In particular,* SBs(L) *has no isolated point.*

PROOF. By virtue of (1.18) we may assume $\mathrm{Bs}|L|=\mathrm{SBs}(L)$. Let $\mathcal{B}$ be the subsheaf of $\mathcal{O}_V[L]$ generated by global sections. Then $\mathcal{J}=\mathcal{B}[-L]$ is an $\mathcal{O}_V$-ideal which defines a subscheme B such that $\mathrm{Supp}(B)=\mathrm{Bs}|L|$. We will derive a contradiction assuming that L_X is ample. Let $\mathcal{I}$ be the $\mathcal{O}_V$-ideal of the subscheme $B-X$ and let $f: T\to V$ be the blowing-up of V with center $\mathcal{I}$. So $\mathcal{E}=f^*\mathcal{I}$ is a principal $\mathcal{O}_T$-ideal defining a Cartier divisor E on T lying over $B-X$. Moreover, $f^*|L|=E+\Lambda$ for some linear system Λ on T such that $\mathrm{Bs}\,\Lambda=f^{-1}(X)\cong X$. Then $[\Lambda]$ is semiample by (1.10). So $\mathrm{Bs}|mf^*L|=E$ for some $m>0$. On the other hand, $\mathrm{Bs}|mf^*L|=\mathrm{Bs}(f^*|mL|)=f^{-1}(\mathrm{Bs}|mL|)=f^{-1}(\mathrm{Supp}(B))\supset f^{-1}(X)$ since V is normal. This contradiction proves the assertion.

(1.20) THEOREM. *Let* $f: V\to W$ *be a surjective morphism from an irreducible space* V *onto a normal variety* W. *Then* $\mathrm{SBs}(f^*L)=f^{-1}(\mathrm{SBs}(L))$ *for any line bundle* L *on* W. *In particular,* L *is semiample if so is* $L_V=f^*L$.

PROOF. It is clear that $\mathrm{SBs}(L_V)\subset f^{-1}(\mathrm{SBs}(L))$. In order to show the converse, let $V\to S\to W$ be the Stein factorization of f. Since $g_*\mathcal{O}_V=\mathcal{O}_S$ for $g: V\to S$, we have $|tL_V|=g^*|tL_S|$ for any t and hence $\mathrm{SBs}(L_V)=g^{-1}(\mathrm{SBs}(L_S))$. Thus we reduce the problem to the case in which f is finite. Moreover, we may assume that V is a normal variety. Indeed, if $\nu: \tilde{V}\to V$ is the morphism from the normalization of V_{red}, then $\mathrm{SBs}(L_{\tilde{V}})=(f\circ\nu)^{-1}(\mathrm{SBs}(L))$ implies $\mathrm{SBs}(L_V)=f^{-1}(\mathrm{SBs}(L))$. Our proof proceeds in several steps.

Step 1, the case in which W is the quotient of V with respect to an action of a finite group G. Suppose that there is a point x on V not in $\mathrm{SBs}(L_V)$ such that $y=f(x)\in\mathrm{SBs}(L)$. For any $\sigma\in G$ we have $\sigma(\mathrm{SBs}(L_V))=\mathrm{SBs}(L_V)$ since $\sigma^*L_V=L_V$. Hence $\sigma(x)\notin\mathrm{SBs}(L_V)$. We may assume $\mathrm{SBs}(L_V)=\mathrm{Bs}|L_V|$ by (1.18). Then, for a general $\varphi\in H^0(V, L_V)$, we have $\varphi(\sigma(x))\neq 0$ for any $\sigma\in G$. Set $\Phi=\bigotimes_{\sigma\in G}\varphi^\sigma\in H^0(V, gL_V)$ where g is the order of G. Then $\Phi=f^*\psi$ for some $\psi\in H^0(W, gL)$ because Φ is G-invariant. By construction $\Phi(x)\neq 0$, hence $\psi(y)\neq 0$, contradicting $y\in\mathrm{SBs}(L)$.

From now on, we treat the algebraic and analytic cases separately. First we consider the problem in the algebraic category. Let $\mathfrak{R}(V)$ and $\mathfrak{R}(W)$ be the fields of rational functions on V and W respectively.

Step 2 in the algebraic case, where $\mathfrak{R}(V)/\mathfrak{R}(W)$ is a separable extension. There is an extension of fields $F/\mathfrak{R}(V)$ such that $F/\mathfrak{R}(W)$ is finite and Galois. Let X be the normalization of W in F. Then the morphism $g: X\to W$ is factored through V since f is finite. We have $\mathrm{SBs}(L_X)=g^{-1}(\mathrm{SBs}(L))$ by Step 1. This implies $\mathrm{SBs}(L_V)=f^{-1}(\mathrm{SBs}(L))$ since the former contains the image of $\mathrm{SBs}(L_X)$.

Step 3, where $\mathfrak{K}(V)/\mathfrak{K}(W)$ is purely inseparable. By a similar technique as above, we reduce the problem to the case in which f is the Frobenius morphism F, that means, V is isomorphic to W as an abstract scheme and F is defined on any affine open subset $U=\mathrm{Spec}(R)$ of this scheme by $F^*\xi=\xi^p$ for every $\xi\in R$. (Warning: Usually V is not isomorphic to W as a $\mathfrak{K}$-scheme.) Then, for any $\varphi\in H^0(V, tL_V)$, we see that $\varphi^{\otimes p}$ comes from $H^0(W, tpL)$ via F^*, where $p=\mathrm{char}(\mathfrak{K})$. This implies $\mathrm{SBs}(L_V)=f^{-1}(\mathrm{SBs}(L))$ similarly as in Step 1.

Step 4, general cases. Let F be the separable closure of $\mathfrak{K}(W)$ in $\mathfrak{K}(V)$ and let Y be the normalization of W in F. Then f is factored through Y. Our assertion is valid for $Y\to W$ by Step 2 and also for $V\to Y$ by Step 3. Therefore it is valid for f.

Thus we complete the proof in the algebraic case. Next we consider the analytic case. Using the desingularization theory we take a non-singular model $g: M\to N$ of $f: V\to W$. Then $\mathrm{SBs}(L_M)$ and $\mathrm{SBs}(L_N)$ are the inverse images of $\mathrm{SBs}(L_V)$ and $\mathrm{SBs}(L_W)$ respectively since V and W are normal. Replacing further M by $\mathcal{S}pec_{\mathrm{an}}(g_*\mathcal{O}_M)$, we reduce the problem to the case in which W is non-singular, V is normal and f is finite. Let B be the branch locus of f and set $U=W-B$. Then $f_U: f^{-1}(U)\to U$ is an étale morphism. Let H be the subgroup of $\pi_1(U)$ corresponding to f_U, that is, the image of $\pi_1(f^{-1}(U))$. Take a subgroup N of H such that N is a normal subgroup of $\pi_1(U)$ and that $G=\pi_1(U)/N$ is a finite group. Let U' be the quotient of the universal covering $\tilde{U}$ of U by the natural action of N on $\tilde{U}$. Then we have a natural étale morphism $f'_U: U'\to U$ which makes U the quotient of U' by the natural action of G. By construction, f'_U factors through $f^{-1}(U)$. Now, thanks to [GR], we have a finite morphism $f': V'\to W$ from a normal compact variety V' such that its restriction over U is nothing but f'_U. Then f' factors through V. Moreover, we see that the action of G on U' is extended holomorphically to V' and f' makes W the quotient of V' with respect to this extended action. Hence $\mathrm{SBs}(L_{V'})=(f')^{-1}\mathrm{SBs}(L)$ by Step 1. This implies $\mathrm{SBs}(L_V)=f^{-1}(\mathrm{SBs}(L))$ similarly as in Step 2 in the algebraic case. Thus we complete the proof.

(1.21) The above assertion would not always be true if W were not normal. In fact, if $f: V\to W$ is the normalization of a rational curve W with a node and if L is a non-torsion element in $\mathrm{Pic}_0(W)\cong \boldsymbol{G}_m$, then $\mathrm{SBs}(L_V)=\emptyset$ while $\mathrm{SBs}(L)=W$.

§2. Numerical semipositivity.

(2.1) Definition. A line bundle L on a space S is said to be *numerically semipositive* (abbr.: n-semipositive) if $L^rX\geqq 0$ for any irreducible reduced subspace X of S, where $r=\dim X$.

(2.2) PROPOSITION. *For any morphism $f: T\to S$, f^*L is n-semipositive if so is L on S.*

This is obvious. It is also clear that any semiample line bundle is n-semipositive.

(2.3) PROPOSITION. *Let $f: T\to S$ be a surjective morphism between algebraic spaces. Then a line bundle L on S is n-semipositive if so is f^*L.*

PROOF. Let X be an irreducible reduced subspace of S. We should show $L^rX\geqq 0$ for $r=\dim X$. Replacing T by $f^{-1}(X)$ if necessary, we may assume that $S=X$. Furthermore, we may assume that T is a variety. By Chow's lemma, there is a projective variety V together with a birational morphism $\pi: V\to T$. If $\dim V>r$, then a general hyperplane section H of V is mapped onto X by $f\circ\pi$. Replacing V by H and repeating this process if necessary, we reduce the problem to the case in which $\dim V=r$. Then $(L_V)^r\{V\}=\deg(f)\cdot L^rX\geqq 0$ since $L_V=\pi^*(f^*L)$ is n-semipositive. So $L^rX\geqq 0$ because $\deg(f)>0$.

(2.4) PROPOSITION. *Suppose that S is projective and let A be an ample line bundle on S. Then $A+L$ is ample for any n-semipositive line bundle L.*

For a proof, use the criterion of Nakai [N].

(2.5) COROLLARY. *Let $L_1, \cdots, L_n$ be n-semipositive line bundles on an algebraic variety S with* $\dim S=n$. *Then $L_1\cdots L_n\{S\}\geqq 0$.*

PROOF. By virtue of Chow's lemma we may assume that S is projective. Take an ample line bundle A on S. Then $A+tL_j$ is ample for any $t\geqq 0$ by (2.4). Hence $(A+tL_1)\cdots(A+tL_n)\{S\}>0$ for any $t\geqq 0$. Letting $t\to\infty$, we infer that $L_1\cdots L_n\geqq 0$.

(2.6) COROLLARY. *Let $L_1, \cdots, L_n, L_1', \cdots, L_n'$ be n-semipositive line bundles on an algebraic variety V with* $\dim V=n$. *Suppose that there exists a positive integer m_j for each $j=1, \cdots, n$ such that $|m_j(L_j-L_j')|\neq\emptyset$. Then $L_1\cdots L_n\{V\}\geqq L_1'\cdots L_n'\{V\}$.*

PROOF. Because of the birational invariance of the intersection numbers we may assume that V is normal. Take a member D of $|m_n(L_n-L_n')|$ and let $D=\sum\delta_iD_i$ be the prime decomposition of D as a Weil divisor. By (2.5) we have $L_1\cdots L_{n-1}\{D\}=\sum\delta_iL_1\cdots L_{n-1}\{D_i\}\geqq 0$. So $L_1\cdots L_n\geqq L_1\cdots L_{n-1}L_n'$. Similarly we obtain $L_1\cdots L_{n-1}L_n'\geqq L_1\cdots L_{n-2}L_{n-1}'L_n'\geqq\cdots\geqq L_1L_2'\cdots L_n'\geqq L_1'\cdots L_n'$. Thus we prove the desired inequality.

(2.7) THEOREM. *A line bundle L on an algebraic variety V is numerically semipositive if $LC \geqq 0$ for any curve C in V.*

This very useful criterion is due to Kleiman [K1]. A proof can be found in [Ha 1; p. 34] too. In fact, all the elementary numerical properties of semipositive line bundles were established by Kleiman.

§ 3. Index of positivity.

(3.1) DEFINITION. Given a numerically semipositive line bundle L on an algebraic variety V, we define $\sigma(L)$ to be the maximum of the dimensions of subvarieties X of V such that $L^r X > 0$ for $r = \dim X$. Clearly $0 \leqq \sigma(L) \leqq n = \dim V$, and $\sigma(L) = n$ if and only if $L^n > 0$. Sometimes $\sigma(L)$ will be called the index of positivity of L. As we will see later, this is a numerical version of the Kodaira dimension $\kappa(L, V)$ (cf. [I] and [F3]).

REMARK. This formulation does not work well on non-algebraic complex analytic varieties.

(3.2) PROPOSITION. *Let $f: W \to V$ be a morphism between algebraic varieties and let L be an n-semipositive line bundle on V. Then $\sigma(L_W) \leqq \sigma(L)$. Moreover, the equality holds if f is surjective.*

PROOF. For any subvariety Y of W with $\dim Y = r > \sigma(L)$, $Z = f(Y)$ is a subvariety of V. If $\dim Z = r$, then $L^r Y = L^r Z \deg(f_Y) = 0$ since $L^r Z = 0$. If $\dim Z < r$, then $L^r Y = 0$ since $(L_Z)^r = 0$ in the Chow ring of Z. In either case $L^r Y = 0$, which proves $\sigma(L_W) \leqq \sigma(L)$.

In order to show the second assertion, let X be a subvariety of V such that $\dim X = \sigma(L) = \sigma$ and $L^\sigma X > 0$. By an argument as in (2.3), we find a variety T together with a morphism $g: T \to W$ such that $\dim T = \sigma$ and $(f \circ g)(T) = X$. Replacing T by $g(T)$ if necessary, we may assume that g is an inclusion. Then $(L_W)^\sigma\{T\} = L_T^\sigma = L^\sigma\{X\} \cdot \deg(T \to X) > 0$. This implies $\sigma(L_W) = \sigma$.

(3.3) THEOREM. *Let L be an n-semipositive line bundle on a projective variety V with $n = \dim V$. Then, for any ample line bundle A on V, $L^r A^{n-r} = 0$ if $r > \sigma(L)$ and $L^r A^{n-r} > 0$ if $r \leqq \sigma(L)$.*

PROOF. Replacing A by mA with $m \gg 0$ if necessary, we may assume that A is very ample. Taking general hyperplane sections successively $(n-r)$-times, we obtain a subvariety Y of V such that $\dim Y = r$ and $L^r A^{n-r} = L^r Y$. So $L^r A^{n-r} = 0$ if $r > \sigma(L)$.

We will prove the second assertion by induction on $n-r$. $L^n > 0$ if $\sigma(L) = n$.

So we may assume $r<n$. We may further assume that V is normal. Let X be a subvariety of V such that $\dim X=\sigma(L)=\sigma$ and $L^{\sigma}X>0$. For $m\gg 0$, we have a member D of $|mA|$ such that $X\subset D$. Let D_j be a prime component of D containing X. Then $\sigma(L_{D_j})\geqq\sigma$. So $mL^rA^{n-r}=L^rA^{n-r-1}\{D\}\geqq L^rA^{n-r-1}\{D_j\}>0$ by the induction hypothesis. Thus we prove $L^rA^{n-r}>0$ for $r\leqq\sigma(L)$.

(3.4) COROLLARY. *Let L and H be n-semipositive line bundles on an algebraic variety V. Suppose that $H^0(V, mL-H)\neq 0$ for some positive integer m. Then $\sigma(L)\geqq\sigma(H)$.*

For a proof, use (2.6). Note that we may assume that V is projective by virtue of Chow's lemma and (3.2).

(3.5) THEOREM. *Let L be a numerically semipositive line bundle on an algebraic variety V. Then $\kappa(L, \mathcal{F})\leqq\sigma(L)$ for any coherent sheaf $\mathcal{F}$ on V. In particular, $\kappa(L, V)\leqq\sigma(L)$.*

For a proof, we recall the following

(3.6) THEOREM. *Let L be a line bundle on a normal variety V such that $\kappa=\kappa(L, V)\geqq 0$, i.e., $|sL|\neq\emptyset$ for some positive integer s. Then, there are a normal variety V' together with a birational morphism $\pi: V'\to V$, an effective Cartier divisor E on V' and a linear system Λ on V' such that*

a) *$\pi^*|mL|=E+\Lambda$ for some positive integer m,*

b) *$\mathrm{Bs}\,\Lambda=\emptyset$ and $\dim W=\kappa(L, V)$, where W is the image of the rational mapping Φ defined by Λ,*

c) *any generic fiber F of Φ is an irreducible space with $\kappa(L_F, F)=0$.*

For a proof, see [F3; (3.8) and (3.11)].

(3.7) PROOF OF (3.5). Similarly as in [F3; (2.5)], we use the Noetherian induction on $\mathrm{Supp}(\mathcal{F})$. In view of the arguments in Steps 2, 3, 4 of [F3; (2.5)], we infer that it suffices to consider the case $\mathcal{F}=\mathcal{O}_V$. Then, we may assume V to be normal by virtue of (3.2).

Let things be as in (3.6) and set $H=[\Lambda]=\Phi^*\mathcal{O}_W(1)$. Then H is n-semipositive and $\sigma(H)=\dim W=\kappa(L, V)$ by (3.2). On the other hand $\sigma(L)=\sigma(\pi^*L)\geqq\sigma(H)$ by (3.4). So $\kappa(L, V)\leqq\sigma(L)$ as required.

(3.8) Suppose that L is semiample. Take $m>0$ such that $\mathrm{Bs}|mL|=\emptyset$ and let W be the image of the rational mapping defined by $|mL|$. Then we have $\sigma(L)=\dim W=\kappa(L, V)$. However, for a general n-semipositive line bundle L, the equality $\sigma(L)=\kappa(L, V)$ is not true.

Problem. Find a good sufficient condition in order that $\sigma(L)=\kappa(L, V)$.
Later we will see that $\kappa(L, V)=\dim V$ if $\sigma(L)=\dim V$.

(3.9) *Example.* Let P and H be as in Example (1.16). Then H is n-semipositive and $\sigma(H)=1$. But $\kappa(H, P)=0$.

(3.10) *Example.* Let C be a non-singular elliptic curve and let L be a non-torsion element in $\mathrm{Pic}_0(C)$. Then $\sigma(L)=0$ and $\kappa(L, C)<0$.

(3.11) LEMMA. *Let L_1 and L_2 be n-semipositive line bundles on algebraic varieties V_1 and V_2 respectively. Let π_i be the projections from $V=V_1\times V_2$ onto V_i and let $L=\pi_1^*L_1+\pi_2^*L_2$. Then L is n-semipositive, $\sigma(L)=\sigma(L_1)+\sigma(L_2)$ and $\kappa(L, V)=\kappa(L_1, V_1)+\kappa(L_2, V_2)$ where we define $\kappa=-\infty$ if $\kappa<0$.*

Proof is easy.

(3.12) LEMMA. *Let L be an n-semipositive line bundle on an algebraic variety V. Let $P=\boldsymbol{P}(L\oplus\mathcal{O}_V)$ and let H be the tautological line bundle on P. Then H is n-semipositive, $\sigma(H)=\sigma(L)+1$ and $\kappa(H, P)=\kappa(L, V)+1$, where we define $\kappa=-1$ if $\kappa<0$.*

PROOF. Let D be the divisor on P corresponding to the subbundle $\mathcal{O}_V$. Then $D\in|H|$ and $(D, H_D)\cong(V, L)$. So H is n-semipositive by (2.7) and $\sigma(H)=\sigma(L)+1$ by (3.3). Using $h^0(P, tH)=h^0(V, S^t(L\oplus\mathcal{O}_V))=\sum_{j=0}^{t}h^0(V, jL)$, we obtain $\kappa(H, P)=\kappa(L, V)+1$.

(3.13) Combining the above examples and lemmas we can construct examples with various values (σ, κ) such that $\kappa\leqq\sigma<\dim V$.

§ 4. Geometrical semipositivity.

In this section we work in the category of complex analytic spaces. Although the results here seem to be more or less known to experts, we present outlines of our proofs for the sake of the convenience of the reader.

(4.1) DEFINITION. Let φ be a real differentiable (1, 1)-form on a complex manifold M. We define a Hermitian form φ_p on the tangent space T_p^M of M at $p\in M$ in the standard way. φ is said to be semipositive (resp. positive) if φ_p is positive semidefinite (resp. definite) at every point p on M. In this case the number of positive eigenvalues of φ_p, denoted by $\sigma(\varphi_p)$, is a lower semicontinuous function on p. We define $\sigma(\varphi)=\mathrm{Max}_{p\in M}\sigma(\varphi_p)$. So, there is a non-empty open subset U of M such that $\sigma(\varphi_p)=\sigma(\varphi)$ for any $p\in U$.

(4.2) DEFINITION. A line bundle L on M is said to be *geometrically semipositive* (abbr. g-semipositive) if the real Chern class $c_1(L)_{\boldsymbol{R}}$ can be represented by a semipositive real (1, 1)-form in terms of De Rham isomorphism. Note that L is ample if and only if $c_1(L)_{\boldsymbol{R}}$ can be represented by a positive (1, 1)-form.

(4.3) LEMMA. *Let $f: N\to M$ be a holomorphic mapping between complex manifolds and let φ be a semipositive real* (1, 1)*-form on M. Then $f^*\varphi$ is semipositive and $\sigma(f^*\varphi)\leqq\sigma(\varphi)$. Moreover, the equality holds if f is surjective.*

PROOF. For any $x\in N$, $(f^*\varphi)_x$ is the pull-back of φ_y by the natural mapping $T_x^N\to T_y^M$, where $y=f(x)$. Hence this is positive semidefinite since so is φ_y. Moreover $\sigma((f^*\varphi)_x)\leqq\sigma(\varphi_y)$. So $\sigma(f^*\varphi)\leqq\sigma(\varphi)$.

If f is surjective, there is an open dense subset V of M over which f is smooth. Take a point $y\in U\cap V$ where U is as in (4.1) and take $x\in f^{-1}(y)$. Then $T_x^N\to T_y^M$ is surjective and $\sigma((f^*\varphi)_x)=\sigma(\varphi_y)=\sigma(\varphi)$. This implies $\sigma(f^*\varphi)=\sigma(\varphi)$.

(4.4) LEMMA. *Let $\varphi_1, \cdots, \varphi_n$ be semipositive* (1, 1)*-forms on M as in* (4.1), *where $n=\dim M$. Then $\int_M \varphi_1\wedge\cdots\wedge\varphi_n\geqq 0$.*

PROOF. For any point y on M, the Hermitian form $(\varphi_j)_y$ on $T=T_y^M$ is positive semidefinite for each j. So the tensor product of them defines a positive semidefinite Hermitian form on $T^{\otimes n}$, the restriction of which to $\overset{n}{\wedge}T\subset T^{\otimes n}$ corresponds to $(\varphi_1\wedge\cdots\wedge\varphi_n)(y)$. Hence $(\varphi_1\wedge\cdots\wedge\varphi_n)(y)\geqq 0$ for every $y\in M$. This implies our assertion.

(4.5) COROLLARY. *Let W be a subvariety of M with $\dim W=r$ and let $L_1, \cdots, L_r$ be geometrically semipositive line bundles on M. Then $L_1\cdots L_r\{W\}\geqq 0$. In particular, any g-semipositive line bundle is n-semipositive.*

PROOF. Take a non-singular model N of W. Then $L_1\cdots L_r\{W\}=L_1\cdots L_r\{N\}$, the latter is non-negative by (4.4) and (4.3).

(4.6) LEMMA. *Let φ be a semipositive* (1, 1)*-form as in* (4.1) *and let ω be a positive* (1, 1)*-form on M. Then $I=\int_M \varphi^r\wedge\omega^{(n-r)}=0$ if $r>\sigma(\varphi)$ and $I>0$ if $r\leqq\sigma(\varphi)$.*

PROOF. Similarly as in (4.4), we prove this lemma by techniques of linear algebra at each point of M.

(4.7) COROLLARY. *Let φ and φ' be semipositive* (1, 1)*-forms as in* (4.1) *on a*

Kähler manifold M. Then $\sigma(\varphi)=\sigma(\varphi')$ if φ and φ' are cohomologous to each other.

(4.8) DEFINITION. A complex analytic space S is called a *Fujiki space* if there exists a surjective morphism $f: M\to S$ from a Kähler manifold M onto S. In particular, any algebraic space is a Fujiki space.

For any g-semipositive line bundle L on a Fujiki manifold N, we define $\sigma(L)$ to be $\sigma(\varphi)$ where φ is a semipositive real (1, 1)-form representing $c_1(L)_{\boldsymbol{R}}$. By (4.3) and (4.7) this is independent of the choice of φ and hence $\sigma(L)$ is well-defined. When N is algebraic, this definition coincides with that in (3.1) by virtue of (3.3) and (4.6).

(4.9) The highlight of this section is the following

THEOREM. *Let L be a g-semipositive line bundle on a Fujiki manifold M. Then $H^q(M, -L)=0$ for any $q<\sigma(L)$.*

We outline a proof based on the harmonic theory. Recall first the following

(4.10) THEOREM. *Let $f: N\to M$ be a surjective morphism between Fujiki manifolds. Then the natural mapping $H^{p,q}(M, E)\to H^{p,q}(N, f^*E)$ is injective for any vector bundle E on M and any integers p, q.*

PROOF. Clearly we may assume that N is Kähler. Let Ω be a Kähler form on N and fix a Hermitian metric on M and a Hermitian norm of E. Let $A^{p,q}(M, E)$ (resp. $\mathscr{H}^{p,q}(M, E)$) denote the space of E-valued differentiable (resp. harmonic) (p, q)-forms on M. Note that the dual bundle $E^\vee$ of E becomes a Hermitian bundle in a natural way, so we have $\mathscr{H}^{m-p,m-q}(M, E^\vee)\subset A^{m-p,m-q}(M, E^\vee)$ for $m=\dim M$. Let ω be the positive (1, 1)-form corresponding to the Hermitian metric on M and let $\|\ \|$ denote the pointwise Hermitian norm induced on $A^{p,q}(M, E)$ (cf. [KM; p. 93]). Then we have a natural conjugate linear mapping $\sharp: \mathscr{H}^{p,q}(M, E)\to\mathscr{H}^{m-p,m-q}(M, E^\vee)$ such that $\varphi\wedge(\sharp\varphi)=\|\varphi(z)\|\omega^m/m!$ (cf. [KM; p. 105]). In particular, $\varphi\wedge(\sharp\varphi)>0$ on a non-empty open subset U of M if $\varphi\neq 0$. Then, by a similar method as in (4.4) and (4.6), we obtain $0<\int_N f^*(\varphi\wedge(\sharp\varphi))\wedge\Omega^{n-m}$ $=\int_N(f^*\varphi)\wedge(f^*(\sharp\varphi))\wedge\Omega^{n-m}$, where $n=\dim N$ and we regard $f^*\varphi\in A^{p,q}(N, f^*E)$, $f^*(\sharp\varphi)\in A^{m-p,m-q}(N, f^*E^\vee)$. It suffices to show that $f^*\varphi$ is not cohomologous to zero in terms of Dolbeault cohomology. If so, we would have $\psi\in A^{p,q-1}(N, f^*E)$ such that $f^*\varphi=\bar\partial\psi$. Then $f^*\varphi\wedge f^*(\sharp\varphi)\wedge\Omega^{n-m}=\bar\partial(\psi\wedge f^*(\sharp\varphi)\wedge\Omega^{n-m})=$ $d(\psi\wedge f^*(\sharp\varphi)\wedge\Omega^{n-m})$ since both $f^*(\sharp\varphi)$ and Ω are $\bar\partial$-closed and $\psi\wedge f^*(\sharp\varphi)\wedge\Omega^{n-m}$ $\in A^{n,n-1}(N)$. Hence the integration of this form over N must be zero by Stokes' theorem. This contradicts the preceding inequality.

(4.11) PROOF OF (4.9). Our method is a modification of the argument in [KM; p. 125, Theorem 7.1]. In fact, we may assume that M is Kähler by virtue of (4.10).

By Serre duality it suffices to show $H^q(M, K^M+L)=0$ for $q>n-\sigma(L)$, where $n=\dim M$. Take a Hermitian norm of L such that its curvature form λ is semi-positive (cf. [KM; p. 128, Theorem 7.4]). Let U be the non-empty open set such that $\sigma(\lambda_x)=\sigma(L)$ for every $x\in U$. Following the calculations as in [KM; p. 126] where λ corresponds to the Hermitian matrix $X_{\tau\bar{\sigma}}-R_{\bar{\sigma}\tau}$, we obtain $a_j q\sum_{A_{q-1},B_{q-1}}\sum_{\sigma,\tau}(X_{\tau\bar{\sigma}}-R_{\bar{\sigma}\tau})g_{\alpha_1\bar{\beta}_1}\cdots g_{\alpha_{q-1}\beta_{q-1}}\varphi^{\tau A_{q-1}}\overline{\varphi^{\sigma B_{q-1}}}\omega^n/n!\leqq 0$ for any $\varphi\in\mathscr{H}^{0,q}(M, K^M+L)$. For each $x\in M$, we take a coordinate system $(z^1, \cdots, z^n)$ on a neighborhood of x such that $g_{\alpha\bar{\beta}}(x)=\delta_{\alpha\beta}$ and $(X_{\tau\bar{\sigma}}-R_{\bar{\sigma}\tau})(x)=\delta_{\sigma\tau}d_\sigma$ with $d_\sigma>0$ for $\sigma\leqq\sigma(\lambda_x)$ and $d_\sigma=0$ for $\sigma>\sigma(\lambda_x)$. The volume element integrated above is of the form $qI(x)\omega^n/n!$, where $I(x)=\sum_{A_{q-1},\sigma}d_\sigma\varphi^{\sigma A_{q-1}}\overline{\varphi^{\sigma A_{q-1}}}\geqq 0$. Hence $I(x)=0$ for every $x\in M$. If $x\in U$, this implies $\varphi^{A_q}(x)=0$ unless $\sigma(L)<\alpha_j$ for each $j=1, \cdots, q$. Therefore, if $q>n-\sigma(L)$, we obtain $\varphi^{A_q}(x)=0$ for every $x\in U$ and $A_q=(\alpha_1, \cdots, \alpha_q)$. This implies $\varphi^{A_q}\equiv 0$ on M because φ is harmonic.

(4.12) CONJECTURE. Let $f: M\to V$ be a surjective morphism from a Fujiki manifold M onto a space V. Then $R^q f_*\omega_M=0$ for any $q>\dim M-\dim V$, where $\omega_M=\mathcal{O}_M(K^M)$ is the dualizing sheaf of M.

(4.13) PROPOSITION. (4.12) *is true when V is projective.*

PROOF. Set $\mathscr{H}^q=R^q f_*\omega_M$ and take an ample line bundle A on V. For any sufficiently large integer t we have $H^p(V, \mathscr{H}^q[tA])=0$ for $p>0$. So, using the Leray spectral sequence, we infer $H^q(M, K^M+tA_M)\cong H^0(V, \mathscr{H}^q[tA])$. By (4.9) and the Serre duality, this vanishes for $q>\dim M-\dim V$ since f^*A is g-semi-positive and $\sigma(f^*A)=\dim V$. Hence $\mathscr{H}^q=0$ because A is ample.

(4.14) CONJECTURE. Any n-semipositive line bundle on an algebraic manifold is g-semipositive.

This would be useful in the study of n-semipositive line bundles because then we can enjoy the power of differential geometry. One of the strongest evidence of this conjecture is the following remarkable result of Kawamata [Ka] and Viehweg [V]:

(4.15) THEOREM. *Let L be an n-semipositive line bundle on an algebraic manifold M. Then $H^q(M, -L)=0$ for $q<\sigma(L)$.*

(4.16) CONJECTURE. For any surjective morphism $f: M\to N$ of complex manifolds and any line bundle L on N, L is g-semipositive if so is f^*L.

Of course, this would follow from (4.14) if things are algebraic.

§ 5. Cohomological semipositivity.

(5.1) The purpose of this section is to prove the following

THEOREM. *For any ample line bundle A and any coherent sheaf $\mathcal{F}$ on a projective scheme S, there exists an integer k such that $H^q(\mathcal{F}[tA+L])=0$ for any $q>0$, $t\geqq k$ and any n-semipositive line bundle L on S.*

Of course, the point is that k can be chosen independently of L.

(5.2) By $\mathrm{V}^q(\mathcal{F}, A)$ we denote the assertion that there is an integer k having the above property. Clearly $\mathrm{V}^q(\mathcal{F}, A)$ is true if so is $\mathrm{V}^q(\mathcal{F}[jA], mA)$ for every $j=0, 1, \cdots, m-1$, where m is a positive integer. Therefore, in order to prove (5.1), it suffices to prove $\mathrm{V}^q(\mathcal{F}, A)$ for any very ample line bundle A.

From now on, we consider the problem for a fixed very ample line bundle A and write $\mathrm{V}^q(\mathcal{F})$ instead of $\mathrm{V}^q(\mathcal{F}, A)$. We say that $\mathrm{V}(\mathcal{F})$ is true if so is $\mathrm{V}^q(\mathcal{F})$ for any positive integer q.

(5.3) LEMMA. *Let $0\to\mathcal{F}\to\mathcal{G}\to\mathcal{H}\to 0$ be an exact sequence of coherent sheaves on S. Then*

1) *$\mathrm{V}^q(\mathcal{G})$ follows from $\mathrm{V}^q(\mathcal{F})$ and $\mathrm{V}^q(\mathcal{H})$.*
2) *$\mathrm{V}^q(\mathcal{H})$ follows from $\mathrm{V}^q(\mathcal{G})$ and $\mathrm{V}^{q+1}(\mathcal{F})$.*
3) *If $q\geqq 2$, $\mathrm{V}^q(\mathcal{F})$ follows from $\mathrm{V}^q(\mathcal{G})$ and $\mathrm{V}^{q-1}(\mathcal{H})$.*

Proof is easy.

(5.4) LEMMA. *Let $\varphi: \mathcal{F}[-A]\to\mathcal{F}$ be a homomorphism. Then $\mathrm{V}^q(\mathcal{F})$ follows from $\mathrm{V}^{q-1}(\mathrm{Coker}(\varphi))$ and $\mathrm{V}^q(\mathrm{Ker}(\varphi))$ for any $q\geqq 2$.*

PROOF. Take an integer k such that $H^{q-1}(\mathrm{Coker}(\varphi)[tA+L])=H^q(\mathrm{Ker}(\varphi)[tA+L])=0$ for any $t\geqq k$ and any n-semipositive line bundle L. Then we have $h^q(\mathcal{F}[(t-1)A+L])\leqq h^q(\mathrm{Im}(\varphi)[tA+L])\leqq h^q(\mathcal{F}[tA+L])$ for every $t\geqq k$. Hence $h^q(\mathcal{F}[tA+L])\leqq h^q(\mathcal{F}[(t+1)A+L])\leqq\cdots\leqq h^q(\mathcal{F}[uA+L])=0$ for $u\gg 0$ because A is ample. This shows $\mathrm{V}^q(\mathcal{F})$.

(5.5) For a subvariety V of $P\cong \boldsymbol{P}^N$ with $\dim V=n$, we define $\omega_V=\mathcal{E}xt^{N-n}_{\mathcal{O}_P}(\mathcal{O}_V, \omega_P)$, where ω_P is the canonical invertible sheaf $\cong\mathcal{O}_P(-N-1)$ of P. It turns out that ω_V depends only on the intrinsic structure of V and not on the embedding $V\subset P$. For a proof, see [F3; § 1]. Indeed, ω_V is nothing but $\mathcal{D}^n(\mathcal{O}_V)$ under the notation in [F3]. For a while ω_V will be called the "dualizing" sheaf of V.

(5.6) LEMMA. 1) *For any coherent sheaf $\mathcal{F}$ on V, $\mathrm{Hom}_V(\mathcal{F}, \omega_V)\cong\mathrm{Ext}_P^{N-n}(\mathcal{F}, \omega_P)$,* *the latter being dual to $H^n(\mathcal{F})$ by the Serre duality on P.*

2) *There is a surjective homomorphism $H^n(V, \omega_V) \to \mathrm{Ext}_P^N(\mathcal{O}_V, \omega_P)$.*

PROOF. We use the spectral sequence with $E_2^{p,q} = H^p(P, \mathcal{E}xt^q_{\mathcal{O}_P}(\mathcal{F}, \omega_P))$ converging to $\mathrm{Ext}_P^{p+q}(\mathcal{F}, \omega_P)$. For any $q < N-n$ we have $\mathcal{E}xt^q_{\mathcal{O}_P}(\mathcal{F}, \omega_P) = 0$. Hence $\mathrm{Ext}_P^{N-n}(\mathcal{F}, \omega_P) \cong H^0(\mathcal{E}xt^{N-n}_{\mathcal{O}_P}(\mathcal{F}, \omega_P))$. Next we use the spectral sequence of sheaves with $\mathcal{E}_2^{p,q} = \mathcal{E}xt^p_{\mathcal{O}_V}(\mathcal{F}, \mathcal{E}xt^q_{\mathcal{O}_P}(\mathcal{O}_V, \omega_P))$ converging to $\mathcal{E}xt^{p+q}_{\mathcal{O}_P}(\mathcal{F}, \omega_P)$ (compare, e.g., [CE; p. 348, Case 4]). We have $\mathcal{E}xt^q_{\mathcal{O}_P}(\mathcal{O}_V, \omega_P) = 0$ for $q < N-n$. So $\mathcal{E}xt^{N-n}_{\mathcal{O}_P}(\mathcal{F}, \omega_P) \cong \mathcal{E}_2^{0, N-n} \cong \mathcal{H}om_{\mathcal{O}_V}(\mathcal{F}, \omega_V)$. Combining these observations we obtain $\mathrm{Ext}_P^{N-n}(\mathcal{F}, \omega_P) \cong \mathrm{Hom}_V(\mathcal{F}, \omega_V)$. This proves 1).

To prove 2), we use the spectral sequence with $E_2^{p,q} = H^p(P, \mathcal{E}xt^q_{\mathcal{O}_P}(\mathcal{O}_V, \omega_P))$ converging to $\mathrm{Ext}_P^{p+q}(\mathcal{O}_V, \omega_P)$. We have $E_2^{p,q} = 0$ for $q < N-n$ similarly as above. On the other hand, [F3; (1.14)] implies $\dim(\mathrm{Supp}(\mathcal{E}xt^{N-n+j}_{\mathcal{O}_P}(\mathcal{O}_V, \omega_P))) < n-j$ for each $j > 0$. So $E_2^{p, N-p} = 0$ unless $p = n$. Hence $\mathrm{Ext}_P^N(\mathcal{O}_V, \omega_P) \cong E_\infty^{n, N-n}$, which is a holomorphic image of $E_2^{n, N-n} \cong H^n(V, \omega_V)$. Thus we prove 2).

(5.7) COROLLARY. *Let $f: V \to W$ be a surjective morphism between projective varieties V, W with* $\dim V = \dim W = n$. *Then there is a homomorphism $\varphi: f_*\omega_V \to \omega_W$ such that* $\mathrm{Supp}(\mathrm{Coker}(\varphi))$ *is a proper subset of W.*

PROOF. By (5.6; 2), we have $H^n(V, \omega_V) \neq 0$ because $\mathrm{Ext}_P^N(\mathcal{O}_V, \omega_P)$ is dual to $H^0(\mathcal{O}_V)$. We consider the Leray spectral sequence with $E_2^{p,q} = H^p(W, R^q f_* \omega_V)$ converging to $H^{p+q}(V, \omega_V)$. $\mathrm{Supp}(R^q f_* \omega_V)$ is contained in the set $X_q = \{w \in W \mid \dim f^{-1}(w) \geqq q\}$. Since $\dim f^{-1}(X_q) < n$ if $q > 0$, we have $\dim X_q < n-q$. So $E_2^{n-q, q} = 0$ unless $q = 0$. This implies $E_2^{n,0} \neq 0$ because $H^n(V, \omega_V) \neq 0$. So $H^n(W, f_*\omega_V) \neq 0$. Hence, by (5.6; 1), $\mathrm{Hom}_W(f_*\omega_V, \omega_W)$ contains a non-zero element φ. Set $\mathcal{J} = \mathrm{Im}(\varphi)$. Then $\mathrm{Hom}_W(\mathcal{J}, \omega_W) \neq 0$ implies $H^n(\mathcal{J}) \neq 0$ by (5.6; 1). Hence $\dim \mathrm{Supp}(\mathcal{J}) = n$. Since W is a variety and ω_W is invertible at any generic point of W, this implies $\mathcal{J} = \omega_W$ on an open dense subset of W. So $\mathrm{Coker}(\varphi)$ is supported on a proper subset of W.

(5.8) LEMMA. *Suppose that* $\mathrm{char}(\mathfrak{K}) = p > 0$ *and let X be a projective variety in $P \cong \boldsymbol{P}^N$. Suppose that* $\mathrm{V}(\mathcal{F})$ *is true for any coherent sheaf $\mathcal{F}$ on X such that* $\mathrm{Supp}(\mathcal{F}) \neq X$. *Then* $\mathrm{V}(\omega_X)$ *is true.*

PROOF. Let $F: X \to X$ be the Frobenius morphism of the abstract scheme X. So, F induces the identity of the underlying topological space of X and $F^*: \mathcal{O}_X \to \mathcal{O}_X$ is defined by $F^* f = f^p$. F is not a morphism of the $\mathfrak{K}$-scheme X. Namely, if $\pi: X \to \mathrm{Spec}(\mathfrak{K})$ is the structure morphism, $\pi \circ F$ defines a different $\mathfrak{K}$-scheme structure from the original one. This new $\mathfrak{K}$-scheme will be denoted by X'. Then F can be viewed as a morphism $X' \to X$ of $\mathfrak{K}$-schemes.

Any coherent sheaf $\mathcal{F}$ on X determines a unique coherent sheaf $\mathcal{F}'$ on X'

which corresponds to the same $\mathcal{O}_X$-module on the abstract scheme X. Then we have $F^*\mathcal{L}=(\mathcal{L}')^{\otimes p}$ for any invertible sheaf $\mathcal{L}$ on X.

We claim $h^q(X', \mathcal{F}')=h^q(X, \mathcal{F})$ for any integer q and any coherent sheaf $\mathcal{F}$ on X. Indeed, $h^q(X', \mathcal{F}')=\dim(R^q(\pi\circ F)_*\mathcal{F}')=\dim(R^q(F_0\circ\pi)_*\mathcal{F})$ where F_0 is the Frobenius of Spec($\mathfrak{K}$). Since $\mathfrak{K}$ is perfect, the latter is equal to $\dim(R^q\pi_*\mathcal{F})=h^q(X, \mathcal{F})$. Thus we prove the claim.

We next claim that ω'_X is the "dualizing" sheaf $\omega_{X'}$ of X'. To see this, we first consider the case $X=P$. Then X' is isomorphic to $\boldsymbol{P}^N$ and $F: X'\to X$ is the mapping defined by taking the p-powers of the homogeneous coordinates. Since $\mathcal{O}_X(1)'=\mathcal{O}_{X'}(1)$, the assertion is obvious in this case. In general we have $\omega_X=\mathcal{E}xt_{\mathcal{O}_P}^{N-n}(\mathcal{O}_X, \omega_P)$. Since $\omega_{P'}=\omega'_P$, the same procedure on the abstract scheme P gives ω_X and $\omega_{X'}$. Hence $\omega_{X'}=\omega'_X$.

Now, applying (5.7), we obtain a homomorphism $\varphi: F_*\omega_{X'}\to\omega_X$ such that V($\mathcal{C}$) is true for $\mathcal{C}=\mathrm{Coker}(\varphi)$. We claim that $\mathrm{V}^q(\mathcal{K})$ is true for any $q\geqq 2$, where $\mathcal{K}=\mathrm{Ker}(\varphi)$. Indeed, we have an injective homomorphism $\psi: \mathcal{K}[-A]\to\mathcal{K}$ since A is very ample. Moreover ψ is bijective at any general point on X and hence V(Coker(ψ)) is true by assumption. Therefore (5.4) applies.

Thus we can find a positive integer k such that $h^q(\mathcal{C}[tA+L])=h^{q+1}(\mathcal{K}[tA+L])=0$ for any $q>0$, $t\geqq k$ and any n-semipositive line bundle L on X. Then we have $h^q(\omega_X[tA+L])\leqq h^q(\mathrm{Im}(\varphi)[tA+L])\leqq h^q(X, F_*\omega'_X[tA+L])=h^q(X', \omega'_X[F^*(tA+L)])=h^q(X', \omega'_X[p(tA'+L')])=h^q(X, \omega_X[ptA+pL])$ for any q, t and L as above. Since $pt\geqq k$, we can repeat similar arguments to obtain $h^q(\omega_X[tA+L])\leqq h^q(\omega_X[p(tA+L)])\leqq\cdots\leqq h^q(\omega_X[p^e(tA+L)])$. The last term vanishes for $e\gg 0$ since $tA+L$ is ample. Consequently $h^q(\omega_X[tA+L])=0$ for any $q>0$, $t\geqq k$ and any n-semipositive line bundle L on X. Thus we prove V(ω_X).

(5.9) LEMMA. *Suppose that* char($\mathfrak{K}$)=0 *and let* $\pi: M\to S$ *be a morphism from a Fujiki manifold* M *to a projective scheme* S *such that* $\dim\pi(M)=\dim M=n$. *Then* V($\pi_*\omega_M$) *is true.*

PROOF. For any positive integer q, we have $R^q\pi_*\omega_M=0$ by (4.13). So $h^q(\pi_*\omega_M[tA+L])=h^q(M, K^M+tA+L)=h^{n-q}(M, -tA-L)=0$ for $t>0$ and any n-semipositive line bundle L on S by virtue of (4.9) and the Serre duality.

(5.10) Now we prove (5.1) by showing V($\mathcal{F}$) by the Noetherian induction on Supp($\mathcal{F}$). $\mathcal{F}$ can be considered to be a sheaf on the subscheme Z of S defined by the $\mathcal{O}_S$-ideal $\mathrm{Ker}(\mathcal{O}_S\to\mathcal{E}nd(\mathcal{F}))$. Using (5.3) we reduce the problem to the case in which Z is irreducible. Let $\mathcal{N}$ be the sheaf of nilpotent functions on Z. Set $\mathcal{F}_j=\mathcal{N}^j\mathcal{F}/\mathcal{N}^{j+1}\mathcal{F}$. Then $\mathcal{F}_j$ is a sheaf on Z_{red}. If V($\mathcal{F}_j$) is true for each j, we can prove V($\mathcal{N}^j\mathcal{F}$) for every j by the descending induction on j using (5.3; 1). So it suffices to consider the case in which Z is a variety.

By virtue of (5.8) and (5.9), there exists a coherent sheaf $\mathcal{T}$ on Z such that $\mathrm{V}(\mathcal{T})$ is valid and that $\mathcal{T}$ is invertible on an open dense subset of Z. Indeed, we set $\mathcal{T}=\omega_Z$ if $\mathrm{char}(\mathfrak{K})>0$. If $\mathrm{char}(\mathfrak{K})=0$, we take a desingularization $\pi: M\to Z$ and set $\mathcal{T}=\pi_*\omega_M$.

Take a sufficiently large integer u such that there is a homomorphism $\psi: \mathcal{T}\to\mathcal{O}_Z[uA]$ which is an isomorphism on an open dense subset of Z. Then $\mathrm{V}(\mathrm{Ker}(\psi))$ and $\mathrm{V}(\mathrm{Coker}(\psi))$ are true by the induction hypothesis. Applying (5.3), we prove $\mathrm{V}(\mathcal{O}_Z[uA])$. This implies $\mathrm{V}(\mathcal{O}_Z)$.

We complete the proof by showing $\mathrm{V}(\mathcal{F})$ by the induction on $r=\mathrm{rank}(\mathcal{F})$, the rank at the generic point of Z. The case $r=0$ is trivial by the Noetherian induction hypothesis. So suppose that $r>0$. Take a sufficiently large integer c such that there is a homomorphism $\iota: \mathcal{O}_Z[-cA]\to\mathcal{F}$ which is injective on an open dense subset of Z. Then $\mathrm{V}(\mathrm{Coker}(\iota))$ is true by the induction hypothesis on r. ι is injective since $\mathcal{O}_Z$ is torsion free. Obviously $\mathrm{V}(\mathcal{O}_Z)$ implies $\mathrm{V}(\mathcal{O}_Z[-cA])$. Hence we obtain $\mathrm{V}(\mathcal{F})$ by (5.3; 1).

§ 6. Consequences of cohomological semipositivity.

(6.1) *Notation.* Given a function $h(t)$ of $t\in\boldsymbol{Z}$, we write $h(t)\leqq O(t^m)$ if there is a positive constant k such that $h(t)\leqq k\cdot t^m$ for any positive t.

(6.2) THEOREM. *Let $\mathcal{F}$ be a coherent sheaf on an algebraic space S and let L be a numerically semipositive line bundle on S. Then $h^q(S, \mathcal{F}[tL])\leqq O(t^m)$, where $m=\mathrm{Min}(\dim(\mathrm{Supp}(\mathcal{F}))-q, \sigma(L))$.*

PROOF. First we consider the case in which S is projective. We use the induction on q in this case. When $q=0$, the assertion follows from (3.5) and [F3; (2.5)]. When $q>0$, thanks to (5.1), we have a very ample line bundle A on S such that $h^q(\mathcal{F}[A+tL])=0$ for any $t\geqq 0$. Let D be a general member of A such that the induced homomorphism $\mathcal{F}[-D]\to\mathcal{F}$ is injective. Then $h^q(\mathcal{F}[tL])\leqq h^{q-1}(\mathcal{F}_D[A+tL])\leqq O(t^m)$ by the induction hypothesis. Thus we prove the assertion.

Next we consider the general case. We use the Noetherian induction on $\mathrm{Supp}(\mathcal{F})$. By a similar method as in [F3; (2.5)], we reduce the problem to the case in which $\mathrm{Supp}(\mathcal{F})=S$ and S is a variety. By Chow's lemma there is a birational morphism $f: V\to S$ from a projective variety V. Set $\mathcal{G}=f^*\mathcal{F}$ and let φ be the natural homomorphism $\mathcal{F}\to f_*\mathcal{G}$. This is an isomorphism on an open dense subset of S. Hence, for $\mathcal{K}=\mathrm{Ker}(\varphi)$ and $\mathcal{C}=\mathrm{Coker}(\varphi)$, we have $h^q(\mathcal{K}[tL])\leqq O(t^m)$ and $h^{q-1}(\mathcal{C}[tL])\leqq O(t^m)$ by the induction hypothesis. So it suffices to show $h^q(f_*\mathcal{G}[tL])\leqq O(t^m)$.

Using the Leray spectral sequence with $E_2^{i,j}=H^i(R^jf_*\mathcal{G}[tL])$ converging to

$H^{i+j}(V, \mathcal{G}[tL])$, we infer $h^q(f_*\mathcal{G}[tL]) \leqq \sum_{j\geqq 1} h^{q-j-1}(R^jf_*\mathcal{G}[tL]) + h^q(V, \mathcal{G}[tL])$. We have $h^q(V, \mathcal{G}[tL]) \leqq O(t^m)$ since V is projective. On the other hand, we have $\dim(\mathrm{Supp}(R^jf_*\mathcal{G})) \leqq \dim S - j - 1$ for any $j \geqq 1$ because $\dim f^{-1}(x) \geqq j$ for every point x on $\mathrm{Supp}(R^jf_*\mathcal{G})$. Therefore $h^{q-j-1}(R^jf_*\mathcal{G}[tL]) \leqq O(t^m)$ by the induction hypothesis. Combining these observations we prove the assertion.

(6.3) COROLLARY. *Let $\mathcal{F}$, L be as above. Then $h^q(\mathcal{F}[-tL]) \leqq O(t^m)$ where $m = \mathrm{Min}(q, \sigma(L))$ and q is any integer.*

PROOF. Using the Noetherian induction on $\mathrm{Supp}(\mathcal{F})$ similarly as in (6.2), we reduce the problem to the case in which $\mathrm{Supp}(\mathcal{F}) = S$ and S is a variety. Moreover, by the same method as above, we see that S may be assumed to be projective.

In view of [F3; (1.7)] and using the notation there, we infer $h^q(\mathcal{F}[-tL]) \leqq \sum_{j\geqq 0} h^j(\mathcal{D}^{q+j}(\mathcal{F}) \otimes [tL])$. By (6.2) and [F3; (1.8)] we have $h^j(\mathcal{D}^{q+j}(\mathcal{F})[tL]) \leqq O(t^m)$ for each $j \geqq 0$. Hence $h^q(\mathcal{F}[-tL]) \leqq O(t^m)$.

(6.4) COROLLARY. *$\chi(\mathcal{F}[L]) = \chi(\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ and any line bundle L which is numerically equivalent to zero, i.e., $LC = 0$ for any curve C in S.*

PROOF. $\chi(\mathcal{F}[tL])$ is a polynomial in t. This is bounded since so is $h^q(\mathcal{F}[tL])$ for each q by (6.2). Hence this is a constant. In particular, $\chi(\mathcal{F}[L]) = \chi(\mathcal{F})$.

(6.5) COROLLARY. *Let L be a numerically semipositive line bundle on an algebraic variety V with $\dim V = n$. Suppose that $L^n > 0$. Then $\kappa(L, \mathcal{F}) = n$ for any coherent sheaf $\mathcal{F}$ on V such that $\mathrm{Supp}(\mathcal{F}) = V$. In particular $\kappa(L, V) = n$.*

PROOF. By the Riemann-Roch theorem (or see [K1]) we have $\chi(\mathcal{F}[tL]) - \mathrm{rank}(\mathcal{F})L^nt^n/n! \leqq O(t^{n-1})$. On the other hand $h^0(\mathcal{F}[tL]) - \chi(\mathcal{F}[tL]) \leqq O(t^{n-1})$ by (6.2). Combining them we obtain the result.

(6.6) COROLLARY. *Let L be an n-semipositive line bundle on an algebraic space S. Then $\sigma(L) = \mathrm{Max}(\kappa(L, \mathcal{F}))$, where $\mathcal{F}$ runs through all the coherent sheaves on S.*

For a proof, use (3.5) and (6.5).

(6.7) COROLLARY. *Let L and V be as in (6.5) and suppose in addition that V is projective. Then, for any coherent sheaf $\mathcal{F}$ on V, one has $h^q(V, \mathcal{F}[tL]) \leqq O(t^{n-q-1})$ for each $q \geqq 1$. This means also $h^n(V, \mathcal{F}[tL]) = 0$ for $t \gg 0$.*

PROOF. Take an ample line bundle A such that $H^i(\mathcal{F}[A+tL]) = 0$ for any $i > 0$, $t \geqq 0$. Since $\kappa(L, V) = n$ by (6.5), there is a positive integer m such that

$|mL-A|\neq\emptyset$. So we have a homomorphism $\delta: \mathcal{F}[-D]\to\mathcal{F}$ induced by $D\in|mL-A|$. Set $\mathcal{C}=\mathrm{Coker}(\delta)$, $\mathcal{K}=\mathrm{Ker}(\delta)$ and $\mathcal{J}=\mathrm{Image}(\delta)$. Then we have $h^q(\mathcal{F}[tL])\leqq h^q(\mathcal{C}[tL])+h^q(\mathcal{J}[tL])$, $h^q(\mathcal{J}[tL])\leqq h^q(\mathcal{F}[A+(t-m)L])+h^{q+1}(\mathcal{K}[tL])\leqq h^{q+1}(\mathcal{K}[tL])$ for $t\geqq m$. By virtue of (6.2) we have $h^{q+1}(\mathcal{K}[tL])\leqq O(t^{n-q-1})$ and $h^q(\mathcal{C}[tL])\leqq O(t^{n-1-q})$ since $\mathrm{Supp}(\mathcal{C})\subset D$. Combining these estimates we obtain the desired one.

(6.8) COROLLARY. *Let V, L and $\mathcal{F}$ be as above and assume in addition that $\mathcal{F}$ is torsion free. Then $h^q(V, \mathcal{F}[-tL])\leqq O(t^{q-1})$ for any $q<n$.*

PROOF. This follows from (6.7) by the Serre duality as was formulated in [F3; (1.7)]. Under the notation in [F3], we have $h^q(\mathcal{F}[-tL])\leqq \sum_{j=0}^{n} h^{n-q-j}(\mathcal{D}^{n-j}(\mathcal{F})[tL])$ by [F3; (1.7)]. Since $\dim(\mathrm{Supp}(\mathcal{D}^{n-j}(\mathcal{F})))<n-j$ for any $j>0$ by [F3; (1.14)], we have $h^{n-q-j}(\mathcal{D}^{n-j}(\mathcal{F})[tL])\leqq O(t^{q-1})$ for $j>0$. By (6.7), this estimate is valid also for $j=0$. Hence $h^q(\mathcal{F}[-tL])\leqq O(t^{q-1})$, as desired.

(6.9) THEOREM. *Let $\mathcal{F}$ be a coherent sheaf and let A be an ample line bundle on a projective scheme S. Let m be a positive integer such that $\mathrm{Bs}|mA|=\emptyset$. Then there exists an integer k such that, for every $t\geqq k$ and any n-semipositive line bundle L,*

1) *the natural mapping $H^0(\mathcal{F}[tA+L])\otimes H^0(S, mA)\to H^0(\mathcal{F}[(t+m)A+L])$ is surjective and*

2) *$\mathcal{F}[tA+L]$ is generated by its global sections.*

PROOF. Take k so large that $H^q(\mathcal{F}[(t-qm)A+L])=0$ for any $q>0$, $t\geqq k$ and any n-semipositive line bundle L, where (5.1) is applied. Then 1) follows from the generalized Castelnuovo's lemma (cf. [Mu; p. 41, Th. 2]). Hence $\mathcal{F}[tA+L]$ is generated by global sections if so is $\mathcal{F}[(t+m)A+L]$. So 2) is proved by the descending induction on t.

(6.10) COROLLARY. *For any ample line bundle A on a projective scheme S, there exists an integer k such that $tA+L$ is very ample for any $t\geqq k$ and any n-semipositive line bundle L on S.*

PROOF. Take positive integers b and m such that mA is very ample and that $\mathrm{Bs}|tA+L|=\emptyset$ for any $t\geqq b$ and any n-semipositive line bundle L. Then $k=m+b$ has the desired property.

(6.11) COROLLARY. *For any n-semipositive line bundle L on a projective scheme S, there exists an ample line bundle A such that $A+tL$ is very ample for any $t\geqq 0$.*

REMARK. Thus, the approximate ampleness and the cohomological semipositivity in the sense of [F2] are equivalent to the numerical semipositivity on projective schemes.

(6.12) THEOREM. *Let L be a numerically semipositive line bundle on a projective variety V with $\dim V=n$ such that $L^n>0$. Then there exist a positive integer k and an effective Cartier divisor D on V such that $\mathrm{Bs}|tL-D|=\emptyset$ for every $t\geqq k$.*

PROOF. Take an ample line bundle A as in (6.10). Since $\kappa(L, V)=n$ by (6.5), there is a positive integer k' such that $|k'L-A|\neq\emptyset$ by an argument due to Kodaira (cf., e.g., [F3; (2.8)]). Then the assertion is true for $D\in|k'L-A|$.

(6.13) COROLLARY. *Any numerically semipositive line bundle L on a normal algebraic variety V with $\sigma(L)=\dim V$ is almost base point free in the sense of Goodman [Go].*

Indeed, using Chow's lemma, we reduce the problem to the case in which V is projective, where (6.12) applies.

(6.14) COROLLARY. *Let things be as in (6.12). Then the graded algebra $G(V, L)=\bigoplus_{t\geq 0} H^0(V, tL)$ is finitely generated if and only if L is semiample.*

PROOF. The "if" part follows from (1.3). So we consider the "only if" part. Let $\xi_1, \cdots, \xi_r$ be a homogeneous generator system of $G(V, L)$. Assume that $\mathrm{SBs}(L)\neq\emptyset$. Take a point x on $\mathrm{SBs}(L)$ and set $m_j=\mu_j/\deg(\xi_j)$ for $j=1, \cdots, r$, where μ_j is the order of the zero of ξ_j at x. Note that $\mu_j>0$ since $x\in\mathrm{SBs}(L)$. Let M be the multiplicity of D at x and let μ be the order of the zero of δ at x, where δ is a general element of $H^0(V, dL)$. Then, by (6.12), $\mu\leqq M$ if $d\geqq k$. On the other hand $\mu\geqq md$ for $m=\mathrm{Min}(m_j)$ because δ is a polynomial combination of ξ_j's. Hence $md\leqq M$ for any such δ, which is impossible if $d\gg 0$. So we infer $\mathrm{SBs}(L)=\emptyset$, hence L is semiample by (1.18).

(6.15) *Example* (Zariski). Let C be a non-singular cubic curve in $\boldsymbol{P}^2\cong P$. Take twelve points $p_1, \cdots, p_{12}$ on C in such a way that $\mathcal{O}_C(4)-[p_1+\cdots+p_{12}]$ is not a torsion in $\mathrm{Pic}(C)$. Let S be the blowing-up of P at these points and let E be the sum of the twelve exceptional curves over them. Set $L=4H-E\in\mathrm{Pic}(S)$, where H is the pull-back of $\mathcal{O}_P(1)$. It is easy to see that L is n-semipositive and $L^2>0$. But L is not semiample because the restriction of L to the proper transform of C is not so. Therefore the graded algebra $G(S, L)$ is not finitely generated by (6.14).

(6.16) *Example.* Let C be a non-singular elliptic curve and let L_1 be a non-torsion element in $\mathrm{Pic}_0(C)$. Then, for any ample line bundle L_2 on C, the tautological line bundle L on $P=\boldsymbol{P}(L_1\oplus L_2)$ is n-semipositive and $L^2>0$. But L is not semiample because L_1 is not so. Hence $G(P, L)$ is not finitely generated.

In this way we can construct various examples of n-semipositive line bundles such that $G(S, L)$ is not finitely generated. Compare also (3.11) and (3.12).

(6.17) Unlike general line bundles, the canonical bundle K^M of an algebraic manifold M seems to have special nice properties. So we make the following conjectures:

1) The canonical ring $G(M, K^M)$ is finitely generated.

2) K^M is semiample if and only if it is n-semipositive.

Both are true if $\dim M\leqq 2$. Moreover, recently, Y. Kawamata proved 2) in case $\sigma(K^M)=0=\mathrm{char}(\mathfrak{K})$.

§ 7. Vanishing theorems of Ramanujam type.

(7.1) The motivation of the study in this section is the following

QUESTION. *Let L be a numerically semipositive line bundle on an algebraic manifold M. Then $H^q(M, -tL)=0$ for any $q<\sigma(L)$ and $t\gg 0$?*

When $\mathrm{char}(\mathfrak{K})=0$, this follows from Kawamata's vanishing theorem (cf. (4.15)). Ramanujam [Ra] obtained a partial result in case $\dim M=2$. Unfortunately, however, the answer is No in general (cf. (7.10) below). Here we will give an affirmative answer in case $q=1$.

(7.2) DEFINITION. A line bundle L on a variety V is said to be *numerically semipositive in codimension one* if there exists a closed subset B of V with $\dim B\leqq\dim V-2$ such that $C\subset B$ for any curve C in V with $LC<0$.

(7.3) LEMMA. *Let L be a line bundle on a normal projective variety V with $n=\dim V\geqq 2$. Suppose that $L^2A^{n-2}>0$ for some ample line bundle A on V and that L is n-semipositive in codimension one. Then $h^0(D, \mathcal{O}_D)=1$ for any member D of $|L|$.*

PROOF. We use the induction on n. When $n=2$, let $f: M\to V$ be a desingularization of V (cf. [Ab]). In this case L is n-semipositive and $L^2>0$. So $C=f^*D$ is a numerically connected divisor by the same reason as in [Ra; Lemma 2]. Hence $h^0(C, \mathcal{O}_C)=1$ by [Ra; Lemma 3]. Taking f_* of the exact sequence $0\to\mathcal{O}_M[-f^*L]\to\mathcal{O}_M\to\mathcal{O}_C\to 0$, we obtain the exact sequence $0\to\mathcal{O}_V[-L]\to\mathcal{O}_V\to f_*\mathcal{O}_C$. This gives an injection $\mathcal{O}_D\subset f_*\mathcal{O}_C$. So $h^0(D, \mathcal{O}_D)\leqq h^0(f_*\mathcal{O}_C)=h^0(C, \mathcal{O}_C)=1$.

When $n\geqq 3$, take a large integer m such that $h^1(V, -mA-L)=0$ and mA is very ample. So we have $h^0(D, -mA)=0$. Let W be a general member of $|mA|$. Then W is normal by [Sd] and L_W is n-semipositive in codimension one. So $h^0(X, \mathcal{O}_X)=1$ for $X=D_W$ by the induction hypothesis. This implies $h^0(D, \mathcal{O}_D)=1$ since $0\to\mathcal{O}_D[-mA]\to\mathcal{O}_D\to\mathcal{O}_X\to 0$ is exact.

(7.4) THEOREM. *Let V and L be as in* (7.3). *Suppose in addition that* $h^0(V, L)>(q/2)^2+1$, *where* $q=h^1(V, \mathcal{O}_V)$. *Then* $H^1(V, -L)=0$.

PROOF. Suppose that $r=h^1(V, -L)>0$. By (7.3), the mapping $H^1(\mathcal{O}_V[-D])\to H^1(\mathcal{O}_V)$ is injective for any $D\in|L|$. Hence we get a morphism from $|L|=\boldsymbol{P}(H^0(V, L))$ to the Grassmann variety G parametrizing r-dimensional subspaces of $H^1(V, \mathcal{O}_V)$. $\dim G=r(q-r)\leqq q^2/4<\dim|L|$. So this morphism must be a constant mapping. Hence the image I of $\varphi(\xi): H^1(V, -L)\to H^1(V, \mathcal{O}_V)$ is independent of $\xi\in H^0(V, L)$ provided $\xi\neq 0$. Fix basis of I and $H^1(V, -L)$ and let $\delta(\xi)$ be the determinant of $\varphi(\xi)$ with respect to these basis. Then $\delta(\xi)$ is a polynomial function on $H^0(V, L)$ which has an isolated zero at the origin. This is impossible. So $r=0$.

(7.5) THEOREM. *Let V be a normal projective variety with* $\dim V=n\geqq 2$ *and let L be a line bundle on V which is numerically semipositive in codimension one. Suppose that there exists a very ample line bundle A such that* $L^2A^{n-2}\{V\}>0$. *Then there exists an integer k such that* $H^1(V, -tL-E)=0$ *for any* $t\geqq k$ *and any effective divisor E which is numerically semipositive in codimension one. Moreover, k can be chosen so that it depends only on the polynomial* $\chi(V, uA+vL)$ *in u, v and on the irregularity* $q'(V, A)=h^1(S, \mathcal{O}_S)$ *of a surface S which is obtained from V by taking general members of* $|A|$ $(n-2)$*-times successively.*

REMARK. Let $\iota: V\to\boldsymbol{P}^N$ be the embedding defined by $|A|$. Then $S=\iota(V)\cap T$ for some linear subspace $T\cong\boldsymbol{P}^{N-n+2}$ in $\boldsymbol{P}^N$. By the upper-semicontinuity theorem, we infer that the irregularity of $\iota(V)\cap T$ is the same for any general $(N-n+2)$-dimensional linear space T in $\boldsymbol{P}^N$. Thus $q'(V, A)$ is well-defined.

PROOF OF THE THEOREM. We use the induction on n. When $n=2$, let g be the genus of a general member C of $|A|$. Since $AL>0$ by (3.3), we can find an integer k such that $kLA\geqq 2g$ and that $\chi(V, tL)\geqq(h^1(V, \mathcal{O}_V)/2)^2+2$ for any $t\geqq k$ by virtue of the Riemann-Roch Theorem. Then $h^1(C, tL+sA)=0$ and $h^2(V, tL+(s-1)A)\leqq h^2(V, tL+sA)$ for any $t\geqq k$, $s\geqq 0$. So $h^2(V, tL)\leqq h^2(V, tL+A)\leqq\cdots\leqq h^2(V, tL+sA)=0$ for $s\gg 0$ for any $t\geqq k$. Hence $\chi(V, tL)\leqq h^0(V, tL)\leqq h^0(V, tL+E)$ for any $t\geqq k$ and any effective divisor E. So (7.4) applies if in addition E is n-semipositive. Moreover the choice of k depends only on $\chi(V, vL+uA)$ and $h^1(V, \mathcal{O}_V)$.

When $n\geqq 3$, we have $\chi(D, uA+vL)=\chi(V, uA+vL)-\chi(V, (u-1)A+vL)$ and

$q'(D, A)=q'(V, A)$ for any general member D of $|A|$, which is a normal variety by [Sd]. So we have an integer k depending only on $q'(V, A)$ and $\chi(V, uA+vL)$ such that $H^1(D, -tL-E')=0$ for any general member D of $|A|$, any $t\geqq k$ and any effective divisor E' on D which is n-semipositive in codimension one. Now, given any effective divisor E on V which is n-semipositive in codimension one, we can find a general member D of $|A|$ such that E_D has the same property. Then $H^1(D, -tL_D-E_D-sA_D)=0$ and $h^1(V, -tL-E-sA)\leqq h^1(V, -tL-E-(s+1)A)$ for any $t\geqq k$, $s\geqq 0$. Hence $h^1(V, -tL-E)\leqq\cdots\leqq h^1(V, -tL-E-sA)$ while the last term vanishes for $s\gg 0$. Therefore $h^1(V, -tL-E)=0$ for any $t\geqq k$, as required.

(7.6) COROLLARY. *Let L be a numerically semipositive line bundle on a normal algebraic variety V with $\sigma(L)\geqq 2$. Then $H^1(V, -tL)=0$ for any sufficiently large integer t.*

Indeed, we can easily reduce the problem to the case in which V is projective, where (7.5) applies.

(7.7) COROLLARY. *Let Λ be a linear system on a normal projective variety V. Suppose that* $\dim \mathrm{Bs}\,\Lambda\leqq\dim V-2$ *and that Λ is not composed of a pencil. Then $H^1(V, -tL)=0$ for $t\gg 0$, where $L=[\Lambda]$.*

Indeed, L is clearly n-semipositive in codimension one and (7.5) applies.

(7.8) THEOREM. *Let things be as in* (7.3) *and suppose that $|L|\neq\emptyset$. Then $H^1(V, -L)=0$ if* $\mathrm{char}(\mathfrak{K})=0$ *or if* $\mathrm{char}(\mathfrak{K})>0$ *and the Frobenius is injective on $H^1(V, \mathcal{O}_V)$.*

PROOF. When $\mathrm{char}(\mathfrak{K})=0$, the assertion follows from Kawamata's vanishing theorem. So we assume $p=\mathrm{char}(\mathfrak{K})>0$. By (7.3), it suffices to show that the image I of the mapping $H^1(V, -X)\to H^1(V, \mathcal{O}_V)$ vanishes for $X\in|L|$. By definition of the Frobenius F on $H^1(V, \mathcal{O}_V)$, $F^e(I)$ comes from $H^1(V, -p^eX)$ for any $e\geqq 0$. Hence $F^e(I)=0$ for $e\gg 0$ by (7.5). So $I=0$ because F is injective.

REMARK. This was proved by Ramanujam [Ra] under the assumption in (7.7).

(7.9) COROLLARY. *Let things be as in* (7.3) *and let q' be the dimension of the maximal subspace of $H^1(V, \mathcal{O}_V)$ on which the Frobenius is nilpotent. Then $H^1(V, -L)=0$ if $h^0(V, L)>(q'/2)^2+1$.*

For a proof, use (7.8) and the method in (7.4).

(7.10) Now we will give a counter example to the question in (7.1). Let S be a Raynaud surface and let A be an ample line bundle on S such that $H^1(S, -A)\neq 0$ (see [Ry]). Let M be the $\boldsymbol{P}^1$-bundle $\boldsymbol{P}(A\oplus\mathcal{O}_S)$ over S and let L be the tautological line bundle $\mathcal{O}_M(1)$. Then L is semiample and $\sigma(L)=\dim M=3$, but $H^2(M, -tL)\neq 0$ for any $t\gg 0$.

To see this, let E be the section of $\pi: M\to S$ corresponding to the quotient bundle $\mathcal{O}_S$. Then $(E, [E]_E)\cong(S, -A)$, and hence we can contract E to a point to obtain a normal variety W from M. We see $W\cong\mathrm{Proj}(G(M, L))$ and L is the pull-back of the ample line bundle $\mathcal{O}_W(1)$ on W via the morphism $M\to W$. Hence L is semiample and $L^3>0$. On the other hand, $h^2(M, -tL)=h^1(M, K^M+tL)=h^1(M, (t-2)L+\pi^*(A+K^S))=h^1(S, S^{t-2}(A\oplus\mathcal{O}_S)\otimes[A+K^S])\geqq h^1(S, A+K^S)=h^1(S, -A)>0$ for any $t\geqq 2$.

(7.11) REMARK. In the above example we have $R^1f_*\omega_M\neq 0$ for the morphism $f: M\to W$. So (4.12) is not always true in case $\mathrm{char}(\mathfrak{K})>0$ even if f is birational. However, we can prove the following

(7.12) PROPOSITION. *Let $f: M\to\boldsymbol{P}^N$ be a morphism from a normal projective locally Gorenstein variety M. Let L be a line bundle on M such that $L+tf^*H$ is effective and n-semipositive in codimension one on M for some integer t, where $H=\mathcal{O}_P(1)$. Suppose that $\dim f(M)\geqq 2$. Then $R^{n-1}f_*(\omega_M[L])=0$ for $n=\dim M$. In particular, $R^{n-1}f_*\omega_M=0$ if $\dim f(M)\geqq 2$.*

PROOF. As in (4.13), it suffices to show that $0=h^{n-1}(M, \omega_M[L+tH])=h^1(M, -L-tH)$ for $t\gg 0$. This follows from (7.5).

References

[EGA] Grothendieck, A., Eléments de géométrie algébrique, Publ. Math. I. H. E. S. **4**, **8**, **11**, **17**, 1960, 1961, 1961 and 1963.

[Ab] Abhyankar, S. S., Resolution of singularities of arithmetical surfaces, in Arithmetical Algebraic Geometry, Proc. Conf. Purdue Univ. 1963, pp. 111-152, Harper & Row, New York, 1965.

[At] Atiyah, M. F., Vector bundles over an elliptic curve, Proc. London Math. Soc. **7** (1957), 414-452.

[CE] Cartan, H. and S. Eilenberg, Homological Algebra, Princeton Univ. Press, 1956.

[F1] Fujita, T., On the hyperplane section principle of Lefschetz, J. Math. Soc. Japan **32** (1980), 153-169.

[F2] Fujita, T., On semipositive line bundles, Proc. Japan Acad. **56** (1980), 393-396.

[F3] Fujita, T., On L-dimension of coherent sheaves, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), 215-236.

[Go] Goodman, J. E., Affine open subsets of algebraic varieties and ample divisors, Ann. of Math. **89** (1969), 160-183.

[GR] Grauert, H. and R. Remmert, Komplexe Räume, Math. Ann. **136** (1958), 245-318.

[Ha 1] Hartshorne, R., Ample Subvarieties of Algebraic Varieties, Lecture Notes in Math. 156, Springer, 1970.

[Ha 2] Hartshorne, R., Algebraic Geometry, Graduate Texts in Math. 52, Springer, 1977.

[Hi] Hironaka, H., Introduction to the theory of infinitely near singular points, Memorias de Matematica del Inst. "Jorge Juan" No. 28, Consejo Superior de Investigaciones Cientificas, Madrid, 1974.

[I] Iitaka, S., On D-dimensions of algebraic varieties, J. Math. Soc. Japan **23** (1971), 356-373.

[Ka] Kawamata, Y., A generalization of Kodaira-Ramanujam's vanishing theorem, Math. Ann. **261** (1982), 43-46.

[Kl] Kleiman, S., Towards a numerical theory of ampleness, Ann. of Math. **84** (1966), 293-344.

[KM] Kodaira, K. and J. Morrow, Complex Manifolds, Holt Rinehart & Winston, New York, 1971.

[M] Mumford, D., Varieties defined by quadratic equations, in Questions on Algebraic Varieties, C.I.M.E., III Ciclo, Varenna, 1969, pp. 29-100, Edizion Cremonese, Rome, 1970.

[N] Nakai, Y., A criterion of an ample sheaf on a projective scheme, Amer. J. Math. **85** (1963), 14-26.

[Ra] Ramanujam, C.P., Remarks on the Kodaira vanishing theorem, (with a supplement), J. Indian Math. Soc. N.S. **36** (1972), 41-51 and **38** (1974), 121-124.

[Ry] Raynaud, M., Contre-exemple au "vanishing theorem" en caractéristique $p>0$, in C.P. Ramanujam — a Tribute, pp. 273-278, Tata Inst. Studies in Math. 8, Springer, 1978.

[Sd] Seidenberg, A., The hyperplane sections of normal varieties, Trans. Amer. Math. Soc. **69** (1950), 357-386.

[Sr] Serre, J.P., Faisceaux algébriques cohérents, Ann. of Math. **61** (1955), 197-278.

[V] Viehweg, E., Vanishing theorems, J. Reine Angew. Math. **335** (1982), 1-8.

[Z] Zariski, O., The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. **76** (1962), 560-615.

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