Discriminant of a holomorphic map and logarithmic vector fields

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To Meg

1. Introduction.

Let $F: U \rightarrow V$ be a flat holomorphic map, where U and V are open domains in $\boldsymbol{C}^m$ and $\boldsymbol{C}^k$ respectively. Let C (or D) be the critical set (resp. the discriminant (=the set of critical values)) of F. Assume that $F|_C: C \rightarrow V$ be finite. Then it is known that D is a hypersurface in V.

Our aim here is to study the discriminant D through the logarithmic vector fields along D, which are holomorphic vector fields tangent to D. Their definition is in general as follows; for a (reduced) hypersurface H defined by an equation $h=0$ in an open domain W in $\boldsymbol{C}^n$, define

$$\mathrm{Der}_W(\log H)_p = \{\theta\,;\ \text{germs at } p \text{ in } \boldsymbol{C}^n \text{ of holomorphic vector fields such that } \theta \cdot h_p \in h_p \cdot \mathcal{O}_{W,p}\},$$

where h_p is the germ at p of h and $\mathcal{O}_{W,p}$ is the ring of germs at p of holomorphic functions. We can introduce the structure of coherent sheaf of $\mathcal{O}_W$-Module into

$$\mathrm{Der}_W(\log H) = \bigcup_{p \in W} \mathrm{Der}_W(\log H)_p.$$

An element of $\Gamma(W, \mathrm{Der}_W(\log H))$ is called a *logarithmic vector field along H*. K. Saito studied this sheaf in [5].

Assume that F is finite and $m=k$. Let θ be a holomorphic vector field on V. Then θ is said to be *liftable by F* if there is a holomorphic vector field ϕ on U with $(F_*)_p \phi(p) = \theta(F(p))$ for any $p \in U$, where $(F_*)_p : T_{U,p} \rightarrow T_{V,F(p)}$ is a $\boldsymbol{C}$-linear map induced from F. Note that ϕ is uniquely determined for each liftable θ, because F is finite. Thus we shall write $\phi = F^{-1}\theta$ from now on.

By (M, p) we denote a germ at p of a complex space M. Let

$$f: (\boldsymbol{C}^m, 0) = X \longrightarrow (\boldsymbol{C}^k, 0) = Y$$

be a flat holomorphic map germ with $f(0)=0$, where 0 is the origin. We can define the germ of critical set $(C, 0)$ and the germ of discriminant $(D, 0)$. Assume that $f|_{(C,0)}$ is finite. We fix this situation in the rest of this section. We can define the notion of a liftable germ of holomorphic vector field by localizing the definition above when f is finite.

In Sect. 2, we shall prove the following Theorems A, B and C.

THEOREM A. *When* $f:(\boldsymbol{C}^m, 0)=X\to(\boldsymbol{C}^m, 0)=Y$ *is finite,*

$$\mathrm{Der}_Y(\log D)_0=\{\textit{germs of holomorphic vector fields on } Y \textit{ liftable by } f\}.$$

Although V.I. Arnold [1] proved this only when the situation is induced from the inclusion

$$\mathcal{O}_{C^m,0}\supset(\mathcal{O}_{C^m,0})^{S_m}\ (S_m\text{-invariant subring}),$$

where S_m is the symmetric group, our proof of Theorem A is similar to his.

In [9], [13], the author studied a germ of hypersurface $(H, 0)\subset(\boldsymbol{C}^n, 0)$ such that $\mathrm{Der}_{C^n}(\log H)_0$ is a free $\mathcal{O}_{C^n,0}$-module. Such a germ of hypersurface is called a *germ of free divisor* (or a *germ of Saito divisor* according to Cartier [2]). The beautiful Shephard-Todd-Brieskorn formula for a free divisor consisting of hyperplanes, which generalizes the formula by Orlik-Solomon [3], was proved in [10], [11]. (For this topic, see Cartier's exposition [2].)

THEOREM B. *Assume that* $f:(\boldsymbol{C}^m, 0)=X\to(\boldsymbol{C}^m, 0)=Y$ *is finite. Then*

i) $f^{-1}\mathrm{Der}_Y(\log D)_0\subset\mathrm{Der}_X(\log f^{-1}(D))_0\subset\mathrm{Der}_X(\log C)_0$,

ii) *if* $(D, 0)$ *is free, then*

$$\mathrm{Der}_X(\log f^{-1}(D))_0\cong f^{-1}\mathrm{Der}_Y(\log D)_0\underset{f^{-1}\mathcal{O}_{Y,0}}{\otimes}\mathcal{O}_{X,0}$$

and thus $(f^{-1}(D), 0)$ *is also free.*

Assume that $f:(\boldsymbol{C}^m, 0)=X\to(\boldsymbol{C}^m, 0)=Y$ comes from the inclusion

$$\mathcal{O}_{X,0}\supset(\mathcal{O}_{X,0})^G\cong\mathcal{O}_{Y,0}$$

such that a finite subgroup G of $\mathrm{GL}(m;\boldsymbol{C})$ acts on $\mathcal{O}_{X,0}$ linearly with respect to some parameter system of $\mathcal{O}_{X,0}$. (It is known that such a G must be a complex reflection group.) Then G naturally acts on $\mathrm{Der}_{X,0}=\mathcal{O}_{X,0}\underset{C}{\otimes}T_{X,0}$ ($T_{X,0}$ is the tangent space of X at 0, on which G naturally acts.). Denote the invariant part of $\mathrm{Der}_{X,0}$ by $(\mathrm{Der}_{X,0})^G$. Then we have

THEOREM C.

i) $(\mathrm{Der}_{X,0})^G=f^{-1}\mathrm{Der}_{Y,0}(\log D)$,

ii) $(D, 0)$ *is free (thus so is* $(C, 0)=(f^{-1}(D), 0)$ *thanks to Theorem B).*

For a finite unitary reflection group G, $(C, 0)$ is nothing other than the germ of the union of the reflecting hyperplanes. Then the part that $(C, 0)$ is free has been already proved by Terao [12] and Cartier [2] independently.

In Sect. 3, f is a deformation of a germ of a hypersurface with isolated singularity. In this case we shall give a sufficient condition for the freeness of the germ of the discriminant $(D, 0)$ (Theorem D). To state Theorem D, we need the concept of the free deformation (3.1) introduced by T. Yano [14]. It follows from Theorem D that the discriminant of semiuniversal deformation of a hypersurface isolated singularity is a free divisor, which was proved by K. Saito [6].

2. Discriminant of a finite map.

In this section let

$$f:(\boldsymbol{C}^m, 0)=X \longrightarrow (\boldsymbol{C}^m, 0)=Y$$

be a finite holomorphic map germ.

DEFINITION 2.1. We say that f is a *folding map germ* if f is expressed by

$$f^*y_1=x_1^n$$
$$f^*y_i=x_i \quad (i=2, 3, \cdots)$$

with respect to appropriate coordinates $(x_1, \cdots, x_m)$ for X and $(y_1, \cdots, y_m)$ for Y for some integer $n\geqq 2$.

LEMMA 2.2. *Let f be a folding map germ. Then*

i) *Theorem A holds true (i.e.,* $\mathrm{Der}_Y(\log D)_0=\{$*germs of holomorphic vector fields on Y liftable by f*$\}$).

ii) *If $\{\theta_1, \cdots, \theta_n\}$ is a free $\mathcal{O}_{Y,0}$-base for* $\mathrm{Der}_Y(\log D)_0$, *then $\{f^{-1}\theta_1, \cdots, f^{-1}\theta_n\}$ makes a free $\mathcal{O}_{X,0}$-base for* $\mathrm{Der}_X(\log C)_0$.

PROOF. We can assume that

$$f^*y_1=x_1^n \quad (n\geqq 2)$$
$$f^*y_i=x_i \quad (i=2, 3, \cdots).$$

Then $(C, 0)$ and $(D, 0)$ are defined by $x_1=0$ and $y_1=0$ respectively. The set $\{y_1(\partial/\partial y_1), \partial/\partial y_2, \cdots, \partial/\partial y_m\}$ is a free $\mathcal{O}_{Y,0}$-base for $\mathrm{Der}_Y(\log D)_0$. Since

$$(*) \quad \begin{cases} f^{-1}(y_1(\partial/\partial y_1))=n^{-1}x_1(\partial/\partial x_1) \\ f^{-1}(\partial/\partial y_i)=\partial/\partial x_i \quad (i=2, 3, \cdots), \end{cases}$$

any element of $\mathrm{Der}_Y(\log D)_0$ is liftable by f. Conversely if $\theta\in\mathrm{Der}_{Y,0}$ is liftable, then

$$f^{-1}(\theta\cdot y_1)=(f^{-1}\theta)\cdot(x_1^n)=nx_1^{n-1}((f^{-1}\theta)\cdot x_1).$$

This implies $\theta\cdot y_1\in y_1\cdot\mathcal{O}_{Y,0}$ and thus $\theta\in\mathrm{Der}_Y(\log D)_0$. Therefore i) is proved.

As for ii), it is sufficient to prove it for some specific base $\{\theta_1, \cdots, \theta_m\}$. Since $\{n^{-1}x_1(\partial/\partial x_1), \partial/\partial x_2, \cdots, \partial/\partial x_m\}$ is a free $\mathcal{O}_{X,0}$-base for $\mathrm{Der}_X(\log C)_0$, $(*)$ proves ii). $\square$

Let U and V be open domains in $\boldsymbol{C}^m$. Let $F: U \to V$ be a finite holomorphic map.

DEFINITION 2.3. A point $p \in U$ is called a *folding point* if the map germ at p of F is a folding map germ.

Define

$$\Sigma = \mathrm{Sing}\, C \cup F^{-1}(\mathrm{Sing}\, D),$$

where $\mathrm{Sing}\, C$ (or $\mathrm{Sing}\, D$) is the singular locus of the critical set C (resp. the discriminant D) of F. Then Σ is an analytic subset of U with $\mathrm{codim}_U \Sigma \geqq 2$.

PROPOSITION 2.4.

$$\{\textit{folding point of } F\} \supset C \backslash \Sigma .$$

PROOF. Take $p \in C \backslash \Sigma$. Choose the coordinates $(x_1, \cdots, x_m)$ (or $(y_1, \cdots, y_m)$) near p (resp. $F(p)$) as follows:

i) p (resp. $F(p)$) is the origin,

ii) C (resp. D) is defined by $x_1 = 0$ (resp. $y_1 = 0$) near p (resp. $F(p)$).

We shall consider only the local situation near p. Denote the germ at p of F by $f: f = F_p$. Put $f^* y_i = f_i(x) \in \boldsymbol{C}\{x_1, \cdots, x_m\} = \mathcal{O}_{X,p}$ $(i=1, \cdots, m)$, then $f^* y_1 = f_1 \in x_1 \cdot \mathcal{O}_{X,p}$ because $f(C) = D$. Denote $(x_2, \cdots, x_m)$ by x' for simplicity. Define $n > 0$ and $g_1(x_1, \cdots, x_m) \in \mathcal{O}_{X,p}$ satisfying $f_1 = x_1^n g_1$ and $g_1(0, x') \neq 0$. We can assume that

$$(*) \qquad u x_1^k = \partial(f_1, \cdots, f_m)/\partial(x_1, \cdots, x_m)$$
$$= \begin{vmatrix} n x_1^{n-1} g_1 + x_1^n(\partial g_1/\partial x_1), & x_1^n(\partial g_1/\partial x_2), & \cdots, & x_1^n(\partial g_1/\partial x_n) \\ * & * & & * \end{vmatrix}$$

($k > 0$, u: a unit). Thus we have $k \geqq n-1$. If $k > n-1$, then from $(*)$ we have

$$(**) \qquad x_1 \cdot \mathcal{O}_{X,p} \ni u x_1^{k-n+1}$$
$$= \begin{vmatrix} n g_1 + x_1(\partial g_1/\partial x_1), & x_1(\partial g_1/\partial x_2), & \cdots, & x_1(\partial g_1/\partial x_m) \\ * & * & & * \end{vmatrix}.$$

Put $x_1 = 0$ in this equality, then

$$0 = n g_1(0, x')(\partial(f_2, \cdots, f_m)/\partial(x_2, \cdots, x_m)|_{x_1=0})$$
$$= n g_1(0, x')(\partial(\tilde{f}_2, \cdots, \tilde{f}_m)/\partial(x_2, \cdots, x_m)),$$

where $\tilde{f}_i(x') = f_i(0, x') \in \boldsymbol{C}\{x'\}$ $(i=2, \cdots, m)$. Since $\tilde{f} = (\tilde{f}_2, \cdots, \tilde{f}_m): (C, p) \to (D, f(p))$ is finite and $\dim_p C = \dim_{f(p)} D$, we obtain

$$\partial(\tilde{f}_2, \cdots, \tilde{f}_m)/\partial(x_2, \cdots, x_m) \neq 0 .$$

This contradicts that $g_1(0, x') \neq 0$. Thus we have $k = n-1$. From $(**)$, we deduce

$$u=ng_1(0,\ x')(\partial(f_2,\ \cdots,\ f_m)/\partial(x_2,\ \cdots,\ x_m)|_{x_1=0}).$$

Therefore both g_1 and $\partial(f_2, \cdots, f_m)/\partial(x_2, \cdots, x_m)$ are units in $\mathcal{O}_{X,p}$. This proves that p is a folding point. $\square$

PROOF OF THEOREM A. Let $F: U\to V$ be a representative for a finite holomorphic map germ $f:(\boldsymbol{C}^m, 0)\to(\boldsymbol{C}^m, 0)$. Assume that F is finite and surjective. Let θ be a holomorphic vector field on V. If θ is logarithmic along the discriminant D of F, then θ is liftable by F at any point outside $F(\Sigma)$ owing to 2.2 i) and 2.4. Since $\operatorname{codim}_U F^{-1}(F(\Sigma))\geqq 2$, θ is liftable anywhere in the light of Hartog's theorem.

Conversely if θ is a holomorphic vector field on V liftable by F, then θ is logarithmic at each point outside $F(\Sigma)$ again by 2.2 i) and 2.4. Let δ_p is a reduced local defining equation for D near $p\in V$. Then $\theta_p\cdot\delta_p$ vanishes on $D\backslash F(\Sigma)$, thus on D near p. This implies that $\theta_p\cdot\delta_p\in\delta_p\cdot\mathcal{O}_{V,p}$. $\square$

For the proof of Theorem B, we need the following

PROPOSITION 2.5. (K. Saito [5]) *Let H be a (reduced) hypersurface defined by an equation: $h=0$ in an open domain $W\subset\boldsymbol{C}^n$. The logarithmic vector fields*

$$\theta_i=\sum_{j=1}^{n} f_{ij}(\partial/\partial z_j)\quad (i=1,\ \cdots,\ n)$$

make a $\Gamma(W, \mathcal{O}_W)$-free base for $\Gamma(W, \mathrm{Der}_W(\log H))$ if and only if

$$|\theta_1,\ \cdots,\ \theta_n| \underset{\mathrm{def}}{=} \det(f_{ij})_{1\leq i,j\leq n}$$

vanishes at the order one along D.

PROOF OF THEOREM B. Let $F: U\to V$ be a representative for a finite holomorphic map germ $f:(\boldsymbol{C}^m, 0)\to(\boldsymbol{C}^m, 0)$ and $\{\theta_1, \cdots, \theta_m\}$ be a $\Gamma(V, \mathcal{O}_V)$-free base for $\Gamma(V, \mathrm{Der}_V(\log D))$. Assume that F is finite and surjective. Put $\bar{C}=f^{-1}(D)$. By Theorem A, $F^{-1}\theta_1, \cdots, F^{-1}\theta_m$ are holomorphic vector fields on U. Thanks to 2.2 ii), $\{F^{-1}\theta_1, \cdots, F^{-1}\theta_m\}$ determines an $\mathcal{O}_{U,p}$-free base for $\mathrm{Der}_U(\log\bar{C})_p$ for each $p\in U\backslash F^{-1}(F(\Sigma))$. Thus we know that $F^{-1}\theta_1, \cdots, F^{-1}\theta_m$ are logarithmic along $\bar{C}$ and that $|F^{-1}\theta_1, \cdots, F^{-1}\theta_m|$, by applying 2.5, vanishes at the order one along $\bar{C}\backslash F^{-1}(F(\Sigma))$, thus along $\bar{C}$. This proves that $\{F^{-1}\theta_1, \cdots, F^{-1}\theta_m\}$ makes a $\Gamma(U, \mathcal{O}_U)$-free base for $\Gamma(U, \mathrm{Der}_U(\log\bar{C}))$ in the light of 2.5 again. $\square$

Next we shall prove Theorem C. We will deal with a specific situation originating from a finite subgroup G of $\mathrm{GL}(m; \boldsymbol{C})$. Denote $\mathcal{O}_{\boldsymbol{C}^m,0}$ simply by $\mathcal{O}$. Naturally G acts $\boldsymbol{C}$-linearly on $\mathcal{O}$ with respect to a parameter system of $\mathcal{O}$. Denote the invariant subring by $\mathcal{O}^G$. Suppose that $\mathcal{O}^G$ is a regular local ring. These induce a finite map germ

$$f:(\boldsymbol{C}^m,\ 0)=X\longrightarrow(\boldsymbol{C}^m,\ 0)=Y.$$

Let $f=(f_1, \cdots, f_m)$ with $f_i \in \boldsymbol{C}\{x_1, \cdots, x_m\}$ $(i=1, \cdots, m)$. Then

$$\boldsymbol{C}\{f_1, \cdots, f_m\}=\boldsymbol{C}\{x_1, \cdots, x_m\}^G.$$

By Theorem A, any element of $\mathrm{Der}_Y(\log D)_0$ is liftable by f. Thus

$$f^{-1}\mathrm{Der}_Y(\log D)_0 \subset (\mathrm{Der}_{X,0})^G.$$

Conversely let $\theta \in (\mathrm{Der}_{X,0})^G$. Then $\theta \cdot f_i \in \boldsymbol{C}\{x_1, \cdots, x_m\}^G = \boldsymbol{C}\{f_1, \cdots, f_m\}$ because f_i is invariant under G. Write $\theta \cdot f_i = g_i(f_1, \cdots, f_m)$ with $g_i \in \boldsymbol{C}\{y_1, \cdots, y_m\}$ $(i=1, \cdots, m)$. Define

$$\phi=\sum_{i=1}^{m} g_i(y)(\partial/\partial y_i) \in \mathrm{Der}_{Y,0},$$

then θ is liftable and

$$\phi=f^{-1}\theta.$$

This shows that $\theta \in \mathrm{Der}_Y(\log D)_0$ because of Theorem A. This proves

$$f^{-1}\mathrm{Der}_Y(\log D)_0 \supset (\mathrm{Der}_{X,0})^G$$

and thus Theorem C i).

Since $\mathcal{O}$ is a finite $\mathcal{O}^G$-module, we have (EGA 0 IV 16.4.8)

$$\begin{aligned}\mathrm{depth}_{\mathcal{O}^G}(\mathrm{Der}_{X,0}) &= \mathrm{depth}_{\mathcal{O}}\mathrm{Der}_{X,0} \\ &= k-(\mathrm{homolog.\,dim}_{\mathcal{O}}\mathrm{Der}_{X,0})=k.\end{aligned}$$

This implies that $\mathrm{Der}_{X,0}$ is $\mathcal{O}^G$-free because $\mathcal{O}^G$ is a regular local ring. Thus the proof of Theorem C ii) reduces to

LEMMA 2.6. *Let R be a local ring and M an R-free module. Assume that a finite group G acts R-linearly on M:*

$$g(\sum_i r_i m_i)=\sum_i r_i g(m_i)$$

with $r_i \in R$, $m_i \in M$ and $g \in G$. Then the invariant submodule M^G is also R-free.

PROOF. It suffices to prove that M^G is a direct summand of M. A map

$$\tau: M \longrightarrow M^G$$

defined by

$$\tau(m)=(\#G)^{-1}\sum_{g \in G} g(m)$$

is an R-homomorphism and identity on M^G. Thus $M \cong M^G \oplus (\ker \tau)$. □

3. Discriminant of a deformation.

In this section we shall deal with a deformation of a germ of a hypersurface with isolated singularity: Let the coordinates of $X=(\boldsymbol{C}^m, 0)$ and $Y=(\boldsymbol{C}^k, 0)$ be $(x_1, \cdots, x_m)$ and $(y_1, \cdots, y_k)$ $(m \geqq k)$ respectively. Put $n=m-k+1$. A flat holomorphic map germ

$$f: X \longrightarrow Y$$

is given by

$$f^* y_1 = f_1(x_1, \cdots, x_m) \in \boldsymbol{C}\{x_1, \cdots, x_m\}$$

$$f^* y_i = x_{n+i-1} \quad (i=2, \cdots, k).$$

Identify $(x_{n+1}, \cdots, x_m)$ with $(y_2, \cdots, y_k)$ by f^*. Define

$$x = (x_1, \cdots, x_n)$$

$$y' = (y_2, \cdots, y_k)$$

$$y = (y_1, \cdots, y_k) = (y_1, y')$$

for simplicity. Then $(x_1, \cdots, x_m) = (x, y')$ and $f_1 \in \boldsymbol{C}\{x, y'\}$.

Assume that the radical of an ideal $(f_1, y_2, \cdots, y_k, \partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n)$ is the maximal ideal of $\mathcal{O}_{X,0}$ (i.e., $C \cap f^{-1}(0) = \{0\}$, or equivalently $f^{-1}(0)$ has an isolated singularity at 0). Define

$$\mathcal{O}_{C,0} = \mathcal{O}_{X,0}/(\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n).$$

The germ of the critical set $(C, 0)$ (as a set) is the support of $\mathcal{O}_{C,0}$. The direct image $f_* \mathcal{O}_{C,0}$ is an $\mathcal{O}_{Y,0}$-module and the germ of the discriminant $(D, 0)$ is defined by the 0-th fitting ideal of $f_* \mathcal{O}_{C,0}$ [8].

Fix f, (x, y') and (y) as above. Call these coordinates the *original coordinates*.

DEFINITION 3.1. The coordinates $(\hat{x}, \hat{y}') = (\hat{x}_1, \cdots, \hat{x}_n, \hat{y}_2, \cdots, \hat{y}_k)$ of X and $(\hat{y}) = (\hat{y}_1, \hat{y}') = (\hat{y}_1, \cdots, \hat{y}_k)$ of Y are said to be *admissible* if $\hat{y}_i = y_i$ $(i=2, \cdots, k)$. The original coordinates are of course admissible.

Let $(\hat{x}, y')$ and $(\hat{y}_1, y')$ be admissible. Put $f^* \hat{y}_1 = \hat{f}_1(\hat{x}, y')$. Notice that

$$(\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n)\mathcal{O}_{X,0} = (\partial \hat{f}_1/\partial \hat{x}_1, \cdots, \partial \hat{f}_1/\partial \hat{x}_n)\mathcal{O}_{X,0}.$$

Define a $\boldsymbol{C}$-algebra homomorphism

$$s: \boldsymbol{C}\{\hat{x}, \hat{y}\} \longrightarrow \boldsymbol{C}\{\hat{x}, y'\} = \mathcal{O}_{X,0}$$

by

$$s(\hat{x}) = \hat{x}, \quad s(\hat{y}_1) = \hat{f}_1, \quad s(y') = y'.$$

Then s does not depend upon the choice of admissible coordinates. Define a $\boldsymbol{C}$-linear map

$$\hat{\varphi}_0 : \mathrm{Der}_{Y,0} \longrightarrow f_* \mathcal{O}_{C,0}$$

by

$$\hat{\varphi}_0(\theta) = [s\{\theta \cdot (f^* \hat{y}_1 - \hat{y}_1)\}],$$

where $\theta \in \mathrm{Der}_{Y,0}$ and $[\ ]$ denotes the residue class in $\mathcal{O}_{C,0} = \mathcal{O}_{X,0}/(\partial f_1/\partial x_1, \cdots, \partial f_1/\partial x_n)$. This definition of $\hat{\varphi}_0$ may depend upon the choice of admissible coordinates. In case that the above admissible coordinates happen to be the original ones, we denote $\hat{\varphi}_0$ by φ_0.

LEMMA 3.2.

$$\hat{\varphi}_0=[v]\varphi_0$$

for some unit $v\in\mathcal{O}_{X,0}$. *Thus neither* $\ker\hat{\varphi}_0$ *nor* $\operatorname{im}\hat{\varphi}_0$ *depends upon the choice of admissible coordinates.*

PROOF. Since we have

$$\begin{aligned} f^*\hat{y}_1-\hat{y}_1&=f^*(\hat{y}_1(y_1, \cdots, y_k))-\hat{y}_1(y_1, \cdots, y_k) \\ &=\hat{y}_1(f^*y_1, y')-\hat{y}_1(y_1, y'), \end{aligned}$$

$f^*\hat{y}_1-\hat{y}_1$ is a multiple of $f^*y_1-y_1$ in $\boldsymbol{C}\{x, y\}$. Similarly, we know that $f^*y_1-y_1$ is a multiple of $f^*\hat{y}_1-\hat{y}_1$ also. Thus

$$f^*\hat{y}_1-\hat{y}_1=u(x, y)(f^*y_1-y_1)$$

with a unit $u(x, y)\in\boldsymbol{C}\{x, y\}$. Then

$$\begin{aligned} \hat{\varphi}_0(\theta)&=[s\{\theta\cdot(f^*\hat{y}_1-\hat{y}_1)\}] \\ &=[s\{\theta\cdot(u(f^*y_1-y_1))\}] \\ &=[s\{u\cdot\theta\cdot(f^*y_1-y_1)+(f^*y_1-y_1)\cdot\theta\cdot u\}] \\ &=[s(u)]\varphi_0(\theta). \end{aligned}$$

□

Define

$$\mathcal{K}_0=\ker\varphi_0, \quad \mathcal{M}_0=\operatorname{im}\varphi_0.$$

Then $\mathcal{M}_0$ is an $\mathcal{O}_{Y,0}$-submodule of $f_*\mathcal{O}_{C,0}$ generated by $\{[g_1], \cdots, [g_k]\}$, where

$$\begin{aligned} &g_1=-1 \\ &g_i=(\partial f_1/\partial y_i)\in\mathcal{O}_{X,0} \quad (i=2, \cdots, k). \end{aligned}$$

Let $Z=(\boldsymbol{C}^{k-1}, 0)$ with its coordinate $(y')=(y_2, \cdots, y_k)$. Let $\pi:(Y, 0)\to(Z, 0)$ be the natural projection induced from the inclusion $\boldsymbol{C}\{y'\}\subset\boldsymbol{C}\{y_1, y'\}$. Assume that $[g_1], \cdots, [g_k]$ are independent over $\mathcal{O}_{Z,0}$ in $\pi_*\mathcal{M}_0$. The following definition is due to T. Yano [13].

DEFINITION 3.1. f is said to be *free* if $\pi_*\mathcal{M}_0$ is $\mathcal{O}_{Z,0}$-free with its base $\{[g_1], \cdots, [g_k]\}$.

Examples of free deformations.
1. A semi-universal deformation (in this case $\mathcal{M}_0=f_*\mathcal{O}_{C,0}$ [6]),
2. the "G-invariant subdeformation" of a free deformation, where G is a finite subgroup of $\mathrm{GL}(m;\boldsymbol{C})$ (see Yano [15]),
3. the subdeformation of a semi-universal deformation of a rational double point obtained through a "folding" of the corresponding Coxeter graph, which was studied by P. Slodowy [7] and Yano [14], [15].

DEFINITION 3.4. We say that f is (TQ) (trivial family of quasi-homogeneous singularity) if

$$f^*\hat{y}_1=u(\hat{x},\ y')g(\hat{x})\in \boldsymbol{C}\{\hat{x},\ y'\}$$
$$f^*y_i=y_i \quad (i=2, \cdots, k)$$

with respect to some admissible coordinates $(\hat{x}, y')$ and $(\hat{y}_1, y')$ with a quasi-homogeneous $g\in \boldsymbol{C}\{\hat{x}\}$ (i.e., $g\in(\partial g/\partial\hat{x}_1, \cdots, \partial g/\partial\hat{x}_n)\mathcal{O}_{X,0}$).

LEMMA 3.5. *Let f be (TQ). Take the admissible coordinates as above. Then*

$$\mathrm{Ann}_{\mathcal{O}_{Y,0}}(f_*\mathcal{O}_{C,0})=\mathrm{Ann}_{\mathcal{O}_{Y,0}}(\mathcal{M}_0)=\hat{y}_1\cdot\mathcal{O}_{Y,0}\,.$$

PROOF. Since $\mathcal{M}_0\ni[1]$, the first equality is obvious.

It is clear that the discriminant is defined by $y_1=0$ as a set. Thus

$$\sqrt{\mathrm{Ann}_{\mathcal{O}_{Y,0}}(\mathcal{M}_0)}=\hat{y}_1\cdot\mathcal{O}_{Y,0}\,.$$

Since g is quasi-homogeneous with an isolated singularity, we can transform g into a weighted homogeneous polynomial [4]. Thus we know that

$$g\in(\hat{x}_1, \cdots, \hat{x}_n)(\partial g/\partial\hat{x}_1, \cdots, \partial g/\partial\hat{x}_n)\,.$$

Assume that

$$g=\sum_i p_i(\hat{x})(\partial g/\partial\hat{x}_i)$$

with $p_i\in(\hat{x}_1, \cdots, \hat{x}_n)$. Then by an easy computation we obtain

$$(u+\sum_i p_i(\hat{x})(\partial u/\partial\hat{x}_i))g=\sum_i p_i(\hat{x})(\partial(ug)/\partial\hat{x}_i)$$

and thus $g\in(\partial(ug)/\partial\hat{x}_1, \cdots, \partial(ug)/\partial\hat{x}_n)$. This implies that $\hat{y}_1\in\mathrm{Ann}_{\mathcal{O}_{Y,0}}(f_*\mathcal{O}_{C,0})$. $\square$

Notice that f is (TQ) if and only if 1) $(\partial\Delta/\partial y_1)(0)\neq 0$ (Δ is a reduced defining equation of the discriminant) and 2) f is represented by a deformation which gives a trivial family of a quasi-homogeneous singularity along the critical set near the origin.

Let

$$F: U \longrightarrow V$$

be a representative for f with the fixed coordinates (x, y') and (y) of U and V respectively.

DEFINITION 3.5. A point $p\in U$ is called a *(TQ)-point* if the map germ F_p is (TQ). (Here F_p is identified with a map germ $(\boldsymbol{C}^{n+k-1}, 0)\to(\boldsymbol{C}^k, 0)$ with the original coordinates given by a translation of the coordinates (x, y') and (y).)

DEFINITION 3.6. We say that F is (GTQ) (generically (TQ)) if there is an analytic subset A of U with $C\supset A\supset \mathrm{Sing}\,C$ and $\mathrm{codim}_U A\geqq 2$ such that each point of $C\backslash A$ is a (TQ)-point.

Examples of (GTQ) deformations.

1. Since a rational double point is quasi-homogeneous and has no parameter, a deformation whose generic singularities are rational double points is (GTQ),
2. a semi-universal deformation is (GTQ), because its generic singularity is an ordinary double point,
3. from 1, any deformation of a rational double point is (GTQ).

DEFINITION 3.7. f is said to be (GTQ) if it is represented by a (GTQ) deformation.

The rest of this article is devoted to the proof of the following

THEOREM D. *For a free and (GTQ) deformation*

$$f:(\boldsymbol{C}^m, 0) \longrightarrow (\boldsymbol{C}^k, 0)$$

the germ of the discriminant $(D, 0)$ is free.

COROLLARY 3.8. *For a free deformation f whose generic fibers are rational double points, $(D, 0)$ is free.*

COROLLARY 3.9. (cf. K. Saito [6]) *For a semi-universal deformation f, $(D, 0)$ is free.*

LEMMA 3.10. *Assume that f is (TQ). Then the sequence*

$$0 \longrightarrow \mathrm{Der}_Y(\log D)_0 \longrightarrow \mathrm{Der}_{Y,0} \xrightarrow{\varphi_0} \mathcal{M}_0 \longrightarrow 0$$

is exact.

PROOF. Because of 3.2, we can replace φ_0 by $\hat{\varphi}_0$ which is obtained from the admissible coordinates $(\hat{x}, y')$ and $(\hat{y}_1, y')$ given in 3.4. Thus

$$\partial(f^*\hat{y}_1)/\partial y_i = \partial(ug)/\partial y_i = g\cdot(\partial u/\partial y_i) \quad (i=2, \cdots, k).$$

Because $[g]=[0]$ in $\mathcal{M}_0$ (3.5),

$$\hat{\varphi}_0(\theta) = [f^*(\theta\cdot\hat{y}_1)] \in \mathcal{M}_0 .$$

Therefore

$$\begin{aligned}\ker(\hat{\varphi}_0) &= \{\theta\in\mathrm{Der}_{Y,0};\ \theta\cdot\hat{y}_1\in\mathrm{Ann}_{\mathcal{O}_{Y,0}}(f_*\mathcal{O}_{C,0})\}\\ &= \{\theta\in\mathrm{Der}_{Y,0};\ \theta\cdot\hat{y}_1\in\hat{y}_1\cdot\mathcal{O}_{Y,0}\}\\ &= \mathrm{Der}_Y(\log D)_0 .\end{aligned}$$

□

PROPOSITION 3.11. *Assume that f is free. Then*

$$\mathrm{homolog.\,dim}_{\mathcal{O}_{Y,0}}(\mathcal{M}_0)=1$$

PROOF. By the assumption,

$$\operatorname{depth}_{\mathcal{O}_{Z,0}}(\pi_*\mathcal{M}_0)=k-1-\operatorname{homolog.dim}_{\mathcal{O}_{Z,0}}(\pi_*\mathcal{M}_0)$$
$$=k-1.$$

Put $I=\operatorname{Ann}_{\mathcal{O}_{Y,0}}(f_*\mathcal{O}_{C,0})$. Then

$$\pi^*:\mathcal{O}_{Z,0}\longrightarrow \mathcal{O}_{Y,0}/I$$

is finite because 1) the locus of I is $(D, 0)$ and 2) $(D, 0)$ is defined by $y_1^l+\Delta_0(y)=0$ $(1\geqq 0)$ with $\Delta_0\in(y_2, \cdots, y_k)\mathcal{O}_{Y,0}$. Notice that

$$I=\operatorname{Ann}_{\mathcal{O}_{Y,0}}(\mathcal{M}_0)$$

because $\mathcal{M}_0\ni[1]=[-g_1]$. Thus we have (EGA 0 IV 16.4.8)

$$k-1=\operatorname{depth}_{\mathcal{O}_{Z,0}}(\pi_*\mathcal{M}_0)$$
$$=\operatorname{depth}_{\mathcal{O}_{Y,0}/I}(\mathcal{M}_0)$$
$$=\operatorname{depth}_{\mathcal{O}_{Y,0}}(\mathcal{M}_0)\,.$$

This implies our assertion. □

Assume that f is free and (GTQ). Choose a sufficiently small representative

$$F:U\longrightarrow V$$

of f such that 1) F is surjective and flat, 2) F is (GTQ) and 3) each germ of F is free. Put $F_1=F^*y_1$. Then $(F_1)_0=f_1$. A coherent sheaf $\mathcal{M}_V$ of $\mathcal{O}_V$-Module is similarly defined to be an $\mathcal{O}_V$-submodule of $F_*\mathcal{O}_C$ generated by $G_1=-1$, $G_i=(\partial F_1/\partial y_i)(x, y')\in\Gamma(U, \mathcal{O}_U)$ $(i=2, \cdots, k)$. Then $\mathcal{M}_{V,0}=\mathcal{M}_0$. Define an $\mathcal{O}_V$-homomorphism

$$\varphi_V:\operatorname{Der}_V\longrightarrow\mathcal{M}_V$$

compatibly with the definition of φ_0:

$$\varphi_V\Big(\sum_{i=1}^k h_i(y)(\partial/\partial y_i)\Big)=\Big[\sum_{i=1}^k(F^*h_i)(x, y')G_i(x, y')\Big]\,.$$

Then $\varphi_{V,0}=\varphi_0$. Put

$$\mathcal{K}=\ker(\varphi_V)\,.$$

By 3.11, $\mathcal{K}$ is a free $\mathcal{O}_V$-Module. Let $p\in V\backslash D$. Then $\mathcal{M}_{V,p}=0$ and thus

$$\mathcal{K}_p=\operatorname{Der}_{V,0}=\operatorname{Der}_V(\log D)_p\,.$$

Next let $p\in D\backslash F(A)$ (A is as in 3.6). Define $\{P_1, \cdots, P_t\}=C\cap F^{-1}(p)$, then C is smooth at each P_i. Denote the component of C containing P_i by C_i. Put $D_i=F(C_i)$ $(i=1, \cdots, t)$. Then we have

$$(D, p)=\bigcup_i(D_i, p)\,.$$

Since the germ at P_i of F is (TQ),

$$\mathcal{K}_p = \bigcap_i \mathrm{Der}_V(\log D_i)_p$$
$$= \mathrm{Der}_V(\log D)_p$$

by applying 3.10. Therefore each element of $\mathcal{K}$ is logarithmic along $D\backslash F(A)$ and thus along D. This shows that

$$\mathcal{K} \subset \mathrm{Der}_V(\log D).$$

Next consider the 0-th fitting ideal of $\mathcal{M}_V$. Then its generator is given by

$$\Delta = |\theta_1, \cdots, \theta_k| \quad \text{(see 2.5)}$$
$$= \det(h_{ij}) \in \Gamma(V, \mathcal{O}_V),$$

where $\{\theta_1, \cdots, \theta_k\}$ is a free base for $\Gamma(V, \mathcal{K})$ with

$$\theta_i = \sum_{j=1}^{k} h_{ij}(y)(\partial/\partial y_j) \quad (i=1, \cdots, k).$$

We easily know that Δ_p is a unit for $p \in V\backslash D$. Moreover Δ_p is reduced for $p \in D\backslash F(A)$ because $\mathrm{Der}_V(\log D)_p = \mathcal{K}_p$ is free (see 2.5). Therefore Δ is reduced on $V\backslash F(A)$ and thus is a reduced defining equation for D entirely on V. Again by 2.5, Theorem D is proved.

REMARK. In the proof above, we have already showed that a generator of the 0-th fitting ideal of $\mathcal{M}_V$ gives a reduced defining equation of the discriminant.

NOTE ADDED IN PROOF. Recently we found that the essential part of Theorem D is proved by K. Saito in [16, (4.3)]. Our (GTQ)-condition is his (2.3.6) and (2.3.7).

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