Kodaira dimension of certain algebraic fiber spaces

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The purpose of this paper is to give lower bounds of the Kodaira dimension of certain algebraic fiber spaces. Concerning this, there is the following conjecture by S. Iitaka [7] (*everything in this paper is considered over the complex number field* $\boldsymbol{C}$):

CONJECTURE. *Let* $f: X \to Y$ *be an algebraic fiber space, i.e.,* X *and* Y *are non-singular projective algebraic varieties and* f *is a surjective morphism with a connected general fiber* X_y. *Then* $\kappa(X) \geq \kappa(Y) + \kappa(X/Y)$ *where* κ *denotes the Kodaira dimension and* $\kappa(X/Y) \underset{\mathrm{def}}{=} \kappa(X_y)$.

For the notation and the definitions see [15], [7], [21] or [8]. In the following cases, it is known that the conjecture is true:

(1) X_y is a curve (Viehweg [18]).

(2) X_y is an abelian variety (Ueno [16]).

(3) $\dim X=3$ (Viehweg [19]).

(4) $\kappa(Y)=\dim Y$ and $\kappa(X) \geq 0$, or $\kappa(Y)=\dim Y$ and $p_g(X_y) \geq 1$ (Kawamata [8]).

(5) $\dim Y=1$, (Kawamata [9]).

In this paper we shall prove that it is also true in the following cases:

(6) $\kappa(X/Y)=0$ and X_y has a finite covering which is birationally equivalent to a non-singular projective algebraic variety with trivial canonical bundle.

(7) X_y is a surface and $\kappa(X/Y)=1$.

We can say something stronger in case (6), and it contains the case (2). From the cases (1) and (5) follows the case (3). Also something more is proved in case X_y is a surface, but the general conjecture is still open.

We have two methods to attack the conjecture: the theory of variations of Hodge structures to prove (6), and the decomposition of the given algebraic fiber space into several algebraic fiber spaces with smaller relative dimensions to prove (7). The construction of this paper is as follows: In Section 1 we collect some results concerning the variations of Hodge structures. The main point is that the so-called canonical extension of some Hodge bundles are semi-positive vector

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bundles. In Section 2 we collect some basic results on the Kodaira dimension. A base change theorem (Theorem 9) is proved. In Section 3 we shall construct algebraic fiber spaces by modifying the period mappings which are originally defined by an analytic method. In Section 4 we prove (6). In Section 5 we prove (7) and some other related results. A generalized canonical bundle formula (Theorem 20) is proved.

Standard notation

For a non-singular projective algebraic variety X of dimension n,

Ω_X^1: the sheaf of holomorphic 1-forms on X

T_X: the tangent bundle on X

$K_X=\Omega_X^n$: the canonical bundle on X

$q(X)=\dim H^0(X, \Omega_X^1)$: the irregularity of X

$P_m(X)=\dim H^0(X, K_X^{\otimes m})$: the m-genus of X for $m\in \boldsymbol{N}$

$p_g(X)=P_1(X)$: the geometric genus of X

$\kappa(X, L)$: the L-dimension of X, where L is a line bundle on X

$\kappa(X, V)=\kappa(\boldsymbol{P}(V), L_V)$, where V is a vector bundle on X and L_V is the tautological line bundle on $\boldsymbol{P}(V)$

$\kappa(X)=\kappa(X, K_X)$: the Kodaira dimension of X.

For an inclusion morphism i: $X\subset Y$,

$N_{X/Y}=i^*T_Y/T_X$: the normal bundle

red: the reduced part.

For a surjective morphism f: $X\to Y$ of non-singular projective algebraic varieties,

$X_y=f^{-1}(y)$: the fiber of f over $y\in Y$

X_y°: an irreducible component of X_y

$\dim X/Y=\dim X-\dim Y=\dim X_y^\circ$ for a general y

$K_{X/Y}=K_X\otimes f^*K_Y^{\otimes -1}$: the relative canonical bundle

$q(X/Y)=q(X_y^\circ)$ for a general y

$P_m(X/Y)=P_m(X_y^\circ)$ for a general y

$p_g(X/Y)=P_1(X/Y)$

$\kappa(X/Y)=\kappa(X_y^\circ)$ for a general y.

The above f is called an algebraic fiber space, if X_y for a general y is irreducible. For other standard notation and definitions we refer the reader to [15] or [8].

1. Period mapping

(1) Let us recall briefly the theory of variation of Hodge structures. We

refer the reader to Griffiths [4] and Schmid [13]. Let n and r be fixed integers. A *variation of Hodge structures* consists of the following data (i) to (iv) with the conditions (v) and (vi):

(i) A base space S, which is assumed to be a non-singular algebraic variety in this paper.

(ii) A local system H_Z of free abelian groups of rank r.

(iii) A non-degenerate bilinear form Q on H_Z, which is symmetric or alternate if n is even or odd, respectively.

(iv) A descending filtration $\{F^p\}_{0\le p\le n}$ on $H \underset{\text{def}}{=} H_Z\otimes\mathcal{O}_S$ by vector subbundles F^p such that $F^{p,q}\underset{\text{def}}{=}\mathrm{Gr}_F^p\,\mathrm{Gr}_{\bar F}^q(H)=0$ if $p+q\neq n$, where $\bar F$ denotes the complex conjugate. This filtration is called the *Hodge filtration*. It follows that $F^{p,n-p}=F^p\cap\bar F^{n-p}$ and $H=\bigoplus_{p=0}^{n}F^{p,n-p}$.

(v) $Q(F^{p,n-p},\bar F^{p',n-p'})=0$ if $p\neq p'$, and $(\sqrt{-1})^n(-1)^pQ(F^{p,n-p},\bar F^{p,n-p})>0$ (positive definite).

(vi) The flat connection on H defined by the lattice H_Z induces a homomorphism D: $F^p\to F^{p-1}\otimes\Omega_S^1$. This is called the *infinitesimal period relation*.

The bilinear form Q induces a positive definite hermitian metric h on $F^n=F^{n,0}$ by $h(v,v')=(\sqrt{-1})^n(-1)^nQ(v,\bar v')$. Let $\Theta=\bar\partial\partial\log h$ be its curvature form. By (v) and (vi), $(\sqrt{-1}/2\pi)\Theta$ is positive semi-definite. Moreover, if the homomorphism $T_{S,s}\to\mathrm{Hom}\,(F_s^{n,0},F_s^{n-1,1})$ induced by D is injective at a point s of S, then $(\sqrt{-1}/2\pi)\,\mathrm{Tr}\,\Theta$ is positive definite at s.

(2) Let $\bar S$ be a non-singular projective algebraic variety which contains S as a Zariski open dense subset and such that $D\underset{\text{def}}{=}\bar S\backslash S$ is a divisor of normal crossing on $\bar S$. For a variation of Hodge structure H on S, we define a vector bundle $\bar H$ on $\bar S$ which is an extension of H. This is called the *canonical extension* of H: Let U be an arbitrary open subset of $\bar S$ with coordinate functions $z_1,\cdots,z_d$ such that $D\cap U=\{z_1\cdots z_e=0\}$ for some $0\le e\le d$. Let γ_i $(i=1,\cdots,e)$ be local monodromies of H_Z corresponding to loops around the z_i-axes. We assume that the γ_i are all *quasi-unipotent*, i.e., the eigenvalues of them are roots of unity. Let $v_1,\cdots,v_r$ be multivalued flat sections of H_Z on $U\backslash D$ which make a basis of H_Z at each point. Then the expressions

$$s_j=\exp\left(-\frac{1}{2\pi\sqrt{-1}}\sum_{i=1}^{e}\log\gamma_i\log z_i\right)v_j,\qquad j=1,\cdots,r$$

give single valued holomorphic sections of H, where the branches of the $\log\gamma_i$

are chosen so that their eigenvalues are in an interval $\sqrt{-1}\,[0, 2\pi)$. Let $\bar{H}|_U$ be a holomorphic vector bundle on U generated by the s_j. Then it can be easily checked that this construction does not depend on the choice of the z_i and the v_j, and gives a vector bundle $\bar{H}$ on $\bar{S}$. Note that $\bar{H}$ has no flat connections nor bilinear forms. If the local monodromies γ_i are all *unipotent*, then the F^p can also be extended to vector subbundles $\bar{F}^p$ of $\bar{H}$ on $\bar{S}$ (p. 235 of Schmid [13]). In this case the canonical extension is compatible with the base change:

PROPOSITION 1. *Let f: $X \to Y$ be a morphism of non-singular projective algebraic varieties and let C and D be divisors of normal crossing on X and Y, respectively, such that $f(X_0) \subset Y_0$, where $X_0 = X \backslash C$ and $Y_0 = Y \backslash D$. Put $f_0 = f|_{X_0}$. Let H_0 be a variation of Hodge structures on Y_0. We assume that all the local monodromies of H_0 around D are unipotent. Let H and H' be the canonical extensions of H_0 and $f_0^* H_0$ on Y and X, respectively. Then there is an isomorphism $f^*H \simeq H'$ which is compatible with the filtrations $\{F^p\}$.*

PROOF. Let $(U;\, z_1, \cdots, z_d)$ and $(V;\, w_1, \cdots, w_{d'})$ be local coordinate neighborhoods on X and Y as above, respectively. If $w_k = \prod z_i^{a_{ik}}$, then $\gamma_{z_i} = \prod \gamma_{w_k}^{a_{ik}}$, where the γ_{z_i} and the γ_{w_k} are local monodromies corresponding to the z_i and the w_k. Since all the local monodromies are unipotent, $\log \prod \gamma_{w_k}^{a_{ik}} = \sum a_{ik} \log \gamma_{w_k}$. Thus $\sum \log \gamma_{z_i} \log z_i = \sum \log \gamma_{w_k} \log w_k$. Therefore, the two canonical extensions coincide. Q.E.D.

(3) By the differential calculus on the Hodge bundles, we obtain the following results:

DEFINITION. Let X be a complete algebraic variety and let L be a line bundle on X. L is said to be *semi-positive*, if for any curve C on X the intersection number $(L \cdot C)$ is non-negative. Let V be a vector bundle on X. V is said to be *semi-positive*, if the tautological line bundle on $\boldsymbol{P}(V)$ is semi-positive.

THEOREM 2. *Let X be a non-singular projective algebraic variety and let D be a divisor of normal crossing on X. Let H_0 be a variation of Hodge structures on $X_0 \underset{\text{def}}{=} X \backslash D$ with unipotent local monodromies and let H be the canonical extension of H_0 on X. Then $F^n = F^n(H)$ is a semi-positive vector bundle on X.*

This theorem is nothing but Theorem 5 of Kawamata [8]. For the proof we use (i) the semi-positivity of the curvature form explained in (1), and (ii) the weight filtrations and the theory of mixed Hodge structures at the boundary D.

THEOREM 3. *Under the conditions in Theorem 2, we moreover assume that the homomorphism $T_{X,x}\to \mathrm{Hom}(F_x^{n,0}, F_x^{n-1,1})$ is injective at a point x of X_0. Then $c_1(F^n)^d>0$, where $d=\dim X$.*

PROOF. Let U_λ be mutually disjoint open subsets of X such that (1) the union of their closures is X, (2) there are coordinate systems $\{z_1^\lambda, \cdots, z_d^\lambda\}$ on the closures of the U_λ such that $D\cap U_\lambda=\{z_1^\lambda\cdots z_{e_\lambda}^\lambda=0\}$ for some $e_\lambda\le d$, and (3) there are nowhere vanishing sections $s_1^\lambda\wedge\cdots\wedge s_r^\lambda$ of $\det F^n$ defined on the closures of the U_λ. Using the partition of unity, we define a metric on X, and let V_ε be the tubular neighborhood of D of radius $\varepsilon>0$ with respect to this metric. The bilinear form Q induces a hermitian metric h on F_0^n. Let h_ε be a hermitian metric on F^n which coincides with h on $X\backslash V_\varepsilon$. We define $T_\varepsilon|_{U_\lambda}=(\sqrt{-1}/2\pi)\bar\partial\partial\log(\det h_\varepsilon(s_i^\lambda, s_j^\lambda))$. Then they do not depend on the choice of the s_i^λ and can be patched together to define a $(1,1)$-form T_ε on X. T_ε represents the first Chern class of F^n. Hence

$$c_1(F^n)^d=\int_X (T_\varepsilon)^d=\int_{X\backslash V_\varepsilon}(T_\varepsilon)^d+\int_{V_\varepsilon}(T_\varepsilon)^d.$$

The first integral is positive by the remark at the end of (1). We shall show that the second integral converges to zero, if $\varepsilon\to 0$. By Stokes' theorem,

$$\int_{V_\varepsilon}(T_\varepsilon)^d=\sum_\lambda\int_{V_\varepsilon\cap U_\lambda}(T_\varepsilon)^d=\left(\frac{\sqrt{-1}}{2\pi}\right)^d\sum_\lambda\int_{\partial(V_\varepsilon\cap U_\lambda)}\partial g_\varepsilon^\lambda\wedge(\bar\partial\partial g_\varepsilon^\lambda)^{d-1},$$

where $g_\varepsilon^\lambda=\log(\det h_\varepsilon(s_i^\lambda, s_j^\lambda))$. $\partial(V_\varepsilon\cap U_\lambda)$ is the union of $\partial V_\varepsilon\cap U_\lambda$ and the common boundaries with the neighboring $V_\varepsilon\cap U_\mu$. On the latter, the difference $\partial g_\varepsilon^\lambda-\partial g_\varepsilon^\mu$ is an innocent holomorphic 1-form. Hence the term which comes from the latter part of the boundary converges to zero, if we apply the following argument to the dimension $d-1$ case instead of d. On $\partial V_\varepsilon\cap U_\lambda$, g_ε^λ coincides with $g^\lambda=\log(\det h(s_i^\lambda, s_j^\lambda))$. We may assume that $\partial V_\varepsilon\cap U_\lambda$ is of the form

$$\bigcup_{i=1}^{e_\lambda}\{|z_i^\lambda|=\varepsilon,\quad \varepsilon\le|z_j^\lambda|\le 1,\quad |z_k^\lambda|\le 1;\quad 1\le j\le e_\lambda,\quad j\ne i,\quad e_\lambda<k\},$$

since we are taking the limit. Thus we have to show that the integral

$$\int_W \partial g^\lambda\wedge(\bar\partial\partial g^\lambda)^{d-1}$$

converges to zero, where W is the first term of the above union by symmetry. By the expression defining the canonical extension, we can write $s_i^\lambda=\sum_{t=1}^r P_t^i v_t$, where the P_t^i are polynomials in the $\log z_k^\lambda$ $(1\le k\le e_\lambda)$ with coefficients of holomorphic functions. Since $Q(v_t, v_{t'})$ are constants, the $h(s_i^\lambda, s_j^\lambda)=(\sqrt{-1})^n(-1)^n Q(s_i^\lambda, s_j^\lambda)$ are polynomials in the $\log z_k^\lambda$ and the $\log \bar{z}_k^\lambda$ with coefficients of combinations of holomorphic and anti-holomorphic functions. Thus g^λ is the logarithm of a polynomial in the $\log z_k^\lambda$ and the $\log \bar{z}_k^\lambda$ with coefficients of combinations of holomorphic and anti-holomorphic functions. Hence the coefficient of $\partial g^\lambda \wedge (\bar{\partial}\partial g^\lambda)^{d-1}$ is $1/z_1^\lambda z_2^\lambda \bar{z}_2^\lambda \cdots z_{e_\lambda}^\lambda \bar{z}_{e_\lambda}^\lambda$ times a function which goes to zero at the boundary on W. Hence the limit is zero. Q.E.D.

(4) Let V_Z be a free abelian group of rank r with a non-degenerate bilinear form Q. We have the classifying space $\mathcal{D}$ of Hodge structures (or the Hodge filtrations) on $V \underset{\text{def}}{=} V_Z \otimes \boldsymbol{C}$. Then $G_Z \underset{\text{def}}{=} \operatorname{Aut}(V_Z, Q)$ acts on $\mathcal{D}$ properly discontinuously. Let H be a variation of Hodge structures on S and we fix a base point s of S and an isomorphism $V_Z \simeq H_{Z,s}$. Then we obtain a complex analytic morphism P: $S \to \mathcal{D}/\Gamma$, where Γ is the image of the homomorphism $\pi_1(S, s) \to G_Z$. This map P is called the *period mapping*. Griffiths (Theorem 9.6 of [4]) showed that if all the local monodromies of H_Z around the boundary are unipotent, then P turns out to be a proper morphism, when we extend H across the part of the boundary where the local monodromy is trivial.

(5) Now, we shall apply the above abstract theory to the geometric situations. Let f: $X \to Y$ be an algebraic fiber space and let L be a fixed relatively ample line bundle on X. Let D be a divisor of normal crossing on Y and put $Y_0 = Y \setminus D$, $X_0 = f^{-1}(Y_0)$ and $f_0 = f|_{X_0}$. We assume that f_0 is smooth. Let $N = \dim X/Y$ and let n be an integer between 1 and N. Let $H_{0,Z}$ be the torsion free part of the kernel of the homomorphism

$$\wedge L^{N-n+1}: \quad R^n f_{0*} \boldsymbol{Z}_{X_0} \longrightarrow R^{2N-n+2} f_{0*} \boldsymbol{Z}_{X_0}.$$

Let $Q(v, v') = \pm(\sqrt{-1})^n L^{N-n} \wedge v \wedge v'$ for $v, v' \in H_{0,Z}$. Since there is a quasi-isomorphism $f_0^{\cdot}\mathcal{O}_{Y_0} \to \Omega^*_{X_0/Y_0}$ the stupid filtration $\{\Omega^{\geq p}_{X_0/Y_0}\}_{0 \leq p \leq n}$ on the complex $\Omega^*_{X_0/Y_0}$ gives a filtration $\{F_0^p\}$ on $H_0 = H_{0,Z} \otimes \mathcal{O}_{Y_0}$. Then the system $(Y_0, H_{0,Z}, Q, \{F_0^p\})$ gives a variation of Hodge structures. In this paper n will be 1 or N.

The monodromy theorem says that the local monodromies of $H_{0,Z}$ in this case are always quasi-unipotent. Hence we can construct the canonical extension H on Y.

THEOREM 4 (Lemma 1 of [9]). *Let $n = N$ in the above construction and let i: $Y_0 \to Y$ be the inclusion morphism. Then we have a natural isomorphism*

$$f_* K_{X/Y} \longrightarrow i_* F_0^n \cap H.$$

PROOF. This is a consequence of the following proposition by Sakai and a deep result by Schmid which gives an estimation of the metric of the Hodge bundle at the boundary.

PROPOSITION 5 (Lemma 1.1 of [12]). *Let X be a compact complex manifold and let D be a divisor of normal crossing on X. Put $X_0 = X \backslash D$. Let $\omega_0 \in H^0(X_0, K_{X_0}^{\otimes m})$ for a positive integer m. Then the integral*

$$\int_{X_0} |(\omega_0 \wedge \bar{\omega}_0)^{1/m}|$$

converges, if and only if ω_0 has a meromorphic extension ω on X such that $\omega \in H^0(X, K_X^{\otimes m} \otimes \mathcal{O}_X((m-1)D))$.

2. Kodaira dimension

For the definition and elementary properties of the Kodaira dimension we refer the reader to Iitaka [21] or Ueno [15]. What are the most important for our purpose will be the following theorems:

(i) (Theorem 8.1 of [15]) The Kodaira dimension measures the growth of the pluri-genera.

(ii) (Theorem 5.10) The Iitaka's fibering theorem.

(iii) (Theorem 5.11) The easy addition theorem by Iitaka.

As a consequence of the easy addition theorem, we have the following:

PROPOSITION 6. *Let X, Y and Z be non-singular projective algebraic varieties, and let $f: X \to Y$ and $g: X \to Z$ be surjective morphisms. Then for a generic point z of Z,*

$$\dim Y - \kappa(Y) \geq \dim(f(X_z^\circ)) - \kappa(f(X_z^\circ)),$$

where $f(X_z^\circ)$ is considered as an irreducible algebraic variety and κ is defined as the Kodaira dimension of its non-singular model.

PROOF. If $\dim f(X_z^\circ) + \dim Z > \dim Y$, then we may replace Z resp. X by its general hyperplane section H (resp. $g^{-1}(H)$), because the restriction of f to $g^{-1}(H)$ is a surjective morphism onto Y. Thus we may assume that $\dim f(X_z^\circ) + \dim Z = \dim Y$. Let Y' be the image of X in $Y \times Z$ by (f, g). Thus, $\dim Y' = \dim Y$, $\kappa(Y') \geq \kappa(Y)$ and $Y_z'^\circ = f(X_z^\circ)$, where we considered the surjective morphism $Y' \to Z$

induced by the second projection. Then by the easy addition theorem, $\dim Y' - \kappa(Y') \geq \dim Y_z'^{o} - \kappa(Y_z'^{o})$. Combinig the above inequalities, we obtain the desired inequality. Q.E.D.

A use of Grothendieck's duality theorem ([6]) gives the following:

PROPOSITION 7. *Let X and Y be Gorenstein algebraic varieties, let j: $Y' \to Y$ be the normalization, and let f': $X \to Y'$ be a proper and birational morphism. Then for any positive integer m, there is a natural injective homomorphism*

$$f'_*\omega(X)^{\otimes m} \longrightarrow j^*\omega(Y)^{\otimes m},$$

where ω denotes the dualizing sheaf.

PROOF. Let $f = f' \circ j$. We choose a closed subvariety E of Y such that

(i) $\operatorname{codim}_Y E \geq 2$, and

(ii) f induces a finite morphism f_0 from $X_0 = X \backslash f^{-1}(E)$ onto $Y_0 = Y \backslash E$.

The duality theorem by Grothendieck says that

$$f_{0*}\boldsymbol{R}\operatorname{Hom}(\omega(X_0),\ f_0^!\omega(Y_0)) = \boldsymbol{R}\operatorname{Hom}(f_{0*}\omega(X_0),\ \omega(Y_0)),$$

where we note that f_0 is finite. Since the left side is just $f_{0*}\mathcal{O}_{X_0}$, we obtain a non-zero homomorphism $f_{0*}\omega(X_0) \to \omega(Y_0)$ which induces the identity homomorphism on an open subset of Y. Since f_0 is an affine morphism, we obtain injective homomorphisms $f'_{0*}\omega(X_0)^{\otimes m} \to j^*\omega(Y_0)^{\otimes m}$ for positive integers m, where f'_0 is the restriction of f' on X_0. Since $\operatorname{codim}_Y E \geq 2$ and $\omega(Y)$ is invertible, we can extend these to the desired homomorphisms $f'_*\omega(X)^{\otimes m} \to j^*\omega(Y)^{\otimes m}$ by Krull's theorem.

Q.E.D.

DEFINITION. Let Y be a non-singular projective algebraic variety and let D be a divisor of normal crossing on Y. A finite and surjective morphism h: $Y' \to Y$ is called a *nice covering* of Y with respect to D, if

(i) Y' is a non-singular and projective algebraic variety and $D' \underset{\text{def}}{=} \operatorname{red} h^{-1}(D)$ is a divisor of normal crossing on Y', and

(ii) There is a divisor of normal crossing D^* on Y which contains both D and the branch locus of h.

THEOREM 8 (Theorem 17 of Kawamata [8]). *Let Y be a non-singular projective algebraic variety, let D be a divisor of normal crossing on Y, and let K be a finite extension of the field $\boldsymbol{C}(Y)$ such that the normalization Y_1 of Y in K ramifies only possibly over D. Then there exists a finite extension L of K*

such that the normalization $h: Y' \to Y$ of Y in L gives a nice covering of Y with respect to D.

THEOREM 9. *Let $f: X \to Y$ be an algebraic fiber space, let $h: Y' \to Y$ be a nice covering, let $\mu: X' \to X \times_Y Y'$ be a resolution of singularities, and let $f': X' \to Y'$ be the algebraic fiber space induced by f:*

$$\begin{tikzcd} X' \arrow[r, "\mu"] \arrow[dr, "f'"'] & X \times_Y Y' \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ & Y' \arrow[r, "h"'] & Y. \end{tikzcd}$$

Then $\kappa(X, K_{X/Y}) \geq \kappa(X', K_{X'/Y'})$.

PROOF. $X \times_Y Y' = \{(x, y') \in X \times Y';\ f(x) = h(y')\}$ is locally complete intersection and hence Gorensten. Since h is flat, $\omega(X \times_Y Y'/Y') = g^*(K_{X/Y})$ by p. 191 of Hartshorne [6]. Let $j: X'' \to X \times_Y Y'$ be the normalization. Then by Proposition 7, $\kappa(X', K_{X'/Y'}) \leq \kappa(X'', j^*\omega(X \times_y Y'/Y')) = \kappa(X, K_{X/Y})$. Q.E.D.

3. Algebraic preparation

THEOREM 10. *Let X be a non-singular and projective algebraic variety and let D be a divisor of normal crossing on X. Let H_0 be a variation of Hodge structures on $X_0 \underset{\text{def}}{=} X \backslash D$. Then there exist a non-singular projective algebraic variety X' and a finite surjective morphism $h: X' \to X$ satisfying the following conditions:*

(i) *$D' \underset{\text{def}}{=} \operatorname{red} h^{-1}(D)$ is a divisor of normal crossing on X'.*

(ii) *Let $X'_0 = X' \backslash D'$, $h_0 = h|_{X'_0}$ and let $H'_0 = h_0^* H_0$ be the variation of Hodge structures on X'_0 induced from H_0. Fix a base point x' of X'_0 and let M be a subgroup of $\boldsymbol{C}^*$ generated by all eigenvalues of the elements of the image of the monodromy representation $\pi_1(X'_0, x') \to \mathrm{GL}(H'_{0, \boldsymbol{Z}, x'})$. Then M is torsion free.*
In particular, if all the local monodromies of H_0 around D are quasi-unipotent, then all the local monodromies of H'_0 around D' are unipotent.

PROOF. Fix a base point x of X_0 and we consider the monodromy representation $\pi_1(X_0, x) \to \mathrm{GL}(H_{0, \boldsymbol{Z}, x})$. By Borel [1] 17.1, there is a congruence subgroup G_0 of $\mathrm{GL}(H_{0, \boldsymbol{Z}, x})$ such that the group generated by the eigenvalues of the elements of G_0 is torsion free. Let π be the subgroup of $\pi_1(X_0, x)$ obtained by the pull-back of G_0. Since the index of G_0 in $\mathrm{GL}(H_{0,Z,x})$ is finite, we get an etale covering X'' of X_0 corresponding to π. By Theorem 8, there is a finite extension L of the field $\boldsymbol{C}(X'')$ such that the normalization X' of X in L is non-singular and that the pull-back D' of D is a divisor of normal crossing. Then X' is the desired variety. Q.E.D.

THEOREM 11. *Let X be an algebraic variety, let Y be an irreducible and reduced complex space and let f: $X \to Y$ be a proper and surjective morphism of complex spaces with connected fibers. Then there exist algebraic varieties X' and Y', a proper and birational morphism μ: $X' \to X$, a proper and bimeromorphic morphism of complex spaces ν: $Y' \to Y$, and a proper and surjective morphism of algebraic varieties f': $X' \to Y'$ such that $f' \circ \nu = \mu \circ f$.*

PROOF. We may assume that X is non-singular and quasi-projective, and Y is normal. Let $\bar{X}$ be a non-singular and projective compactification of X. Put $D = \bar{X} \backslash X$.

CLAIM 1. There is a reduced closed analytic subset E of Y, which is different from Y, such that $Y \backslash E$ is smooth and that f is smooth over $Y \backslash E$.

PROOF. This is an easy consequence of the proper mapping theorem. See Corollary 1.8, Chapter I of Ueno [15]. Q.E.D.

Let y be a point on $Y \backslash E$ and let $\hat{Z}$ be an irreducible component of the Hilbert scheme of $\bar{X}$ which contains a point corresponding to $X_y = f^{-1}(y)$. Let p: $\hat{X} \to \hat{Z}$ be the universal family and let q be the projection from $\hat{X} \subset \bar{X} \times \hat{Z}$ to $\bar{X}$. Since the restriction of f from $X \backslash f^{-1}(E) \subset \bar{X} \times (Y \backslash E)$ to $Y \backslash E$ is proper and flat, we have a complex analytic morphism j: $Y \backslash E \to \hat{Z}$ which sends $y_1 \in Y \backslash E$ to the point of $\hat{Z}$ corresponding to X_{y_1}. Since $N_{X_y/X} = \overset{r}{\bigoplus} \mathcal{O}_{X_y}$ for $r = \dim Y$, j is an open immersion and hence $\hat{Z}$ is generically reduced. Let $Z_1 = \{z \in \hat{Z};\ \hat{Z}$ is reduced at z, $p^{-1}(z)$ is smooth and $p^{-1}(z) \not\subset D \cup f^{-1}(E)\}$ and $Z_0 = \{z \in \mathrm{red}\,\hat{Z};\ p^{-1}(z) \cap X \neq \emptyset\}$. Then Z_1 is an open dense subset of $\mathrm{red}\,\hat{Z}$ with respect to the transcendental topology and Z_0 is a Zariski open dense subset of $\mathrm{red}\,\hat{Z}$ which contains Z_1.

CLAIM 2. j is an isomorphism from $Y \backslash E$ onto Z_1.

PROOF. Suppose that $j(Y \backslash E) \neq Z_1$ and let z be a point in the closure of $j(Y \backslash E)$ in Z_1. If $z \notin j(Y \backslash E)$, then $f(p^{-1}(z) \cap X) \cap Y \backslash E$ contains at least two distinct points y_1 and y_2. Let U_1 and U_2 be disjoint open neighborhoods of y_1 and y_2 in $Y \backslash E$, respectively. Then there is an open neighborhood V of z in Z_1 such that for any

z_1 in V, $f(p^{-1}(z_1)\cap X)\cap U_i\neq\emptyset$ for $i=1,2$. Thus $z_1\notin j(Y\backslash E)$, a contradiction. Therefore, $j(Y\backslash E)=Z_1$. Q.E.D.

Let $X_0'=(\mathrm{red}\,p)^{-1}(Z_0)$.

CLAIM 3. q induces a proper and birational morphism μ_0: $X_0'\to X$.

PROOF. By the same argument as above, we deduce that if $z\in Z_0$, then $f(p^{-1}(z)\cap X)$ contains only one point. Hence $p^{-1}(z)$ is contained in X. Thus X_0' coincides with $(\mathrm{red}\,q)^{-1}(X)$, and hence μ_0 is proper. Since Z_1 is open and dense, μ_0 is birational. Q.E.D.

Let X' and Y' be the normalizations of X_0' and Z_0, respectively, and let f': $X'\to Y'$ and μ: $X'\to X$ be the morphisms induced by p and μ_0, respectively.

CLAIM 4. There is a proper and bimeromorphic morphism ν: $Y'\to Y$ of complex spaces inducing j^{-1} on Z_1.

PROOF. Assigning $z\in Z_0$ to $f(p^{-1}(z))\in Y$, we obtain a set theoretic map ν_0: $Z_0\to Y$, which is an extension of j^{-1}. Let ν: $Y'\to Y$ be the map induced by ν_0. Thus we have the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\mu"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "\nu"] & Y \end{tikzcd} .$$

Since f, f' and μ are proper, ν is proper. Let t be a germ of complex analytic functions on Y. Since $f'_*\mu^*f^*t$ is a germ of complex analytic functions on Y', ν is complex analytic. Q.E.D.

Combining the above arguments, we conclude the proof of the theorem. Q.E.D.

Let X', D' and H_0' be as in Theorem 10. We assume that all the local monodromies are unipotent. Let H' be the canonical extension of H_0' on X'. As in (4) of Section 1, we have a period mapping P: $X_0'\to\mathcal{D}/\Gamma$, which is a proper morphism of complex spaces. By the Stein factorization, we obtain a proper morphism P_0': $X_0'\to Y_0'$. Then by Theorem 11, there are an algebraic fiber space P'': $X''\to Y''$, divisors of normal crossing D'' and E'' on X'' and Y'', respectively, a proper birational morphism μ: $X_0'' \to X_0'$, and a proper bimeromorphic morphism ν: $Y_0'' \to Y_0'$, where $X_0'' = X'' \setminus D''$ and $Y_0'' = Y'' \setminus E''$, such that the following diagram

$$\begin{array}{ccccc} X'' & \supset & X_0'' & \xrightarrow{\mu} & X_0' \\ {\scriptstyle P''}\downarrow & & {\scriptstyle P_0''}\downarrow & & {\scriptstyle P_0'}\downarrow \\ Y'' & \supset & Y_0'' & \xrightarrow[\nu]{} & Y_0' \end{array}$$

is commutative, where $P_0'' = P''|_{X_0''}$. Let H_0'' be a variation of Hodge structures on X_0'' obtained by the pull-back of H_0' by μ. Let $x \in X_0''$ and $y = P''(x) \in Y_0''$ and we shall consider the local system $H_{0,Z}''|_{P_0''^{-1}(y)}$. Since the stabilizer subgroup G_y of $\mathrm{Aut}(\mathcal{D})$ at a point of $\mathcal{D}$ corresponding to y is compact and $\mathrm{Aut}(H_{0,Z,x}'', Q)$ is a discrete subgroup of $\mathrm{Aut}(\mathcal{D})$, the monodromy group of $H_{0,Z}''|_{P_0''^{-1}(y)}$ is of finite order, and hence is a trivial group by the condition (ii) of Theorem 10. Therefore, $H_{0,Z}''$ on X_0'' induces a local system $H_{0,Z}^\#$ on Y_0'' such that $P_0''^* H_{0,Z}^\# = H_{0,Z}''$, and we obtain a variation of Hodge structures $H_0^\#$ on Y_0''. Let H'' and $H^\#$ be the canonical extensions of H_0'' and $H_0^\#$ on X'' and Y'', respectively. By Proposition 1, $P''H^\# = H'' = \mu^* H'$. Thus, in particular, $\kappa(Y'', F^n(H^\#)) = \kappa(X', F^n(H'))$.

4. Parabolic fiber spaces

An algebraic variety X is said to be *parabolic*, if the Kodaira dimension $\kappa(X)$ is zero. In this section we shall study algebraic fiber spaces whose general fibers are parabolic. Let X be a non-singular projective parabolic algebraic variety. Thus $\max_{m \geq 1} P_m(X) = 1$. Let m be the smallest positive integer such that $P_m(X) \neq 0$. We have a non-zero homomorphism φ: $K_X^{\otimes -m} \to \mathcal{O}_X$. Let $X^\#$ be the closed subvariety of $\mathrm{Spec}\left(\bigoplus_{k \geq 0} K_X^{\otimes -k}\right)$, the total space of the line bundle K_X, defined by the ideal generated by $u - \varphi(u)$ for $u \in K_X^{\otimes -m}$. Then $X^\#$ is an irreducible finite covering of X of degree m which ramifies only possibly along $\{\varphi = 0\}$. We can easily show that $\kappa(X^\#) = 0$ and $p_g(X^\#) = 1$, where κ and p_g of a singular variety are defined as those of its non-singular model (see [8] Sect. 3, Claim 2). We call $X^\#$ the *canonical covering* of X. Let π: $X^\# \to X$ be the canonical projection.

PROPOSITION 12. *Let π: $X^\# \to X$ be as above and let X' be another non-singular projective algebraic variety such that $\kappa(X') = 0$ and $p_g(X') = 1$. Suppose that there is a generically finite and surjective morphism μ: $X' \to X$. Then μ is factored* *through by π and a morphism $\mu^{\sharp}: X' \to X^{\sharp}$.*

PROOF. Let ω and s be non-zero global sections of $K_X^{\otimes m}$ and $K_{X'}$, respectively. Since $\kappa(X')=0$, $\mu^*\omega = cs^{\otimes m}$ for some $c \in \boldsymbol{C}^*$. Thus $X^{\sharp} \times_X X'$ breaks into m irreducible components, and the projection from one of its components to $X^{\sharp}$ gives μ^*.

Q.E.D.

The main theorem in this section is the following:

THEOREM 13. *Let f: $X \to Y$ be an algebraic fiber space with parabolic general fibers. We assume that the canonical covering of the general fiber is birationally equivalent to a non-singular projective algebraic variety with trivial canonical bundle. Then there is an algebraic fiber space f_1: $X_1 \to Y_1$ which is birationally equivalent to the given f: $X \to Y$ such that $\kappa(X_1, K_{X_1/Y_1}) \geq p\text{-}\dim(f)$, where $p\text{-}\dim(f)$ is a non-negative integer called the period dimension of f, which is defined as the dimension of the image of the period mapping associated to f (for the precise definition see the proof).*

COROLLARY 14. *Under the above conditions, if $\kappa(Y) \geq 0$, then $\kappa(X) \geq \max(\kappa(Y),\ p\text{-}\dim(f))$. For example, if the general fibers are birationally equivalent to abelian varieties, or to surfaces with $\kappa=0$, then the above inequality holds.*

PROOF. We remark the following. First, $\kappa(X, K_{X/Y})$ does not depend on the choice of birational models of X, but of Y. Second, if we replace Y by a nice covering Y' and f by the induced fiber space f': $X' \to Y'$ as in Section 2, then $\kappa(X', K_{X'/Y'}) \leq \kappa(X, K_{X/Y})$.

Step 1. We assume first that the general fiber X_y has itself a birational model with trivial canonical bundle. Then there is a surjective morphism $Y' \to Y$ such that the major component of the pull-back $X \times_Y Y'$ is birationally equivalent to some algebraic fiber space $X' \to Y'$ whose general fiber has a trivial canonical bundle. If $\dim Y' > \dim Y$, then we may replace Y' by its general hyperplane section using Bertini's theorem. Thus we may assume that $\dim Y' = \dim Y$. Replacing Y by its suitable birational model, we obtain a nice covering $Y' \to Y$ by Theorem 8. Therefore, we may assume from the first that $K_{X_y} \cong \mathcal{O}_{X_y}$. For this type of fiber spaces, the local Torelli theorem holds, i.e., the homomorphism

$$H^1(X_y, T_{X_y}) \longrightarrow \mathrm{Hom}(H^0(X_y, K_{X_y}),\ H^1(X_y, \Omega_{X_y}^{n-1}))$$

is an isomorphism, where $n = \dim X_y$. Let Y_0 be a Zariski open dense subset of

Y over which f is smooth. Put $X_0=f^{-1}(Y_0)$. The variation of Hodge structures in $R^n f_* C_{X_0}$ gives a period mapping from Y_0 as in Section 1. After replacing Y by a nice covering of a suitable birational model, we may moreover assume that this period mapping gives a new algebraic fiber space P: $Y \to Z$ as in Section 3. We have (the canonical extension of) a variation of Hodge structures H on Z whose pull-back to Y by P is just the given one (see the argument at the end of Section 3). The dimension of the image of P does not depend on the choice of the birational models of Y nor that of coverings, but only on the birational equivalence class of f. We shall call it the *period dimension* of f and denote it by p-dim(f).

Let z be a general point of Z and let y be a general point of Y lying over z. Suppose that there is a tangent vector t in $T_{Z,z}$ such that the homomorphism $F^n \to F^{n-1}/F^n$ given by the connection D on H and t is trivial. Then a tangent vector t' in $T_{Y,y}$ lying over t is sent to zero by the composition of homomorphisms

$$T_{Y,y} \xrightarrow{\rho} H^1(X_y, T_{X_y}) \longrightarrow \mathrm{Hom}(H^0(X_y, K_{X_y}),\ H^1(X_y, \Omega^{n-1}_{X_y})),$$

where ρ denotes the Kodaira-Spencer map. By the local Torelli theorem, we deduce that $\rho(t')$ is itself zero, and hence all the homomorphisms $F^p \to F^{p-1}/F^p$ given by t are trivial for $0 \leq p \leq n$. This means that the Hodge structure does not move along the direction of t, a contradiction. Therefore, we proved that the assumption of Theorem 3 holds for H. By Theorem 2 and the following theorem, we conclude that

$$\kappa(X, K_{X/Y}) \geq \kappa(Y, F^n) = \kappa(Z, F^n) = \dim Z = p\text{-}\dim(f). \qquad \text{Q.E.D. for Step 1.}$$

THEOREM 15. *Let L be a semi-positive line bundle on a proper algebraic variety X of dimension d. Then the self-intersection number (L^d) is positive, if and only if $\kappa(X, L)=d$, i.e.,* $\dim H^0(X, L^{\otimes m}) \geq cm^d$ *for some positive number c and for $m \gg 0$.* (κ denotes the L-dimension of X.)

PROOF (due to A. Sommese). We may assume that X is non-singular and projective. Let H be an ample line bundle on X such that $H \otimes K_X$ is very ample. Let Y be a divisor on X corresponding to a general section of $H \otimes K_X$. We consider the following short exact sequence:

$$0 \longrightarrow L^{\otimes m} \longrightarrow L^{\otimes m} \otimes H \otimes K_X \longrightarrow L^{\otimes m} \otimes H \otimes K_X \otimes \mathcal{O}_Y \longrightarrow 0$$

for any positive integer m. Since $L^{\otimes m} \otimes H$ is ample, $H^i(X, L^{\otimes m} \otimes H \otimes K_X)=0$ for $i>0$, by the Kodaira vanishing theorem. Thus

$$\dim H^0(X, L^{\otimes m}\otimes H\otimes K_X)=\chi(X, L^{\otimes m}\otimes H\otimes K_X)=\frac{1}{d!}(L^d)m^d+\text{lower terms.}$$

Since $\dim Y=d-1$,

$$\dim H^0(Y, L^{\otimes m}\otimes H\otimes K_X\otimes \mathcal{O}_Y)\leq bm^{d-1}$$

for some positive number b. Thus by the following exact sequence

$$0\longrightarrow H^0(X, L^{\otimes m})\longrightarrow H^0(X, L^{\otimes m}\otimes H\otimes K_X)\longrightarrow H^0(Y, L^{\otimes m}\otimes H\otimes K_X\otimes\mathcal{O}_Y),$$

we complete the proof. Q.E.D.

Step 2. We shall treat the general case. After changing birational models of X and Y and after a base change by a nice covering of Y, we come to the following situation:

(i) There is an algebraic fiber space $f^\sharp\colon X^\sharp\to Y$ which is factored through by f and which satisfies the first assumption in Step 1:

$$\begin{tikzcd} X^\sharp \arrow[d, "f^\sharp"'] \arrow[dr, "\pi"] & \\ Y & X \arrow[l, "f"] \end{tikzcd}$$

(ii) There are divisors of normal crossing $C, C^\sharp$ and D on $X, X^\sharp$ and Y, respectively, such that $C=\operatorname{red} f^{-1}(D)$ and $C^\sharp=\operatorname{red} f^{\sharp -1}(D)$. Put $X_0=X\backslash C$ and $X_0^\sharp=X^\sharp\backslash C^\sharp$.

(iii) f and $f^\sharp$ are smooth on X_0 and $X_0^\sharp$, respectively, and the fibers of $f^\sharp$ over Y_0 are birationally equivalent to the canonical coverings of those of f.

It is easily shown that p-$\dim(f^\sharp)$ depends only on f, and we shall denote it by p-$\dim(f)$. Let F^n be the line bundle $f^\sharp_* K_{X^\sharp/Y}$ considered in Step 1. We showed that $\kappa(Y, F^n)=p$-$\dim(f)$. Let V be an open subset of Y with a coordinate system $\{y_1,\cdots,y_d\}$ such that $D\cap V=\{y_1\cdots y_e=0\}$ for some $e\leq d$. Put $U=f^{-1}(V)$ and $U^\sharp=f^{\sharp -1}(V)$. Let s be a local section of F^n on V and let m be the covering degree of π. Then $s^{\otimes m}|_{U^\sharp\backslash C^\sharp}$ is a pull-back by π of some section ω of $f_*K_{X/Y}^{\otimes m}$ on $U\backslash C$. Since

$$\begin{aligned}\int_U\left|\left(\prod_{k=1}^{e} y_k^{1-m}\omega\wedge\prod_{k=1}^{e}\bar{y}_k^{1-m}\bar{\omega}\right)^{1/m}\right| &=\frac{1}{m}\int_{U^\sharp}\left|\left(\prod_{k=1}^{e} y_k^{1-m}s^{\otimes m}\wedge\prod_{k=1}^{e}\bar{y}_k^{1-m}\bar{s}^{\otimes m}\right)^{1/m}\right|\\ &=\frac{1}{m}\int_{U^\sharp}\left|\prod_{k=1}^{e}(y_k\bar{y}_k)^{1/m-1}s\wedge\bar{s}\right|\end{aligned}$$

is integrable, $\prod_{k=1}^{e} y_k^{1-m}\omega$ has poles on $C\cap U$ of order at most $m-1$. Hence ω can be extended to a holomorphic section of $f_*K_{X/Y}^{\otimes m}$ on U. Thus we obtained an injective homomorphism $(F^n)^{\otimes m} \to f_*K_{X/Y}^{\otimes m}$. Therefore, $\kappa(X, K_{X/Y}) \geq \kappa(Y, F^n) = p\text{-}\dim(f)$, and we complete the proof of Theorem 13. Q.E.D.

REMARK. By the above proof the same theorem holds, if (i) $p_g(X/Y)=1$, and (ii) the local Torelli theorem holds or $\dim(X/Y)=2$. For the related topics to (ii), we refer the reader to [2] and [17].

5. Surface case

In this section we shall prove the following theorems:

THEOREM 16. *Let f: $X \to Y$ be an algebraic fiber space. If* $\dim X/Y=2$ *and* $\kappa(X/Y)=1$, *then* $\kappa(X) \geq \kappa(Y)+1$.

THEOREM 17. *Let f: $X \to Y$ be an algebraic fiber space and let α_y: $X_y \to A(X_y)$ be the Albanese mapping of the general fiber X_y of f. If* $\dim X/Y=2$, $\kappa(X/Y)=2$ *and* $\dim(\operatorname{Im}\alpha_y)=1$, *then* $\kappa(X) \geq \kappa(Y)+2$.

In the curve case we have the following theorem by Viehweg [18]:

THEOREM 18. *Let f: $X \to Y$ be an algebraic fiber space such that* $\dim X/Y=1$. *Then $\kappa(X) \geq \kappa(Y)+\kappa(X/Y)$. Moreover, if $\kappa(Y) \geq 0$, then $\kappa(X) \geq$* $\dim \varphi(Y)$, *where φ denotes the rational map from Y to the coarse moduli space of curves of given genus induced by f.*

In both cases of Theorems 16 and 17, we have a decomposition of f into two algebraic fiber spaces g and h such that $f=h \circ g$,

$$\begin{tikzcd} X \arrow[dd, "f"'] \arrow[dr, "g"] & \\ & Z \arrow[dl, "h"] \\ Y & \end{tikzcd},$$

where g: $X \to Z$ is obtained by the Iitaka fibering of the general fiber X_y of f in the case of Theorem 16, and by the Stein factorization of α_y in the case of Theorem 17. Thus we can use the above Viehweg's theorem in the proof as follows.

LEMMA 19 (Induction lemma). *To prove Theorems* 16 *and* 17, *it is enough to prove that $\kappa(X)>0$ if $\kappa(Y) \geq 0$.*

PROOF. If $\kappa(Y)=-\infty$, then there is nothing to prove. Suppose $\kappa(Y)\geq 0$. Let Φ: $X\rightarrow W$ be the Iitaka fibering of X. By assumption, $\dim W>0$ and hence $X_w\neq X$, where X_w is the general fiber of Φ. By Proposition 6, $\dim Y-\kappa(Y)\geq \dim f(X_w)-\kappa(f(X_w))$. By the Stein factorization of $X_w\rightarrow f(X_w)$, we obtain an algebraic fiber space f_w: $X_w\rightarrow Y_w$. Since $\kappa(Y)\geq 0$, $\kappa(f(X_w))\geq 0$, and hence $\kappa(Y_w)\geq 0$. Then since $\kappa(X_w)=0$, $\dim X_w/Y_w\leq 1$ by the assumption. If $\dim X_w/Y_w=1$, then $\kappa(Y_w)=0$ and the general fiber of f_w is an elliptic curve by the Viehweg's theorem. If $\dim X_w/Y_w=0$, then $\kappa(f(X_w))\leq\kappa(Y_w)\leq\kappa(X_w)=0$. Thus in any case, $\kappa(f(X_w))=0$. Therefore, $\dim Y-\kappa(Y)\geq \dim X_w-1=\dim X-\kappa(X)-1$, which proves Theorem 16. If $\kappa(X/Y)=2$, then X_y cannot be an elliptic fiber space and hence $\dim X_w/Y_w=0$. Thus, $\dim Y-\kappa(Y)\geq\dim X_w=\dim X-\kappa(X)$, which proves Theorem 17. Q.E.D.

PROOF OF THEOREM 17. Since $\kappa(X_y)=2$, the general fiber of g must be a curve of genus greater than 1. Therefore, using the Viehweg's theorem twice, $\kappa(X)\geq\kappa(Z)+1\geq\kappa(Y)+1$. Q.E.D.

PROOF OF THEOREM 16. Let q be the genus of the general fiber of h. If $q\geq 2$, then using Theorem 18 twice, $\kappa(X)\geq\kappa(Z)\geq\kappa(Y)+1$ and we are done. By the same way, if $q=1$ and if the fibers of either g or h have non-constant moduli, then $\kappa(X)>0$. Also, if $q=1$ and $\kappa(Y)>0$, then $\kappa(X)>0$. Thus, what are remaining are the following two cases:

(i) $q=1$, and g and h have constant moduli, or

(ii) $q=0$.

To treat these cases, we need the following generalization of Kodaira's canonical bundle formula for elliptic surfaces (In case there are no multiple fibers, this was obtained by Ueno [14] Theorem 6.1.).

First we recall Kodaira's theory of elliptic surfaces. An *elliptic surface* f: $X\rightarrow Y$ is an algebraic fiber space whose general fiber is an elliptic curve such that $\dim X=2$. It is called *minimal*, if any fiber contains no exceptional curves of the first kind. Kodaira classified all the singular fibers Q which can appear in minimal elliptic surfaces ([10, II] Theorem 6.2): namely, there are fibers of types $k\mathrm{I}_b$, I_b^*, II, II*, III, III*, IV and IV*, where $k\mathrm{I}_b$ is a multiple fiber of multiplicity k and b is any non-negative integer. We attach to these fibers rational numbers $a=a(Q)=:$ 0, 1/2, 1/6, 5/6, 1/4, 3/4, 1/3 and 2/3, respectively (see Table I in [10, II] P. 604, Table II in [10, III] p. 14, and [11] Theorem 12). Then Kodaira's canonical bundle formula says that

$$K_X^{\otimes 12}=f^*(K_Y^{\otimes 12}\otimes\mathcal{O}_Y(\sum 12a(Q)P)\otimes J^*\mathcal{O}_{P^1}(1))\otimes\mathcal{O}_X\Big(12\sum_{k\mathrm{I}_b}(k-1)Q'\Big),$$

where the P are the points of Y corresponding to the singular fibers Q, the second summation is taken over all the multiple fibers $Q=kQ'$, and J: $Y\to\boldsymbol{P}^1$ is the J-invariant function. Since there is no concept of minimality in higher dimensional cases, we shall state the following theorem only in terms of the direct image sheaves.

THEOREM 20. *Let f: $X\to Y$ be an algebraic fiber space whose general fiber is an elliptic curve. Let Y_0 be the Zariski-open subset of Y over which f is smooth. Put $X_0=f^{-1}(Y_0)$ and $f_0=f|_{X_0}$. We assume that $D\underset{\mathrm{def}}{=}Y\backslash Y_0$ is a divisor of normal crossing on Y. Let $D=\sum D_i$ be the decomposition into irreducible components. Then*

(1) *The J-invariants of the fibers of f_0 give a morphism J: $Y\to\boldsymbol{P}^1$.*

(2) *$f_*K_{X/Y}$ is a line bundle on Y.*

(3) *We have the following isomorphism*

$$(f_*K_{X/Y})^{\otimes 12}\cong\mathcal{O}_Y(\sum 12a_iD_i)\otimes J^*\mathcal{O}_{P^1}(1),$$

where the a_i are the rational numbers corresponding to the types of the singularities over the general points of the D_i given before the theorem.

Moreover, if $D'=\sum' D_i$ is a sum of disjoint irreducible components of D, then

(4) *The isomorphism in* (3) *induces an inclusion*

$$f_*K_{X/Y}^{\otimes m}\supseteq\mathcal{O}_Y(\sum' ma_iD_i+\sum' m(k_i-1)/k_iD_i)\otimes J^*\mathcal{O}_{P^1}(m/12),$$

where the k_i are the multiplicities of the fibers over the general points of the D_i, if they are of types $k_i\mathrm{I}_{b_i}$, and m is a positive common multiple of the k_i and 12.

PROOF. The statement (1) follows from Schmid [13] Theorem 4.12. By Theorem 4, $f_*K_{X/Y}$ is a reflexive sheaf. Thus we have (2) by Hartshorne [20] Proposition 1.9. Now we shall prove (3). We shall give an explicit description of the variation of Hodge structures given by the algebraic fiber space f.

The local system $H_{0,Z}=R^1f_{0*}\boldsymbol{Z}_{X_0}$ gives a variation of Hodge structures on Y_0 as in Section 1. Let $H_0=H_{0,Z}\otimes\mathcal{O}_{Y_0}$ and let H be the canonical extension of H_0 on Y. The period domain $\mathcal{D}$ in this case is the upper half plane $\{z\in\boldsymbol{C};\ \mathrm{Im}\,z>0\}$. Let $V_Z=\boldsymbol{Z}\oplus\boldsymbol{Z}$ and let $\{v_1, v_2\}$ be a basis of V_Z. The alternate bilinear form Q is given by $Q(v_1, v_2)=1$. The arithmetic group $G_Z=\mathrm{Aut}(V_Z, Q)=\mathrm{SL}(2, \boldsymbol{Z})$ acts in the following way:

(i) For a basis $\{v_1, v_2\}$ of V_Z, $g\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = {}^t\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and

(ii) for $z \in \mathcal{D}$, $g(z) = (az+b)/(cz+d)$, where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_Z.$$

The quotient space $\mathcal{D}/G_Z$ has a one point compactification $\boldsymbol{P}^1 = \mathcal{D}/G_Z \cup \{\infty\}$.

A Hodge structure on $V = V_Z \otimes \boldsymbol{C}$ is given by a 1-dimensional linear subspace $F = F^1$ of V such that $-(\sqrt{-1})Q(F, \bar{F}) > 0$. Then $Q(F, v_1) \neq 0$. Let $\omega = \omega_F$ be an element of F such that $Q(\omega, v_1) = -1$ and let $z = Q(\omega, v_2)$. Thus $\omega = zv_1 + v_2$ and $\operatorname{Im} z = -1/2(\sqrt{-1})Q(\omega, \bar{\omega}) > 0$. The period mapping is given by $(F;\ v_1, v_2) \mapsto z$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_Z$ and let $\{v_1', v_2'\}$ be another basis of V_Z such that $g\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}$. According to this basis, we define $\omega' = \omega_F' \in F$ and $z' \in \mathcal{D}$ as above. Then $\omega = (cz+d)\omega'$ and $z' = (az+b)/(cz+d)$. Thus we obtain the period mapping P given by $F \mapsto z \bmod G_Z$.

An automorphic form $q = q(z)$ of weight d is, by definition, a holomorphic function on $\mathcal{D}$ such that $q(g(z)) = (cz+d)^d q(z)$, for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_Z$. Then the formula $q(z)\omega^{\otimes m}$ is invariant under the action of G_Z. Hence pulling it back, we obtain a section of $(f_* K_{X_0/Y_0})^{\otimes m} = f_* K_{X_0/Y_0}^{\otimes m}$. It is known that there is an automorphic form Δ of weight 12 which has the only simple zero at infinity. Thus the automorphic factor $(cz+d)^{12}$ defines a line bundle on $\boldsymbol{P}^1 = \mathcal{D}/G_Z \cup \{\infty\}$ which is isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}(1)$.

Therefore, we have an injective homomorphism $J^*\mathcal{O}_{\boldsymbol{P}^1}(1) \to (f_* K_{X/Y})^{\otimes 12}$, because $|Q(\omega, \bar{\omega})| = 2 \operatorname{Im} z$ has a logarithmic growth at the boundary. Since it is also surjective on Y_0, we have an isomorphism $(f_* K_{X/Y})^{\otimes 12} = J^*\mathcal{O}_{\boldsymbol{P}^1}(1) \otimes \mathcal{O}_Y(\sum b_i D_i)$ for some non-negative integers b_i. The b_i are determined by estimating the growth of the metric $h(s_i, s_i)$ of generating sections s_i of $f_* K_{X/Y}$ at general points of the D_i. The eigenvalues of the local monodromies around the D_i corresponding to the subbundle $f_* K_{X/Y}$ of the canonical extension H are given in Table I in [10, II]: they are 1, -1, $e^{\pi i/3}$, $e^{5\pi i/3}$, $e^{\pi i/2}$, $e^{3\pi i/2}$, $e^{2\pi i/3}$ and $e^{4\pi i/3}$, if the types of the singularities at the general points of the D_i are $k\mathrm{I}_b$, I_b^*, II, II*, III, III*, IV and IV*, respectively. Hence the growth is logarithmic function times $|z|^{-2a_i}$, where $z = 0$ is the local equation of D_i. Therefore, our b_i coincides with a_i for each i.

Finally, we shall prove (4). Since $f_* K_{X/Y}^{\otimes m}$ does not depend on the choice of the birational models of X, we may assume that $C \underset{\text{def}}{=} \operatorname{red} f^{-1}(D)$ is a divisor of normal crossing on X. By the above argument, $f^*K_{X/Y}^{\otimes m}$ contains $J^*\mathcal{O}_{P^1}(m/12)$. What we have to show is thus the following: Let D_1 be an irreducible component of D such that the fiber of f over the general point of D_1 is of type $k_1\mathrm{I}_{b_1}$, and let $f^*D_1=\sum r_jC_j$ be the decomposition into irreducible components. Then any holomorphic local section of $J^*\mathcal{O}_{P^1}(m/12)$ gives a holomorphic section of $K_{X/Y}^{\otimes m}$ which has zeros at the C_j of order at least $mr_j(k_1-1)/k_1$.

Let y_0 be a general point of the image $f(C_j)$, let V be an open neighborhood of y_0 in Y with a coordinate system $\{y_1, \cdots, y_d\}$ such that $D_1\cap V=\{y_1=0\}$ and $f(C_j)\cap V=\{y_1=\cdots=y_e=0\}$ for some $e\leq d$, and let s be a generating local section of $J^*\mathcal{O}_{P^1}(m/12)$ on V. Since $\int_{f^{-1}(y)}|(s\wedge\bar{s})^{1/m}|$ has only logarithmic poles as a function of y on V, the integral $\int_{f^{-1}(V)}|(\theta_{m'}\wedge\bar{\theta}_{m'})^{1/mm'}|$ converges, where $\theta_{m'}=\prod_{k=1}^{e}y_k^{1-mm'}s^{\otimes m'}$ for a positive integer m'. Hence by Proposition 5, $\theta_{m'}$ has a pole of order at most $mm'-1$ on C_j. If $f(C_j)=D_1$ and $e=1$, then $r_j\geq k_1$ and hence $y_1^{mm'/k_1}\theta_{m'}=y_1^{1+mm'/k_1-mm'}s^{\otimes m'}$ is holomorphic at C_j. Since m' is arbitrary, we conclude that $y_1^{m(1-k_1)/k_1}s$ is holomorphic at C_j. Thus we are done in this case. If $f(C_j)\neq D_1$ and $e\geq 2$, then $y_2^{mm'-1}\theta_{m'}$ is holomorphic at C_j, and hence $y_1^{1-mm'}s^{\otimes m'}$ is holomorphic at C_j. Thus $y_1^{-m}s$ is holomorphic at C_j and we are done also in this case. Therefore, we complete the proof of Theorem 20. Q.E.D.

PROOF OF THEOREM 16 CONTINUED.

Case (i). We shall prove that $\kappa(X, K_{X/Y})>0$ for a suitable birational model of f: $X\to Y$. Using Theorems 8 and 9, we reduce it to the case where Z is birationally equivalent to a product $Y\times E$ for an elliptic curve E. Thus we come to the following situation:

$$\begin{tikzcd}
X \arrow[dr, "g"] \arrow[ddd, "f"'] & \\
& Z \arrow[d, "\nu"] \\
& Y\times E \arrow[dl, "h_1"] \\
Y &
\end{tikzcd},$$

where

(a) E is an elliptic curve,

(b) f is the given fiber space, g is an elliptic fiber space with constant moduli, h_1 is the projection, and ν is a proper birational morphism,

(c) there is a divisor of normal crossing D on Z such that g is smooth outside of D, and

(d) if D' is the collection of all the irreducible components of D which project onto Y, then D' is smooth.

By (4) of Theorem 20, we have an inclusion $g_*K_{X/Z}^{\otimes m}\supset\mathcal{O}_Z(\sum' ma_i'D_i)$, for any common multiple m of 12 and the k_i, where $a_i'=\max(a_i, (k_i-1)/k_i)$. In particular $H^0(Z, g_*K_{X/Z}^{\otimes m})\neq 0$. Kodaira's canonical bundle formula applied to an elliptic surface $X_y\to Z_y$ for a general point y of Y says that the above inclusion is an isomorphism when restricted on Z_y. Since $\kappa(X_y)=1$, we conclude that there is an irreducible component D_1 of D' such that $a_1'\neq 0$.

Let $D_1^\flat$ be the image of D_1 on $Y\times E$. Then $\kappa(Y\times E, D_1^\flat)>0$. On the other hand, since any irreducible component of $\nu^*D_1^\flat-D_1$ is exceptional with respect to ν, $H^0(Z, \mathcal{O}_Z(ma_1(D_1-\nu^*D_1^\flat))\otimes K_{Z/Y}^{\otimes mk})\neq 0$ for some positive integer k, where we note that $K_{Y\times E}=h_1^*K_Y$. Combining the above, $H^0(Z, g_*K_{X/Y}^{\otimes mk}\otimes\mathcal{O}_Z(-ma_1\nu^*D_1^\flat))\neq 0$. Thus, $\kappa(X, K_{X/Y})\geq\kappa(Z, \nu^*D_1^\flat)=\kappa(Y\times E, D_1^\flat)>0$. Q.E.D.

Case (ii). By the same argument as in case (i), we reduce it to the following situation:

$$\begin{tikzcd}
X \arrow[dr, "g"] \arrow[ddd, "f"'] & \\
& Z \arrow[d, "\nu"] \\
& Y\times \boldsymbol{P}^1 \arrow[dl, "h_1"] \\
Y &
\end{tikzcd},$$

where

(a) f is the given algebraic fiber space, g is an elliptic fiber space, h_1 is the projection, and ν is a proper birational morphism,

(b) there is a divisor of normal crossing D on Z such that g is smooth outside of D, and

(c) if D' is the collection of all the irreducible components of D which project onto Y, then D' is smooth.

By Theorem 20 (4), we have an inclusion $g_*K_{X/Z}^{\otimes m}\supset\mathcal{O}_Z(\sum' ma_i'D_i)\otimes J^*\mathcal{O}_{P^1}(m/12)$ as above. Let D'' be a general divisor on Z corresponding to $J^*\mathcal{O}_{P^1}(1)$, which can be chosen to be reduced. Let $D''=\sum D_j''$ be the decomposition into irreducible components. We may assume that D' and D'' have no common irreducible components. Using Theorems 8 and 9 again, we may assume that $\deg_Y D_i \leq 1$ and $\deg_Y D''_j \leq 1$ for all i and j. Since $\kappa(X_y)=1$, we have $\deg_{Z_y} \mathcal{O}_{Z_y}(\sum' ma'_i D_i + m/12 \sum D''_j) > 2m$, for a general point y of Y. Since $ma'_i < m$ and $m/12 < m$, we can make a sequence (C_1, C_2, C_3), $(C_4, C_5), \cdots, (C_{2m}, C_{2m+1})$ of divisors on Z such that

(1) C_k is an irreducible component of either D' or D'', and $\deg_Y C_k = 1$, for $1 \leq k \leq 2m+1$,

(2) $C_1 \neq C_2 \neq C_3 \neq C_1$ and $C_{2k'} \neq C_{2k'+1}$ for $2 \leq k' \leq m$, and

(3) $\sum_{k=1}^{2m+1} C_k \leq \sum' ma'_i D_i + m/12 \sum D''_j$.

We shall show that $\dim H^0(Z, K_{Z/Y} \otimes \mathcal{O}_Z(C_1+C_2+C_3)) \geq 2$ and $\dim H^0(Z, K_{Z/Y} \otimes \mathcal{O}_Z(C_{2k'}+C_{2k'+1})) \geq 1$ for $2 \leq k' \leq m$. If these are shown, then since $f_* K_{X/Y}^{\otimes m} = h_*(g_* K_{X/Z}^{\otimes m} \otimes K_{Z/Y}^{\otimes m})$, it follows that $\dim H^0(X, K_{X/Y}^{\otimes m}) \geq 2$ and we are done.

Replacing the birational models of Y and Z, if necessary, and using Theorems 8 and 9 again, we may assume the following conditions:

(i) There is a divisor of normal crossing E on Y such that, if we put $Y_0 = Y \backslash E$ and $Z_0 = h^{-1}(Y_0)$, then h is smooth on Z_0.

(ii) Any pair out of $\{C_1, \cdots, C_{2m+1}\}$ are disjoint if not coincide, and $\bigcup_{k=1}^{2m+1} C_k \cup (Z \backslash Z_0)$ is a divisor of normal crossing on Z. Since $H^0(\boldsymbol{P}^1, K_{\boldsymbol{P}^1} \otimes \mathcal{O}_{\boldsymbol{P}^1}(\{0\}+\{\infty\}))$ is isomorphic to $\boldsymbol{C}$ with a generator dz/z for some inhomogeneous coordinate z on $\boldsymbol{P}^1$, giving the residues $+1$ and -1 on C_1 and C_2, respectively, we obtain a global section ω_1 of $K_{Z_0/Y_0} \otimes \mathcal{O}_{Z_0}(C_1+C_2)$.

CLAIM. ω_1 can be extended to a section of $K_{Z/Y} \otimes \mathcal{O}_Z(C_1+C_2)$.

PROOF. Let V be an open subset of Y with a coordinate system $\{y_1, \cdots, y_d\}$ such that $V \cap E = \{y_1 \cdots y_e = 0\}$ for some $e \leq d$. Put $V_0 = V \cap Y_0$, $U = h^{-1}(V)$, $U_0 = h^{-1}(V_0)$, and $\eta = dy_1 \wedge \cdots \wedge dy_d$. Then $\theta = \omega_1 \wedge \eta$ gives a section of $K_{U_0} \otimes \mathcal{O}_{U_0}(C_1+C_2)$. The form ω_1 gives an isomorphism $\alpha: U_0 \to V_0 \times \boldsymbol{P}^1$ which sends $C_1 \cap U_0$ and $C_2 \cap U_0$ to $V_0 \times \{0\}$ and $V_0 \times \{\infty\}$, respectively. We can choose positive real-valued C^∞-functions r_1 and r_2 on V_0 with algebraic growth at E such that $r_1 < r_2$ and that the closure W of the set $\{z \in U_0;\ |\mathrm{pr}_{\boldsymbol{P}^1}(\alpha(z))| < r_1(h(z))$ or $|\mathrm{pr}_{\boldsymbol{P}^1}(\alpha(z))| > r_2(h(z))\}$ in U becomes a neighborhood of $(C_1 \cup C_2) \cap U$ which contains no divisors in $h^{-1}(V \backslash V_0)$. Then,

$$\int_{U_0 \backslash W} |\theta \wedge \bar{\theta}| = \int_{V_0} 2\pi \log \frac{r_2}{r_1} |\eta \wedge \bar{\eta}|$$

converges. Therefore, by Proposition 5, θ can be extended to a section of $K_U \otimes \mathcal{O}_U(C_1+C_2)$ on $(U \backslash W) \cup U_0$, and hence on U by Hartogs' theorem. Thus ω_1 can be extended on U and we complete the proof. Q.E.D. for the Claim.

By the same way, giving residues $+1$ and -1 on C_1 and C_3, respectively, we obtain another section of $K_{Z/Y}\otimes\mathcal{O}_Z(C_1+C_3)$. Hence $\dim H^0(Z, K_{Z/Y}\otimes\mathcal{O}_Z(C_1+C_2+C_3))\geq 2$. Similarly, $\dim H^0(Z, K_{Z/Y}\otimes\mathcal{O}_Z(C_{2k'}+C_{2k'+1}))\geq 1$ for $2\leq k'\leq m$. Thus we complete the proof of Theorem 16. Q.E.D.

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