Discrete reflection groups in the parabolic subgroup of $SU(n, 1)$ and generalized Cartan matrices of Euclidean type

By Masaaki YOSHIDA

(Communicated by N. Iwahori)

§ 0. Introduction

Let $B_n=\{(z_1, \cdots, z_n) \in \boldsymbol{C}^n \mid \sum_{j=1}^{n} |z_j|^2<1\}$ be the unit ball in the complex euclidean space $\boldsymbol{C}^n$ $(n \geq 2)$, and $\mathrm{Aut}(B_n)$ the holomorphic automorphism group of B_n. Let $\mathcal{P}$ be a parabolic subgroup of $\mathrm{Aut}(B_n)$ and P the corresponding boundary point of B_n. For a discrete subgroup Γ of locally finite volume of $\mathcal{P}$, we have the following conjecture: "The factor space $B_n/\Gamma \cup \{P\}$ added by the point P (the Satake compactification of B_n/Γ) is non-singular if and only if Γ is generated by quasi-reflections". This is true for $n=2$ (Yoshida-Hattori [11]). In this paper, we give the following partial answer to the conjecture.

MAIN THEOREM. *If Γ is generated by reflections and if the point group $W(\Gamma)$ of Γ is a Coxeter group, then the variety $B_n/\Gamma \cup \{P\}$ is non-singular.*

We list up every conjugacy class of discrete subgroup of $\mathcal{P}$ of locally finite volume such that it is generated by reflections and that the point group is a Coxeter group (Table I in Theorem 1). Let A be a generalized Cartan matrix of Euclidean type (Table II), W_A the Weyl group of A, and Q the root lattice of A. We put $\tilde{W}_A = W \ltimes Q$. A bijective correspondence between the groups in Table I and the groups $\{\tilde{W}_A\}$ is obtained (Theorem 2). In accordance with Theorems 1 and 2, the main theorem is reduced to the theorem of Looijenga ([3]), which is reviewed in 5.3.

Here we want to state the motivation of the conjecture. Picard ([6]) and Terada ([8]) studied the monodromy groups of Lauricella's hypergeometric differential equations $F_D(\alpha, \beta_1, \cdots, \beta_n, \gamma; x_1, \cdots, x_n)$ defined in the n dimentional complex projective space. They found some conditions for parameters $\alpha, \beta_1, \cdots, \beta_n, \gamma$ such that the monodromy groups are discrete subgroups of $\mathrm{Aut}(B_n)$. Roughly speaking, this implies that the Satake compactification of the quotient space of B_n by the monodromy groups are non-singular. We notice that the monodromy groups in question are generated by quasi-reflections. On the fixed points in the interior of B_n, the above statement is a consequence of Chevalley's theorem. When we observe the cusps, we naturally arrive at the conjecture. For more details see [10] and [11].

The author wishes to express his deepest appreciation to the members of Ropponmatsu Seminar who gave him unceasing encouragement and valuable comments.

Table of contents

§ 1. Structure of discrete subgroups of G of locally finite volume

1.1. Let Y be an $(l+1)$ dimensional vector space over $\boldsymbol{C}$ with a fixed coordinate system $(z, u_1, \cdots, u_l)$ and

$$D=\left\{(z, u_1, \cdots, u_l) \in Y \,\middle|\, \operatorname{Im} z> \sum_{j=1}^{l} |u_j|^2\right\}$$

be a domain in Y. The domain D can be regarded as a domain in the $(l+1)$ dimensional complex projective space $\boldsymbol{P}^{l+1}(\boldsymbol{C})$ by the natural embedding of Y into $\boldsymbol{P}^{l+1}(\boldsymbol{C})$. If $v={}^t(v_0, v_1, \cdots, v_{l+1})$ is a homogeneous coordinate of $\boldsymbol{P}^{l+1}(\boldsymbol{C})$ related by $(z, u_1, \cdots, u_l)$ by $z=v_0/v_{l+1}, u_1=v_1/v_{l+1}, \cdots, u_l=v_l/v_{l+1}$ and if

$$H=\begin{pmatrix} & & i \\ & -2I_l & \\ -i & & \end{pmatrix}, \quad (I_l=l\times l \text{ identity matrix})$$

then the domain is expressible as $\{v \in \boldsymbol{P}^{l+1}(\boldsymbol{C}) \mid {}^t\bar{v}Hv>0\}$, where ${}^t\bar{v}$ is the transpose of the complex conjugate of v. The closure of D in $\boldsymbol{P}^{l+1}(\boldsymbol{C})$ meets the hyperplane at infinity $v_{l+1}=0$ at the unique point $P={}^t(1, 0, \cdots, 0)$. Remark that the domain D is projectively equivalent to the unit ball $B_{l+1}=\left\{(z_0, z_1, \cdots, z_l) \in \boldsymbol{C}^{l+1} \mid \sum_{j=0}^{n} |z_j|^2<1\right\}$.

The complex analytic automorphism group $\operatorname{Aut}(D)$ of D is identified with the quotient group of the subgroup of $GL(l+2, \boldsymbol{C})$:

$$\{X \in GL(l+2, \boldsymbol{C}) \mid {}^t\bar{X}HX=kH, \text{ for some } k>0\}$$

by the multiplicative group $\boldsymbol{C}^{\times}$ of $\boldsymbol{C}$. For the sake of simplicity we express an element of $\operatorname{Aut}(D)$ by a suitable matrix belonging to the corresponding coset. Under this convention, an element g of $\operatorname{Aut}(D)$ keeps the point P fixed in a geodesic sense (for the definition see [7]) if and only if g is of the form

$$[U, \beta, \gamma]=\begin{bmatrix} 1 & 2i{}^t\bar{\beta}U & \gamma+i{}^t\bar{\beta}\beta \\ 0 & U & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

where $\beta \in \boldsymbol{C}^l$, $\gamma \in \boldsymbol{R}$ and U is an $l\times l$ unitary matrix. The subgroup of $\operatorname{Aut}(D)$ which consists of every element of the form $[U, \beta, \gamma]$ is denoted by G. The group which consists of every element of the form

$$\begin{bmatrix} 1 & \beta & \gamma \\ 0 & A & \alpha \\ 0 & 0 & 1 \end{bmatrix} \quad A \in GL(l, \boldsymbol{C}), \quad \alpha, {}^t\beta \in \boldsymbol{C}^l, \quad \gamma \in \boldsymbol{C},$$

is denoted by $\hat{G}$. Let us define for $N>0$ the subdomain $D(N):=\Big\{(z, u_1, \cdots, u_l) \in$

$D\,\Big|\,\mathrm{Im}\, z-\sum_{j=1}^{l}|u_j|^2>N\Big\}.$

DEFINITION. A subgroup Γ of G is said to be of locally finite volume (at P) if the quotient space $D(N)/\Gamma$ has finite volume with respect to the $\mathrm{Aut}(D)$-invariant measure of D for sufficiently large $N>0$.

Let G_1 be the normal subgroup of G consisting of every element of the form $[I_l, \alpha, \gamma]$, $\alpha\in \boldsymbol{C}^n$, $\gamma\in \boldsymbol{R}$. The center of G, which is also the center of G_1 is given by

$$Z:=\{[I_l, 0, \gamma]\,|\,\gamma\in \boldsymbol{R}\}.$$

We prepare some notations:

$F:=\{(z, 0, \cdots, 0)\in Y\,|\,z\in \boldsymbol{C}\}\subset Y$,

$Y':=Y/F$: l dimensional complex vector space with the coordinate $(u_1, \cdots, u_n)$ with the natural inner product,

$\pi: Y\to Y'$: natural projection,

$A(Y'):=\left\{(A|\beta):=\begin{pmatrix} A & \beta \\ 0 & 1\end{pmatrix}\,\middle|\, A\in GL(l, \boldsymbol{C}), \beta\in \boldsymbol{C}^l\right\}$: affine transformation group,

$U(l)$: $l\times l$ unitary group,

$\boldsymbol{E}(Y'):=\{(U|\beta)\,|\,U\in U(l),\ \beta\in \boldsymbol{C}^l\}$: complex motion group on Y',

π_*: $G\to \boldsymbol{E}(Y')$: surjective homomorphism given by $[U, \beta, \gamma]\to(U|\beta)$.

DEFINITION. A crystallographic group on Y' is a discrete subgroup of $\boldsymbol{E}(Y')$ with compact quotient.

1.2. Let Γ be a discrete subgroup of G of locally finite volume.

LEMMA 1.1. (i) *$\pi_*(\Gamma)$ is a crystallographic group on Y'.* (ii) *The group defined by $\Gamma_1(\Gamma):=\Gamma\cap G_1$ is a normal subgroup of Γ of finite index.* (iii) *There exists a positive number $q(\Gamma):=\mathrm{Inf}\,\{|\gamma|\,|\,[I_l, 0, \gamma]\in \Gamma_1(\Gamma)\}$, and the center of Γ is given by $Z(\Gamma):=\{[I_l, 0, \gamma]\,|\,\gamma\in q(\gamma)\boldsymbol{Z}\}$.*

PROOF. Same as [11; Proposition 1.1].

By (i) and the Bieberbach's theorem (cf. [13]), there exists a lattice $L(\Gamma)\subset Y'$ such that $\pi_*(\Gamma_1(\Gamma))=\{(I_l|\alpha)\,|\,\alpha\in L(\Gamma)\}$. We shall identify the group $\pi_*(\Gamma_1(\Gamma))$ and the lattice $L(\Gamma)$. Under this convention, we define the point group of $\pi_*(\Gamma)$ by $W(\Gamma):=\pi_*(\Gamma)/L(\Gamma)$. We shall also call the finite group $W(\Gamma)$ the point group of Γ. Then we have the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & Z(\Gamma) \arrow[r] \arrow[d] & \Gamma_1(\Gamma) \arrow[r] \arrow[d] & L(\Gamma) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & Z(\Gamma) \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & \pi_*(\Gamma) \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & W(\Gamma) \arrow[r] \arrow[d] & W(\Gamma) \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

1.3. In this section, we consider the exact sequence:

$$1 \longrightarrow Z(\Gamma) \longrightarrow \Gamma_1(\Gamma) \longrightarrow L(\Gamma) \longrightarrow 1.$$

LEMMA 1.2. *There exists a function* c_Γ: $L(\Gamma) \to \boldsymbol{R}$ *such that* $[I_l, \alpha, \gamma] \in \Gamma_1(\Gamma)$ *if and only if* $\alpha \in L(\Gamma)$ *and* $\gamma \equiv c_\Gamma(\alpha) \bmod q(\Gamma)\boldsymbol{Z}$. *The function* c_Γ *satisfies, for* $\alpha, \alpha' \in L(\Gamma)$,

$$c_\Gamma(\alpha+\alpha') \equiv c_\Gamma(\alpha) + c_\Gamma(\alpha') - 2\,\mathrm{Im}\,{}^t\bar{\alpha}\alpha' \quad \mathit{mod}\ q(\Gamma)\boldsymbol{Z}.$$

PROOF. We have only to recall that

$$[I_l, \alpha, \gamma][I_l, \alpha', \gamma'] = [I_l, \alpha+\alpha', \gamma+\gamma'-2\,\mathrm{Im}\,{}^t\bar{\alpha}\alpha'].$$ Q. E. D.

LEMMA 1.3. *We have, for all* $\alpha, \alpha' \in L(\Gamma)$,

$$\frac{4}{q(\Gamma)}\,\mathrm{Im}\,{}^t\bar{\alpha}\alpha' \in \boldsymbol{Z}.$$

PROOF. The definition of $q(\Gamma)$ and the identity

$$[I_l, \alpha, \gamma][I_l, \alpha', \gamma'][I_l, \alpha, \gamma]^{-1}[I_l, \alpha', \gamma']^{-1} = [I_l, 0, -4\,\mathrm{Im}\,{}^t\bar{\alpha}\alpha']$$

lead to the conclusion. Q.E.D.

COROLLARY 1.1. *The correspondence* $(x, y) \to \frac{1}{q(\Gamma)}\,\mathrm{Im}\,{}^t xy$ *gives a non-degenerate alternating* $\boldsymbol{R}$*-bilinear form* $Y' \times Y' \to \boldsymbol{R}$ *which induces* $L(\Gamma) \times L(\Gamma) \to \boldsymbol{Z}$. *That is,* $Y'/L(\Gamma)$ *is a canonically poralized abelian variety.*

COROLLARY 1.2. *The quotient space* $Y/\Gamma_1(\Gamma)$ *is the total space of the* C^**-bundle (determined by the above Riemann form) over the abelian variety* $Y'/L(\Gamma)$.

COROLLARY 1.3. *The quotient space* $(Y/\Gamma_1(\Gamma)) \cup (Y'/L(\Gamma))$ *added by the abelian variety* $Y'/L(\Gamma)$ *gives a smooth compactification of the quotient* $Y/\Gamma_1(\Gamma)$.

1.4.

LEMMA 1.4. *There exists a function* b_Γ: $W(\Gamma)\to Y'$ *such that* $(w|\beta)\in\pi_*(\Gamma)$ *if and only if* $w\in W(\Gamma)$ *and* $\beta\equiv b_\Gamma(w) \mod L(\Gamma)$. *The function* b_Γ *satisfies, for* $w, w'\in W(\Gamma)$,

$$b_\Gamma(ww')\equiv b_\Gamma(w)+wb_\Gamma(w') \quad \mathrm{mod}\, L(\Gamma).$$

PROOF. We have only to recall that

$$(w|\beta)(w'|\beta')=(ww'|\beta+w\beta'). \qquad \text{Q.E.D.}$$

LEMMA 1.5. *There exists a function* d_Γ: $W(\Gamma)\to \boldsymbol{R}$ *such that* $[w,\beta,\gamma]\in\Gamma$ *if and only if* $w\in W(\Gamma)$ *and* $\beta\equiv b_\Gamma(w) \mod L(\Gamma)$ *and* $\gamma\equiv d_\Gamma(w)+c_\Gamma(\alpha)-2\,\mathrm{Im}\,{}^t\bar{\alpha}b_\Gamma(w)$ $\mathrm{mod}\, q(\Gamma)Z$, *where* $\alpha=\beta-b_\Gamma(w)\in L(\Gamma)$. *The function* d_Γ *satisfies, for* $w, w'\in W(\Gamma)$,

$$d_\Gamma(ww')\equiv d_\Gamma(w)+d_\Gamma(w')-2\,\mathrm{Im}\,{}^t\overline{b_\Gamma(w)}wb_\Gamma(w')+c_\Gamma(\alpha)-2\,\mathrm{Im}\,{}^t\bar{\alpha}b_\Gamma(ww') \quad \mathrm{mod}\, q(\Gamma)Z,$$

where $\alpha=b_\Gamma(ww')-b_\Gamma(w)-wb_\Gamma(w')\in L(\Gamma)$.

PROOF. We have only to recall that

$$[w,\beta,\gamma][w',\beta',\gamma']=[ww',\beta+w\beta',\gamma+\gamma'-2\,\mathrm{Im}\,{}^t\bar{\beta}w\beta']. \qquad \text{Q.E.D.}$$

LEMMA 1.6. *We have, for* $w\in W(\Gamma)$ *and* $\alpha\in L(\Gamma)$,

$$c_\Gamma(w\alpha)\equiv c_\Gamma(\alpha)-4\,\mathrm{Im}\,{}^t\overline{b_\Gamma(w)}w\alpha \quad \mathrm{mod}\, q(\Gamma)\boldsymbol{Z}.$$

PROOF. Consider the exact sequence

$$1\to\Gamma_1(\Gamma)\to\Gamma\to W(\Gamma)\to 1.$$

Since the group $\Gamma_1(\Gamma)$ is invariant under the action of $W(\Gamma)$ and the lifting of $w\in W(\Gamma)$ to Γ is given by $[w, b_\Gamma(w), d_\Gamma(w)]$, the following identity leads to the assertion:

$$[w,b_\Gamma(w),d_\Gamma(w)][I_l,\alpha,c_\Gamma(\alpha)][w,b_\Gamma(w),d_\Gamma(w)]^{-1}=[I_l,w\alpha,c_\Gamma(\alpha)-4\,\mathrm{Im}\,{}^t\overline{b_\Gamma(w)}w\alpha],$$

Q.E.D.

1.5. On the contrary, starting from a finite subgroup W of $U(\Gamma)$, we shall construct discrete subgroups of G of locally finite volume:

PROPOSITION 1. *Let* W *be a finite subgroup of* $U(l)$. *Assume that there exists a* W*-invariant lattice* $L\subset Y'$. *Let* q *be a positive number such that* $(4/q)\mathrm{Im}\,{}^t\bar{\alpha}\alpha'\in\boldsymbol{Z}$ *for all* $\alpha,\alpha'\in L$, b *be a function* $W\to Y'$ *satisfying, for* $w, w'\in W$, $b(ww')\equiv b(w)+wb(w')$ $\mathrm{mod}\, L$, c *be a function* $L\to\boldsymbol{R}$ *satisfying, for* $\alpha,\alpha'\in L$ *and* $w\in W$,

$$c(\alpha+\alpha')\equiv c(\alpha)+c(\alpha')-2\,\mathrm{Im}\,{}^t\bar{\alpha}\alpha' \quad \mathit{mod}\ q\boldsymbol{Z},$$

and

$$c(w\alpha)\equiv c(\alpha)-4\,\mathrm{Im}\,{}^t\overline{b(w)}w\alpha \quad \mathit{mod}\ q\boldsymbol{Z},$$

and d be a function $W\to\boldsymbol{R}$ satisfying, for $w, w'\in W$, $d(ww')\equiv d(w)+d(w')-2\,\mathrm{Im}\,{}^t\overline{b(w)}\times wb(w')+c(\alpha)-2\,\mathrm{Im}\,{}^t\bar{\alpha}\,b(ww')$ mod $q\boldsymbol{Z}$, where $\alpha=b(ww')-b(w)-wb(w')\in L$. Then the group Γ defined by

$$\Gamma=\{[w,\beta,\gamma]\,|\,w\in W, \beta\equiv b(w) \quad \mathit{mod}\ L,$$
$$\gamma\equiv d(w)+c(\alpha)-2\,\mathrm{Im}\,{}^t\bar{\alpha}\,b(w) \quad \mathit{mod}\ q\boldsymbol{Z},\ \mathit{where}\ \alpha=\beta-b(w)\}$$

is a discrete subgroup of G of locally finite volume such that $W(\Gamma)=W$, $L(\Gamma)=L$, $q(\Gamma)=q$, $c_\Gamma=c$, $b_\Gamma=b$ and $d_\Gamma=d$.

PROOF. Easy.

§ 2. Conjecture and main theorem

2.1.

DEFINITION. $w\in U(l)$ is called a quasi-reflection if w is of finite order, $w\neq I_l$, and has exactly $l-1$ eigenvalues equal to 1. The unique nontrivial eigenvalue of w is denoted by $\mu(w)$. A root $r(w)$ of w is a base of the eigen space corresponding to the eigenvalue $\mu(w)$.

A quasi-reflection $w\in U(l)$ is represented by

$$w=I_l+(\mu(w)-1)r(w)\,{}^t\overline{r(w)}/{}^t\overline{r(w)}r(w).$$

DEFINITION. An element g of $E(Y')$ or G is called a quasi-reflection if g is of finite order, $g\neq$identity, and keeps a hyperplane in Y' or in D, respectively, pointwisely fixed.

The following is easy and well-known.

LEMMA 2.1. (i) *$(w|\beta)\in E(Y')$ is a quasi-reflection if and only if $w\in U(l)$ is a quasi-reflection and β is parallel to $r(w)$.* (ii) *$[w,\beta,\gamma]\in G$ is a quasi-reflection if and only if $(w|\beta)\in E(Y')$ is a quasi-reflection and*

$$\gamma=i|\beta|^2(\mu(w)+1)/(\mu(w)-1).$$

2.2. CONJECTURE. For a subgroup Γ of G of locally finite volume, the quotient space $D/\Gamma\cup\{P\}$ added by the point P is non-singular if and only if Γ is generated by quasi-reflections.

This conjecture is valid if $l=1$ (Yoshida-Hattori [11]). Here we give another proof. Mumford's criterion ([5]) asserts that a point P on a two dimensional normal variety is a regular point if and only if there exists a neighbourhood U of P such that $U-\{P\}$ is simply connected. The variety $D/\Gamma\cup\{P\}$ is normal, because it is the quotient by the finite group $W(\Gamma)$ of the variety obtained by blowing down an elliptic curve, with negative self-intersection number, on a non-singular surface (Corollary 1.3.). On the other hand, it is known that the quotient of a simply connected manifold by a properly discontinuous group is also simply connected if and only if it is generated by transformations with a fixed point ([1]). Remark that an element of G has a fixed point in D if and only if it is a quasi-reflection. Thus $D/\Gamma\cup\{P\}$ is non-singular if and only if Γ is generated by quasi-reflections.

2.3. DEFINITION. A quasi-reflection of order 2 is called a reflection. A finite group $W\subset GL(l, \boldsymbol{C})$ is called a Coxeter group if W is generated by l reflections $w_1, \cdots, w_l$ and the relations are generated by $(w_i w_j)^{m_{ij}}=1$, $1\leq i, j\leq l$ for some integers $\{m_{ij}\}$.

MAIN THEOREM. *For a discrete subgroup Γ of G of locally finite volume, if Γ is generated by reflections and if the point group $W(\Gamma)$ of Γ is an irreducible Coxeter group, then the variety $D/\Gamma\cup\{P\}$ is non-singular.*

§ 3. Crystallographic reflection groups whose point groups are Coxeter groups

If Γ is generated by (quasi-) reflections, then the crystallographic group $\pi_*(\Gamma)$ and the point group $W(\Gamma)$ is generated by (quasi-) reflections. Hereafter, we shall restrict ourselves to consider a finite Coxeter group as a point group. For a finite Coxeter group $W\subset U(l)$, we shall find every W-invariant lattice $L\subset Y'$ such that the crystallographic group $W\ltimes L$ is generated by reflections.

3.1.

LEMMA 3.1. *For an irreducible finite Coxeter group $W\subset GL(l, \boldsymbol{C})$, there exists a W-invariant lattice in $\boldsymbol{C}^l$ if and only if W belongs to one of the following types*: A_l $(l\geq 1)$, B_l $(l\geq 2)$, D_l $(l\geq 4)$, $G_2^{(6)}$, E_l $(l=6, 7, 8)$ *and* F_4.

PROOF. It is well-known that if the Coxeter diagram has a subgraph $\circ\overset{m}{\text{—}}\circ$ $(m\neq 2, 3, 4, 6)$, then there exists on W-invariant lattice. Q.E.D.

3.2. Since any finite subgroup of $GL(l, \boldsymbol{C})$ is conjugate to a subgroup of $U(l)$, we fix an integral matrix representation of each Coxeter group as follows. Let W be an irreducible finite Coxeter group in Lemma 3.1 and C the Cartan matrix of a Lie algebra of which Weyl group is isomorphic to the group W. If C is symmetric, then C is uniquely determined by W, up to reordering. If C is not symmetric, we choose a diagonal matrix D so that DC may be a symmetric positive definite matrix. (We put $D=I_l$ if C is symmetric.) Let $Y'_{DC}=\sum_{j=1}^{l}\boldsymbol{C}e_j$ be a complex vector space with an inner product DC, and $U(DC)$ be group consists of linear transformations which preserve DC: $U(DC)=\{w\in GL(l,\boldsymbol{C})\mid {}^t\bar{w}\,DC\,w=DC\}$. Recall that a reflection $w\in U(DC)$ with the root r is represented by $w=I_l-P(r)$, where $(1/2)P(r)$ is the orthogonal projection of Y'_{DC} to $\boldsymbol{C}r$.

Then the representation $W\to U(DC)$ defined by $w_j\mapsto I_l-P(e_j)$ gives a faithfull representation of W. Furthermore we have $P(e_j)=e_j{}^te_jC$ where we identified e_j and ${}^t(0,\cdots,0,\overset{\overset{j}{\smile}}{1},0,\cdots,0)$. The image of this embedding shall be denoted by $W(C)$ and called the Coxeter group with Cartan matrix C.

3.3.

LEMMA 3.2. *If $w\in U(DC)$ is a reflection with the root r and if $L\subset Y'_{DC}$ is a w-invariant lattice, then we have $P(r)L\subset\boldsymbol{C}r\cap L$.*

PROOF. Obvious.

LEMMA 3.3. *Let L be a $W(C)$-invariant lattice, and assume that $L\cap\boldsymbol{C}e_j=(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j$.* (i) *If the Coxeter diagram of $W(C)$ has a subgraph ⓙ—ⓚ, then $L\cap\boldsymbol{C}e_k=(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k$.* (ii) *If the Coxeter diagram of $W(C)$ has a subgraph ⓙ═ⓚ, and if $C_{jk}=-2$, $C_{kj}=-1$, then we have $L\cap\boldsymbol{C}e_k=2(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k$ or $2(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k+\boldsymbol{Z}\omega e_k$ $(\omega=1,\tau,1+\tau)$ or $(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k$.*

PROOF. (i) Since $C_{jk}=C_{kj}=-1$, we have $P(e_k)(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j=(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k$ and $P(e_j)(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k=(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j$. Thus Lemma 3.2 leads to the assertion. To show the claim (ii), we have only to remark that $P(e_k)(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j=2(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k$ and $P(e_j)(\boldsymbol{Z}+\tau\boldsymbol{Z})e_k=(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j$. Q.E.D.

3.4.

PROPOSITION 2. *Every crystallographic reflection group on Y' whose point group is a Coxeter group is conjugate in $A(Y')$ to one of the following groups. Since every group is a semi-direct product $W(C)\ltimes L$ of the point group $W(C)$ and the lattice L, we list up C and L.*

Type of $W(C)$	Cartan matrix C	Lattice L
B_l $(l\geq 2)$	$C(B_l)=\begin{bmatrix}2&-1&&\\-1&\ddots&&\\&&2&-1\\&&-2&2\end{bmatrix}$	$L_1(B_l)=\sum_{j=1}^{l-1}L(\tau)e_j+2L(\tau)e_l$ $L_2(B_l)=\sum_{j=1}^{l-1}L(\tau)e_j+(\boldsymbol{Z}+2\tau\boldsymbol{Z})e_l$ $L_3(B_l)=\sum_{j=1}^{l}L(\tau)e_j$
F_4	$C(F_4)=\begin{bmatrix}2&-1&&\\-1&2&-1&\\&-2&2&-1\\&&-1&2\end{bmatrix}$	$L_1(F_4)=\sum_{j=1}^{2}L(\tau)e_j+\sum_{j=3}^{4}2L(\tau)e_j$ $L_2(F_4)=\sum_{j=1}^{2}L(\tau)e_j+\sum_{j=3}^{4}(\boldsymbol{Z}+2\tau\boldsymbol{Z})e_j$
$G_2^{(6)}$	$C(G_2^{(6)})=\begin{bmatrix}2&-1\\-3&2\end{bmatrix}$	$L_1(G_2^{(6)})=L(\tau)e_1+3L(\tau)e_2$ $L_2(G_2^{(6)})=L(\tau)e_1+(\boldsymbol{Z}+3\tau\boldsymbol{Z})e_2$
A_l $(l\geq 1)$	$C(A_l)$	$\sum_{j=1}^{l}L(\tau)e_j$
D_l $(l\geq 4)$	$C(D_l)$	$\sum_{j=1}^{l}L(\tau)e_j$
E_l $(l=6,7,8)$	$C(E_l)$	$\sum_{j=1}^{l}L(\tau)e_j$

Here $L(\tau)=\boldsymbol{Z}+\tau\boldsymbol{Z}$ $(\operatorname{Im}\tau>0)$. *We omitted the Cartan matrix of* A_l, D_l *and* E_l, *which are uniquely determined.*

REMARK 3.1. For the types B_l, F_4 and $G_2^{(6)}$, alternative choise of a Cartan matrix may be permitted. The corresponding lattices are the followings:

B_l $(l\geq 2)$	$C'(B_l)=\begin{bmatrix}2&-1&&\\-1&\ddots&&\\&&2&-2\\&&-1&2\end{bmatrix}$	$L_1'(B_l)=\sum_{j=1}^{l}L(\tau)e_j$ $L_2'(B_l)=\sum_{j=1}^{l-1}L(\tau)e_j+\left(\boldsymbol{Z}+\frac{\tau}{2}\boldsymbol{Z}\right)e_l$ $L_3'(B_l)=\sum_{j=1}^{l-1}L(\tau)e_j+\frac{1}{2}L(\tau)e_l$
F_4	$C'(F_4)=\begin{bmatrix}2&-1&&\\-1&2&-2&\\&-1&2&-1\\&&-1&2\end{bmatrix}$	$L_1'(F_4)=\sum_{j=1}^{4}L(\tau)e_j$ $L_2'(F_4)=\sum_{j=1}^{2}L(\tau)e_j+\sum_{j=3}^{4}\left(\boldsymbol{Z}+\frac{\tau}{2}\boldsymbol{Z}\right)e_j$
$G_2^{(6)}$	$C'(G_2^{(6)})=\begin{bmatrix}2&-3\\-1&2\end{bmatrix}$	$L_1'(G_2^{(6)})=L(\tau)e_1+L(\tau)e_2$ $L_2'(G_2^{(6)})=L(\tau)e_1+(\boldsymbol{Z}+(\tau/3)\boldsymbol{Z})e_2.$

We have that the groups $W(C(B_l))\ltimes L_k(B_l)$, $W(C(F_4))\ltimes L_k(F_4)$ and $W(C(G_2^{(6)}))\ltimes L_k(G_2^{(6)})$ are conjugate to the groups $W(C'(B_l))\ltimes L_k'(B_l)$, $W(C'(F_4))\ltimes L_k'(F_4)$ and $W(C'(G_2^{(6)})\ltimes L_k'(G_2^{(6)})$ respectively, by $\begin{bmatrix}1&&\\&1&\\&&2\end{bmatrix}$, $\begin{bmatrix}1&&&\\&1&&\\&&2&\\&&&2\end{bmatrix}$ and $\begin{bmatrix}1&\\&3\end{bmatrix}$ respectively.

REMARK 3.2. The group $W(C(B_2))\ltimes L_1(B_2)$ is conjugate to the group $W(C(B_2))\ltimes L_3(B_2)$. Thus for $l=2$, we have five crystallographic groups $W(C(B_2))\ltimes L_1(B_2)$, $W(C(B_2))\ltimes L_2(B_2)$, $W(C(G_2^{(6)})\ltimes L_1(G_2^{(6)})$, $W(C(G_2^{(6)})\ltimes L_2(G_2^{(6)})$ and $W(C(A_2))\ltimes L$ which correspond to the groups $(2.1)_0$, $(2.1)_1$, $(6.6)_0$, $(6.6)_1$ and $(3.3)_0$ in [9; Theorem 5.1], respectively.

PROOF OF PROPOSITION 2. It is known that if the point group W of a crystallographic reflection group Γ is a Coxeter group, then Γ is the semi-direct product of W and the lattice of Γ.

LEMMA 3.4. *Let W be a Coxeter group and L a W-invariant lattice, then the group $\Gamma=W\ltimes L$ is generated by reflections if and only if*

$$L=\sum_{w\in R_W} L\cap Cr(w),$$

where R_W is the set of reflections in W and $r(w)$ is a root of w.

PROOF. The set of reflections in Γ is given by

$$R=\{(w\,|\,a)\,|\,w\in R_W,\ a\in L,\ a\in Cr(w)\}.$$

The set of all parallel displacements in the group generated by R is given by

$$\{(1\,|\,a)\,|\,a\in L,\ a\in r(w) \text{ for some } w\in R_W\}.$$

Thus we have the lemma. Q.E.D.

Let $\tilde{L}$ be a $W(C)$-invariant lattice.

LEMMA 3.5. *If we transform $\tilde{L}$ by multiplying a suitable constant to $\tilde{L}$, then $\tilde{L}$ contains one of the $W(C)$-invariant lattices stated in the proposition.*

PROOF. By Lemma 3.2 we can assume that $\tilde{L}\cap Ce_1=(\boldsymbol{Z}+\tau\boldsymbol{Z})e_1$ for some τ ($\operatorname{Im}\tau>0$). We repeatedly apply Lemma 3.3 (i). Since we have

$$\frac{1}{\tau}\{(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j+(2\boldsymbol{Z}+\tau\boldsymbol{Z})e_k\}=\left(\boldsymbol{Z}+\frac{-1}{\tau}\boldsymbol{Z}\right)e_j+\left(\boldsymbol{Z}+\frac{-2}{\tau}\boldsymbol{Z}\right)e_k$$

and

$$\frac{1}{1+\tau}\{(\boldsymbol{Z}+\tau\boldsymbol{Z})e_j+(2\boldsymbol{Z}+2\tau\boldsymbol{Z}+(1+\tau)\boldsymbol{Z})e_k\}=\left(\boldsymbol{Z}+\frac{-1}{1+\tau}\boldsymbol{Z}\right)e_j+\left(\boldsymbol{Z}+\frac{-2}{1+\tau}\boldsymbol{Z}\right)e_k,$$

we apply Lemma 3.3 (ii) by putting $\omega=1$. Then the lemma is proved except for the types F_4 and $G_2^{(6)}$. The group $W(C(F_4))\ltimes\left(\sum_{j=1}^{4}L(\tau)e_j\right)$ is conjugate to $W(C'(F_4))\ltimes L_1'(F_4)$ by $\begin{bmatrix} &&&1\\ &&1&\\ &1&&\\ 1&&& \end{bmatrix}$. Thus Remark 3.1 proves the lemma for F_4. The analogous proof is available for the type $G_2^{(6)}$. Q.E.D.

Let L be a lattice in the proposition such that $L \subset \tilde{L}$. Notice that we have $L = \sum_{j=1}^{l} L \cap Ce_j$ and, by the proof of Lemma 3.5, $\tilde{L} \cap Ce_j = L \cap Ce_j$. It is known that every $w \in R_{W(C)}$ is conjugate to one of the fundamental reflections w_j $(j=1, \cdots, l)$. That is, there exist $x \in W$ and j such that $w = x w_j x^{-1}$ and so, we have $r(w) = x(e_j)$ and $\tilde{L} \cap Cr(w) = x(\tilde{L} \cap Ce_j)$. Suppose that the group $W(C) \ltimes \tilde{L}$ is generated by reflections, then by Lemma 3.4, we have

$$\tilde{L} = \sum_{w \in R_{W(C)}} \tilde{L} \cap Cr(w) \subset \sum_{x \in W} \sum_{j=1}^{l} x(\tilde{L} \cap Ce_j) = W(\sum L \cap Ce_j) = W(L) = L.$$

The proof of the proposition is now complete. Q.E.D.

§4. Reflection group Γ in G whose point group is a Coxeter group

The goal of this chapter is to find every discrete subgroup of G of locally finite volume generated by reflections such that its point group is an irreducible Coxeter group.

4.1. Let C be a Cartan matrix and D be a diagonal matrix such that DC is symmetric and positive definite. We put

$$[w, \beta, \gamma]_{DC} = \begin{bmatrix} 1 & 2i\,{}^t\bar{\beta} DCw & \gamma + i\,{}^t\bar{\beta} DC\beta \\ 0 & w & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$G_{DC} = \{[w, \beta, \gamma]_{DC} \mid w \in U(DC), \quad \beta \in Y'_{DC} \cong \boldsymbol{C}^l, \quad \gamma \in \boldsymbol{R}\}.$$

If K_{DC} is an $l \times l$ matrix such that ${}^t\bar{K}_{DC} K_{DC} = DC$, then the correspondence

$$[w, \beta, \gamma]_{DC} \mapsto [K_{DC} w K_{DC}^{-1}, K_{DC}\beta, \gamma] = \begin{bmatrix} 1 & & \\ & K_{DC} & \\ & & 1 \end{bmatrix} [w, \beta, \gamma]_{DC} \begin{bmatrix} 1 & & \\ & K_{DC}^{-1} & \\ & & 1 \end{bmatrix}$$

gives the isomorphism $K_{DC} : G_{DC} \to G$. We shall reformulate Proposition 1 for b identically zero and for $W(C) \subset U(DC)$ instead of $W \subset U(l)$.

LEMMA 4.1. *Let* $W(C) \subset U(DC)$ *be a Coxeter group and* $L \subset Y'_{DC}$ *a* $W(C)$-*invariant lattice. Let q be a positive number such that* $(4/q)\mathrm{Im}\,{}^t\bar{\alpha} DC\alpha' \in \boldsymbol{Z}$, $\alpha, \alpha' \in L$, c_{DC} *a function* $L \to \boldsymbol{R}$ *such that*

$$c_{DC}(\alpha + \alpha') \equiv c_{DC}(\alpha) + c_{DC}(\alpha') - 2\,\mathrm{Im}\,{}^t\bar{\alpha} DC\alpha' \quad \mathrm{mod}\ q\boldsymbol{Z}$$
$$c_{DC}(w\alpha) \equiv c_{DC}(\alpha) \quad \mathrm{mod}\ q\boldsymbol{Z}, \quad \alpha, \alpha' \in L, \quad w \in W(C),$$

and d_{DC} *a function* $W(C) \to R$ *such that*

$$\begin{aligned} d_{DC}(ww') &\equiv d_{DC}(w) + d_{DC}(w') && \mathrm{mod}\ q\boldsymbol{Z}, \\ 2d_{DC}(w_j) &\equiv 0 && \mathrm{mod}\ q\boldsymbol{Z}, \\ m_{jk}(d_{DC}(w_j) + d_{DC}(w_k)) &\equiv 0 && \mathrm{mod}\ q\boldsymbol{Z}. \end{aligned}$$

Then the group defined by

$$\Gamma_{DC} = \{[w, \alpha, \gamma]_{DC} \mid w \in W(C),\ \alpha \in L,\ \gamma \equiv c_{DC}(\alpha) + d_{DC}(w) \quad \mathrm{mod}\ q\boldsymbol{Z}\}$$

is transformed by K_{DC} *to a discrete subgroup of* G *of locally finite volume.*

4.2. If the group Γ_{DC} is a reflection group, so is the crystallographic group $W(C) \ltimes L$. Thus we shall consider each subgroup Γ_{DC} such that the point group $W(C)$ and the lattice L are in the list of Proposition 2. For a Cartan matrix C, we choose and fix a diagonal matrix D so that DC is equal to the following symmetric matrix:

Type	DC
$B_l \quad (l \geq 2)$	$\begin{bmatrix} 2 & -1 & & & \\ -1 & \ddots & & & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$
F_4	$\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 1 & -1/2 \\ & & -1/2 & 1 \end{bmatrix}$
$G_2^{(6)}$	$\begin{bmatrix} 2 & -1 \\ -1 & 2/3 \end{bmatrix}$
A_1	1
$A_l \quad (l \geq 2), D_l \quad (l \geq 4)$ $E_l \quad (l = 6, 7, 8)$	C

For a $W(C)$-invariant lattice L, we define a positive number q_0 by

$$q_0 = \mathrm{Max}\left\{ q \,\middle|\, \frac{4}{q} \operatorname{Im} {}^t\bar{\alpha} DC \alpha' \in \boldsymbol{Z}, \text{ for all } \alpha, \alpha' \in L \right\}.$$

LEMMA 4.2. *For* $L_1(B_2)$, *we have* $q_0 = 8 \operatorname{Im} \tau$. *For any other lattice in Proposition* 2, *we have* $q_0 = 4 \operatorname{Im} \tau$.

PROOF. Easy.

4.3. We shall denote by R_W the set of all reflections in W. For a $W(C)$-

invariant lattice L, a natural number p and the functions c_{DC} and d_{DC}, satisfying the conditions in Proposition 1, we put

$$\Gamma(C, L, p, c_{DC}, d_{DC}) = \{[w, \alpha, \gamma]_{DC} \mid w \in W(C), \alpha \in L, \gamma \equiv c_{DC}(\alpha) + d_{DC}(w) \mod q\boldsymbol{Z}, q = q_0/p\}$$

and

$$R(C, L, p, c_{DC}, d_{DC}) = \{[w, \alpha, 0]_{DC} \mid w \in R_{W(C)}, \alpha \in L \cap \boldsymbol{C}r(w), c_{DC}(\alpha) + d_{DC}(w) \equiv 0 \mod q\boldsymbol{Z},\ q = q_0/p\}.$$

LEMMA 4.3. *The set $R(C, L, p, c_{DC}, d_{DC})$ coincides with the set of all reflections in the group $\Gamma(C, L, p, c_{DC}, d_{DC})$.*

PROOF. Direct consequence of Lemma 2.1 and Lemma 4.1. Q.E.D.

For each pair $(W(C), L)$ in Proposition 2, we seek for functions c_{DC} and d_{DC} and a natural number p so that the set $R(C, L, p, c_{DC}, d_{DC})$ generates the group $\Gamma(C, L, p, c_{DC}, d_{DC})$.

LEMMA 4.4. *If the set $R_\Gamma = R(C, L, p, c_{DC}, d_{DC})$ generates the group $\Gamma = \Gamma(C, L, p, c_{DC}, d_{DC})$, then for every $w \in R_{W(C)}$, there exists $\alpha \in L$ such that $[w, \alpha, 0]_{DC} \in R_\Gamma$.*

PROOF. If R_Γ generates Γ, then the set $\{w \in W(C) \mid [w, \alpha, 0]_{DC} \in R_\Gamma$, for some $\alpha \in L\}$ generates $W(C)$. On the other hand, following is well-known. If $\{w_1, \cdots, w_l\}$ is any system of generating reflections of a finite reflection group W, then every $w \in R_W$ is conjugate to one of w_j's. Completion of the proof is now immediate.

Q.E.D.

4.4. Till the end of this chapter, we shall omit the subscript DC if there is no danger of confusion. e.g. $c = c_{DC}$, $d = d_{DC}$, $[w, \alpha, \gamma] = [w, \alpha, \gamma]_{DC}$, etc.

LEMMA 4.5. *Let $C, DC, L, q = q_0/p, c = c_{DC}$ and $d = d_{DC}$ as above. If the Coxeter diagram of $W(C)$ has a subdiagram ⓙ—ⓚ, then* (i) *$d(w_j) \equiv d(w_k) \mod q\boldsymbol{Z}$,* (ii) *if furthermore we have*

$$L \cap \boldsymbol{C}e_j = (\omega_1\boldsymbol{Z} + \omega_2\boldsymbol{Z})e_j,$$
$$L \cap \boldsymbol{C}e_k = (\omega_1\boldsymbol{Z} + \omega_2\boldsymbol{Z})e_k,$$

then $c(\omega_\nu e_j) \equiv c(\omega_\nu e_k) \equiv 0 \mod q\boldsymbol{Z}$ $(\nu = 1, 2)$.

PROOF. Since $w_j = (w_j w_k w_j) w_k (w_j w_k w_j)$, we have

$$d(w_j) \equiv d(w_k) + 2d(w_j w_k w_j) \mod q\boldsymbol{Z}.$$

Lemma 4.1 asserts $2d(w_j w_k w_j) \equiv 0 \mod q\boldsymbol{Z}$. These prove (i). Recall that the function c is $W(C)$-invariant mod $q\boldsymbol{Z}$ (Lemma 4.1). We have

$$\begin{aligned} c(\omega_\nu e_j) &\equiv c(\omega_\nu w_k e_j) && \bmod q\boldsymbol{Z} \\ &\equiv c(\omega_\nu(e_j+e_k)) && \bmod q\boldsymbol{Z} \\ &\equiv c(\omega_\nu e_j)+c(\omega_\nu e_k) && \bmod q\boldsymbol{Z} \end{aligned}$$

and so $c(\omega_\nu e_j)\equiv 0 \bmod q\boldsymbol{Z}$. Q.E.D.

LEMMA 4.6. *Let $(W(C), L)$ be in the list of Proposition 2. If the Coxeter group $W(C)$ is of type A_l $(l\geq 3)$, D_l $(l\geq 4)$ or E_l $(l=6,7,8)$, then the set $R_\Gamma = R(C, L, p, c, d)$ generates the group $\Gamma=\Gamma(C, L, p, c, d)$ if and only if $p=1$, $d\equiv 0$ mod $q\boldsymbol{Z}$ and $c(e_j)\equiv c(\tau e_j)\equiv 0$ mod $q\boldsymbol{Z}$.*

PROOF. Put

$$R_\Gamma(w_j)=\{[w, \alpha, 0]\in R_\Gamma \mid w=w_j\}.$$

Then we have

$$R_\Gamma(w_j)=\{[w_j, (n+m\tau)e_j, 0] \mid c((n+m\tau)e_j)+d(w_j)\equiv 0 \quad \bmod q\boldsymbol{Z},\ n, m\in \boldsymbol{Z}\}.$$

By Lemma 4.2 and 4.5 we have

$$c((n+m\tau)e_j)\equiv nc(e_j)+mc(\tau e_j)-2nm \operatorname{Im} C_{jj}\tau\equiv 0 \quad \bmod q\boldsymbol{Z},$$

because $C_{jj}=2$ and $q=4 \operatorname{Im}\tau/p$. On the other hand, by Lemma 4.4, the set $R_\Gamma(w_j)$ is not empty. Thus we have $d(w_j)\equiv 0$ mod $q\boldsymbol{Z}$, $1\leq j\leq l$. This implies that

$$R_\Gamma=\{[w, \alpha, 0] \mid w\in R_{W(C)},\ \alpha\in L\cap Cr(w)\}.$$

In particular R_Γ is independent of p $(p=1, 2, \cdots)$ and generates the group Γ for $p=1$. Q.E.D.

LEMMA 4.7. *If the Coxeter group $W(C)$ is of type F_4, then the group $\Gamma(C(F_4), L_\nu(F_4), p, c, d)$ $(\nu=1, 2)$ is generated by reflections if and only if $p=1, d\equiv 0$ mod $q\boldsymbol{Z}$ and*

$$\begin{aligned} &\text{for } \nu=1, && c(e_j)\equiv c(\tau e_j)\equiv 0 && \bmod q\boldsymbol{Z} && j=1, 2, \\ & && c(2e_j)\equiv c(2\tau e_j)\equiv 0 && \bmod q\boldsymbol{Z} && j=3, 4, \\ &\text{for } \nu=2, && c(e_j)\equiv c(\tau e_j)\equiv 0 && \bmod q\boldsymbol{Z} && j=1, 2, \\ & && c(e_j)\equiv c(2\tau e_j)\equiv 0 && \bmod q\boldsymbol{Z} && j=3, 4. \end{aligned}$$

PROOF. Analogous to that of Lemma 4.6. Q.E.D.

LEMMA 4.8. *If the Coxeter group $W(C)$ is of type B_l $(l\geq 3)$, then the group*

$\Gamma(C(B_l),\ L_\nu(B_l),\ p, c, d)$ $(\nu=1,2,3)$ is generated by reflection if and only if

$$\begin{aligned} &c(e_j)\equiv c(\tau e_j)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z}, && 1\leq j\leq l-1,\\ &d(w_j)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z}, && 1\leq j\leq l-1, \end{aligned}$$

and one of the following conditions is satisfied:

(1) $\nu=1,\ p=1,\ d(w_l)\equiv c(2e_l)\equiv c(2\tau e_l)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z},$

(2) $\nu=2,\ p=1,\ d(w_l)\equiv c(e_l)\equiv c(2\tau e_l)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z},$

(3-1) $\nu=3,\ p=1,\ d(w_l)\equiv c(e_l)\equiv c(\tau e_l)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z},$

(3-2) $\nu=3,\ p=2,\ d(w_l)\equiv c(e_l)\equiv c(\tau e_l)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z},$

(3-3) $\nu=3,\ p=1,\ d(w_l)\equiv 0,\ \ c(e_l)\equiv 0,\ \ c(\tau e_l)\equiv q/2 \quad \mathit{mod}\ q\boldsymbol{Z},$

(3-4) $\nu=3,\ p=1,\ d(w_l)\equiv 0,\ \ c(e_l)\equiv q/2,\ \ c(\tau e_l)\equiv 0 \quad \mathit{mod}\ q\boldsymbol{Z},$

(3-5) $\nu=3,\ p=1,\ d(w_l)\equiv c(e_l)\equiv c(\tau e_l)\equiv q/2 \quad \mathit{mod}\ q\boldsymbol{Z}.$

PROOF. By the same reasoning as Lemma 4.6, we conclude that $c(e_j)\equiv c(\tau e_j)\equiv d(w_j)\equiv 0$ mod $q\boldsymbol{Z}$, $1\leq j\leq l-1$. Moreover we see that $2c(\omega e_l)\equiv 0$ mod $q\boldsymbol{Z}$ for $\omega e_l\in L$, in fact we have $c(\omega e_l)\equiv c(w_l\omega e_l)\equiv -c(\omega e_l)$ mod $q\boldsymbol{Z}$. Thus for each lattice L_ν $(\nu=1, 2, 3)$, we seek for every possible values p $(p=1,2,\cdots)$, $d(w_l)\equiv 0, q/2$ and $c(\omega e_l)\equiv 0, q/2$ mod $q\boldsymbol{Z}$ where $\omega e_l\in L\cap Ce_l$, so that the group Γ is generated by reflections. We shall study the set $R_\Gamma(w_l)$. Recall that

$$R_\Gamma(w_l)=\{[w_l, \omega e_l, 0]\,|\,\omega e_l\in L\cap Ce_l,\ c(\omega e_l)+d(w_l)\equiv 0 \quad \mathrm{mod}\, q\boldsymbol{Z}\}.$$

For each lattice L_ν $(\nu=1,2,3)$, we calculate the value $c(\omega e_l)$:

1) $$\begin{aligned} c(2(n+m\tau)e_l)&\equiv nc(2e_l)+mc(2\tau e_l)-2nm4(DC)_{ll}\,\mathrm{Im}\,\tau\\ &\equiv nc(2e_l)+mc(2\tau e_l) \quad \mathrm{mod}\, q\boldsymbol{Z} \end{aligned}$$

2) $$\begin{aligned} c((n+2m\tau)e_l)&\equiv nc(e_l)+mc(2\tau e_l)-2nm2(DC)_{ll}\,\mathrm{Im}\,\tau\\ &\equiv nc(e_l)+mc(2\tau e_l) \quad \mathrm{mod}\, q\boldsymbol{Z}, \end{aligned}$$

3) $$\begin{aligned} c((n+m)e_l)&\equiv nc(e_l)+mc(\tau e_l)-2nm(DC)_{ll}\,\mathrm{Im}\,\tau\\ &\equiv nc(e_l)+mc(\tau e_l)-2nm\,\mathrm{Im}\,\tau \quad \mathrm{mod}\, q\boldsymbol{Z}. \end{aligned}$$

Thus for each lattice L_ν $(\nu=1,2,3)$, the set $R_\Gamma(w_l)$ is represented by

1) $\{[w_l, 2(n+m\tau)e_l, 0]\,|\,d(w_l)+nc(2e_l)+mc(2\tau e_l)\equiv 0 \quad \mathrm{mod}\, q\boldsymbol{Z}\}$, for every p,

2) $\{[w_l, (n+2m\tau)e_l, 0]\,|\,d(w_l)+nc(e_l)+mc(2\tau e_l)\equiv 0 \quad \mathrm{mod}\, q\boldsymbol{Z}\}$, for every p,

3) $$\begin{aligned} &\{[w_l, (n+m\tau)e_l, 0]\,|\,d(w_l)+nc(e_l)+mc(\tau e_l)-(q/2)nmp\equiv 0 \quad \mathrm{mod}\, q\boldsymbol{Z}\}\\ &=\{[w_l, (n+m\tau)e_l, 0]\,|\,d(w_l)+nc(e_l)+mc(\tau e_l)-nm(q/2)\equiv 0 \quad \mathrm{mod}\, q\boldsymbol{Z}\} \quad \text{if } p \text{ is odd},\\ &\{[w_l, (n+m\tau)e_l, 0]\,|\,d(w_l)+nc(e_l)+mc(\tau e_l)\equiv 0 \ \mathrm{mod}\ q\boldsymbol{Z}\} \quad \text{if } p \text{ is even}. \end{aligned}$$

If Γ is generated by reflections, then the set

$$\{\omega e_l - w' e_l \mid [w_l, \omega e_l, 0], [w_l, \omega' e_l, 0] \in R_\Gamma(w_l)\}$$

must generate the lattice $L \cap \boldsymbol{C} e_l$. Now we have only to calculate this set to show the lemma. Q.E.D.

4.5.

LEMMA 4.9. *Let x be an element of Y'_{DC} such that $x - wx \in L$ for all $w \in W(C)$. If we take conjugate by $[I_l, x, 0]$, the group $\Gamma(C, L, p, c, d)$ is transformed into the group $\Gamma(C, L, p, c', d')$, where*

$$\begin{aligned} &c'(\alpha) \equiv c(\alpha) - 4\,\mathrm{Im}\,{}^t\bar{x}DC\alpha \quad \mathrm{mod}\ q\boldsymbol{Z}, \ \alpha \in L \\ &d'(w) \equiv d(w) - c(x - wx) - 4\,\mathrm{Im}\,{}^t\bar{x}DC(wx) \quad \mathrm{mod}\ q\boldsymbol{Z}, \ w \in W(C). \end{aligned}$$

PROOF. The definition of the functions c and d with the identities

$$[I_l, x, 0]\,[I_l, \alpha, \gamma]\,[I_l, x, 0]^{-1} = [I_l, \alpha, \gamma - 4\,\mathrm{Im}\,{}^t\bar{x}(DC)d]$$

and

$$[I_l, x, 0]\,[w, \alpha, \gamma]\,[I_l, x, 0]^{-1} = [w, \alpha + x - wx, \gamma - 2\,\mathrm{Im}\,{}^t\bar{x}(DC)\alpha + 2\,\mathrm{Im}\,{}^t\overline{(x+\alpha)}DC(wx)],$$

lead to the assertion. Q.E.D.

LEMMA 4.10. *Among the reflection groups obtained in Lemma* 4.8, *the groups corresponding to* (3-3), (3-4) *and* (3-5) *are conjugate in $\hat{G}$ to the group corresponding to* (3-1).

PROOF. Set $$x = \frac{1}{2}\left(e_1 + 2e_2 + \cdots + (l-1)e_{l-1} + \frac{l}{2}e_l\right).$$

Remark that we have

$$x - w_j x = \begin{cases} 0 & 1 \le j \le l-2, \\ e_j & j = l-1,\ l, \end{cases}$$

$$\mathrm{Im}\,{}^t\bar{x}(DC)\alpha = 0 \quad \text{for all} \quad \alpha \in \sum_{j=1}^{l} \boldsymbol{R} e_j,$$

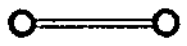
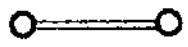
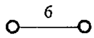
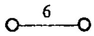
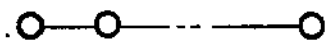
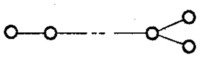
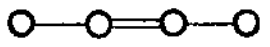
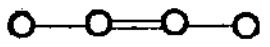
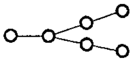
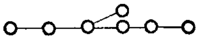
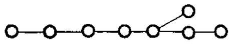
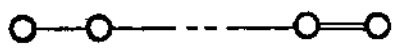
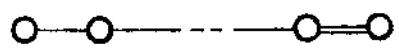
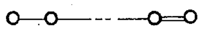
$$\mathrm{Im}\,{}^t\bar{x}(DC)(w_j x) = 0,$$

and

$$4\,\mathrm{Im}\,{}^t\bar{x}(DC)\tau e_j = \begin{cases} 0 & 1 \le j \le l-1, \\ 2\,\mathrm{Im}\,\tau & j = l. \end{cases}$$

If we take conjugate by $[I_l, x, 0]$, then the group with the condition (3-1) and (3-4) are transformed to those with (3-3) and (3-5), respectively. In the same way, we

Table I

Name		Cartan matrix C	Coxeter diagram of $W(C)$	Lattice L	p
$\Gamma(A_2^{(2)})$	1	$C(A_1)$		$L(\tau)e_1$	1
$\Gamma(A_2^{(1)})$	1	$C(A_1)$		$L(\tau)e_1$	2
$\Gamma(C_2^{(1)})$	2	$C(B_2)$		$L_2(B_2)$	1
$\Gamma(A_4^{(2)})$	2	$C(B_2)$		$L_3(B_2)$	1
$\Gamma(A_3^{(2)})$	2	$C(B_2)$		$L_3(B_2)$	2
$\Gamma(D_4^{(3)})$	2	$C(G_2^{(6)})$		$L_1(G_2^{(6)})$	1
$\Gamma(G_2^{(1)})$	2	$C(G_2^{(6)})$		$L_1(G_2^{(6)})$	1
$\Gamma(A_l^{(1)})$	l $(l\geq 2)$	$C(A_l)$		$\sum_{j=1}^{l} L(\tau)e_j$	1
$\Gamma(D^{(1)})$	l $(l\geq 4)$	$C(D_l)$		$\sum_{j=1}^{l} L(\tau)e_j$	1
$\Gamma(E_6^{(2)})$	4	$C(F_4)$		$L_1(F_4)$	1
$\Gamma(F_4^{(1)})$	4	$C(F_4)$		$L_2(F_4)$	1
$\Gamma(E_6^{(1)})$	6	$C(E_6)$		$\sum_{j=1}^{6} L(\tau)e_j$	1
$\Gamma(E_7^{(1)})$	7	$C(E_7)$		$\sum_{j=1}^{7} L(\tau)e_j$	1
$\Gamma(E_8^{(1)})$	8	$C(E_8)$		$\sum_{j=1}^{8} L(\tau)e_j$	1
$\Gamma(A_{2l-1}^{(2)})$	l $(l\geq 3)$	$C(B_l)$		$L_1(B_l)$	1
$\Gamma(B_l^{(1)})=\Gamma(C_l^{(1)})$	l $(l\geq 3)$	$C(B_l)$		$L_2(B_l)$	1
$\Gamma(A_{2l}^{(2)})$	l $(l\geq 3)$	$C(B_l)$		$L_3(B_l)$	1
$\Gamma(D_{l-1}^{(2)})$	l $(l\geq 3)$	$C(B_l)$		$L_3(B_l)$	2

take conjugate by $[I_l, \tau x, 0]$ to convert (3-1) into (3-4). Q.E.D.

4.6.

THEOREM 1. *Let Γ be a discrete subgroup of G of locally finite volume such that Γ is generated by reflections and that the point group $W(\Gamma)$ of Γ is an irreducible Coxeter group. Then Γ is conjugate in $\hat{G}$ to one of the groups in*

Table I. *Each low denotes the group* $\Gamma(C, L, p, c, d)$ *such that the function* c *and* d *satisfy the following conditions*: $d(w)\equiv 0 \bmod qZ$ $w\in W(C)$, *and* $c(\omega_1 e_j)\equiv c(\omega_2 e_j)\equiv 0 \bmod qZ$, *where* $L\cap Ce_j=(\omega_1 Z+\omega_2\tau Z)e_j$, $\omega_1, \omega_2\in \boldsymbol{R}$.

REMARK. The reason for these curious naming will be recognized in Theorem 2.

REMARK. The groups $\Gamma(A_1^{(1)})$ and $\Gamma(A_2^{(2)})$ are conjugate to the groups $\Gamma_{II}(\tau;1,0,0;0)$ and $\Gamma_{II}(\tau;2,0,0;0)$ in [11], respectively.

PROOF. We have already completed the proof of the theorem for $l\geq 3$. A proof for $l=1$ is covered in [11]. By Remark 3.2, we can modify Lemma 4.6 and 4.8 so that they are available for the types A_2 and B_2, respectively. Only the remaining type is $G_2^{(6)}$, which we treat separately. Since the proof is analogous to those of another group, we omit it. Q.E.D.

§5. Weyl groups of Euclidean Lie algebras

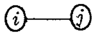
5.1. In this and the next section, we shall review some fundamental definitions and facts about the generalized Cartan matrices of Euclidean type and their Weyl groups, and prepare some notations.

DEFINITION. An $(l+1)\times(l+1)$ matrix $A=(A_{ij})_{0\leq i,j\leq l}$ is called a generalized Cartan matrix if (i) $A_{ij}\in \boldsymbol{Z}$, (ii) $A_{jj}=2$, $A_{ij}\leq 0$ for $i\neq j$ and (iii) $A_{ij}=0$ if and only if $A_{ji}=0$.

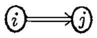
DEFINITION. We say that a generalized Cartan matrix A is of Euclidean type if (i) it is indecomposable, (ii) $\det A=0$, (iii) for each k $(0\leq k\leq l)$, the $l\times l$ matrix $(A_{ij})_{i,j\neq k}$ is a (classical) Cartan matrix.

Complete classification of generalized Cartan matrices of Euclidean type is known. Instead of giving the matrices, we list up their Dynkin diagrams with the coefficients of the null root (cf. 5.2).

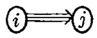
DEFINITION. The Dynkin diagram of a generalized Cartan matrix A is a graph on the vertices $\{0,1,\cdots,l\}$ with the following edges:



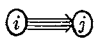
if $A_{ij}=A_{ji}=-1$,



if $A_{ij}=-1$, $A_{ji}=-2$,



if $A_{ij}=-1$, $A_{ji}=-3$,



if $A_{ij}=-1$, $A_{ji}=-4$,

if $A_{ij}=A_{ij}=-2$.

The meaning of the notation ◎ will be explained in 5.3.

5.2. Let V be an $(l+2)$ dimensional real vector space with bases $\{\alpha_0, \alpha_1, \cdots, \alpha_l, \delta\}$, and V^* the dual space of V. For a Euclidean generalized Cartan matrix $A=(A_{ij})_{0\le i,\, j\le l}$, we define the subset $\{\alpha_0^\vee, \alpha_1^\vee, \cdots, \alpha_l^\vee\}$ of V^* by the following conditions:

$$<\alpha_i^\vee, \alpha_j> = A_{ij} \qquad 0\le i, j\le l,$$

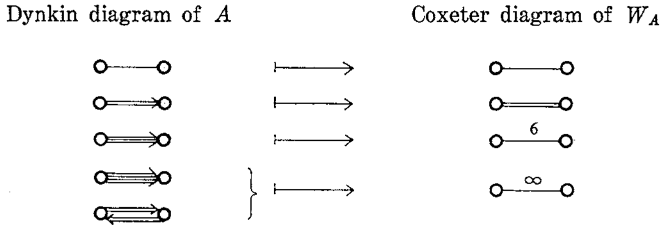
$$<\alpha_i^\vee, \delta> = \begin{cases} -1 & i=0, \\ 0 & 1\le i\le l, \end{cases}$$

where $<,>$denotes the dual pairing of V^* and V. There exists a unique vector $n=\sum_{j=0}^{l} n_j\alpha_j \in V$, called the null root of A, such that (i) $<\alpha_j^\vee, n>=0, 0\le j\le l$, (ii) n_j is a positive integer and (iii) one of the n_j's is equal to 1.

Any α_j $(0\le j\le l)$ determines a fundamental reflection s_j in V defined by

$$s_j(x)=x-<\alpha_j^\vee,\ x>\alpha_j.$$

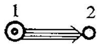
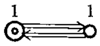
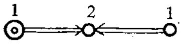
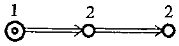
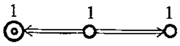
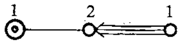
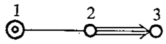
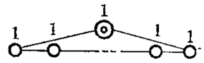
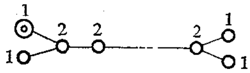
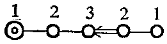
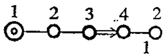
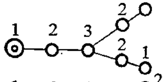
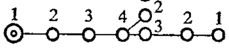
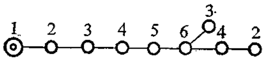
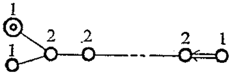
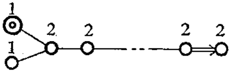
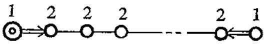
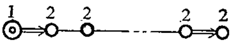
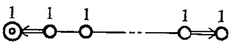
The fundamental reflections generate a subgroup W_A of Aut(V), called the Weyl group of A. The pair $(W_A, \{s_0, s_1, \cdots, s_l\})$ is a Coxeter system. Its Coxeter diagram is obtained from the Dynkin diagram of A by the following operation:



5.3. We shall recall the work of Looijenga ([3]). Put

$$F_B := \mathbf{R}n \subset V,$$

Table II

Type X	number of vertices	Dynkin diagram
$A_2^{(2)}$	2	
$A_1^{(1)}$	2	
$C_2^{(1)}$	3	
$A_4^{(2)}$	3	
$A_3^{(2)}$	3	
$D_4^{(3)}$	3	
$G_2^{(1)}$	3	
$A_l^{(1)}$	$l+1$ $(l\geq 2)$	
$D_l^{(1)}$	$l+1$ $(l\geq 4)$	
$E_6^{(2)}$	4	
$F_4^{(1)}$	4	
$E_6^{(1)}$	7	
$E_7^{(1)}$	8	
$E_8^{(1)}$	9	
$A_{2l-1}^{(2)}$	$l+1$ $(l\geq 3)$	
$B_l^{(1)}$	$l+1$ $(l\geq 3)$	
$C_l^{(1)}$	$l+1$ $(l\geq 3)$	
$A_{2l}^{(2)}$	$l+1$ $(l\geq 3)$	
$D_{l+1}^{(2)}$	$l+1$ $(l\geq 3)$	

$$\begin{aligned} V_B &:= \boldsymbol{R}\alpha_0+\boldsymbol{R}\alpha_1+\cdots+\boldsymbol{R}\alpha_l \subset V, \\ V' &:= V/F_B, \\ N &:= \boldsymbol{Z}n, \\ Q &:= \boldsymbol{Z}\alpha_0+\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l: \text{ Root lattice.} \end{aligned}$$

Since the subspaces F_B and V_B is W_A-invariant, the group $W_A \subset GL(V)$ operates on V_B/F_B. The consequent group is denoted by $\overline{W}_A \subset GL(V_B/F_B)$. $\overline{W}_A$ is a finite Coxeter group. We choose α_0 so that the Coxeter graph of $\overline{W}_A$ is equal to that of W_A which is taked off the vertex α_0. In Table II, we marked the corresponding vertex of the Dynkin diagram of A by ◎. In particular, the coefficient n_0 of the null root n is equal to 1. Thus we have

$$Q=N+\boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l.$$

Put

$$\begin{aligned} Q' &:= Q/N \\ &= \boldsymbol{Z}\alpha_1+\cdots+\boldsymbol{Z}\alpha_l. \end{aligned}$$

In the sequel we fix the bases $n, \alpha_1, \cdots, \alpha_l, \delta$ of V. Then every element of W_A has a form

$$\begin{bmatrix} 1 & b & c \\ 0 & w & a \\ 0 & 0 & 1 \end{bmatrix} \qquad w \in \overline{W}_A, \quad a, {}^t b \in \boldsymbol{Z}^l, \quad c \in \boldsymbol{Z}.$$

The natural homomorphism ρ: $W_A \to GL(V')$ is an isomorphism into. Let T be the kernel of the natural homomorphism $W_A \to \overline{W}_A$ (cf. [3;5.1]). By the correspondence $T \to Q'$ defined by

$$\begin{bmatrix} 1 & \beta & \gamma \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \overset{\rho}{\longmapsto} \begin{bmatrix} 1 & t \\ & 1 \end{bmatrix} \longmapsto t,$$

we shall identify the group T and the corresponding sublattice of Q' of finite index. Since ρ is an isomorphism, the values β and γ are determined by t, which we shall denote them by $\beta=\beta(t)$ and $\gamma=\gamma(t)$.

The group $W_A \subset GL(V)$ acts properly discontinuously on the half space

$$I := \boldsymbol{R}n+\boldsymbol{R}\alpha_1+\cdots+\boldsymbol{R}\alpha_l+\boldsymbol{R}^+\delta \subset V,$$

where $\boldsymbol{R}^+$ denote the set of positive numbers. Consider the domain $\Omega = V+\sqrt{-1}\,I$ in the complexification of V. The lattice Q acts on Ω as parallel displacements and the group $\tilde{W}_A = W_A \ltimes Q$ acts on Ω properly discontinuously. We define the surjection

$$f: \Omega \to H = \{\tau \in \boldsymbol{C} \mid \operatorname{Im} \tau > 0\}$$

by

$$zn + \sum_{j=1}^{l} x_j \alpha_j + \tau\delta \mapsto \tau,$$

and put $\Omega(\tau) = f^{-1}(\tau)$. The action of $\tilde{W}_A$ preserves the fibration of f. Let $W_A(\tau)$ and $\tilde{W}_A(\tau)$ be the restriction of W_A and $\tilde{W}_A$ to $\Omega(\tau)$, respectively.

THEOREM (Looijenga [3]). *The factor space $\Omega(\tau)/\tilde{W}_A(\tau)$ added by a point is non-singular.*

§6. Correspondence between A and Γ

The purpose of this chapter is to establish a correspondence between $\{A\}$ and $\{\Gamma(X)\}$.

6.1. Let C be a Cartan matrix and L a $W(C)$-invariant lattice. We shall transform the group

$$\Gamma(C, L, p, c_{DC}, d_{DC}) = \{[w, \alpha, \gamma]_{DC} \mid w \in W(C),\ \alpha \in L,$$
$$\gamma \equiv c_{DC}(\alpha) + d_{DC}(w) \mod q\boldsymbol{Z},\ q = q_0/p\}.$$

Let $(z, u_1, \cdots, u_l)$ be the coordinate of $Y = F + Y'_{DC}$. We introduce new coordinate $(z', u_1, \cdots, u_l)$ of Y:

$$z' = \frac{1}{q}\{z - i(u_1, \cdots, u_l) DC^t(u_1, \cdots, u_l)\},$$

which is essential in the proof of Theorem 2.

LEMMA 6.1. *Under the coordinate $(z', u_1, \cdots, u_l)$, the element $[w, \alpha, \gamma]_{DC}$ is represented by*

$$[w, \alpha, \gamma]'_{DC} = \begin{bmatrix} 1 & \left(\dfrac{4 \operatorname{Im} {}^t\alpha}{q}\right) DCw & \dfrac{\gamma}{q} + \left(\dfrac{2 \operatorname{Im} {}^t\alpha}{q}\right) DC\alpha \\ 0 & w & \alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

PROOF. Recall that

$$[w, \alpha, \gamma]_{DC} = \begin{bmatrix} 1 & 2i{}^t\bar{\alpha} DCw & \gamma + i{}^t\bar{\alpha} DC\alpha \\ 0 & w & \alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

Set $u = {}^t(u_1, \cdots, u_l)$. Since DC is symmetric and ${}^t w DCw = DC$, we have

$$\begin{aligned} & z+2i{}^t\bar{\alpha}DCwu+\gamma+i{}^t\bar{\alpha}DC\alpha-i{}^t(wu+\alpha)DC(wu+\alpha) \\ &= z-i{}^tu{}^twDCwu+i\{2{}^t\bar{\alpha}-2{}^t\alpha\}DCwu+\gamma+i\{{}^t\bar{\alpha}-{}^t\alpha\}DC\alpha \\ &= qz'+(4\operatorname{Im}{}^t\alpha)DCwu+\gamma+(2\operatorname{Im}{}^t\alpha)DC\alpha. \qquad \text{Q.E.D.} \end{aligned}$$

We put $\Gamma'(C, L, p, c_{DC}, d_{DC})=\{[w, \alpha, \gamma]_{DC} \mid [w, \alpha, \gamma]_{DC} \in \Gamma(C, L, p, c_{DC}, d_{DC})\}$.

6.2. Let the notations be as in §5. We identify the fibre $\Omega(\tau)$ and the complexification $(V_B)_C$ of V_B, and fix the bases $n, \alpha_1, \cdots, \alpha_l$. Then the groups $W_A(\tau)$, Q and $\tilde{W}_A(\tau)$ are represented by

$$W_A(\tau)=\left\{\begin{bmatrix} 1 & b & c\tau \\ 0 & w & a\tau \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & b & c \\ 0 & w & a \\ 0 & 0 & 1 \end{bmatrix} \in W_A\right\}, \quad Q=\begin{bmatrix} 1 & 0 & Z \\ 0 & 1 & Z^l \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \tilde{W}_A(\tau) &= W_A(\tau) \ltimes Q \\ &= \left\{\begin{bmatrix} 1 & b & c\tau+Z \\ 0 & w & a\tau+Z^l \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; \begin{bmatrix} 1 & b & c \\ 0 & w & a \\ 0 & 0 & 1 \end{bmatrix} \in W_A\right\}. \end{aligned}$$

Let $\tilde{\rho}\colon \tilde{W}_A(\tau) \to A((V_B)_C)$ and $\varphi\colon \tilde{W}_A(\tau) \to \overline{\overline{W}}_A$ be the homomorphisms defined by

$$\begin{bmatrix} 1 & \mu & \nu \\ 0 & w & \omega \\ 0 & 0 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} w & \omega \\ 0 & 1 \end{bmatrix},$$

and

$$\begin{bmatrix} 1 & \mu & \nu \\ 0 & w & \omega \\ 0 & 0 & 1 \end{bmatrix} \longmapsto w,$$

respectively, and put

$$(TQ)(\tau)=\left\{\begin{bmatrix} 1 & \beta(t) & \tau\gamma(t)+Z \\ 0 & 1 & \tau t+Z^l \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; t \in T\right\}.$$

If we identify the lattice N with the group

$$\begin{bmatrix} 1 & 0 & Z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then we have the following commutative diagram of exact sequences.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & N \arrow[r] \arrow[d] & (TQ)(\tau) \arrow[r] \arrow[d] & \tilde{\rho}((TQ)(\tau)) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & N \arrow[r] \arrow[d] & \tilde{W}_A(\tau) \arrow[r, "\tilde{\rho}"] \arrow[d, "\varphi"'] & \tilde{\rho}(\tilde{W}_A(\tau)) \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & \bar{W}_A \arrow[r] \arrow[d] & \bar{W}_A \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

6.3. Let A be a generalized Cartan matrix of Euclidean type. We shall show that the matrix representation of $\tilde{W}_A(\tau)$ given in 6.2 coincides with the group $\Gamma'(C, L, p, c_{DC}, d_{DC})$ for suitable C, L, p, c_{DC} and d_{DC}. We cut off the first low, corresponding to α_0, and the first column, corresponding to α_0, from the matrix A and obtain a (classical) Cartan matrix $(A_{ij})_{1\leq i,\, j\leq l}$. We let $C=(A_{ij})_{1\leq i,\, j\leq l}$. Recall that we chose the root α_0 in 5.3 so that $\bar{W}_A$ may coincides with $W(C)$. Let T be the sublattice of Q' defined in 5.3. We put $L=Z^l+\tau T$. Then we have $\tilde{\rho}((TQ)(\tau))\cong L$ and $\tilde{\rho}(\tilde{W}_A(\tau))\cong W(C)\ltimes L$.

We shall identify the group W_A and the image of $W_A \hookrightarrow \tilde{W}_A \xrightarrow{\sim} \tilde{W}_A(\tau)$, and denote the image of the fundamental reflections $s_0, s_1, \cdots, s_l \in W_A$ by capital letters: $S_0, S_1, \cdots, S_l$. Let $w_1, \cdots, w_l$ be the fundamental reflections of the Cartan matrix C. Then the matrices S_j $(1\leq j\leq l)$ is represented by

$$S_j=\begin{bmatrix} 1 & & \\ & w_j & \\ & & 1 \end{bmatrix} \in \tilde{W}_A(\tau).$$

Thus we let $d(w)=0$ for all $w\in W(C)=\bar{W}_A$.

For each A, we shall calculate the matrix S_0 to determine the lattice L and find an integer p and a function $c: L\to \boldsymbol{R}$ such that $\tilde{W}_A(\tau)$ coincides with $\Gamma'(C, L, p, c, 0)$. First, we work at the type $A_l^{(1)}$. We have $C=C(A_l)$. Let $x=u_0 n+\sum_{j=1}^{l} u_j\alpha_j+\tau\delta$ be an element of V. The fundamental reflection $s_0\in W_A$ acts on V as follows:

$$\begin{aligned} s_0(x) &= x-\langle\alpha_0^\vee, x\rangle\alpha_0 \\ &= x-\{-u_1-u_l-\tau\}\alpha_0 \\ &= (u_0+u_1+u_l+\tau)n+\sum_{j=1}^{l}(u_j-u_1-u_l-\tau)\alpha_j+\tau\delta. \end{aligned}$$

Thus $S_0 \in \tilde{W}_A(\tau)$ is represented by

$$S_0 = \begin{bmatrix} 1 & 1 & & & 1 & \tau \\ & & & & -1 & -\tau \\ & -1 & 1 & & -1 & -\tau \\ & \vdots & & \ddots & \vdots & \vdots \\ & -1 & & 1 & -1 & -\tau \\ & -1 & & & & -\tau \\ & & & & & 1 \end{bmatrix}.$$

Since we have $\varphi(S_0)=\varphi(S_1\cdots S_{l-1}\,S_l\,S_{l-1}\cdots S_1)$, we calculate $S_1S_0S_1$, $S_2S_1S_0S_1S_2, \cdots,$ $S_{l-1}\cdots S_1S_0S_1\cdots S_{l-1}$, and conclude that the matrices

$$\begin{bmatrix} 1 & -2 \quad 1 \qquad & \tau \\ & \boxed{\qquad I_l \qquad} & -\tau \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \quad 1 \; \overset{j}{-2} \; 1 & \tau \\ & \boxed{\qquad I_l \qquad} & -\tau \\ & & 1 \end{bmatrix} \quad (j \qquad 2\leq j\leq l-1$$

and

$$\begin{bmatrix} 1 & \qquad 1 \quad -2 & \tau \\ & \boxed{\qquad I_l \qquad} & \\ & & -\tau \\ & & 1 \end{bmatrix}$$

belong to $\tilde{W}_A(\tau)$. Thus we have $T=Q'=\sum_{j=1}^{l} Z\alpha_j$, and so $L=\sum_{j=1}^{l} L(\tau)e_j$, where e_j stands for ${}^t(0,\cdots,0,\overset{j}{1},0,\cdots,0)$. By the matrix representation of the root lattice Q in 6.2, we let $c(e_j)=0, 1\leq j\leq l$. The following system of equations

$$\frac{4}{q}\operatorname{Im}{}^t(-\tau e_1)DC=(-2,1,0,\cdots,0),$$

$$\frac{4}{q}\operatorname{Im}{}^t(-\tau e_j)DC=(0,\cdots,0,1,\overset{j}{-2},1,0,\cdots,0) \qquad 2\leq j\leq l-1,$$

$$\frac{4}{q}\operatorname{Im}{}^t(-\tau e_l)DC=(0,\cdots,0,1,-2)$$

and

$$\frac{\gamma_j}{q}+\frac{2}{q}(\operatorname{Im}{}^t(-\tau e_j))DC(-\tau e_j)=\tau \qquad 1\leq j\leq l$$

has solutions: $q=4\operatorname{Im}\tau$, $\gamma_j=0$. We put $p=1$ and $c_{DC}(\tau e_j)=0$. Then Lemma 6.1 asserts that the group $\tilde{W}_A(\tau)$ coincides with $\Gamma(C(A_l), L, 1, c_{DC}, 0)$.

Similar proof is available for types $D_l^{(1)}$ $(l\geq 4)$ and $E_l^{(1)}$ $(l=6,7,8)$. We have $C=C(D_l)$ and $C(E_l)$ respectively, and $L=\sum_{j=1}^{l} L(\tau)e_j$. We put $p=1$ and $c_{DC}(e_j)=$

$c_{DC}(\tau e_j)=0$ $(1\leq j\leq l)$ and see that $\tilde{W}_A(\tau)$ is equal to $\Gamma(C(X), L, 1, c_{DC}, 0)$.

For other types we omit the details, and list up C and T;

X	C	T
$A_2^{(2)}$	$C(A_1)$	$4\boldsymbol{Z}\alpha_1$
$A_1^{(1)}$	$C(A_1)$	$2\boldsymbol{Z}\alpha_1$
$C_2^{(1)}$	$C'(B_2)$	$2\boldsymbol{Z}\alpha_1+\boldsymbol{Z}\alpha_2$
$A_4^{(2)}$	$C(B_2)$	$2\boldsymbol{Z}\alpha_1+2\boldsymbol{Z}\alpha_2$
$A_3^{(2)}$	$C(B_2)$	$\boldsymbol{Z}\alpha_1+\boldsymbol{Z}\alpha_2$
$D_4^{(3)}$	$C'(G_2^{(6)})$	$\boldsymbol{Z}\alpha_1+\boldsymbol{Z}\alpha_2$
$G_2^{(1)}$	$C(G_2^{(6)})$	$\boldsymbol{Z}\alpha_1+3\boldsymbol{Z}\alpha_2$
$E_6^{(2)}$	$C'(F_4)$	$\sum_{j=1}^{4}\boldsymbol{Z}\alpha_j$
$F_4^{(1)}$	$C(F_4)$	$\sum_{j=1}^{2}\boldsymbol{Z}\alpha_j+\sum_{j=3}^{4}2\boldsymbol{Z}\alpha_j$
$A_{2l-1}^{(2)}$	$C'(B_l)$	$\sum_{j=1}^{l}\boldsymbol{Z}\alpha_j$
$B_l^{(1)}$	$C(B_l)$	$\sum_{j=1}^{l-1}\boldsymbol{Z}\alpha_j+2\boldsymbol{Z}\alpha_l$
$C_l^{(1)}$	$C'(B_l)$	$\sum_{j=1}^{l-1}2\boldsymbol{Z}\alpha_j+\boldsymbol{Z}\alpha_l$
$A_{2l}^{(2)}$	$C(B_l)$	$\sum_{j=1}^{l}2\boldsymbol{Z}\alpha_j$
$D_{l-1}^{(2)}$	$C(B_l)$	$\sum_{j=1}^{l}\boldsymbol{Z}\alpha_j$.

For the type $A_2^{(2)}$, we let $\tau\to\tau/4$ and for the types $A_1^{(1)}, C_2^{(1)}, A_4^{(2)}, C_l^{(1)}$ and $A_{2l}^{(2)}$, we let $\tau\to\tau/2$. We calculate the number q and obtain that $q=2\operatorname{Im}\tau$ for $A_1^{(1)}, A_3^{(2)}, B_{l+1}^{(2)}$, and $4\operatorname{Im}\tau$ for others. Since we know the value q_0 by Lemma 4.2, we have the number p.

Now we are in the position to state the theorem.

6.4.

THEOREM 2. *There exists a surjective correspondence from the set of generalized Cartan matrices of Euclidean type into the set of* $\hat{G}$*-conjugate classes of the discrete subgroup, with parameter* τ *on* H, *of* G *of locally finite volume such that it is generated by reflections and that its point group is an irreducible Coxeter group. The correspondence is given as follows: Let* A *be a generalized Cartan matrix of type* X (*Table* II), *and* $\tilde{W}_A(\tau)$ *the group defined in* § 5. *Then the group*

$\tilde{W}_A(\tau)$ is transformed, by the inverse operation defined in 6.1, *into the group $\Gamma(X)$ (Table* I).

REMARK. The correspondence $X \mapsto \Gamma(X)$ is one to one except for $\Gamma(B_l^{(1)}) = \Gamma(C_l^{(1)})$.

Acknowledgement. The author wishes to express his deepest appreciation to Professor T. Yokonuma who improved the proof of Proposition 2 of which original proof was long and complicated.

References

[1] Armstrong, M.A., On the fundamental group of an orbit space, Proc. Cambridge Philos Soc. **61** (1965), 639-646.

[2] Bourbaki, N., Groupes et Algèbres de Lie, Ch. 4, 5 et 6. Hermann, Paris, 1968.

[3] Looijenga, E., Invariant theory for generalized root systems, Invent. Math. **61** (1980), 1-31.

[4] Iwahori, N. and T. Yokonuma, Kac-Moody algebra and Macdonald identity, Sûgaku **33** (1981), 193-212.

[5] Mumford, D., The topology of normal singularities of an algebraic surface and a criterion for simplicity, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 229-246.

[6] Picard, E., Sur les fonctions de deux variables indépendantes analogues aux fonctions modulaires, Acta Math. **2** (1883), 114-126.

[7] Piatetskii-Shapiro, I.I., Automorphic Functions and the Geometry of Classical Domains, Gordon and Breach, New York, 1969.

[8] Terada, T., Probléme de Riemann et fonctions automorphes provenant des fonctions hypergéométriques de plusieurs variables, J. Math. Kyoto Univ. **13** (1973), 557-578.

[9] Tokunaga, S. and M. Yoshida, Complex crystallographic groups I, J. Math. Soc. Japan **34** (1982), 581-593.

[10] Yoshida, M., Local theory of Fuchsian systems with certain discrete monodromy groups I, Funkcial. Ekvac. **21** (1978), 105-137.

[11] Yoshida, M. and S. Hattori, Local theory of Fuchsian systems with certain discrete monodromy groups III, Funkcial. Ekvac. **22** (1979), 1-40.

[12] Yoshida, M., Discrete reflection groups in a parabolic subgroup of $Sp(2, R)$ and symmetrizable hyperbolic generalized Cartan matrices of rank 3, preprint.

[13] Wolf, J.A., Spaces of Constant Curvature, 1967, McGraw-Hill, New York, 1967.

(Received March 1, 1982)

Department of Mathematics
Faculty of Science
Kyushu University
Hakozaki, Fukuoka
812 Japan