On the separation principle of stochastic control

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§ 0. Introduction

In 1968, Wonham [12] showed that the separation principle would effect when we want to control a partially observable system disturbed by some noise.

We want to minimize the cost $J[u]=E\left[\int_0^T L(t, x_t^u, u_t)dt\right]$, where x_t^u follows the stochastic differential equation

$$\begin{cases} dx_t=A_t x_t dt+b(t, u_t)dt+C_t dw_{1t} \\ dy_t=F_t x_t dt+G_t dw_{2t} \\ x_0;\ \text{non-degenerated Gaussian},\ y_0=0. \end{cases}$$

Here A_t, C_t, F_t and G_t are non-random matrices, $\{x_0, w_{1s}, w_{2s};\ 0\leqq s\leqq T\}$ are mutually independent, and w_1, w_2 are Brownian motions. x_t and y_t are called respectively the system and the observed processes. u_t is a control. The data accumulated up to the time t are represented as $\sigma(y_s;\ 0\leqq s\leqq t)=\mathscr{Y}_t$, so the control must be $\mathscr{Y}_t$-adapted.

Wonham showed that the optimal control is represented as $u_t^0=\phi_0(t, \hat{x}_t)$, where $\hat{x}_t=E[x_t^u | \mathscr{Y}_t]$, and on the way, he also said that the conditional probability of x_t given $\mathscr{Y}_t$ is $N(\hat{x}_t, Q_t)$ (Q_t is non random and independent of controls). This says that the above problem is separated to two steps, the filtering and the control problem of a completely observable system.

Then what could we say when the system equation is non-linear? In a deterministic case or a completely observable case of Markov type, the optimal control are often represented as $u_t^0=\phi_0(t, x_t)$. That is to say, the necessary information is only that about the present state and we could discard others. In our case, the data at hand is $\mathscr{Y}_t$ and that about the present state is $x_t/\mathscr{Y}_t$, the conditional probability of x_t given $\mathscr{Y}_t$. So if we write its density function as $\pi_t(.)$, u_t^0 must be represented as $\phi_0(t, \pi_t(.))$. Though we have more to be done for completeness, it is proved in this paper that this fact is true when there exists a smooth solution for a certain quasi linear partial differential equation (called Bellman equation).

When we look at the linear case again from our standpoint, their information about the present was concentrated to $\hat{x}_t$, because the conditional distribution of

x_t is $N(\hat{x}_t, Q_t)$.

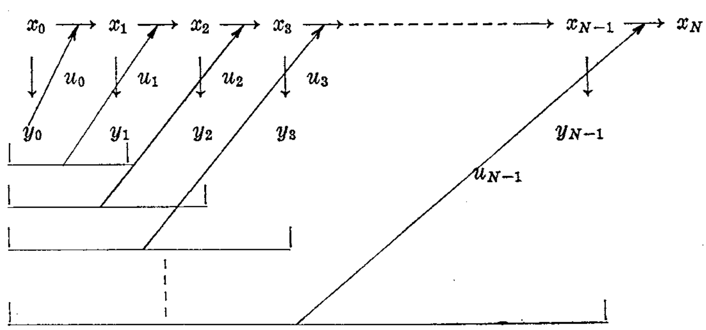
First in the next section, we treat the case in which the state spaces of x_t and y_t are finite, and see that the optimal control can be expressed as a function of π_t but not of $\hat{x}_t$ in general. In Section 2, the case when the system process is of a jump type is considered, and in Section 3, the case when the system and the observed processes are expressed as a solution of a non linear stochastic differential equation with the Gaussian white noise.

Accordingly, we deal with the case when the data at hand at time t is $\mathscr{Y}_{t-h}$. As supposed, in the linear case, the optimal control is proved in Section 4 to be represented as $u_t^h = \phi_h(t, \hat{x}_t^h)$ where $\hat{x}_t^h = E[x_t^u | \mathscr{Y}_{t-h}]$, but in the non linear case, it is seen in Section 5 that $x_t/\mathscr{Y}_{t-h}$ is not sufficient for the optimal control. This means that the above consideration may be incomplete.

Finally, Prof. Y. Okabe and Prof. S. Kusuoka as well as Prof. S. Itô supported our work. The author owes the information on the separation principle to Prof. H. Kunita in his book [6]. We would have pleasure in recording here our grateful acknowledgement to them.

§1. The finite state space, discrete time case

Let the number of the times be N and the state spaces of x, y $\{1, \cdots, M_1\}$, $\{1, \cdots, M_2\}$, respectively. We consider the following system;



$$J[u] = E\left[\sum_{k=1}^{N} L_k(x_k, u_{k-1})\right]$$

That is to say, we are given the following quantities,

(i) the distribution of x_0,

(ii) $P^u(x_{n+1}=k | x_n=i, x_0, \cdots, x_{n-1}, y_0, \cdots, y_{n-1}) \equiv P_n^{ik}(u_n) = P^u(x_{n+1}=k | x_n=i)$

$$i=1,\cdots,M_1,\ k=1,\cdots,M_1,\ n=0,\cdots,N-1.$$

(From now on, we write "$x_0,\cdots,x_n$" as "x_0^n".)

(iii) $P^u(y_n=j|x_n=k, x_0^{n-1}, y_0^{n-1})=P^u(y_n=i|x_n=k)\equiv q_n^{kj}$

(independent of the controls)

$$k=1,\cdots,M_1,\quad j=1,\cdots,M_2,\quad n=0,\cdots,N-1.$$

As for the admissible controls, we consider all $\mathscr{Y}_k$-adaptive processes taking values in U, where $\mathscr{Y}_k=\sigma(y_0,\cdots,y_k)$ and U is the control region.

Then, if we write $P^u(x_n=i|y_0^n)$ as $\pi_n^i(y_0^n, u_0^{n-1})$, the separation principle in this case is stated as follows;

PROPOSITION 1.1. *The optimal control is represented as* $u_n^0=\phi_n(\pi_n^1,\cdots,\pi_n^M)$, $n=0,1,\cdots,N-1$. (*Here* ϕ_n; $[0,1]^M\to U$ *are measurable.*)

PROOF. First, we note that

$$\begin{aligned}\pi_{n+1}^k(y_0^{n+1}, u_0^n)&=P^u(x_{n+1}=k, y_0^{n+1})/P^u(y_0^{n+1})\\&=P^u(x_{n+1}=k, y_{n+1}|y_0^n)P^u(y_0^n)/P^u(y_0^{n+1})\end{aligned}$$

and

$$P^u(x_{n+1}=k, y_{n+1}=j|y_0^n)=q_{n+1}^{k,j}\sum_{i=1}^{M_1}P_n^{ik}(u_n)\pi_n^i(y_0^n, u_0^{n-1}),$$

$$P^u(y_{n+1}=j|y_0^n)=\sum_{k=1}^{M_1}q_{n+1}^{kj}\sum_{i=1}^{M_1}P_n^{ik}(u_n)\pi_n^i(y_0^n, u_0^{n-1}).$$

So, we can express the π_{n+1}^k as a following form;

$$\pi_{n+1}^k(y_0^{n+1}, u_0^n)=f_{n+1}^k(u_n, \pi_n, y_{n+1})$$

where f is a function depending on q_{n+1} and P_n. Using the dynamic programming method, we prove the statement inductively.

Step 1; Assuming that $u_0,\cdots,u_{N-2}$ are given, we look for the optimal u_{n-1}. Now, the distribution of (x_0^{N-1}, y_0^{N-1}) is given. From above

$$E^u[L_N(x_N, u_{N-1})|y_0^{N-1}]=\sum_{k=1}^{M_1}L_N(k, u_{N-1})\sum_{i=1}^{M}P_{N-1}^{ik}(u_{N-1})\pi_{N-1}^i(y_0^{N-1}, u_0^{N-2})$$

so, we get

$$u_{N-1}^0=\phi_{N-1}(\pi_{N-1}^1(y_0^{N-1}, u_0^{N-2}),\cdots,\pi_{N-1}^{M_1}(y_0^{N-1}, u_0^{N-2}))$$

minimizing the above quantity.

In this way, we suppose that for $m=n+1, n+2,\cdots,N-1$ u_m^0 are expressed as $\phi_m(\pi_m^1, \cdots, \pi_m^{M_1})$, and define $r_m^*(\pi_m^1, \cdots, \pi_m^{M_1})$ as follows;

$$r_m^*(\pi_m^1, \cdots, \pi_m^{M_1}) = E\left[\sum_{t=m+1}^{N} L_t(x_t, u_{t-1}^0) \mid y_0^m\right].$$

Step 2; Assuming that $u_0, \cdots, u_{n-1}$ are given, we find the u_n^0 minimizing

$$E^u\left[\sum_{m=n+2}^{N} L_m(x_m, u_{m-1}^0) + L_{n+1}(x_{n+1}, u_n^0) \mid y_0^n\right].$$

Noting that

$$E^u\left[\sum_{m=n+2}^{N} L_m(x_m, u_{m-1}^0) \mid y_0^n\right] = E^u\left[E^u\left[\sum_{m=n+2}^{N} L_m(x_m, u_{m-1}^0) \mid y_0^{n+1}\right] \Big| \, y_0^n\right]$$

$$= E^u[r_{n+1}^*(\pi_{n+1}^1(y_0^{n+1}, u_0^n), \cdots, \pi_{n+1}^{M_1}(y_0^{n+1}, u_0^n)) \mid y_0^n],$$

the above quantity is written as

$$\sum_{j=1}^{M_2} r_{n+1}^*(f_{n+1}^1(u_n, \pi_n, j), \cdots, f_{n+1}^{M_1}(u_n, \pi_n, j)) \sum_{m,h=1}^{M_1} q_{n+1}^{mj} P_n^{mh}(u_n) \pi_n^h(y_0^n, u_0^{n-1})$$
$$+ \sum_{k=1}^{M_1} L_{n+1}(k, u_n) \sum_{i=1}^{M_1} P_n^{ik}(u_n) \pi_n^i(y_0^n, u_0^{n-1}).$$

So u_n^0 is expressed as

$$u_n^0 = \phi_n(\pi_n^1(y_0^n, u_0^n), \cdots, \pi_n^{M_1}(y_0^n, u_0^{n-1}))$$

and $r_n^*(\pi_n^1(y_0^n, u_0^{n-1}), \cdots, \pi_0^{M_1}(y_0^n, u_0^{n-1}))$ is defined. When n reaches 0, the proof is concluded. (Q.E.D.)

§ 2. x_t; finite state space, continuous time case

We show that the separation principle effects even if the time is continuous in Section 1. Now let the state space of the system process x_t be $\{1, \cdots, M\}$. We want to control the 'transition probability' of x_t, based on the observed data. Let us construct two typical models, and discuss them in each case.

[2.1] *The case when the noise of the observed process is a Brownian motion.*

The problem in this section is to find the control u_t^0 minimizing the cost

$$J[u] = E\left[\int_0^T L(t, x_t, u_t) dt\right] \tag{2.1}$$

under the system equation

$$\begin{cases} x_t = x_0 + \int_0^{t+} \int_R F(s, x_{s-}, u_{s-}, v) N(ds, dv) \\ y_t = \int_0^t g(s, x_s) dt + w_t. \end{cases} \tag{2.2}$$

Here

$$F(t, x, u, v)=\sum_{k=1}^{M} I_{\{x=k\}} \sum_{j(\neq k),1\leqq j\leqq M} (j-k) I_{\{0<v-(kM+j)K<\lambda_{kj}(t,u)\}}.$$

u_t is a control, and we will soon later mention about the admissible controls. K is a constant appearing in the following assumptions. x_0, w, N are independent and w is a 1-dimensional Brownian motion, N is a stationary Poisson point process with the characteristic measure $dv(\subset R^1)$. We suppose that

$$E[x_0^2]<\infty,\ \pi_0^i \equiv P(x_0=i)>0,\ i=1, \cdots, M.$$

The formulation of (2.2) is derived from the expression for Markov processes (Skorokhod [11]). The function F decides the 'transition probability'. The factor $(j-k)$ means the jump from k to j and the last factor means its conditional probability.

Let the control region be $U(\subset R^m)$. We assume the followings about the functions

$$\begin{cases}\lambda_{ik}: [0,1]\times U \longrightarrow R, & i,k=1,\cdots,M \\ g\ : [0,1]\times\{1,\cdots,M\} \longrightarrow R \\ L\ : [0,1]\times\{1,\cdots,M\}\times U \longrightarrow R; \end{cases}$$

i) $\sum_{k=1}^{M} \lambda_{ik}(t,u)=0$ for all t in $[0,1]$, u in U.

ii) for $i\neq k$, $0\leqq \lambda_{ik}(t,u)\leqq K$ for all t and u.

There exists some function ρ: $[0,T]\to R_+$, $\rho(t)\downarrow 0$ $(t\downarrow 0)$ such that

$$|\lambda_{ik}(t,u)-\lambda_{ik}(s,u)|\leqq \rho(|t-s|) \text{ for all } t,s \text{ in } [0,T] \text{ and } u \text{ in } U.$$
$$|\lambda_{ik}(t,u)-\lambda_{ik}(t,v)|\leqq K|u-v| \text{ for all } t \text{ in } [0,T],\ u,v \text{ in } U.$$

iii) $g(t,k)\equiv g_{tk}$ are continuous in t, and then particularly bounded; $|g_{tk}|\leqq K$.

iv) $|L(t,k,u)|\leqq K$ for all t in $[0,T]$, u in U, $k=1,\cdots,M$, they are continuous in t and

$$|L(t,k,u)-L(t,k,v)|\leqq K|u-v| \quad \text{for } t \text{ in } [0,T],\ u,v \text{ in } U,\ k=1,\cdots,M.$$

v) We define

$$\lambda\colon [0,T]\times\left\{x\in R^{M-1};\ \sum_{k=1}^{M-1} x_k\leqq 1,\ x_i\geqq 0, \quad i=1,\cdots,M-1\right\}\times R^{M-1}\times U \longrightarrow R$$

as

$$\lambda(t,x,p,z)\equiv \sum_{i=1}^{M-1} p_i\left\{\lambda_{Mi}(t,z)+\sum_{k=1}^{M-1}(\lambda_{ki}(t,z)-\lambda_{Mi}(t,z))x_k\right\}$$
$$+\sum_{k=1}^{M-1}(L(t,k,z)-L(t,M,z))x_k+L(t,M,z).$$

We assume the existence of $z_0 \in U$ which minimizes λ for each t, x, p, and write it as $\phi(t, x, p)$. Then ϕ is α-Hölder continuous in t, Lipshitz continuous in x, p.

REMARK 2.1. Though v) is rather technical, we will show a reasonable example in the case $M=2$ at the end of this section. And if we assume the control region to be compact, the assumption of the existence of z_0 is not necessary.

Next, we would talk about the admissible controls (whose definition will be applied to Subsection 2.2). We will write the set of all of those as $\mathfrak{A}$. Let $C=\{f;\ [0, T]\to R,\ \text{continuous}\}$ and $\mathcal{B}_t(C)$ be a σ-field generated by the cylindrical sets up to t.

DEFINITION 2.1. u is an admissible control if and only if there is a $\mathcal{B}_t(C)$-progressively measurable function ϕ having the following property (a) or b)) and u_t can be represented as $\phi(t, y)$.

a) (i) There exist $0=t_0<t_1<\cdots<t_m=T$ and a function ρ: $[0, t]\to R_+$, $\rho(t)\downarrow 0$ $(t\downarrow 0)$ such that $|\phi(t, y)-\phi(s, y)|\leqq\rho(|t-s|)$ $t_i\leqq t, s<t_{i+1}$, $i=0, \cdots, m-1$.

(ii) There exists $\bar{K}>0$ such that $|\phi(t, y)-\phi(t, y)|\leqq\bar{K}\sup_{0\leqq s\leqq t}|y_s-y'_s|$ for t in $[0, T]$, y, y' in C.

b) There exist $0<t_1<\cdots<t_n=T$ such that $\phi(t, y)=\phi(t_i, y), t_i\leqq t<t_{i+1}, i=0, \cdots, n-1$, y in C.

REMARK 2.2. For each u in $\mathfrak{A}$, the stochastic differential equation (2.2) has a unique strong solution.

1°) The case when ϕ satisfies a)

We can show this by successive approximation method. But looking at the form of $F(t, x, u, v)$, we should estimate the first order moments instead of the second order ones. Then we can use the Borel-Cantelli's lemma as usual.

2°) The case when ϕ satisfies b)

We decompose the interval $[0, T]$ into the $[t_i, t_{i+1}]$'s, and solve (2.1) in each interval one after another.

First of all, we prepare two lemmas. The former gives the interpretation of (2.2), and the latter is a tool for the filtering. Now we define $\mathcal{F}_t$ as

$$\mathcal{F}_t\equiv\sigma(x_0, w_s, N((r_1, r_2], A);\ 0\leqq s\leqq t,\ 0\leqq r_1<r_2\leqq t,\ A\in\mathcal{B}(R)).$$

Then we get the following,

LEMMA 2.1. *For* $i\neq k$,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}P(x_{t+\varepsilon}=k\,|\,\mathcal{F}_t, x_t=i)=\lambda_{ik}(t, u_t)\qquad a.s.$$

Moreover

$$|P(x_{t+\varepsilon}=k|\mathcal{F}_t, x_t=i)-\lambda_{ik}(t, u_t)\varepsilon|\leqq o(\varepsilon)$$

and the rapidity of the convergence of the right side can be taken independent of t, ω.

PROOF.

1°) The case when ϕ satisfies b)

Let $t_j \leqq t < t+\varepsilon < t_{j+1}$,

$$\begin{aligned}&P(x_{t+\varepsilon}=k|\mathcal{F}_t, x_t=i)\\ &=P(x \text{ goes from } i \text{ to } k \text{ by one jump in } [t, t+\varepsilon]|\mathcal{F}_t, x_t=i)\\ &\quad+P(x \text{ goes from } i \text{ to } k \text{ by more than two jumps in } [t, t+\varepsilon]|\mathcal{F}_t, x_t=i).\end{aligned}$$

The first term

$$=\exp\left(-\int_t^{t+\varepsilon}\lambda_{ik}(s, u_{s-})\,ds\right)\int_t^{t+\varepsilon}\lambda_{ik}(s, u_{s-})ds\times\prod_{j(\neq k)}\exp\left(-\int_t^{t+\varepsilon}\lambda_{ij}(s, u_{s-})ds\right).$$

In the same way

$$\text{The second term}\leqq\sum_{n=2}^{\infty}\{M(M-1)\}^n(\varepsilon K)^n,$$

hence

$$\leqq\frac{\varepsilon^2\{M(M-1)K\}^2}{1-\varepsilon M(M-1)K}.$$

2°) The case when ϕ satisfies a)

We define x'_r as

$$x'_r\equiv\begin{cases}x_r & \text{if } r\leqq t\\ x_t+\displaystyle\int_t^{r+}\int_R F(t, x_{t-}, u_{t-}, v)N(ds, dv) & \text{if } r>t\end{cases}$$

$$\begin{aligned}&|P(x_{t+\varepsilon}=k|\mathcal{F}_t, x_t=i)-\lambda_{ik}(t, u_t)\varepsilon|\\ &\leqq|P(x'_{t+\varepsilon}=k|\mathcal{F}_t, x'_t=i)-\lambda_{ik}(t, u_t)\varepsilon|\\ &\quad+|P(x_{t+\varepsilon}=k|\mathcal{F}_t, x_t=i)-P(x'_{t+\varepsilon}=k|\mathcal{F}_t, x'_t=i)|.\end{aligned}$$

As to the first term, we mentioned in 1°). That the second term$\leqq o(\varepsilon)$ can be shown by Chebichev's inequality and taking moments. (Q. E. D.)

LEMMA 2.2.
$$V_t^i\equiv\delta_{x_t i}-\delta_{x_0 i}-\int_0^t\lambda_{x_{s-}i}(s, u_{s-})ds$$

(where δ_{ij} is a Kronecker's delta)

are right continuous $\mathcal{F}_t$-martingales. Furthermore, $\langle V^i, w\rangle_t\equiv 0$.

PROOF. As x_t takes values only on N, we can use the Ito's formula for δ_{xi}. Then from (2.2),

$$\delta_{x_t i}-\delta_{x_0 i}=\int_0^{t+}\int_R \{\delta_{i,x_{s-}+F(s,x_{s-},u_{s-},v)}-\delta_{i,x_{s-}}\}N(ds,dv)$$
$$=\mathcal{F}_t\text{-mart.}+\int_0^{t+}\int_R \{\delta_{i,x_{s-}+F(s,x_{s-},u_{s-},v)}-\delta_{i,x_{s-}}\}dsdv.$$

From the form of the F, we can show that the second term is zero.

If we consider $V_t^i w_t$, the second statement is proved in the same way.

(Q. E. D.)

With the above notes and lemmas, we proceed to the first theorem.

THEOREM 2.1. *We assume* i)–v) *for the control system* (2.1), (2.2). *If the Bellman equation described below (a degenerated non-linear parabolic equation) has a smooth solution W, then the following separation principle is proved. (W should be continuously differentiable up to first order in t, second order in x, and further W_x should be β-Hölder continuous in t, Lipschitz continuous in x.)*

That is to say, for all $\varepsilon>0$ there exists an ε-optimal control u^ε represented as follows;

there exist $0=t_0<t_1<\cdots<t_{m-1}<t_m=T$ and measurable functions

$$\phi_i\equiv\phi(t_i,\cdot):(]0,1[)^{M-1}\longrightarrow U \text{ such that}$$
$$u_t^\varepsilon=\phi(t_i,\pi_{t_i})\quad t_i\leqq t<t_{i+1}\quad i=0,1,\cdots,m-1$$

where

$$\pi_t^k\equiv P^u(x_t=k\,|\,\mathcal{Y}_t)$$
$$\mathcal{Y}_t\equiv\sigma(y_s;\ 0\leqq s\leqq t)$$
$$\pi_t\equiv(\pi_t^1,\cdots,\pi_t^{M-1}).$$

The Bellman equation is

$$(2.3)\quad \begin{cases} W_t(t,x)+\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{M-1}\Big[(g_{ti}-g_{tM})-\sum_{k=1}^{M-1}(g_{tk}-g_{tM})x_k\Big] \\ \times\Big[(g_{tj}-g_{tM})-\displaystyle\sum_{k'=1}^{M-1}(g_{tk'},-g_{tM})x_{k'}\Big]x_ix_jW_{x_ix_j}(t,x) \\ +\lambda(t,x,W_x(t,x),\phi(t,x,W_x(t,x)))=0 \\ \qquad \text{in}\quad [0,T)\times\Big\{x\in R^{M-1};\ \displaystyle\sum_{k=1}^{M-1}x_k<1,\ x_i>0,\ i=1,\cdots,M-1\Big\} \\ W(T,x)\equiv0\quad \text{in}\quad \Big\{x\in R^{M-1};\ \displaystyle\sum_{k=1}^{M-1}x_k<1,\ x_i>0,\ i=1,\cdots,M-1\Big\}. \end{cases}$$

REMARK 2.3. We will show in the proof of Theorem 2.1 that π_t^k take values in $]0,1[$.

PROOF. From the theory of filtering (Fujisaki, Kallianpur and Kunita [2], Liptser-Shiryayev [8]), we get

$$\widehat{\delta_{i,x_t}}-\widehat{\delta_{i,x_0}}=\int_0^t \widehat{\lambda_{x_{s-},i}(s,u_{s-})}ds+\int_0^t(\widehat{\delta_{i,x_s}g(s,x_s)}-\widehat{\delta_{ix_s-}}\widehat{g(s,x_s)})dI_s.$$

Here I_t is called the innovation process and a $\mathscr{Y}_t$-Brownian motion. Using the fact $\sum_{i=1}^{M}\pi_t^i\equiv 1$, this amounts to the following,

$$\pi_t^i-\pi_0^i=\int_0^t\left\{\sum_{k=1}^{M-1}\lambda_{ki}(s,u_s)\pi_s^k+\lambda_{Mi}(s,u_s)\left(1-\sum_{k=1}^{M-1}\pi_s^k\right)\right\}ds$$
$$+\int_0^t\pi_s^i\left(g(s,i)-\sum_{k=1}^{M-1}g(s,k)\pi_s^k-g(s,M)\left(1-\sum_{k=1}^{M-1}\pi_s^k\right)\right)dI_s \tag{2.4}$$
$$i=1,\cdots,M-1.$$

Corresponding to this, we consider the following stochastic differential equation

$$\tilde{\pi}_t^i-\tilde{\pi}_0^i=\int_0^t\left\{\sum_{k=1}^{M-1}\lambda_{ki}(s,\phi(s,\tilde{\pi}_s))\tilde{\pi}_s^k+\lambda_{Mi}(s,\phi(s,\tilde{\pi}_s))\left(1-\sum_{k=1}^{M-1}\tilde{\pi}_s^k\right)\right\}ds$$
$$+\int_0^t\tilde{\pi}_s^i\left\{g(s,i)-\sum_{k=1}^{M-1}g(s,k)\tilde{\pi}_s^k-g(s,M)\left(1-\sum_{k=1}^{M-1}\tilde{\pi}_s^k\right)\right\}dB_s \tag{2.5}$$
$$i=1,\cdots,M-1.$$

Here $\phi(t,x)\equiv\phi(t,x,W_x(t,x))$, and (although formally) we extend $\lambda_{ki}(t,\phi(t,x))$ to $[0,T]\times R^{M-1}$ preserving the Lipschitz continuity in x.

By the local Lipschitz continuity, taking the explosion into account, it has a unique solution. But in fact, we claim:

For each $u\in\mathfrak{A}$, $\pi_t^i>0$ a.s. $i=1,\cdots,M$. And

$$\tilde{\pi}_t^i>0\quad i=1,\cdots,M-1,\qquad \sum_{k=1}^{M-1}\tilde{\pi}_t^k<1\qquad\text{a.s.}$$

as we assumed that $\pi_0^i>0$ $i=1,\cdots,M$, the results follow from the theory of non-attainability (Friedman [1]) for the domain

$$G\equiv\left\{x\in R^{M-1};\ \sum_{k=1}^{M-1}x_k<1,\quad x_i>0\quad i=1,\cdots,M-1\right\}.$$

We should observe that the normal diffusion is zero and Fichera drift points inner at ∂G. The details are omitted.

Then if we consider about

$$u_t^\Delta \equiv \phi(t_i, \pi_{t_i}^\Delta), \quad t_i \leqq t < t_{i+1} \quad i=0, \cdots, m-1, \quad 0=t_0<t_1<\cdots<t_m=T,$$

we can show $E[|\tilde{\pi}_t-\pi_t^\Delta|^2]\to 0$ as $|\Delta|\equiv \max_i |t_{i+1}-t_i|$ tends to zero. (Here π_t^Δ corresponds to u_t^Δ, and I_t in (2.4) are replaced by B_t.) On the other hand, applying Ito's formula to (2.3), we get

$$\begin{aligned} W(T,\tilde{\pi}_T)-W(0,\tilde{\pi}_0) &= \int_0^T \Big\{ W_t(t,\tilde{\pi}_t) + \frac{1}{2}\sum_{i,j=1}^{M-1} \tilde{\pi}_t^i \tilde{\pi}_t^j \Big((g_{ti}-g_{tM}) \\ &\quad - \sum_{k=1}^{M-1}(g_{tk}-g_{tM})\tilde{\pi}_t^k \Big)\Big((g_{tj}-g_{tM}) - \sum_{k'=1}^{M-1}(g_{tk'}-g_{tM})\tilde{\pi}_t^{k'} \Big) W_{x_i x_j}(t,\tilde{\pi}_t) \\ &\quad + \sum_{i=1}^{M-1}\Big(\lambda_{Mi}(t,\phi(t,\tilde{\pi}_t)) + \sum_{k=1}^{M-1}(\lambda_{ki}(t,\phi(t,\tilde{\pi}_t)) \\ &\quad - \lambda_{Mi}(t,\phi(t,\tilde{\pi}_t)))\tilde{\pi}_t^k \Big) W_{x_i}(t,\tilde{\pi}_t) \Big\} dt + \text{mart.} \\ &= \text{mart.} - \int_0^T \sum_{k=1}^{M} L(t,k,\phi(t,\tilde{\pi}_t))\tilde{\pi}_t^k \, dt \end{aligned}$$

so

$$E\left[\int_0^T \sum_{k=1}^{M} L(t,k,\phi(t,\tilde{\pi}_t))\tilde{\pi}_t^k dt \right] = W(0,\pi_0),$$

where $\tilde{\pi}_t^M = 1-\sum_{k=1}^{M-1}\tilde{\pi}_t^k$. On the other hand, for all $u\in\mathfrak{A}$,

$$\begin{aligned} W(T,\pi_T)-W(0,\pi_0) &= \text{mart.} + \int_0^T \Big\{ W_t(t,\pi_t) + \frac{1}{2}\sum_{i,j=1}^{M-1} \pi_t^i \pi_t^j \Big((g_{ti}-g_{tM}) \\ &\quad - \sum_{k=1}^{M-1}(g_{tk}-g_{tM})\pi_t^k \Big)\Big((g_{tj}-g_{tM}) - \sum_{k'=1}^{M-1}(g_{tk'}-g_{tM})\pi_t^{k'} \Big) W_{x_i x_j}(t,\pi_t) \\ &\quad + \sum_{i=1}^{M-1}\Big(\lambda_{Mi}(t,u_t) + \sum_{k=1}^{M-1}(\lambda_{ki}(t,u_t)-\lambda_{Mi}(t,u_t))\pi_t^k \Big) W_{x_i}(t,\pi_t) \Big\} dt. \end{aligned}$$

From the definition of $\phi(t,x,p)$ and (2.3),

$$\begin{aligned} \text{The integrand} &\geqq W_t(t,\pi_t) + \frac{1}{2}\sum_{i,j=1}^{M-1} \pi_t^i \pi_t^j \Big((g_{ti}-g_{tM}) - \sum_{k=1}^{M-1}(g_{tk}-g_{tM})\pi_t^k \Big) \\ &\quad \times \Big((g_{tj}-g_{tM}) - \sum_{k'=1}^{M-1}(g_{tk'}-g_{tM})\pi_t^{k'} \Big) W_{x_i x_j}(t,\pi_t) \\ &\quad + \sum_{i=1}^{M-1}\Big(\lambda_{Mi}(t,\phi(t,\pi_t)) + \sum_{k=1}^{M-1}(\lambda_{ki}(t,\phi(t,\pi_t)) \\ &\quad - \lambda_{Mi}(t,\phi(t,\pi_t)))\pi_t^k \Big) W_{x_i}(t,\pi_t) \\ &= 0. \end{aligned}$$

So,
$$J[u]=E\int_0^T \sum_{k=1}^{M} L(t,k,u_t)\pi_t^k dt \geqq W(0,\pi_0).$$

This concludes the proof of the theorem. (Q. E. D.)

COROLLARY 2.1. *We assume* i)-v) *for the control system* (2.1), (2.2). *Let the partial differential operator of* (2.3) *be* L, *we can say the following. If there exists a solution of the following partial differential equation*

$$(2.3)' \quad \begin{cases} |LW(t,x)| \leqq \varepsilon_0 \\ \qquad \text{in } [0,T]\times\left\{x\in R^{M-1};\ \sum_{k=1}^{M-1} x_k<1,\ x_i>0,\ i=1,\cdots,M-1\right\} \\ |W(T,x)| \leqq \varepsilon_0 \quad \text{in } \left\{x\in R^{M-1};\ \sum_{k=1}^{M-1} x_k<1,\ x_i>0,\ i=1,\cdots,M-1\right\}, \end{cases}$$

then to any $\varepsilon>4\varepsilon_0$, *we have an* ε-*optimal control as in the theorem.*

Example 2.1. Let $M=2$ and $U=[0,1]$,

$$\begin{pmatrix} \lambda_{11}(t,z) & \lambda_{12}(t,z) \\ \lambda_{21}(t,z) & \lambda_{22}(t,z) \end{pmatrix} = \begin{pmatrix} -a(t)z & a(t)z \\ b(t)z & -b(t)z \end{pmatrix}$$

$$L(t,k,z)=c_k(t)z^2 \qquad k=1,2.$$

We assume the followings for the coefficients.

i) They are continuously differentiable in t.

ii) Let $\alpha(t)$ be $g(1,t)-g(2,t)$. There exist $\delta>0$, $K(>\delta)$ and $\varepsilon>0$ such that

$$K>|\alpha(t)|\geqq\delta$$
$$0\leqq a(t)\leqq K, \qquad 0\leqq b(t)\leqq K$$
$$\varepsilon\leqq c_k(t)\leqq K \qquad k=1,2.$$

iii) $0<\pi_0^1<1$.

Then the statement of the theorem follows.

PROOF. In this case,

$$\lambda(t,x,p,z)=p(-a(t)x+b(t)(1-x))z+(c_1(t)x+c_2(t)(1-x))z^2$$

and

$$\phi(t,x,p)=\begin{cases} \theta(t,x,p) & \text{if } 0\leqq\theta(t,x,p)\leqq1 \\ 1 & \text{if } \theta(t,x,p)\geqq1 \\ 0 & \text{if } \theta(t,x,p)\leqq0 \end{cases}$$

where

$$\theta(t,x,p)=\frac{-p(-a(t)x+b(t)(1-x))}{2(c_1(t)x+c_2(t)(1-x))}.$$

In this case, (2.3) becomes

$$(2.3)'' \quad \begin{cases} W_t(t,x)+\dfrac{1}{2}\alpha(t)^2x^2(1-x)^2W_{xx}(t,x)+\lambda(t,x,W_x(t,x),\phi(t,x,W_x(t,x)))=0 \\ W(T,x)=0 \quad 0\leqq t\leqq T, \quad 0<x<1. \end{cases}$$

We transform the variables as follows;

$$x'=f(x)=\begin{cases} \log x & \text{if } 0<x\leqq\dfrac{1}{3} \\ -\log(1-x) & \text{if } \dfrac{2}{3}\leqq x<1 \\ \text{and in } \dfrac{1}{3}\leqq x\leqq\dfrac{2}{3}, \text{ these are connected so that } f(x) \text{ is} \\ \text{smooth and strictly increasing.} \end{cases}$$

Letting the inverse function of f be $\tilde{f}$ and $\tilde{W}(t,x')=W(t,f(x'))$, (2.3)″ becomes

$$(2.3)''' \quad \begin{cases} \tilde{W}_t(t,x')+\dfrac{1}{2}\alpha(t)^2\tilde{f}(x')^2(1-\tilde{f}(x'))^2\left(\dfrac{d\tilde{f}}{dx'}\right)^{-2}\tilde{W}_{x'x'}(t,x') \\ \quad -\dfrac{d^2\tilde{f}}{dx'^2}\left(\dfrac{d\tilde{f}}{dx'}\right)^{-1}\tilde{W}_{x'}(t,x') \\ \quad +\lambda\left(t,\tilde{f}(x'),\left(\dfrac{d\tilde{f}}{dx'}\right)^{-1}\tilde{W}_{x'}(t,x'),\phi\left(t,\tilde{f}(x'),\left(\dfrac{d\tilde{f}}{dx'}\right)^{-1}\tilde{W}_{x'}(t,x')\right)\right)=0 \\ \tilde{W}(T,x)=0 \qquad 0\leqq t\leqq T, \qquad -\infty<x'<\infty. \end{cases}$$

This is a uniformly parabolic quasi-linear partial differential equation. From Ladyjenskaya, Solonnikov and Uraltseva [7], there exists a solution to (2.3)‴.

By the way, because W and ϕ do not satisfy the property v) uniformly, we treat them as follows. Let the processes z_t^1, z_t^2 be

$$(2.6) \quad \begin{cases} z_t^1=\pi_0^1+\displaystyle\int_0^t K(1-z_s^1)ds+\int_0^t\alpha(s)z_s^1(1-z_s^1)dB_s \\ z_t^2=\pi_0^1+\displaystyle\int_0^t(-Kz_s^2)ds+\int_0^t\alpha(s)z_s^2(1-z_s^2)dB_s. \end{cases}$$

And the same Brownian motion as the above one would be adopted to (2.4), (2.5). Then we claim that

$$\begin{cases} 0<z_t^2\leqq\pi_t^{\Delta 1}\leqq z_t^1<1 & 0\leqq t\leqq T \quad \text{a.s.} \\ 0<z_t^2\leqq\tilde{\pi}_t^1\leqq z_t^1<1 & 0\leqq t\leqq T \quad \text{a.s.} \end{cases}$$

At first, $0<z_t^i<1$, $i=1,2$, is easily obtained from the theory of non-attainability. Let $f(x)$ be $(x\wedge 0)^4$ and x_t be $\pi_t^{\Delta 1}-z_t^2$. Then by Ito's formula, $f(x_t)\leqq(4K+6K^2)\times\int_0^t f(x_s)ds+\text{mart.}$, hence $E[f(x_t)]\leqq(4K+6K^2)\int_0^t E[f(x_s)]ds$. Then $f(x_t)\equiv 0$, i.e.

$\pi_t^{\Delta 1} \geqq z_t^2 \; 0 \leqq t \leqq T$ a.s. The others are proved in the same way.

Now we define τ_n as $\inf\left\{t;\; z_t^1 \geqq 1-\frac{1}{n} \text{ or } z_t^2 \leqq \frac{1}{n}\right\}$, then $\tau_n \uparrow T \; (n\to\infty)$ a.s. Using the usual cut-off technique and the bounded convergence theorem, we can show

$$J[u] \geqq W(0, \pi_0^1) \quad \text{for all } u \in \mathfrak{A}$$

$$E\left[\int_0^T \sum_{k=1}^2 L(t, k, \phi(t, \tilde{\pi}_t^1))\tilde{\pi}_t^k dt\right] = W(0, \pi_0^1)$$

and

$$\pi_t^{\Delta 1} \longrightarrow \tilde{\pi}_t^1,\; u_t^{\Delta} \longrightarrow \phi(t, \tilde{\pi}_t^1) \;\; (|\Delta| \to 0) \text{ in prob. for all } t \text{ in } [0, T].$$

This concludes the proof. (Q. E. D.)

[2.2] *The counting observation case.*

In this subsection, we consider the following system.

$$\text{(2.7)} \qquad \begin{cases} x_t = x_0 + \displaystyle\int_0^{t+}\int_R F(s, x_{s-}, u_{s-}, v) N(ds, dv) \\ y_t = \displaystyle\int_0^{t+}\int_R G(s, x_{s-}, v) N(ds, dv) \end{cases}$$

$$\text{(2.8)} \qquad J[u] = E\left[\int_0^T L(t, x_t, u_t) dt\right]$$

where F and N were defined in the previous section, and

$$G(t, x, v) = \sum_{k=1}^M I_{\{x=k\}} I_{\{0 \leqq v + kK \leqq \rho_k(t)\}}.$$

We assume that $E[x_0^2] < \infty$, $\pi_0^i > 0 \; i=1, \cdots, M$, and define λ, ϕ as in the previous section. Then in addition to i), iv), v) in the previous section, we assume

ii)′ $\lambda_{ik}(t, u)$ are continuous as in ii) in the previous section, and there exists an $\varepsilon_0 > 0$ such that $\varepsilon_0 \leqq \lambda_{ik}(t, u) \leqq K$,

iii)′ $\rho_k(t)$ are continuous in t and $\varepsilon_0 \leqq \rho_k(t) < K$.

Replacing C, $\mathcal{B}_t(C)$ with C_+, $\mathcal{B}_t(C_+)$, the class of admissible controls is defined analogously as Definition 2.1. Here C_+ is the class of all real right continuous functions from $[0, T]$.

REMARK 2.4. As in the previous subsection, for each $u \in \mathfrak{A}$ (2.7) has a unique strong solution.

Since y_t is a random point process, from Liptser-Shiryayev [8], we obtain the equation satisfied by $\pi_t^k \equiv P(x_t=k \mid \mathscr{Y}_t)$. Let the conpensators of $(y_t, \mathscr{F}_t)$, $(y_t, \mathscr{Y}_t)$ be A_t and $\bar{A}_t$. Then

$$A_t = \sum_{k=1}^{M} \int_0^t I_{\{x_{s-}=k\}} \rho_k(s-) ds$$

and

$$\bar{A}_t = \int_0^t \int_R E[G(s, x_{s-}, v) \mid \mathscr{Y}_{s-}]\, ds dv = \sum_{k=1}^{M} \int_0^t \rho_k(s-) \pi_{s-}^k ds.$$

Corresponding to Lemma 2.2, we get the following

LEMMA 2.3.

$$V_t^i \equiv \delta_{ix_t} - \delta_{ix_0} - \int_0^t \lambda_{x_{s-}i}(s-, u_{s-}) ds \qquad i=1, \cdots, M$$

are right continuous $\mathscr{F}_t$*-martingales, and* $\langle v^i, m \rangle_t \equiv 0$. *Here* $m_t \equiv y_t - A_t$ ($\mathscr{F}_t$*-martingale*).

PROOF. By the Ito's formula,

$$V_t^i = \int_0^{t+} \int_R (\delta_{i, x_{s-}+F(s, x_{s-}, u_{s-}, v)} - \delta_{ix_{s-}}) \tilde{N}(ds, dv),$$

$$\begin{aligned} v_t^i m_t = & \int_0^{t+} \int_R \{(V_{s-}^i + \delta_{i, x_{s-}+F(s, x_{s-}, u_{s-}, v)} - \delta_{ix_{s-}}) \\ & \times (m_{s-} + G(s, x_{s-}, v)) - V_{s-}^i m_{s-}\} \tilde{N}(ds, dv) \\ & + \int_0^{t+} \int_R \{(V_s^i + \delta_{i, x_{s-}+F(s, x_{s-}, u_{s-}, v)} - \delta_{ix_{s-}})(m_s + G(s, x_s, v)) \\ & - V_s^i m_s - \delta_{ix_s})\}\, ds dv, \end{aligned}$$

where $N(ds, dv) = N(ds, dv) - dsdv$.

The first term is an $\bar{\mathscr{F}}_t$-martingale and

$$\text{the second term} = \int_0^{t+} \int_R (\delta_{i, x_s+F(s, x_s, u_s, v)} - \delta_{ix_s}) G(s, x_s, v)\, ds dv.$$

From the definition of F and G, $F(s, x_s, u_s, v)=0$ when $G(s, x_s, v) \neq 0$. (Q.E.D.)

Therefore writing $y_t - \bar{A}_t$ as $\bar{m}_t^u$, from [8] we get

$$\pi_t^i - \pi_0^i = \sum_{k=1}^{M} \int_0^t \lambda_{ki}(s, u_s) \pi_s^k ds + \int_0^{t+} \pi_{s-}^i \left(\frac{\rho_i(s-)}{\sum_{k=1}^{M} \rho_k(s-) \pi_{s-}^k} - 1 \right) d\bar{m}_s \tag{2.9}$$

$i=1, \cdots, M.$

Noting that $\sum_{k=1}^{M} \pi_t^k \equiv 1$, this proceeds to

THEOREM 2.2. *We assume* i), ii)′, iii)′, iv) *and* v) *for the control system* (2.7), (2.8). *If the Bellman equation described below (a non-linear first order partial differential-difference equation) has a smooth solution* W, *the following is proved.* (W *should be continuously differentiable in* (t, x), *and* W_x *be* β-*Hölder continuous in* t, *Lipschitz continuous in* x.)

That is to say, for all $\varepsilon>0$, *there exists an* ε-*optimal control* u^ε *represented as follows; there exist* $0=t_0<t_1<\cdots<t_{m-1}<t_m=T$ *and* $\phi_i=\phi(t_i,\cdot):(]0,1[)^{M-1}\to U$ *measurable such that* $u_t^\varepsilon=\phi(t_i,\pi_{t_i})\equiv\phi(t_i,\pi_{t_i}^1,\cdots,\pi_{t_i}^{M-1})$ *for* $t_i\leqq t<t_{i+1}$ $i=1,\cdots,M-1$. *The Bellman equation is*

$$
(2.10)\quad \left\{\begin{aligned}
& W_t(t,x)+\lambda(t,x,W_x(t,x),\ \phi(t,x,W_x(t,x)))-\left\{\sum_{k=1}^{M-1}(\rho_k(t)-\rho_M(t))x_k+\rho_M(t)\right\}\\
&\times\left\{W\left(t,\ \cdots,\frac{\rho_i(t)x_i}{\sum_{k'=1}^{M-1}(\rho_{k'}(t)-\rho_M(t))x_{k'}+\rho_M(t)},\cdots\right)-W(t,x)\right.\\
&\left.-\sum_{i=1}^{M-1}x_i\left(\frac{\rho_i(t)}{\sum_{k'=1}^{M-1}(\rho_{k'}(t)-\rho_M(t))x_{k'}+\rho_M(t)}-1\right)W_{x_i}(t,x)\right\}\\
&=0\\
&\qquad \text{in } [0,T]\times\left\{x\in R^{M-1};\ \sum_{k=1}^{M-1}x_k<1,\quad x_i>0\quad i=1,\cdots,M-1\right\}\\
& W(T,x)\equiv 0\quad \text{in } \left\{x\in R^{M-1};\ \sum_{k=1}^{M-1}x_k<1,\quad x_i>0\quad i=1,\cdots,M-1\right\}.
\end{aligned}\right.
$$

REMARK 2.5. We will show in the proof of Theorem 2.2 that π_t^k's take values in $]0,1[$.

PROOF. At first, we consider the following system,

$$
(2.11)\quad \left\{\begin{aligned}
&\tilde{x}_t=x_0+\int_0^{t+}\int_R F(s,\tilde{x}_{s-},\phi(s-,\tilde{\pi}_{s-}),v)N(ds,dv)\\
&\tilde{\pi}_t^i=\pi_0^i+\sum_{k=1}^{M}\int_0^t\lambda_{ki}(s,\phi(s,\tilde{\pi}_s))\tilde{\pi}_s^k ds\\
&\qquad+\int_0^{t+}\int_R\tilde{\pi}_{s-}^i\left(\frac{\rho_i(s)}{\sum_{k=1}^{M}\rho_k(s)\tilde{\pi}_s^k}-1\right)G(s,\tilde{x}_{s-},v)N(ds,dv)\\
&\qquad-\int_0^t\tilde{\pi}_s^i\left(\frac{\rho_i(s)}{\sum_{k=1}^{M}\rho_k(s)\tilde{\pi}_s^k}-1\right)\sum_{k'=1}^{M}\rho_{k'}(s)\tilde{\pi}_s^{k'}ds\qquad i=1,\cdots,M.
\end{aligned}\right.
$$

For this, we claim that

(2.11) has a strong solution, and

$$\bar{\pi}_t^i>0, \quad i=1, \cdots, M, \quad \sum_{k=1}^M \bar{\pi}_t^k \equiv 1 \quad \text{a.s.}$$

We define $g: R \to R$ Lipschitz continuously as follows

$$g(x)=\begin{cases} 1 & \text{if } x \geqq \varepsilon_0 \\ 0 & \text{if } x \leqq \frac{\varepsilon_0}{2} \\ 0 \leqq g(x) \leqq 1 & \text{otherwise,} \end{cases}$$

and corresponding to (2.11), consider

$$\text{(2.12)} \quad \begin{cases} \bar{x}_t = x_0 + \int_0^{t+}\int_R F(s, \bar{x}_{s-}, \phi(s, \bar{\pi}_{s-}), v) N(ds, dv) \\ \bar{\pi}_t^i = \pi_0^i + \sum_{k=1}^M \int_0^t \lambda_{ki}(s, \phi(s, \bar{\pi}_s)) \bar{\pi}_s^k ds \\ \qquad + \int_0^{t+}\int_R \bar{\pi}_{s-}^i \left(\frac{\rho_i(s-)}{\sum_{k=1}^M \rho_k(s-) \bar{\pi}_{s-}^k} - 1 \right) g\left(\sum_{k=1}^M \rho_k(s-) \bar{\pi}_{s-}^k \right) \\ \qquad \times G(s, \bar{\pi}_{s-}, v) N(ds, dv) \\ \qquad - \int_0^t \bar{\pi}_s^i \left(\frac{\rho_i(s)}{\sum_{k=1}^M \rho_k(s) \bar{\pi}_s^k} - 1 \right) g\left(\sum_{k=1}^M \rho_k(s) \bar{\pi}_s^k \right) \sum_{k'=1}^M \rho_{k'}(s) \bar{\pi}_s^{k'} ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=1, \cdots, M. \end{cases}$$

Clearly this has a unique strong solution. If we show that $\bar{\pi}_t$ satisfies the statement of the lemma, the proof is completed.

Writing the jump times of $(\bar{x}_t, \bar{\pi}_t)$ as $0<\sigma_1<\sigma_2<\cdots(\uparrow\infty)$, we define τ as

$$\tau \equiv \inf\{t>0;\ \bar{\pi}_t^i \leqq 0 \text{ for some } i\}.$$

We have only to see $\tau=\infty$ a.s.

1°) $\sigma_1 \leqq \tau$ a.s.

In fact, suppose $\tau(\omega)<\sigma_1(\omega)$. In $0 \leqq t \leqq \tau$, there exists a $K'(\geqq \varepsilon_0)$ such that

$$\frac{d\bar{\pi}_t^i}{dt} = \sum_{k=1}^M \lambda_{ki}(t, \phi(t, \bar{\pi}_t)) \bar{\pi}_t^k - \bar{\pi}_t^i \left(\frac{\rho_i(t)}{\sum_{k=1}^M \rho_k(t) \bar{\pi}_t^k} - 1 \right) g\left(\sum_{k=1}^M \rho_k(t) \bar{\pi}_t^k \right) \sum_{k'=1}^M \rho_{k'}(t) \bar{\pi}_t^{k'}$$

$$\geqq -K' \bar{\pi}_t^i + \sum_{k(\neq i)} \lambda_{ki}(t, \phi(t, \bar{\pi}_t)) \bar{\pi}_t^k.$$

So if $0 \leqq \bar{\pi}_t^i \leqq \varepsilon_0/4k$, then $d\bar{\pi}_t^i/dt \geqq \varepsilon_0/2>0$. This is a contradiction.

2°) $\sigma_1<\tau$ a.s.

From 1°) $\bar{\pi}^i_{\sigma_1 -}>0$ $i=1,\cdots,M$, $\sum_{k=1}^{M}\bar{\pi}^k_{\sigma_1 -}=1$. Therefore $\bar{\pi}^i_{\sigma_1}\geqq\bar{\pi}^i_{\sigma_1 -}+(\varepsilon_0/K-1)\bar{\pi}^i_{\sigma_1 -}$ $=(\varepsilon_0/K)\bar{\pi}^i_{\sigma_1 -}>0$.

3°) In the same way, we can see $\tau=\infty$ a.s. finally.

Now we return to the proof of the theorem. Corresponding to (2.11), we define $\tilde{y}_t$ as $\tilde{y}_t\equiv\int_0^{t+}\int_R G(s,\tilde{x}_{s-},v)N(ds,dv)$, then it is easily checked that $\mathscr{Y}_t=\tilde{\mathscr{Y}}_t\equiv\sigma(\tilde{y}_s;\ 0\leqq s\leqq t)$.

The rest of the proof is similar to that in the previous subsection, so will be omitted. (Q.E.D.)

§3. The general case

We now consider the general case. In this section, because we discuss the story in the space of all continuous functions, we will adopt as admissible any control for which the system equation has a weak solution.

Here we will pose quite strong hypotheses, and moreover, have not yet found satisfactory examples except for few facts.

Now the state space of x_t and y_t are R^n, R^m. When we want to minimize the cost

$$J[u]=E\left[\int_0^t L(t,x_t,u_t)dt\right] \tag{3.1}$$

under the system

$$\begin{cases} x_t=x_0+\int_0^t a(s,x_s,u_s)ds+w_{1t} \\ y_t=\int_0^t g(s,x_s)ds+w_{2t}, \end{cases} \tag{3.2}$$

the separation principle will effect also in this case. This is what we want to say.

First of all, we make the following assumptions;

i) $\{x_0, w_{1t}, w_{2t};\ t\geqq 0\}$ are mutually independent, and w_1, w_2 are respectively n-dim., m-dim. Brownian motions.

ii) x_0 has a density, and if we write it as $\phi: R^n\to R$, there exists s $\left(>\frac{n}{2}+2\right)$ such that $\phi\in W_2^s$. (Here W_2^s is the Sobolev space.) Further it is twice continuously differentiable, and bounded as well as its first and second derivatives.

iii) Let us write the control region as $U\subset R^d$.

$$a:[0,T]\times R^n\times U\longrightarrow R^n$$
$$g:[0,T]\times R^n\longrightarrow R^m$$

are both s-time differentiable, and they are continuous in (t, x) and bounded as well as all their derivatives. Furthermore there exists $K>0$ such that

$|a(t, x, z_1)-a(t, x, z_2)| \leqq K|z_1-z_2|$ for x in R^n, t in $[0, T]$, z_1, z_2 in U.

iv) The second derivatives of ϕ (with respect to x) and the first derivatives of a, g are locally Hölder continuous.

v) L: $[0, T]\times R^n\times U\to R$ is bounded.

vi) Let us define $V\equiv W_2^1\equiv H^1(R^n)$, $V_+\equiv\{u\in V;\ u\geqq 0 \text{ a.s.}\}$, $V_+^1\equiv V_+\cap L^1(R^n)$, $H\equiv L^2(R^n)$, and the norm of V, H would be $\|\cdot\|$, $|\cdot|$ respectively. Further we define $\lambda: [0, T]\times V_+^1\times V\times U\to R$ as

$$\lambda(t, x, p, z)\equiv\int_R q(x)\left[\sum_{i=1}^n a_i(t, x, z)\frac{\partial}{\partial x_i}p(x)+L(t, x, z)\right]dx.$$

The $z\in U$ minimizing λ —— which we will write as $\psi(t, x, p)$ —— is measurable in t, and there exists $K>0$ such that

$$|\psi(t, q_1, p_1)-\psi(t, q_2, p_2)|\leqq K\{|q_1-q_2|+\|p_1-p_2\|\}$$
$$\text{for } t \text{ in } [0, T],\ q_1, q_2 \text{ in } V_+^1,\ p_1, p_2 \text{ in } V.$$

Let $C_m\equiv\{f: [0, T]\to R^m \text{ continuous}\}$, $\mathcal{B}_t(C_m)$ are defined as before and $\mathcal{B}(C_m)=\mathcal{B}_T(C_m)$. Suppose that μ_0 is the Wiener measure on C_m, then $\bar{\mathcal{B}}(C_m)$ and $\bar{\mathcal{B}}_t(C_m)$ are the completion of $\mathcal{B}(C_m)$ and $\mathcal{B}_t(C_m)$ with respect to μ_0.

u is an admissible control (i.e. $u\in\mathfrak{A}$) if and only if there exists

$$\phi: [0, T]\times C_m\longrightarrow U\ (\bar{\mathcal{B}}_t(C_m)\text{-progressively measurable})$$

and u_t is represented as $\phi(t, y)$.

REMARK 3.1. From the boundedness condition of a and g, it is well-known that (3.2) has a unique distributional solution for each $u\in\mathfrak{A}$. And if we write the probability measures generated on C_{n+m} by (x, y), (w_1, w_2) as μ_1^u, μ_0, $\mu_1^u\sim\mu_0$.

Before stating the theorem, let us remember some results about filtering. Let the representative element of C_{n+m} be (ξ, η), and

$$\begin{cases} \mathcal{Y}_t\equiv\sigma(y_s: 0\leqq s\leqq t) \\ \quad p_0(t, x)\equiv\int_{R^n}(2\pi t)^{-n/2}\exp(-|x-z|^2/2t)\phi(z)dz \\ \Lambda_t^u(\xi_0^t, \eta_0^t)\equiv d\mu_1^u/d\mu_0(\xi_0^t, \eta_0^t) \\ \tilde{\Lambda}_t^u(x, \eta_0^t)\equiv E_{\mu_0}[\Lambda_t(\xi_0^t, \eta_0^t)\,|\,\sigma(\xi_t, \eta_0^t)]_{\xi_t=x}. \\ \Phi^u(t, x, \eta)\equiv p_0(t, x)\tilde{\Lambda}_t(x, \eta_0^t). \end{cases}$$

Then, following Rozovskii [10], Zakai [13], $\pi_t(x)$ is represented as

$$\pi_t(x)=\frac{\Phi^u(t,x)}{\int_R \Phi^u(t,z)dz}$$

where $\Phi^u(t,x)$ satisfies the following stochastic partial differential equation;

$$(3.3)\quad \begin{cases} d\Phi^u(t,x)=\left\{1/2\sum_{i=1}^{n} D_{x_ix_i}\Phi^u(t,x)-\sum_{i=1}^{n} D_{x_i}(a_i(t,x,\tilde{\phi}(t,\eta))\Phi^u(t,x))\right\}dt \\ \qquad +\sum_{i=1}^{m} g_i(t,x)\Phi^u(t,x)d\eta_t^i \\ \Phi^u(0,x)=\phi(x) \qquad \text{for } t \text{ in } [0,T],\ x \text{ in } R \end{cases}$$

where $\tilde{\phi}$ corresponds to $u\in\mathfrak{A}$.

On the other hand, from the functional analytic point of view, this is said as follows. From now on, to avoid confusion, the control will be written as α. We define $A^\alpha(t): V\to V'$, $B_k(t): H\to H$ as

$$<A^\alpha(t)u,v>\equiv\frac{1}{2}\sum_{i=n}^{n}\int\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx-\sum_{i=1}^{n}\int_{R^n} a_i(t,x,\alpha_t)u(x)\frac{\partial v(x)}{\partial x_i}dx, \qquad u,v \text{ in } V,$$

$$(B_k(t)u)(x)\equiv g_k(t,x)u(x) \qquad u \text{ in } H \qquad k=1,\cdots,m,$$

where $V\equiv W_2^1(R^n)$, $H\equiv L^2(R^n)$, $V'\equiv W_2^{-1}(R^n)$. Letting Ω be C_m and $\tilde{\mu}_0$ be the Wiener measure on C_m, we consider on $(\Omega,\tilde{\mu}_0)$

$$(3.4)\quad \begin{cases} d\tilde{\Phi}^\alpha(t)+A^\alpha(t)\tilde{\Phi}^\alpha(t)dt=B(t)\tilde{\Phi}^\alpha(t)d\eta_t \\ \tilde{\Phi}^\alpha(0)=\phi \end{cases}$$

(Krylov-Rozovskii [5], Pardoux [9]). Then, using Sobolev's lemma, $\tilde{\Phi}^\alpha(t,x)=\Phi^\alpha(t,x)$ is obtained.

With the above preparation at hand, we proceed to

THEOREM 3.1. *We assume* i)-v) *for the control system* (3.1), (3.2). *If the Bellman equation* (3.5) (*a non-linear partial differential equation on the infinite dimensional space*) *has a solution* $W:[0,T]\times H\to R$ *satisfying* $(W\text{-}1)$-$(W\text{-}6)$ *below, then does the separation principle effects in the following meaning.*

That is to say, there exists a $\theta:[0,T]\times V\to U$ *measurable such that* $u_t^0=\theta(t,\phi(t))$. *Here the Bellman equation is*

$$(3.5)\quad \begin{cases} \inf_{z\in U}[W_t(t,x)-<A^\alpha(t)u,W_u(t,u)>+(W_{uu}(t,u)B(t)u,B(t)u)_{(L^2)^m} \\ \qquad +(L(t,\cdot,z),u)_{L^2}]=0 \\ W(T,u)=0 \qquad\qquad \textit{for } t \textit{ in } [0,T], u \textit{ in } V_+^1 \end{cases}$$

where for $z\in U, A^z(t): V\to V'$ *is defined as*

$$<A^z(t)u, v>\equiv\frac{1}{2}\sum_{i=1}^{n}\int_{R^n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx-\sum_{i=1}^{n}\int_{R^n}a_i(t,x,z)u(x)\frac{\partial v(x)}{\partial x_i}dx$$
$$\text{for } u,v \text{ in } V.$$

And the conditions to be satisfied by W are;

(*W*-1) *it is differentiable in t, and twice differentiable in u,*

(*W*-2) W, W_t, W_u *are bounded,*

(*W*-3) $W:[0,T]\times H\to R,\ W_t:[0,T]\times H\to R,\ W_u:[0,T]\times H\to H$ *are continuous,*

(*W*-4) *for all Q in* $\mathcal{L}^1(H)$, $\mathrm{Tr}[W_{uu}(\cdot,\cdot)\circ Q]:[0,T]\times H\to R$ *is continuous,*

(*W*-5) $W_u|_{[0,T]\times V}$ *takes the value on V and there exists* $K>0$ *such that*

(*W*-6) $\|W_u(t,u)-W_u(t,v)\|\leqq K|u-v|$ *for t in* $[0,T]$, u,v *in V*
$\|W_u(t,u)\|\leqq K(1+\|u\|)$ u *in V.*

PROOF. First of all, we extend the $\phi(t,u,v)$ defined in vi) to $[0,T]\times H\times V$, preserving Lipschitz continuity in the second and third variables. (For example, first, extend it to $[0,T]\times(V\cap L^1)\times V$ by $\hat{\phi}(t,u,v)\equiv\phi(t,u^+,v)$, where $u^+(x)\equiv u(x)\vee 0$. Then it can be extended uniquely to $[0,T]\times H\times V$, because $V\cap L^1$ is dense in H.) The extended one will be written as ϕ again. Now, we define $\bar{A}$: $[0,T]\times V\to V'$ as

$$<\bar{A}(t,u),v>\equiv\frac{1}{2}\sum_{i=1}^{n}\int_{R^n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx$$
$$-\sum_{i=1}^{n}\int_{R^n}a_i(t,x,\phi(t,u,W_u(t,u)))u(x)\frac{\partial v(x)}{\partial x_i}dx$$
$$u,v \text{ in } V.$$

(We have expressed as above to emphasize non-linearity, and corresponding to this, $B_k:[0,T]\times H\to H$ will be written as $B_k(t)u=B_k(t,u)$.)

Then we claim that the stochastic partial differential equation

$$\left\{\begin{array}{l} d\Phi^0(t)+\bar{A}(t,\Phi^0(t))dt=B(t,\Phi^0(t))d\eta_t \\ \Phi^0(0)=\phi \end{array}\right. \tag{3.6}$$

has a unique H-valued well measurable solution in $L^2(\Omega\times]0,T[;V)\cap L^2(\Omega;C([0,T];H))$; we will make use of Pardoux [9].

We note that there exist $\lambda>0$, $K>0$ such that

$$<\bar{A}(t,u),u>\geqq\frac{1}{4}\|u\|^2-\lambda|u|^2,\quad |<\bar{A}(t,u),v>|\leqq K\|u\|\,\|v\|.$$

Defining $\bar{A}_j$ $j=1,2,\cdots$ as below, to each of these (3.6) has a solution Φ^j. Then Φ^j converge to what we want.

First, we define a Lipschitz continuous function

$$f_j: R_+ \longrightarrow [0,1] \qquad j=1,2,\cdots \qquad \text{as}$$

$$f_j(x)=\begin{cases} 1 & \text{if} \quad 0\leqq x\leqq j \\ 0 & \text{if} \quad x\geqq j \\ 0\leqq f_j(x)\leqq i & \text{otherwise.} \end{cases}$$

Then let $\rho_j: R_+\to[0,1]$ $j=1,2,\cdots$ be defined as $\rho_j(u)=f_j(|u|_H)$ $(u\in H)$. These are obviously Lipschitz continuous. With these, $\bar{A}_j:[0,T]\times V\to V'$ $j=1,2,\cdots$ are defined as

$$<\bar{A}_j(t,u),v>\equiv\frac{1}{2}\sum_{i=1}^{n}\int_{R^n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx-\rho_j(u)\sum_{i=1}^{n}\int_{R^n}a_i(t,x,\phi(t,u,W_u(t,u)))u(x)\frac{\partial v}{\partial x_i}dx.$$

Then there exist $\lambda_j>0$ $j=1,2,\cdots$ such that

$$<\bar{A}_j(t,u)-\bar{A}_j(t,v),\ u-v>\geqq\frac{1}{4}\|u-v\|^2-\lambda_j|u-v|^2.$$

So from Pardoux [9], there exists a unique solution Φ^j to

$$\text{(3.7)} \qquad \begin{cases} d\Phi^j(t)+\bar{A}_j(t,\Phi^j(t))dt=B(t,\Phi^j(t))d\eta_t \\ \Phi^j(0)=\phi. \end{cases}$$

Now for $\tau_j\equiv\inf\{t;|\Phi^j(t)|\geqq j\}$, we define $\Phi^0(t)$ as $\Phi^0(t)\equiv\Phi^j(t)$ $t\leqq\tau_j$. From the energy inequality, this is proved to be the one that we want.

We return to the proof of the theorem. As Φ^0 can be constructed by the successive approximation method, there exists a $\bar{\mathcal{B}}_t(C_m)$-progressively measurable function $F:[0,T]\times C_m\to V$ such that $\Phi^0(t)=F(t,\eta)$. Then if we define u^0 as

$$u_t^0=\phi(t,\Phi^0(t),\ W_u(t,\Phi^0(t)))=\tilde{\phi}(t,\eta),$$

this belongs to $\mathfrak{A}$. And Φ^{u^0} corresponding to this u^0 must satisfy

$$\text{(3.8)} \qquad \begin{cases} d\Phi^{u^0}(t)+A^{u^0}(t,\Phi^{u^0}(t))dt=B(t,\Phi^{u^0}(t))d\eta_t \\ \Phi^{u^0}(0)=\phi. \end{cases}$$

On the other hand, (3.8) is satisfied also by Φ^0, so $\Phi^0=\Phi^{u^0}$ by the uniqueness.

Now we show that this u^0 is in fact the optimal control. (We write $\phi(t,v,W_v(t,v))$ as $\theta(t,v)$.) First

$$J[u^0]=E\left[\int_0^T L(t,x_t,u_t^0)dt\right]$$

$$=E_{\mu 1}^{u^0}\left[\int_0^T L(t,\xi_t,\tilde{\phi}(t,\eta))dt\right]$$
$$=\int_0^T E_{\mu_0}\left[\int_{R^n} \Phi^0(t,x)L(t,x,\tilde{\phi}(t,\eta))dx\right]dt.$$

(Here the last equality is obtained by the independence of ξ and η with respect to μ_0.)

Then, by the Bellman equation (3.5),

$$J[u^0]=-E_{\mu_0}\left[\int_0^T \{W_t(t,\Phi^0(t))-<A^{u^0}(t,\Phi^0(t)),W_u(t,\Phi^0(t))>\right.$$
$$\left.+(W_{uu}(t,\Phi^0(t))B(t,\Phi^0(t)),\ B(t,\Phi^0(t)))_{(L^2)^m}\}dt\right],$$

and so, applying Ito's formula (Pardoux [9]), we obtain

$$=W(0,\phi)-W(T,\Phi^0(T))=W(0,\phi).$$

In the same way, we see that for each $u\in\mathfrak{A}$, $J[u]\geqq W(0,\phi)$. This concludes the proof of the theorem. (Q.E.D.)

Example 3.1. $U=[-1,1]$, $n=m=1$.

We consider the following system

$$\begin{cases} x_t=x_0+\int_0^t\{\beta(s,x_s)u_s+\gamma(s,x_s)\}ds+w_{1t} \\ y_t=\int_0^t g(s,x_s)ds+w_{2t} \end{cases}$$
$$J[u]=E\left[\int_0^T\{\eta(t,x_t)u_t+\delta(t,x_t)\}dt\right].$$

If we assume the following,

i) β,γ,δ,η belong to $H^{k,k/2}(R\times[0,T])$ $(k\geqq 3)$,

(Here $H^{k,k/2}$ is the Hölder space.)

ii) there exists $M>0$ independent of s such that $\|\delta(s,\cdot)-\eta(s,\cdot)\|_{L^1}\leqq M$

iii) η is large enough,

then the optimal control is $u_t^0\equiv -1$. But we can see that the optimal control for the following completely observable system

$$x_t=x_0+\int_0^t\{\beta(s,x_s)u_s+\gamma(s,x_s)\}ds+w_t$$
$$J[u]=E\left[\int_0^T\{\eta(t,x_t)u_t+\delta(t,x_t)\}dt\right]$$

is also $u_t^0 \equiv -1$, so the above result might be a matter of course.

Example 3.2. $x_t, y_t \in R^n$, and A_t, B_t, F_t, M_t, N_t are matrices. $M_t \geqq 0, N_t > 0$. We consider the following linear system,

$$\begin{cases} dx_t = A_t x_t dt + B_t u_t dt + dw_{1t} \\ dy_t = F_t x_t dt + dw_{2t} \\ x_0 : \text{a non degenerated Gaussian, } y_0 = 0 \end{cases}$$

$$J[u] = E\left[\int_0^T (x_t' M_t x_t + u_t' N_t u_t) dt\right].$$

As we were obliged to suppose boundedness of the coefficients, this is not included in the above discussion. But if we formally solve (3.5) (i.e. we regard it as the equation on $\mathcal{S}$), it is obtained that

$$W(t, u) = \int_{R^n} (x' P_t x + q_t) u(x) dx,$$

where

$$\begin{cases} dP_t/dt = -2A_t' P_t - M_t + P_t' B_t N_t^{-1} B_t P_t \\ dq_t/dt = \mathrm{Tr}\{-P_t - P_t' B_t N_t^{-1} B_t' P_t Q_t\} \\ P_T = 0, \quad q_T = 0. \end{cases}$$

(Here Q_t is the error matrix satisfying the well-known Riccati equation.) This is equivalent to that in Wonham [12]. And we can also see that

$$u_t^0 = -N_t^{-1} B_t' P_t \hat{x}_t.$$

Further, we have got the following fact. Let $n=1$. We consider two systems, to each of which corresponds F_t^1 and F_t^2 and the other coefficients are the same. Let $Q_t^i, u_t^i, P_t^i, q_t^i$, $i=1,2$ be the corresponding errors, optimal controls, P_t's and q_t's. If $|F_t^1| \leqq |F_t^2|$, then $Q_t^1 \geqq Q_t^2$ (Ihara [2]). This is quite reasonable. So

$$P_t^1 = P_t^2, \quad dq_t^1/dt = dq_t^2/dt.$$

Then we conclude $J[u^1] \geqq J[u^2]$ as expected.

§4. The case when only $\mathcal{Y}_{t-h}$ is available at time t (linear case)

Here the control system is the same as that of Wonham [12]. But we consider the case when at time t, because of the calculation time or the time for information transmitting, only $\mathcal{Y}_{t-h}$ is available. We show the effectiveness of the separation principle in this case, and then, that the system is stable when $h \downarrow 0$.

The proof is similar to that of Wonham, so will be omitted.

We consider the following linear system

$$(4.1)\qquad \begin{cases} dx_t = A_t x_t dt + b(t, u_t)dt + C_t dw_{1t} \\ dy_t = F_t x_t dt + G_t dw_{2t} \\ x_0 : \text{non degenerated Gaussian}, \quad y_0 = 0 \end{cases}$$

$$(4.2)\qquad J[u] = E\left[\int_0^T L(t, x_t, u_t)dt\right].$$

We assume (A. 1)-(A. 7) in Wonham [12].

Let the control region $U \subset R^m$ be compact and convex.

$$\mathcal{B}_t^h(C_n) \equiv \begin{cases} \{C_n, \phi\} & \text{if } 0 \leqq t \leqq h \\ \mathcal{B}_{t-h}(C_n) & \text{if } t > h. \end{cases}$$

DEFINITION 4.1. We fix some $a \in U$. u is an admissible control (i.e. $u \in \mathfrak{A}^h$) if and only if there exists a $\mathcal{B}_t^h(C_n)$-progressively measurable function ϕ: $[0, T] \times C_n \to U$ such that

$$u_t = \begin{cases} \phi(t, y) & \text{if } t \geqq h \\ a & \text{if } 0 \leqq t < h. \end{cases}$$

Here

i) There exists $K > 0$ such that

$$|\phi(t, f) - \phi(t, g)| \leqq K \sup_{0 \leqq s \leqq t-h} |f(s) - g(s)| \qquad \text{for } t \text{ in } [h, T], \quad f, g \text{ in } C_n.$$

ii) ϕ is α-Hölder continuous in t. We define

$$\mathcal{Y}_t^h \text{ and } \hat{x}_t^h \text{ as } \mathcal{Y}_t^h \equiv \begin{cases} \{\Omega, \phi\} & \text{if } 0 \leqq t < h \\ \sigma(y_s;\ 0 \leqq s \leqq t-h) & \text{if } t \geqq h, \end{cases}$$

$$\hat{x}_t^h \equiv E[x_t | \mathcal{Y}_t^h].$$

Then we get the following

THEOREM 4.1. *We assume* (A. 1)-(A. 8) *in Wonham* [12]. *Then*

i) *There exists a function* $\hat{\phi}^h : [0, T] \times R^n \to U$ *such that the optimal control* $u^h \in \mathfrak{A}^h$ *is represented as* $u_t^h = \hat{\phi}^h(t, x_t^h)$. $\hat{\phi}^h$ *is obtained by solving the Bellman equation.*

ii) *As* $h \downarrow 0$,

$$u_t^h \longrightarrow u_t^0, \quad x_t^{u^h} \longrightarrow x_t^{u^0}, \quad \widehat{x_t^{u^h, h}} \longrightarrow \widehat{x_t^{u^0}} \equiv \widehat{x_t^{u^0, 0}} \quad in\ L^2.$$

(Then in particular, $J[u^h]\to J[u^0]$.)

REMARK 4.1. Let us assume

$$U=R^n, \quad b(t,u)=B_t u$$
$$L(t,x,u)=x'M_t x+u'N_t u \qquad M_t\geqq 0, \quad N_t>0.$$

This is not included in the above case, but we easily obtain that

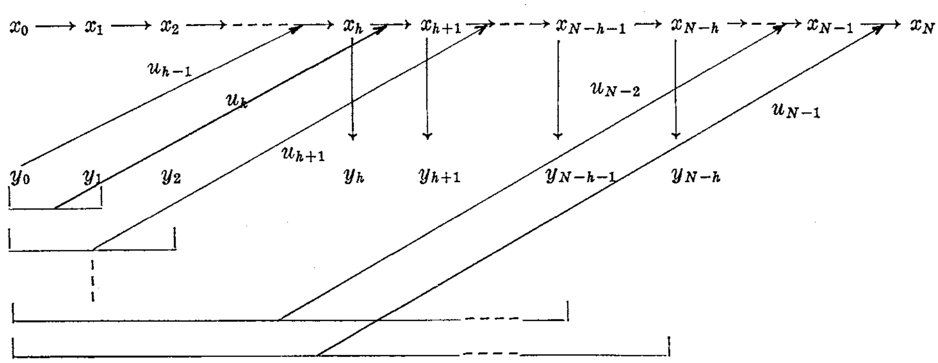
$$\hat{\phi}^h(t,\xi)=-N_t^{-1}B_t'P_t\xi \quad \text{(independent of } h).$$

§ 5. Consideration about the non-linear case of § 4

The result obtained in the previous section is consistent with our claim that we can abandon the data except for that about the present state.

But in fact, this has come from a particular reason, and for the general case, the more careful consideration will be needed. We will conclude the story by outlining this below.

Let us reconstruct the model of § 1 for § 5, then it becomes as follows.



$$J[u]=E\left[\sum_{k=h}^{N} L_k(x_k, u_{k-1})\right]$$

When formulated, the next quantities are given;

i) The distribution of x_0

ii) $P^u(y_n=j\,|\,x_n=k,\ x_0^{n-1},\ y_0^{n-1})=P^u(y_n=j\,|\,x_n=k)\equiv q_n^{kj}$ (independent of controls)

$$k=1,\cdots,M_1, \quad j=1,\cdots,M_2, \quad n=0,1,\cdots,N-h$$

iii) $P^u(x_{n+1}=k\,|\,x_n=i,\ x_0^{n-1},\ y_0^{n-h+1})=P^u(x_{n+1}=k\,|\,x_n=i)=P_n^{ik}(u_n)$

$$i=1,\cdots,M_1, \quad k=1,\cdots,M_1, \quad n=h-1,\cdots,N-1.$$

iv) $P(x_{n+1}=k|x_n=i)=P_n^{ik}$ (independent of controls)

$$i=1,\cdots,M_1,\quad k=1,\cdots,M_1,\quad n=0,\cdots,h-2.$$

And corresponding to Section 1, $\mathcal{F}_{k-h}$-adapted processes are adopted as admissible controls.

We define $\pi_n^i, \pi_n^{h,i}$ as

$$\pi_n^i \equiv P^u(x_n=i|y_0^n),\quad \pi_n^{h,i} \equiv P^u(x_n=i|y_0^{n-h+1}).$$

Then the optimal control is proved to be of the form

$$u_n^h=\phi_n(\pi_{n-h+1}, u_{n-h+1}^h, \cdots, u_{n-1}^h),$$

and not represented as

$$u_n^h=\tilde{\phi}_n(\pi_n^{h,1}, \cdots, \pi_n^{h,M_1})$$

in general. This says that we must collect the data about the state at time $t-h$, and observe the adopted control since then.

It may be expected that this is the case even in the continuous time, but at present we cannot even formulate it.

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