On the absolute continuity of the law of a system of multiple Wiener integral

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1. Introduction.

Let $\{B_t;\ 0\leq t\leq 1\}$ be a standard Brownian motion starting from the origin, $L^2(B)$ be the set of all square integrable functionals of the Brownian motion $\{B_t;\ 0\leq t\leq 1\}$, and $\mathscr{H}_n$ be the set of all multiple Wiener integrals of degree n, that is to say,

$$\mathscr{H}_n=\left\{\int_0^1\cdots\int_0^1 g(s_1,\ \cdots,\ s_n)dB_{s_1}\cdots dB_{s_n};\ g\in\hat{L}^2([0,1]^n)\right\},$$

where

$$\int_0^1\cdots\int_0^1 g(s_1,\cdots,s_n)dB_{s_1}\cdots dB_{s_n}$$

is a multiple Wiener integral of g and $\hat{L}^2([0,1]^n)$ is the set of all symmetric Lebesgue square integrable functions on $[0,1]^n$. It was proved by Ito [2] that $L^2(B)=\sum_{n=0}^{\infty}\oplus\mathscr{H}_n$.

Let $\mathscr{H}$ denote $\bigcup_{m=0}^{\infty}\sum_{n=0}^{m}\mathscr{H}_m$, then $\mathscr{H}$ must be considered a set of polynomials in Brownian motion. Our interest is in studying the probability law ν_Y on $\boldsymbol{R}^n$ induced by $Y=(Y_1,\cdots,Y_n)$ such that all Y_k's are elements of $\mathscr{H}$. Recently Shigekawa [4] obtained a sufficient condition for ν_Y to be absolutely continuous by using Malliavin's calculus. In this paper we give a necessary and sufficient condition for that by algebraic method.

Our theorem is the following.

THEOREM. *Let $Y_1,\cdots,Y_n$ be elements of $\mathscr{H}$. Let $I=\{f;\ f$ is a polynomial in $y_1,\cdots,y_n$ satisfying $f(Y_1,\ \cdots,\ Y_n)=0$ with probability one$\}$ and V be a real algebraic variety in $\boldsymbol{R}^n$ given by $V=\{(y_1,\ \cdots,y_n)\in\boldsymbol{R}^n;\ f(y_1,\ \cdots,y_n)=0$ for all $f\in I\}$. Then ν_Y is absolutely continuous relative to the canonical measure on V.*

In particular, ν_Y is absolutely continuous if and only if $I=\{0\}$.

The definition of the canonical measure on V will be given in Section 2.

The author wishes to thank Professor G. Maruyama and Professor Y. Okabe for useful conversation.

*) Partially supported by the SAKKOKAI FOUNDATION.

2. Preliminary material: real algebraic varieties.

Let $\boldsymbol{R}[y_1, \cdots, y_n]$ be the set of all polynomials in $y_1, \cdots, y_n$ with real coefficients, and let $I(Q)$ denote the set of polynomials which vanish in Q for each point set $Q \subset \boldsymbol{R}^n$. It is easy to see that $I(Q)$ is an ideal of $\boldsymbol{R}[y_1, \cdots, y_n]$. For any subset S of $\boldsymbol{R}[y_1, \cdots, y_n]$, let $V(S)$ denote the set of common zeros of elements of S and let us define the rank $\mathrm{rnk}_p(S)$ of S at a point $p \in \boldsymbol{R}^n$ by the dimension of the vector subspace in $\boldsymbol{R}^n$ generated by $\left\{\left(\frac{\partial f}{\partial y_1}(p), \cdots, \frac{\partial f}{\partial y_n}(p)\right);\ f \in S\right\}$. Furthermore let us define the rank $\mathrm{rnk}_p(Q)$ of a point set Q at $p \in Q$ by $\mathrm{rnk}_p(I(Q))$ and the rank $\mathrm{rnk}(Q)$ of Q by $\max\{\mathrm{rnk}_p(Q);\ p \in Q\}$.

The following is due to Whitney [5].

PROPOSITION 2.1. *Let V be an algebraic variety in $\boldsymbol{R}^n$, i.e. $V = V(S)$ for some $S \subset \boldsymbol{R}[y_1, \cdots, y_n]$, and let $M(V) = \{p \in V;\ \mathrm{rnk}_p(V) = \mathrm{rnk}(V)\}$. Then $M(V)$ is a real analytic submanifold in $\boldsymbol{R}^n$ of dimension $n - \mathrm{rnk}(V)$, and $V_1 = V \backslash M(V)$ is void or is a proper algebraic subvariety of V.*

Since $M(V)$ is regarded as a Riemannian submanifold in $\boldsymbol{R}^n$ with induced metric by $\boldsymbol{R}^n$, we define *the canonical measure on V* by the Riemannian volume of $M(V)$.

The following is also due to Whitney [5].

PROPOSITION 2.2. *For any proper real subvariety V of $\boldsymbol{R}^n$, there exist finite analytic submanifolds $M_1, \cdots, M_s$ in $\boldsymbol{R}^n$ such that the dimensions of $M_1, \cdots, M_s$ are less than n and $V = M_1 \cup \cdots \cup M_s$.*

The following is an immediate consequence of Proposition 2.2.

PROPOSITION 2.3. *Every proper real algebraic subvariety of $\boldsymbol{R}^n$ is of Lebesgue measure zero.*

3. Lemmas.

Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal base of $L^2([0,1])$ and let X_k be a random variable given by $X_k(\omega) = \int_0^1 e_k(t) dB_t(\omega)$, $k = 1, 2, \cdots$. Then $\{X_k(\omega)\}_{k=1}^{\infty}$ are independent and normally distributed. Let $A = \{\alpha = (\alpha_1, \alpha_2, \cdots) \in \boldsymbol{Z}^{\infty};\ \alpha_j \geq 0$ and $|\alpha| = \sum_{j=1}^{\infty} \alpha_j < \infty\}$, and let $H_k(x)$ be the modified Hermite polynomial of order k given by $H_k(\sqrt{2}\,x) = \left(\frac{1}{2^k k!}\right)^{1/2} (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$.

Note that $H_0(x)=1$. Then the following proposition is well-known (see Hida [1] for example).

PROPOSITION 3.1. $\left\{\prod_{j=1}^{\infty} H_{\alpha_j}(X_j(\omega));\ \alpha\in A,\ |\alpha|=m\right\}$ *is a complete orthonormal base of* $\mathcal{H}_m$.

By this proposition we may assume without loss of generality that $\Omega=\boldsymbol{R}^{\infty}$, $X_j(\omega)=\omega_j$ the j-th coordinate of $\omega\in\Omega$, and $p(d\omega)=\bigotimes_{j=1}^{\infty}\rho(d\omega_j)$, where $\rho(du)$ is the normal distribution on $\boldsymbol{R}$, i.e. $\rho(du)=\left(\frac{1}{2\pi}\right)^{1/2}\exp\left(-\frac{1}{2}u^2\right)du$.

For any measurable function $F:\Omega\to\boldsymbol{R}^n$, let F_m be a measurable function from $\Omega\times\Omega$ to $\boldsymbol{R}^n$ given by

$$F_m(\omega',\omega)=F(\omega_1',\cdots,\omega_m',\omega_{m+1},\omega_{m+2},\cdots)\quad\text{for each } \omega',\omega\in\Omega.$$

Then the following holds.

LEMMA 3.1. *There exist a measurable subset* Ω_F *of* Ω *and an increasing sequence* $\{m_k\}_{k=1}^{\infty}$ *of integers such that* $P(\Omega_F)=1$ *and* $F_{m_k}(\omega',\omega)\to F(\omega')$, $k\to\infty$, *in probability with respect to* $P(d\omega')$ *for each* $\omega\in\Omega_F$.

PROOF. First suppose that F is bounded, and let $G^{(m)}=E[F|X_1(\omega),\cdots,X_m(\omega)]$. Then it is obvious that $G_m^{(m)}(\omega',\omega)=G^{(m)}(\omega')$ for $P(d\omega')\times P(d\omega)$-a.e.$(\omega',\omega)$. So we get

$$\begin{aligned}\int_\Omega\int_\Omega|F(\omega')-F_m(\omega',\omega)|P(d\omega')P(d\omega)&\leq\int_\Omega\int_\Omega|F(\omega')-G^{(m)}(\omega')|P(d\omega')P(d\omega)\\&\quad+\int_\Omega\int_\Omega|F_m(\omega',\omega)-G_m^{(m)}(\omega',\omega)|P(d\omega')P(d\omega)\\&=2E[|F-G^{(m)}|]\longrightarrow0,\ \text{as } m\to\infty.\end{aligned}$$

Thus there exists an increasing sequence $\{m_k\}_{k=1}^{\infty}$ such that

$$\int_\Omega P(d\omega)\int_\Omega|F(\omega')-F_{m_k}(\omega',\omega)|P(d\omega')<\frac{1}{4^k}\quad\text{for each } k.$$

Let

$$\Omega_F=\bigcup_{j=1}^{\infty}\left\{\omega\in\Omega;\int_\Omega|F(\omega')-F_{m_k}(\omega',\omega)|P(d\omega)<\frac{1}{2^k}\quad\text{for any } k>j\right\}.$$

Then it is easy to see that $P(\Omega_F)=1$ by the Borel-Cantelli lemma. This implies our lemma. In the case when F is unbounded, we can prove our lemma similarly by considering $\frac{F}{1+|F|}$. This completes the proof.

Suppose that Z is an element of $\sum_{k=0}^{N} \mathcal{H}_k$ for some integer N, then it follows from Proposition 3.1 that there exist real numbers c_α, $\alpha \in A$, such that $c_\alpha=0$ if $|\alpha|>N$, $\sum_\alpha c_\alpha^2<\infty$ and $Z(\omega)=\sum_\alpha c_\alpha \prod_{j=1}^{\infty} H_{\alpha_j}(X_j(\omega))$. For each integer m, let $C_{m,\beta}$, $\beta=(\beta_1, \cdots, \beta_m) \in \{0, 1, \cdots, N\}^m$, be random variables given by

$$C_{m,\beta}(\omega)=\sum_{\alpha \in A_\beta} c_\alpha \prod_{j=m+1}^{\infty} H_{\beta_j}(X_j(\omega)), \text{ where } A_\beta=\{\alpha \in A;\ \alpha_j=\beta_j,\ j=1, \cdots, m\}.$$

Then we obtain

$$Z(\omega)=\sum_\beta C_{m,\beta}(\omega) \prod_{j=1}^{m} H_{\beta_j}(X_j(\omega))$$

and

$$C_{m,\beta_1,\cdots,\beta_m}(\omega)=\sum_{\gamma=0}^{N} C_{m,\beta_1,\cdots,\beta_m,\gamma}(\omega) H_\gamma(X_{m+1}(\omega))$$

for P-a.e. ω and any m and β. Thus we have the following.

LEMMA 3.2. *For any element Z of $\mathcal{H}$, there exist an integer N and a set of elements of $\mathcal{H}$, $\{C_{m,\beta}(\omega);\ m=1, 2, \cdots,\ \beta \in \{0, 1, \cdots, N\}^m\}$, such that each $C_{m,\beta}(\omega)$ is $\sigma\{X_{m+1}(\omega),\ X_{m+2}(\omega), \cdots\}$ measurable, $Z(\omega)=\sum_{\beta \in \{0,\cdots,N\}^m} C_{m,\beta}(\omega) \prod_{j=1}^{m} H_{\beta_j}(X_j(\omega))$ for P-a.e. ω and any m, and*

$$C_{m,\beta_1,\cdots,\beta_m}(\omega)=\sum_{\gamma=0}^{N} C_{m,\beta_1,\cdots,\beta_m,\gamma}(\omega) H_\gamma(X_{m+1}(\omega)) \text{ for } P\text{-a.e.}\omega \text{ and any } m \text{ and } \beta.$$

Using this lemma, we can prove the following.

LEMMA 3.3. *$P\{\omega;\ Z(\omega)=0\}=0$ or 1 for any element Z of $\mathcal{H}$.*

PROOF. Let $\Omega_0=\{\omega \in \Omega;\ Z(\omega)=0\}$ and let $C_{m,\beta}(\omega)$'s be random variables as in Lemma 3.2. Let $A_m(\omega)$ be an algebraic subvariety of $\boldsymbol{R}^m$ given by

$$A_m(\omega)=\left\{(z_1, \cdots, z_m) \in \boldsymbol{R}^m;\ \sum_\beta C_{m,\beta}(\omega) \prod_{j=1}^{m} H_{\beta_j}(z_j)=0\right\}.$$

Then $f_m(\omega)=\rho^{\otimes m}(A_m(\omega))=0$ or 1 by Proposition 2.3. This shows that

$$P[(\Omega_0 \backslash \{f_m(\omega)=1\}) \cup (\{f_m(\omega)=1\} \backslash \Omega_0)]=0.$$

However, it is obvious that $f_m(\omega)$ is $\sigma\{X_{m+1}(\omega),\ X_{m+2}(\omega), \cdots\}$ measurable. So we get $P(\Omega_0)=0$ or 1 by Kolmogorov's 0-1 law. This completes the proof.

LEMMA 3.4. *Suppose that $Z_1(\omega)$ and $Z_2(\omega)$ are elements of $\mathcal{H}$. Then $Z_1(\omega) \cdot Z_2(\omega)$ is also an element of $\mathcal{H}$. Moreover, $Z_1(\omega) \cdot Z_2(\omega)=0$ for P-a.e.ω, if and only if $Z_1(\omega)=0$ for P-a.e.ω or $Z_2(\omega)=0$ for P-a.e.ω.*

The first assertion is well-known (see Shigekawa [4] for example) and the latter assertion is an easy consequence of Lemma 3.3.

4. Proof of Theorem.

Let $Y_1, \cdots, Y_n$ be elements of $\mathcal{H}$ and ν_Y be the probability law on $\boldsymbol{R}^n$ induced by $Y=(Y_1, \cdots, Y_n)$, and let Q be the support of ν_Y and $r=\text{rnk}(Q)$. Furthermore let $\{C^k_{m,\beta}(\omega)\}_{m,\beta}$ be the set of random variables for Y_k as in Lemma 3.2, $k=1, \cdots, n$. That is to say, $Y_k(\omega)=\sum_\beta C^k_{m,\beta}(\omega)\prod_{j=1}^m H_{\beta_j}(X_j(\omega))$ for P-a.e.ω and $C^k_{m,\beta}(\omega)$ is $\sigma\{X_{m+1}(\omega), X_{m+2}(\omega), \cdots\}$ measurable. It is obvious that

$$I(Q)=I=\{f\in \boldsymbol{R}[y_1, \cdots, y_n];\ f(Y_1, \cdots, Y_n)=0 \text{ with probability one}\}.$$

Let $g^k_m(z;\omega)$ denote $\sum_\beta C^k_{m,\beta}(\omega)\prod_{j=1}^m H_{\beta_j}(z_j)$ for any $z=(z_1, \cdots, z_m)\in \boldsymbol{R}^m$ and $\omega\in\Omega$. Let $G_m(z;\omega)=(g^1_m(z;\omega), \cdots, g^n_m(z;\omega))\in \boldsymbol{R}^n$ for each $z\in \boldsymbol{R}^m$ and $\omega\in\Omega$, and let

$$I_m(\omega)=\{f\in \boldsymbol{R}[y_1, \cdots, y_n];\ f(G_m(z;\omega))=0 \text{ for any } z\in \boldsymbol{R}^m\}$$

for each $\omega\in\Omega$. Lemma 3.1 shows that there exist a measurable subset Ω_Y of Ω and an increasing sequence $\{m_k\}$ of integers such that $P(\Omega_Y)=1$ and $G(X_1(\omega'), \cdots, X_{m_k}(\omega');\omega)\to Y(\omega')$, $k\to\infty$, in probability with respect to $P(d\omega')$ for each $\omega\in\Omega_Y$. Let Ω_0' be the set of $\omega\in\Omega$ satisfying

$$Y(\omega)=G_m(X_1(\omega), \cdots, X_m(\omega);\omega),$$
$$G_m(z_1, \cdots, z_m;\omega)=G_{m+1}(z_1, \cdots, z_m, X_{m+1}(\omega);\omega)$$

and $G_m(z;\omega)\in V(I)$ for any m and $z=(z_1, \cdots, z_m)\in \boldsymbol{R}^m$. Then it is easy to see that $P(\Omega_0')=1$ by the similar argument to the proof of Lemma 3.2. Let $\Omega_0=\Omega_0'\cap\Omega_Y$, then it is obvious that $I_1(\omega)\supset I_2(\omega)\supset\cdots\supset I$ and $I_m(\omega)$'s are prime ideals in $\boldsymbol{R}[y_1, \cdots, y_n]$ for each $\omega\in\Omega_0$.

Now let us pause to prove the following Lemmas 4.1 and 4.2.

LEMMA 4.1. *There exists some integer $m(\omega)$ for any $\omega\in\Omega_0$ such that $I_m(\omega)=I$ for any $m\geq m(\omega)$.*

PROOF. Let $\omega\in\Omega_0$. Since the Krull dimension of $\boldsymbol{R}[y_1, \cdots, y_n]$ is n (see Nagata [3] for example), there exists an integer $m(\omega)$ such that $I_m(\omega)=I_{m(\omega)}(\omega)$ for any $m\geq m(\omega)$. So it is sufficient to prove $I_{m(\omega)}(\omega)\subset I$. Suppose that $f\in I_{m(\omega)}(\omega)\backslash I$, then $f\in I_m\backslash I$ for any m. This implies that $f(G_m(X_1(\omega'), \cdots, X_m(\omega');\omega))=0$ for any $\omega'\in\Omega$. On the other hand, $f(G_m(X_1(\omega'), \cdots, X_m(\omega');\omega))\to f(Y(\omega'))$ in probability with respect to $P(d\omega')$. Thus we have shown that $f(Y(\omega'))=0$ for P-a.e.ω', but this contradicts our assumption that $f \notin I$. So we have proved our lemma.

Let $\{u_i\}_{i=1}^{\infty}$ be a dense subset of $\boldsymbol{R}^m$. It is easy to see that for any $\omega \in \Omega_0$, $I_m(\omega)=I$ if and only if the rank of

$$\left(\frac{\partial g_m^k(z;\omega)}{\partial z_j}\bigg|_{z=u_i}\right)_{\substack{k=1,\cdots,n\\ j=1,\cdots,m}}$$

is equal to $n-r$ for some i. So we obtain

$$\begin{aligned}&\{\omega \in \Omega_0;\ I_m(\omega)=I\}\\ &=\Bigg\{\omega \in \Omega_0;\ \det\left(\frac{\partial g_m^{k_s}(z;\omega)}{\partial z_{j_t}}\bigg|_{z=u_i}\right)_{s,t=1,\cdots,n-r}\neq 0\\ &\qquad \text{for some } i,\ \{k_1,\cdots,k_{n-r}\}\subset\{1,\cdots,n\} \text{ and } \{j_1,\cdots,j_{n-r}\}\subset\{1,\cdots,m\}\Bigg\}.\end{aligned}$$

Since

$$\det\left(\frac{\partial g_m^{k_s}(z;\omega)}{\partial z_{j_t}}\bigg|_{z=u_i}\right)_{s,t=1,\cdots,n-r}$$

is an element of $\mathcal{H}$ by Lemma 3.4, it follows from Lemma 3.3 that $P\{\omega \in \Omega_0;\ I_m(\omega)=I\}=0$ or 1. On the other hand, Lemma 4.1 asserts that $P(\bigcup_m\{\omega \in \Omega_0;\ I_m(\omega)=I\})=1$. Thus we have got the following.

LEMMA 4.2. *There exist an integer* m_0, $u_0 \in \boldsymbol{R}^{m_0}$ *and a measurable subset* $\Omega_1 \subset \Omega_0$ *such that* $P(\Omega_1)=1$, $I_{m_0}(\omega)=I$ *and the rank of*

$$\left(\frac{\partial g_{m_0}^k(z;\omega)}{\partial z_j}\bigg|_{z=u_0}\right)_{\substack{k=1,\cdots,n\\ j=1,\cdots,m_0}}$$

is equal to $n-r$ *for any* $\omega \in \Omega_1$.

We now return to the proof of Theorem.

It is easy to see that the probability law of $G(z;\omega)$ under $\rho^{\otimes m_0}(dz)\otimes P(d\omega)$ is ν_Y. Let ν_ω be the probability law of $G(z;\omega)$ under $\rho^{\otimes m_0}(dz)$ for each $\omega \in \Omega_1$, then $\nu_\omega(V)=1$, where $V=V(I)$. Let $A_\omega=\{z \in \boldsymbol{R}^{m_0};\ G(z;\omega) \in V\backslash M(V)\}$ for each $\omega \in \Omega_1$, then A_ω is a subvariety of $\boldsymbol{R}^{m_0}$. Since the image of the neighborhood of u_0 through $G(\cdot,\omega)$ must be an $n-r$ dimensional submanifold in $\boldsymbol{R}^n$ for any $\omega \in \Omega_1$ and the dimension of the variety $V\backslash M(V)$ is less than $n-r$, A_ω is a proper subvariety of $\boldsymbol{R}^{m_0}$. So we get $\nu_\omega(V\backslash M(V))=\rho^{\otimes m_0}(A_\omega)=0$ for any $\omega \in \Omega_1$ by Proposition 2.3.

Similarly we can prove for each $\omega \in \Omega_1$ that

$$\rho^{\otimes m_0}\left(\left\{z \in \boldsymbol{R}^{m_0};\ \text{the rank of}\left(\frac{\partial g_{m_0}^k(z;\omega)}{\partial z_j}\right)_{\substack{k=1,\cdots,n\\ j=1,\cdots,m_0}} \text{is less than } n-r.\right\}\right)=0.$$

This shows that the measure ν_ω is absolutely continuous relative to the canonical measure on V.

However, it is obvious that $\nu_Y(dv)=\int_{\Omega_1} P(d\omega)\nu_\omega(dv)$. Thus we have proved the first assertion of our theorem. The latter assertion is an immediate conclusion. This completes the proof.

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(Received October 30, 1981)

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