The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity (II)

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1. Introduction.

Let (μ, H, B) be an abstract Wiener space in the sense of the previous paper [4]. Let $F: B\to B$ be a Borel map such that $I_B - F: B\to B$ is bijective, and let $\nu=(I_B-F)^{-1}\mu$ be the image measure on B of μ under $(I_B-F)^{-1}: B\to B$. In the previous paper [4], the author gave some sufficient condition on F for the image measure ν to be absolutely continuous relative to μ. However, in the case where B is a Banach space included in $\mathcal{S}'(\boldsymbol{R}^d)$, the space of tempered distributions over $\boldsymbol{R}^d$, and μ and ν can be regarded as stationary ergodic probability measures on $\mathcal{S}'(\boldsymbol{R}^d)$, μ and ν are identical or mutually singular. Therefore we can not expect that ν is absolutely continuous to μ.

But even in the case where μ and ν are mutually singular, there sometimes exists a sub-σ-field $\mathcal{F}$ of $\mathcal{B}(B)$, the Borel field over B, such that $\mathcal{F}$ is not so small and the restricted measures $\mu|_{\mathcal{F}}$ and $\nu|_{\mathcal{F}}$ of μ and ν to the σ-field $\mathcal{F}$ are mutually absolutely continuous. The purpose of the present paper is to find such a σ-field $\mathcal{F}$.

Now let us show the results in our paper. Let $H_1\oplus H_2$ be an orthogonal decomposition of H, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the sub-σ-fields of $\mathcal{B}(B)$ generated by Borel functions

$$\{{}_{B^*}\langle u, \cdot\rangle_B;\ u\in H_1\cap B^*\} \text{ and } \{{}_{B^*}\langle u, \cdot\rangle_B;\ u\in H_2\cap B^*\}$$

respectively. We will show in Theorem 1 that on some condition for F, H_1 and H_2, there exist Borel maps $\pi_1: B\to B$ and $\pi_2: B\to B$, and an $\mathcal{F}_1\times\mathcal{F}_2$-measurable function $\tilde{H}: B\times B\to \boldsymbol{R}$ such that for any bounded Borel function f defined on B, the conditional expectation $E_\nu[f|\mathcal{F}_2]$ of f relative to the σ-field $\mathcal{F}_2$ under the image measure $\nu=(I_B-F)^{-1}\mu$ is represented by

$$E_\nu[f|\mathcal{F}_2](z)=\int_B f(\pi_1\tilde{z}+\pi_2 z)\frac{\exp \tilde{H}(\tilde{z}, z)}{\int_B \exp \tilde{H}(\tilde{\tilde{z}}, z)\,\mu(d\tilde{\tilde{z}})}\mu(d\tilde{z})$$

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for ν-a.e. z. We will also give the explicit form of $\tilde{H}(\tilde{z}, z)$.

In Section 5, we will consider the stochastic non-linear pseudo-differential equation introduced in the author [5]:

$$p(D_x)X-b(q_1(D_x)X, \cdots, q_n(D_x)X)=W,$$

where W is a Gaussian white noise with d-dimensional parameter. We will set some assumption for $p, q_i, j=1, \cdots, n$, and b. It has been shown in [5] that there exists a unique solution X for the equation on some assumption. Let $Y=p(D_x)^{-1}W$. Then X and Y are $\mathcal{S}'(\boldsymbol{R}^d)$-valued random variables. Let $\tilde{\mu}$ and $\tilde{\nu}$ denote the probability laws of Y and X respectively.

For any domain D in $\boldsymbol{R}^d$, let $\mathcal{F}_D$ denote the sub-σ-field of the Borel field $\mathcal{B}(\mathcal{S}'(\boldsymbol{R}^d))$ over $\mathcal{S}'(\boldsymbol{R}^d)$ generated by Borel functions $\{{}_S\langle u, \cdot\rangle_{S'};\ u\in\mathcal{S}(\boldsymbol{R}^d)$ and the support of u is contained in $D\}$, and $\mathcal{J}_D$ denote the sub-σ-field of $\mathcal{B}(\mathcal{S}'(\boldsymbol{R}^d))$ generated by Borel functions $\{{}_S\langle u, \cdot\rangle_{S'};\ u\in\mathcal{S}(\boldsymbol{R}^d)$ and the support of $(p(D_x)p(-D_x))^{-1}u$ is contained in $D\}$, where $\mathcal{S}(\boldsymbol{R}^d)$ denotes the space of rapidly decreasing smooth functions. In the case where $p(D_x)p(-D_x)$ is a differential operator, $\{\mathcal{J}_D;\ D$ is a domain in $\boldsymbol{R}^d\}$ is an innovating system for $\{\mathcal{F}_D;\ D$ is a domain in $\boldsymbol{R}^d\}$ under the probability measure $\tilde{\mu}$ in the sense of Dobrushin and Surgailis [2].

Now let D be a bounded domain in $\boldsymbol{R}^d$ with smooth boundary, and let D^e denote the exterior of D. Moreover let $\tilde{\nu}(\cdot|\mathcal{J}_{D^e})$ denote the conditional probability measure of relative to the σ-field $\mathcal{J}_{D^e}$. Then we will show in Theorem 2 that

(1) the restricted measures $\tilde{\mu}|_{\mathcal{F}_D}$ and $\tilde{\nu}|_{\mathcal{F}_D}$ are mutually absolutely continuous, and

(2) there exists an $\mathcal{F}_D\times\mathcal{J}_{D^e}$-measurable function $\tilde{H}: \mathcal{S}'(\boldsymbol{R}^d)\times\mathcal{S}'(\boldsymbol{R}^d)\to\boldsymbol{R}$ such that for any $E\in\mathcal{F}_D$,

$$\tilde{\nu}(E|\mathcal{J}_{D^e})(w)=\frac{\int_E \exp\tilde{H}(\tilde{w}, w)\ \tilde{\mu}(d\tilde{w})}{\int_{S'} \exp\tilde{H}(\tilde{w}, w)\ \tilde{\mu}(d\tilde{w})} \qquad \text{for } \tilde{\nu}\text{-a.e. } w.$$

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Notation.

For any Banach space E, E^* denotes the dual Banach space of E and I_E denotes the identity map in E.

For any Hilbert spaces H and $K, \mathcal{L}^\infty(H, K)$ denotes the Banach space consisting of all bounded linear maps from H into K with the operator norm, and $\mathcal{L}^2(H, K)$ denotes the Hilbert space consisting of all Hilbert-Schmidt operators with Hilbert-Schmidt norm.

For any σ-fields $\mathcal{F}$ and $\mathcal{F}'$, $\mathcal{F}\vee\mathcal{F}'$ denotes the σ-field generated by $\mathcal{F}\cup\mathcal{F}'$.

$$\langle x\rangle=\sqrt{1+\sum_{j=1}^{d}x_j^2}\quad \text{for any } x=(x_1,\cdots,x_d)\in \boldsymbol{R}^d.$$

2. The Borel maps π_1 and π_2.

Let (μ, H, B) denote an abstract Wiener space throughout this paper. Let H_1 and H_2 be mutually orthogonal closed linear subspaces of H satisfying $H=H_1\oplus H_2$. Let P_1 (resp. P_2) denote the orthogonal projection defined in H onto H_1 (resp. H_2), and let B_1 and B_2 be the closure of H_1 and H_2 in B respectively. Now let B_0 be a real separable reflexive Banach space such that H is densely, continuously included in B_0 and B_0 is also densely, continuously included in B. Then it is clear that B_0^* is densely included in H, and that $B^*\subset B_0^*\subset H\subset B_0\subset B$.

We assume the following two assumptions through this section, Sections 3 and 4:

(A-1) $B_0\cap B_1\cap B_2=\{0\}$, and

(A-2) the orthogonal projection $P_1: H\to H_1$ is extensible to a bounded linear map $\bar{P}_1: B_0\to H_1$.

Then we get the following.

PROPOSITION 2.1. *$B^*\cap H_1+B^*\cap H_2$ is dense in B_0^*. Therefore $B^*\cap H_1+B^*\cap H_2$ is dense in H.*

PROOF. It is obvious that

$$B^*\cap H_1=\{u\in B^*;\ {}_{B^*}\langle u, z\rangle_B=0 \text{ for any } z\in B_2\},$$

and

$$B^*\cap H_2=\{u\in B^*;\ {}_{B^*}\langle u, z\rangle_B=0 \text{ for any } z\in B_1\}.$$

Then it is easy to see that

$$B_1=\{z\in B;\ {}_{B^*}\langle u, z\rangle_B=0 \text{ for any } u\in B^*\cap H_2\}, \tag{2.1}$$

and

$$B_2=\{z\in B;\ {}_{B^*}\langle u, z\rangle_B=0 \text{ for any } u\in B^*\cap H_1\}. \tag{2.2}$$

(See Yosida [7, Appendix to Chapter 5] for instance.)

Now suppose that $B^*\cap H_1+B^*\cap H_2$ is not dense in B_0^*. Then the Hahn-Banach theorem and the reflexivity of B_0 imply that there exists some $z\in B_0$

such that $z\neq 0$ and ${}_{B_0^*}\langle u, z\rangle_{B_0}={}_{B^*}\langle u, z\rangle_B=0$ for any $u\in B^*\cap H_1+B^*\cap H_2$. Thus it follows from (2.1) and (2.2) that $z\in B_0\cap B_1\cap B_2$. But this contradicts the assumption (A-1). This completes the proof.

For any subspace E of H, let $\mathcal{P}(E)$ denote the set of all orthogonal projections on H with a finite dimensional range contained in E. It is easy to see that any projection P, $P\in\mathcal{P}(B^*)$, is extensible to a bounded linear map from B into B^*, which will be denoted by $\tilde{P}$.

Take such sequences $\{P_1^{(n)}\}_{n=1}^\infty$ and $\{P_2^{(n)}\}_{n=1}^\infty$ of increasing orthogonal projections on H that $\{P_1^{(n)}\}_{n=1}^\infty\subset\mathcal{P}(B^*\cap H_1)$ and $\{P_2^{(n)}\}_{n=1}^\infty\subset\mathcal{P}(B^*\cap H_2)$, and that $P_1^{(n)}\uparrow P_1$ and $P_2^{(n)}\uparrow P_2$ strongly as $n\to\infty$, and fix them through this paper. The existence of such sequences are guaranteed by Proposition 2.1.

DEFINITION 2.1. We define a Borel subset $\mathcal{D}(\pi_1)$ of B by

$$\mathcal{D}(\pi_1)=\{z\in B;\ \{\tilde{P}_1^{(n)}z\}_{n=1}^\infty \text{ is convergent in } B\},$$

and a Borel map $\pi_1:\mathcal{D}(\pi_1)\to B_1$ by $\pi_1 z=\lim_{n\to\infty}\tilde{P}_1^{(n)}z$ for each $z\in\mathcal{D}(\pi_1)$.

We define a Borel subset $\mathcal{D}(\pi_2)$ of B by

$$\mathcal{D}(\pi_2)=\{z\in\mathcal{D}(\pi_1);\ z-\pi_1 z\in B_2\},$$

and a Borel map $\pi_2:\mathcal{D}(\pi_2)\to B_2$ by $\pi_2 z=z-\pi_1 z$ for each $z\in\mathcal{D}(\pi_2)$.

PROPOSITION 2.2. (1) *$\mathcal{D}(\pi_1)$ and $\mathcal{D}(\pi_2)$ are linear subspaces of B, and $\pi_1:\mathcal{D}(\pi_1)\to B_1$ and $\pi_2:\mathcal{D}(\pi_2)\to B_2$ are linear.*

(2) *$B_0\subset\mathcal{D}(\pi_2)\subset\mathcal{D}(\pi_1)$ and $\pi_1 z=\bar{P}_1 z$ for each $z\in B_0$.*

(3) *$B_2\subset\mathcal{D}(\pi_2)$, and $\pi_1 z=0$ and $\pi_2 z=z$ for each $z\in B_2$.*

(4) *If $z\in\mathcal{D}(\pi_1)$, then $\pi_1 z\in\mathcal{D}(\pi_2)$, $\pi_1\pi_1 z=\pi_1 z$ and $\pi_2\pi_1 z=0$.*

PROOF. Our assertion (1) is obvious. It is clear that $P_1^{(n)}h=P_1^{(n)}\bar{P}_1 h$ and ${}_{B_0^*}\langle u, h-\bar{P}_1 h\rangle_{B_0}=0$ for any $h\in H$ and $u\in B^*\cap H_1$. Thus we get for any $z\in B_0$,

$$\lim_{n\to\infty}\tilde{P}_1^{(n)}z=\lim_{n\to\infty}\tilde{P}_1^{(n)}\bar{P}_1 z=\bar{P}_1 z \quad\text{and}\quad {}_{B_0^*}\langle u, z-\bar{P}_1 z\rangle_{B_0}=0$$

for any $u\in B^*\cap H_1$. Therefore by (2.2) we see that $z\in\mathcal{D}(\pi_1)$, $\pi_1 z=\bar{P}_1 z$ and $z-\bar{P}_1 z\in B_2$ for any $z\in B_0$. This proves our assertion (2).

It is obvious that $\tilde{P}_1^{(n)}z=0$, $n=1,2,\cdots$, for any $z\in B_2$. This proves our assertion (3). Let $z\in\mathcal{D}(\pi_1)$. Then we see that

$$\tilde{P}_1^{(n)}\pi_1 z=\lim_{m\to\infty}\tilde{P}_1^{(n)}\tilde{P}_1^{(m)}z=\tilde{P}_1^{(n)}z,\quad n=1,2,\cdots,$$

which shows our assertion (4). This completes the proof.

PROPOSITION 2.3. (1) *$\mu(\mathcal{D}(\pi_1))=\mu(\mathcal{D}(\pi_2))=1$ and $\tilde{P}_2^{(n)}z\to\pi_2 z$ in B, $n\to\infty$, for μ-a.e. $z\in\mathcal{D}(\pi_2)$.*
(2) *The probability law on B of $\pi_1 z_1+\pi_2 z_2$ under $\mu(dz_1)\otimes\mu(dz_2)$ is equal to μ. That is,*

$$\int_{B\times B} f(\pi_1 z_1+\pi_2 z_2)\mu(dz_1)\otimes\mu(dz_2)=\int_B f(z)\mu(dz)$$

for any bounded Borel function f on B.

PROOF. By virtue of Carmona [1], we see that $\{\tilde{P}_1^{(n)}z\}_{n=1}^\infty$ and $\{\tilde{P}_2^{(n)}z\}_{n=1}^\infty$ are convergent in B for μ-a.e. z, and that $\tilde{P}_1^{(n)}z+\tilde{P}_2^{(n)}z\to z$ in B, $n\to\infty$, for μ-a.e. z. Thus we have $\mu(\mathcal{D}(\pi_1))=\mu(\mathcal{D}(\pi_2))=1$ and $\tilde{P}_2^{(n)}z\to\pi_2 z$ in B, $n\to\infty$, for μ-a.e. $z\in\mathcal{D}(\pi_2)$. This proves our assertion (1).

Let f be a bounded continuous function defined on B. Since $\tilde{P}_1^{(n)}z$ and $\tilde{P}_2^{(n)}z$ are independent under $\mu(dz)$, we obtain

$$\int_{B\times B} f(\tilde{P}_1^{(n)}z_1+\tilde{P}_2^{(n)}z_2)\mu(dz_1)\otimes\mu(dz_2)=\int_B f(\tilde{P}_1^{(n)}z+\tilde{P}_2^{(n)}z)\mu(dz).$$

Letting $n\to\infty$, we have got

$$\int_{B\times B} f(\pi_1 z_1+\pi_2 z_2)\mu(dz_1)\otimes\mu(dz_2)=\int_B f(z)\mu(dz).$$

This completes the proof.

The probability measure on B_1 (resp. B_2) induced by μ through $\pi_1:\mathcal{D}(\pi_1)\to B_1$ (resp. $\pi_2:\mathcal{D}(\pi_2)\to B_2$) will be denoted by μ_1 (resp. μ_2).

3. The σ-fields $\mathcal{F}_1$ and $\mathcal{F}_2$.

Let $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) denote the sub-σ-field of $\mathcal{B}(B)$, the Borel field over B, generated by Borel functions $\{{}_{B^*}\langle u,\cdot\rangle_B : B\to\boldsymbol{R};\ u\in B^*\cap H_1\}$ (resp. $\{{}_{B^*}\langle u,\cdot\rangle_B : B\to\boldsymbol{R};\ u\in B^*\cap H_2\}$). For each probability measure ν on B, $\mathcal{N}_\nu$ will denote the σ-field generated by ν-null sets, i.e. $\mathcal{N}_\nu=\{A;\ A$ is a subset of B and there exists a Borel subset C of B of ν-measure zero such that $A\subset C$ or $B\backslash A\subset C\}$.

PROPOSITION 3.1. (1) *If $g:B\to\boldsymbol{R}$ is $\mathcal{F}_1$-measurable, then $g(z+z')=g(z)$ for any $z\in B$ and $z'\in B_2$.*
(2) *If $g:B\to\boldsymbol{R}$ is $\mathcal{F}_2$-measurable, $g(z+z')=g(z)$ for any $z\in B$ and $z'\in B_1$.*

PROOF. It is clear that if $u\in B^*\cap H_1$, then ${}_{B^*}\langle u,z+z'\rangle_B={}_{B^*}\langle u,z\rangle_B$ for any

$z\in B$ and $z'\in B_2$. Thus we get our assertion (1) by the definition of the σ-field $\mathcal{F}_1$. The proof of our assertion (2) goes similarly. This completes the proof.

Let $F: B\to B_0$ be a Borel map such that $I_B-F: B\to B$ is bijective, and let $\nu=(I_B-F)^{-1}\mu$ be the image probability measure of μ under $(I_B-F)^{-1}: B\to B$. Then we have the following.

PROPOSITION 3.2. $\nu(\mathcal{D}(\pi_1))=\nu(\mathcal{D}(\pi_2))=1.$

PROOF. It is clear that

$$(I_B-F)^{-1}z=z+F(I_B-F)^{-1}z \quad \text{for any } z\in B. \tag{3.1}$$

Thus it follows from Proposition 2.2 (1) and (2) that $(I_B-F)^{-1}\mathcal{D}(\pi_2)=\mathcal{D}(\pi_2)$. This and Proposition 2.3 lead to our assertion.

PROPOSITION 3.3. (1) *If $f: B_1\to \mathbf{R}$ is a Borel function, then $f(\pi_1\cdot): \mathcal{D}(\pi_1)\to\mathbf{R}$ is $\mathcal{F}_1$-measurable.*
(2) *If $f: B_2\to\mathbf{R}$ is a Borel function, then $f(\pi_2\cdot): \mathcal{D}(\pi_2)\to\mathbf{R}$ is $\mathcal{F}_2\vee\mathcal{N}_\nu$-measurable.*
(3) $\mathcal{F}_1\vee\mathcal{F}_2\vee\mathcal{N}_\nu=\mathcal{B}(B)\vee\mathcal{N}_\nu$.

PROOF. It is obvious that $\mathcal{D}(\pi_1)\in\mathcal{F}_1$ and $f(\pi_1 z)=\lim_{n\to\infty} f(\tilde{P}_1^{(n)}z)$ for any $z\in\mathcal{D}(\pi_1)$ and any bounded continuous function f defined on B. This shows our assertion (1).

Next let us prove our assertion (2). By virtue of Proposition 3.2, we see that $\mathcal{D}(\pi_2)\in\mathcal{N}_\nu$. Let $\tilde{u}$ be an arbitrary element of B_2^* and $g: B_2\to\mathbf{C}$ be a continuous function given by $g(w)=\exp(\sqrt{-1}\,{}_{B_2^*}\langle\tilde{u}, w\rangle_{B_2})$ for each $w\in B_2$. The Hahn-Banach theorem implies that there exists some $u\in B^*$ such that $g(w)=\exp(\sqrt{-1}\,{}_{B^*}\langle u, w\rangle_B)$ for any $w\in B_2$. Observing $u\in B^*\subset B_0^*$, we see by Proposition 2.1 that there exist sequences $\{v_n\}_{n=1}^\infty\subset B^*\cap H_1$ and $\{u_n\}_{n=1}^\infty\subset B^*\cap H_2$ such that

$$v_n+u_n \longrightarrow u \quad \text{in } B_0^*, n\to\infty. \tag{3.2}$$

It is easy to see by (2.1) and (2.2) that

$${}_{B^*}\langle v_n+u_n, \pi_2 z\rangle_B={}_{B^*}\langle u_n, \pi_2 z\rangle_B={}_{B^*}\langle u_n, z\rangle_B \tag{3.3}$$

for any $z\in\mathcal{D}(\pi_2)$.

Let $g_n: B\to\mathbf{C}$ be a function given by $g_n(z)=\exp(\sqrt{-1}\,{}_{B^*}\langle u_n, z\rangle_B)$ for any $z\in B$. Then it is obvious that g_n is $\mathcal{F}_2$-measurable. It follows from Proposition 2.2, Proposition 3.2, (3.1) and (3.3) that

$$\int_B |g(\pi_2 z)-g_n(z)|\nu(dz)$$

$$\leq\int_{\mathscr{D}(\pi_2)} |\exp(\sqrt{-1}\,{}_{B^*}\langle u-(u_n+v_n),\pi_2 z\rangle_B)-1|\nu(dz)$$
$$=\int_{\mathscr{D}(\pi_2)} |\exp(\sqrt{-1}\,{}_{B^*}\langle u-(u_n+v_n),\pi_2 z+(I_{B_0}-\bar{P}_1)F(I_B-F)^{-1}z\rangle_B)-1|\mu(dz)$$
$$\leq\int_{\mathscr{D}(\pi_2)} |\exp(\sqrt{-1}\,{}_{B^*}\langle u-(u_n+v_n),\pi_2 z\rangle_B)-1|\mu(dz)$$
$$+\int_B |\exp(\sqrt{-1}\,{}_{B_0^*}\langle u-(u_n+v_n),(I_{B_0}-\bar{P}_1)F(I_B-F)^{-1}z\rangle_{B_0})-1|\mu(dz).$$

Proposition 2.3 implies that

$$\int_{\mathscr{D}(\pi_2)} |\exp(\sqrt{-1}\,{}_{B^*}\langle u-(u_n+v_n),\pi_2 z\rangle_B)-1|\mu(dz)$$
$$\leq\left\{\int_{\mathscr{D}(\pi_2)} |{}_{B^*}\langle u-(u_n+v_n),\pi_2 z\rangle_B|^2\mu(dz)\right\}^{1/2}=\|P_2(u-(u_n+v_n))\|_H.$$

Therefore by (3.2) we see that

$$\int_B |g(\pi_2 z)-g_n(z)|\nu(dz)\longrightarrow 0 \quad \text{as } n\to\infty.$$

This shows that $g(\pi_2\cdot):\mathscr{D}(\pi_2)\to\boldsymbol{C}$ is $\mathscr{F}_2\vee\mathscr{N}_\nu$-measurable.

Let V be the set of linear combinations of

$$\{\cos({}_{B_2^*}\langle u,\cdot\rangle_{B_2}),\ \sin({}_{B_2^*}\langle u,\cdot\rangle_{B_2});\ u\in B_2^*\}.$$

Then $g(\pi_2\cdot):\mathscr{D}(\pi_2)\to\boldsymbol{R}$ is $\mathscr{F}_2\vee\mathscr{N}_\nu$-measurable for any $g\in V$. Let $f:B_2\to\boldsymbol{R}$ be a bounded continuous function and let $C=\sup\{|f(w)|;\ w\in B_2\}$. Since the image measure $\pi_2\nu$ on B_2 of ν under $\pi_2:\mathscr{D}(\pi_2)\to B$ is a Radon measure, there exists a sequence $\{K_m\}_{m=1}^\infty$ of increasing compact subsets of B_2 such that $\pi_2\nu(B_2\backslash K_m)\to 0$ as $m\to\infty$.

By virtue of the Stone-Weierstrass theorem, we see that there exists a sequence $\{\tilde{f}_n\}_{n=1}^\infty\subset V$ such that $\tilde{f}_n(w)\to f(w)$, $n\to\infty$, uniformally for $w\in K_m$, $m=1,2,\cdots$. Let $f_n:B_2\to\boldsymbol{R}$, $n=1,2,\cdots$, be functions given by $f_n(w)=\min\{C,\max\{-C,\tilde{f}_n(w)\}\}$ for each $w\in B_2$. Then it is obvious that $f(\pi_2\cdot):\mathscr{D}(\pi_2)\to\boldsymbol{R}$ is $\mathscr{F}_2\vee\mathscr{N}_\nu$-measurable, and we get

$$\int_{\mathscr{D}(\pi_2)} |f(\pi_2 z)-f_n(\pi_2 z)|\nu(dz)=\int_{B_2}|f(w)-f_n(w)|\pi_2\nu(dw)\longrightarrow 0,\quad n\to\infty.$$

Therefore $f(\pi_2\cdot):\mathscr{D}(\pi_2)\to\boldsymbol{R}$ is $\mathscr{F}_2\vee\mathscr{N}_\nu$-measurable. This proves our assertion (2). Our assertion (3) follows immediately from our assertions (1), (2) and the fact that $z=\pi_1 z+\pi_2 z$ for any $z\in\mathscr{D}(\pi_2)$. This completes the proof.

4. Gibbs representation of $(I_B-F)^{-1}\mu$.

In this section we assume that a Borel map $F: B\to B_0$ satisfies the following five assumptions.

(F-1) $F(z+h)-F(z)\in H$ for any $z\in B$ and $h\in H$, and there exists a map $DF: B\to \mathcal{L}^{\infty}(H, H)$ (not necessarily Borel) such that $\|F(z+h)-F(z)-DF(z)h\|_H=o(\|h\|_H)$, $\|h\|_H\to 0$, and $DF(z+\cdot): H\to\mathcal{L}^{\infty}(H, H)$ is continuous for any $z\in B$.

(F-2) $I_B-F: B\to B$ is bijective and $I_H-DF(z): H\to H$ is invertible for any $z\in B$.

(F-3) $P_1DF(z): H\to H$ and $DF(z)P_1: H\to H$ are Hilbert-Schmidt operators for any $z\in B$, and $P_1DF(z+\cdot): H\to\mathcal{L}^2(H, H)$ and $DF(z+\cdot)P_1: H\to\mathcal{L}^2(H, H)$ are continuous for any $z\in B$.

(F-4) $I_B-F_2: B\to B$ is bijective, where F_2 denotes a Borel map $(I_{B_0}-\bar{P}_1)F: B\to B_0$, and $I_H-P_2DF(z): H\to H$ is invertible for any $z\in B$.

(F-5) For any $z\in B$ and $x\in B_1$, $F(x+z)-F(z)\in H$ and $DF(x+z)-DF(z): H\to H$ is a Hilbert-Schmidt operator, and moreover $DF(x+z+\cdot)-DF(z+P_2\cdot): H\to\mathcal{L}^2(H, H)$ is continuous.

REMARK 4.1. Since $F: B\to B_0$ is a Borel map and $\mathcal{L}^2(H, H)$ is a separable Hilbert space, $P_1DF(\cdot): B\to\mathcal{L}^2(H, H)$, $DF(\cdot)P_1: B\to\mathcal{L}^2(H, H)$ and $DF(x+\cdot)-DF(\cdot): B\to\mathcal{L}^2(H, H), x\in B_1$, are Borel maps.

Let $H_0^{(n)}: B_1\times B_2\to\boldsymbol{R}$, $n=1, 2, \cdots$, be Borel functions given by

$$\begin{aligned}H_0^{(n)}(x, y)={}_{B^*}\langle P^{(n)}(F(x+y)-F_2(y)), x+y-F_2(y)\rangle_B \\ -\operatorname{trace}_H P^{(n)}(DF(x+y)-P_2DF(y)P_2)(I_H-P_2DF(y)P_2)^{-1}\end{aligned}$$

for any $x\in B_1$ and $y\in B_2$, where $P^{(n)}=P_1^{(n)}+P_2^{(n)}$. And let $\nu=(I_B-F)^{-1}\mu$.

The following is our main result.

THEOREM 1. (1) *There exists a Borel function* $H_0: B_1\times B_2\to\boldsymbol{R}$ *such that* $H_0^{(n)}(x, y)\to H_0(x, y), n\to\infty$, *in probability with respect to* $\pi_1\mu(dx)\otimes\pi_2\nu(dy)$.

(2) *For any bounded function* $f: B\to\boldsymbol{R}$, *the conditional expectation* $E_\nu[f|\mathcal{F}_2]$ *of* f *relative to the* σ*-field* $\mathcal{F}_2$ *under the probability measure* ν *is given by*

$$E_\nu[f|\mathcal{F}_2](z)=\int_B f(\pi_1\tilde{z}+\pi_2 z)\frac{\exp(H(\pi_1\tilde{z}, \pi_2 z))}{\int_B \exp(H(\pi_1\tilde{\tilde{z}}, \pi_2 z))\mu(d\tilde{\tilde{z}})}\mu(d\tilde{z})$$

for ν-a.e. z, *where*

$$H(x, y)=H_0(x, y)-\frac{1}{2}\|F(x+y)-F_2(y)\|_H^2+\log|\delta_H((I_H-DF(x+y))(I_H-P_2DF(y)P_2)^{-1})|$$

for each $x\in B_1$ *and* $y\in B_2$.

Here $\delta_H(A)$ *denotes the Carleman-Fredholm determinant of an operator* $A: H \to H$ (see [4, Definition 6.1] for the detail).

In particular, the restricted measures $\mu|_{\mathcal{F}_1}$ and $\nu|_{\mathcal{F}_1}$ of μ and ν to the σ-field $\mathcal{F}_1$ are mutually absolutely continuous.

REMARK 4.2. Suppose that

$$\|(DF(x+y)-P_2DF(y)P_2)(I_H-P_2DF(y)P_2)^{-1}\|_{\mathcal{L}^\infty(H,H)}<1$$

for any $x \in B_1$ and $y \in B_2$. Noting that

$$\delta_H(I_H-K)=\exp\left(-\sum_{n=2}^{\infty}\frac{1}{n}\operatorname{trace}_H K^n\right),\ K\in\mathcal{L}^2(H,H)$$

such that $\|K\|_{\mathcal{L}^\infty(H,H)}<1$, we get

$$H(x,y)=H_0(x,y)-\frac{1}{2}\|F(x+y)-F_2(y)\|_H^2$$
$$-\sum_{n=2}^{\infty}\frac{1}{n}\operatorname{trace}_H[(DF(x+y)-P_2DF(y)P_2)(I_H-P_2DF(y)P_2)^{-1}]^n.$$

We will prove Theorem 1 in several steps.

Step 1. First we prove the following.

PROPOSITION 4.1. (1) *The image of* $F_2: B \to B_0$ *is contained in* B_2.
(2) $I_{B_2}-F_2: B_2 \to B_2$ *is bijective.*

PROOF. (1) is obvious. For any $u \in B_2$, there exists $v \in B$ such that $(I_B-F_2)v=u$ by the assumption (F-4). Since $v=F_2v+u \in B_2$, we see that $I_{B_2}-F_2: B_2 \to B_2$ is surjective. On the other hand, the injectivity of $I_B-F_2: B \to B$ leads to that of $I_{B_2}-F_2: B_2 \to B_2$. This completes the proof.

Let $G_1: B_1 \oplus B_2 \to B_1 \oplus B_2$ and $G_2: B_1 \oplus B_2 \to B_1 \oplus B_2$ be Borel maps given by

$$G_1(x,y)=(x,y-F_2(y)) \tag{4.1}$$

and

$$G_2(x,y)=(G_2^{(1)}(x,y),G_2^{(2)}(x,y))=(x-\bar{P}_1F(x+y),y-F_2(x+y)) \tag{4.2}$$

for each $(x,y) \in B_1 \oplus B_2$.

Then we have the following

PROPOSITION 4.2. (1) $G_1: B_1 \oplus B_2 \to B_1 \oplus B_2$ *is bijective.*
(2) $G_2: B_1 \oplus B_2 \to B_1 \oplus B_2$ *is bijective and*

$$G_2^{-1}(x, y)=(x+\bar{P}_1F(I_B-F)^{-1}(x+y),\ y+F_2(I_B-F)^{-1}(x+y))$$

for any $(x, y)\in B_1\oplus B_2$.

(3) $G_2^{(1)}(x, y)+G_2^{(2)}(x, y)=(I_B-F)(x+y)$ *for any* $(x, y)\in B_1\oplus B_2$.

PROOF. The assertions (1) and (3) are obvious. Let us prove our assertion (2). Let $J: B_1\oplus B_2\to B_1\oplus B_2$ be a Borel map given by

$$J(x, y)=(J^{(1)}(x, y), J^{(2)}(x, y))=(x+\bar{P}_1F(I_B-F)^{-1}(x+y),\ y+F_2(I_B-F)^{-1}(x+y))$$

for any $(x, y)\in B_1\oplus B_2$. Then it is obvious that

$$J^{(1)}(x, y)+J^{(2)}(x, y)=(I_B-F)^{-1}(x+y).$$

Therefore we get

$$J\circ G_2(x, y)=(G_2^{(1)}(x, y)+\bar{P}_1F(x+y),\ G_2^{(2)}(x, y)+F_2(x+y))=(x, y),$$

and

$$G_2\circ J(x, y)=(J^{(1)}(x, y)-\bar{P}_1F(I_B-F)^{-1}(x+y),\ J^{(2)}(x, y)-F_2(I_B-F)^{-1}(x+y))=(x, y).$$

This completes the proof.

Step 2. It is clear that $(\mu_1\otimes\mu_2, H_1\oplus H_2, B_1\oplus B_2)$ is an abstract Wiener space. Let $K: B_1\oplus B_2\to B_1\oplus B_2$ be a Borel map given by

$$K(x, y)=(x, y)-G_2\circ G_1^{-1}(x, y)\quad \text{for each } (x, y)\in B_1\oplus B_2.$$

Then it is obvious that

$$K(x, y)=(\bar{P}_1F(x+(I_{B_2}-F_2)^{-1}y),\ F_2(x+(I_{B_2}-F_2)^{-1}y)-F_2((I_{B_2}-F_2)^{-1}y)) \tag{4.3}$$

for each $(x, y)\in B_1\oplus B_2$. Thus by the assumption (F-5), we see that K is a Borel map defined on $B_1\oplus B_2$ into $H_1\oplus H_2$. For each $(x, y)\in B_1\oplus B_2$, let $DK(x, y): H_1\oplus H_2\to H_1\oplus H_2$ be a bounded linear operator given by

$$DK(x, y)(h_1, h_2)=(DK^{(1)}(x, y)(h_1, h_2),\ DK^{(2)}(x, y)(h_1, h_2)) \tag{4.4}$$

for each $(h_1, h_2)\in H_1\oplus H_2$, where

$$DK^{(1)}(x, y)(h_1, h_2)=P_1DF(x+(I_{B_2}-F_2)^{-1}y)(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2),$$

and

$$\begin{aligned}DK^{(2)}(x, y)(h_1, h_2)=&P_2DF(x+(I_{B_2}-F_2)^{-1}y)(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2)\\&-P_2DF((I_{B_2}-F_2)^{-1}y)P_2(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2).\end{aligned}$$

Then it is easy to see that

$$\|K(x+h_1, y+h_2)-K(x, y)-DK(x, y)(h_1, h_2)\|_{H_1\oplus H_2}=o(\|h_1\|_{H_1}+\|h_2\|_{H_2}),$$
$$\|h_1\|_{H_1}+\|h_2\|_{H_2}\longrightarrow 0,$$

for each $(x, y)\in B_1\oplus B_2$. By the assumptions on F, we also see that $DK(x, y): H_1\oplus H_2\to H_1\oplus H_2$ is a Hilbert-Schmidt operator and $DK(x+\cdot, y+\cdot): H_1\oplus H_2\to \mathcal{L}^2(H_1\oplus H_2, H_1\oplus H_2)$ is continuous for each $(x, y)\in B_1\oplus B_2$.

Note that

$$(4.5)\qquad (DK^{(1)}(x, y)+DK^{(2)}(x, y))(h_1, h_2)=(DF(x+(I_{B_2}-F_2)^{-1}y)-P_2DF((I_{B_2}-F)^{-1}y)P_2)\cdot(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2),$$

and that

$$(4.6)\qquad (h_1+h_2)-(DK^{(1)}(x, y)+DK^{(2)}(x, y))(h_1, h_2)=(I_H-DF(x+(I_{B_2}-F_2)^{-1}y))(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}(h_1+h_2)$$

for each $(x, y)\in B_1\oplus B_2$ and $(h_1, h_2)\in H_1\oplus H_2$. Thus $I_{H_1\oplus H_2}-DK(x, y): H_1\oplus H_2\to H_1\oplus H_2$ is invertible for any $(x, y)\in B_1\oplus B_2$.

Let $\bar{H}_0^{(n)}: B_1\oplus B_2\to \boldsymbol{R}, n=1, 2, \cdots$, be Borel functions given by

$$\bar{H}_0^{(n)}(x, y)={}_{B_1^*\oplus B_2^*}\langle(P_1^{(n)}, P_2^{(n)})K(x, y), (x, y)\rangle_{B_1\oplus B_2}-\mathrm{trace}_{H_1\oplus H_2}(P_1^{(n)}, P_2^{(n)})DK(x, y)$$

for each $(x, y)\in B_1\oplus B_2$, where $(P_1^{(n)}, P_2^{(n)})$ denotes the orthogonal projection on $H_1\oplus H_2$ such that $(P_1^{(n)}, P_2^{(n)})(h_1, h_2)=(P_1^{(n)}h_1, P_2^{(n)}h_2)$ for each $(h_1, h_2)\in H_1\oplus H_2$. Then we have

$$\bar{H}_0^{(n)}(x, y)={}_{B^*}\langle P^{(n)}(F(x+(I_{B_2}-F_2)^{-1}y)-F_2((I_{B_2}-F_2)^{-1}y)), x+y\rangle_B-\mathrm{trace}_H P^{(n)}(DF(x+(I_{B_2}-F_2)^{-1}y)-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)\cdot(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1}.$$

Therefore we have got

$$(4.7)\qquad \bar{H}_0^{(n)}(x, y)=H_0^{(n)}(x, (I_{B_2}-F_2)^{-1}y)$$

for each $(x, y)\in B_1\oplus B_2$ and $n=1, 2, \cdots$. According to [4, Corollary to Theorem 4.2], we see that there exists a Borel function $\bar{H}_0: B_1\oplus B_2\to \boldsymbol{R}$ such that

$$(4.8)\qquad \bar{H}_0^{(n)}(x, y)\longrightarrow \bar{H}_0(x, y),\quad n\to\infty,$$

in probability with respect to $\mu_1(dx)\otimes\mu_2(dy)$. Furthermore by virtue of [4, Theorem 6.4], we see that $(I_{B_1\oplus B_2}-K)^{-1}\mu_1\otimes\mu_2$ and $\mu_1\otimes\mu_2$ are mutually absolutely continuous, and that

$$(I_{B_1\oplus B_2}-K)^{-1}\mu_1\otimes\mu_2(dx\times dy)$$
$$=|\delta_{H_1\oplus H_2}(I_{H_1\oplus H_2}-DK(x,y))|\exp\left(\bar{H}_0(x;y)-\frac{1}{2}\|K(x,y)\|_H^2\right)\mu_1(dx)\otimes\mu_2(dy).$$

Thus by (4.3) and (4.6), we obtain

$$\begin{aligned}(4.9)\quad & G_1\circ G_2^{-1}\mu_1\otimes\mu_2(dx\times dy)\\ &=|\delta_H((I_H-DF(x+(I_{B_2}-F_2)^{-1}y))(I_H-P_2DF((I_{B_2}-F_2)^{-1}y)P_2)^{-1})|\\ &\quad\times\exp\left(\bar{H}_0(x,y)-\frac{1}{2}\|F(x+(I_{B_2}-F_2)^{-1}y)-F_2((I_{B_2}-F_2)^{-1}y)\|_H^2\right)\mu_1(dx)\otimes\mu_2(dy).\end{aligned}$$

So it is easy to see that

$$(4.10)\qquad G_2^{-1}\mu_1\otimes\mu_2(dx\times dy)=\rho(x,y)\mu_1(dx)\otimes(I_{B_2}-F_2)^{-1}\mu_2(dy),$$

where

$$\begin{aligned}\rho(x,y)&=|\delta_H((I_H-DF(x+y))(I_H-P_2DF(y)P_2)^{-1})|\\ &\quad\times\exp\left(\bar{H}_0(x,(I_{B_2}-F_2)y)-\frac{1}{2}\|F(x+y)-F_2(y)\|_H^2\right).\end{aligned}$$

Note that (4.7) and (4.8) imply that

$$(4.11)\qquad H_0^{(n)}(x,y)\longrightarrow\bar{H}_0(x,(I_{B_2}-F_2)y),\quad n\to\infty,$$

in probability with respect to $\mu_1(dx)\otimes(I_{B_2}-F_2)^{-1}\mu_2(dy)$.

Step 3. Let us prove the following.

PROPOSITION 4.3.

$$(1)\qquad \int_{B_1\oplus B_2}f(x+y)G_2^{-1}\mu_1\otimes\mu_2(dx\times dy)=\int_B f(z)\nu(dz)$$

for any bounded Borel function $f: B\to\boldsymbol{R}$.

$$(2)\qquad \int_{B_1\oplus B_2}g(x,y)G_2^{-1}\mu_1\otimes\mu_2(dx\times dy)=\int_B g(\pi_1z,\pi_2z)\nu(dz)$$

for any bounded Borel function $g: B_1\oplus B_2\to\boldsymbol{R}$.

PROOF. Let $f: B\to\boldsymbol{R}$ be a bounded Borel function. By Proposition 2.3 (2) and Proposition 4.2 (2), we see that

$$\begin{aligned}\int_{B_1\oplus B_2}f(x+y)G_2^{-1}\mu_1\otimes\mu_2(dx\otimes dy)&=\int_{B_1\oplus B_2}f(x+y+F(I_B-F)^{-1}(x+y))\mu_1(dx)\otimes\mu_2(dy)\\ &=\int_B f((I_B-F)^{-1}z)\mu(dz)=\int_B f(z)\nu(dz).\end{aligned}$$

This proves our assertion (1).

Now let $g_1 : B_1 \to \boldsymbol{R}$ and $g_2 : B_2 \to \boldsymbol{R}$ be bounded Borel functions. Then it follows from Propositions 2.2, 3.2, (4.10) and our assertion (1) that $\pi_1(x+y)=x$ and $\pi_2(x+y)=y$ for $G_2^{-1}\mu_1 \otimes \mu_2$-a.e. (x, y). Thus we have got by Proposition 2.2 and our assertion (1),

$$\int_{B_1 \oplus B_2} g_1(x) g_2(y) G_2^{-1}\mu_1 \otimes \mu_2(dx \times dy)$$
$$= \int_{B_1 \oplus B_2} g_1(\pi_1(x+y)) g_2(\pi_2(x+y)) G_2^{-1}\mu_1 \otimes \mu_2(dx \times dy) = \int_B g_1(\pi_1 z) g_2(\pi_2 z) \nu(dz).$$

This proves our assertion (2). This completes the proof.

Now we will complete the proof of Theorem 1. It follows from Proposition 4.3 (2) and (4.10) that $\pi_2\nu$ and $(I_{B_2}-F_2)^{-1}\mu_2$ are mutually absolutely continuous. Therefore (4.11) implies that $H_0^{(n)}(x, y) \to \bar{H}_0(x, (I_{B_2}-F_2)y)$, $n \to \infty$, in probability with respect to $\pi_1\mu(dx) \otimes \pi_2\nu(dy)$. This shows Theorem 1 (1) and $H_0(x, y) = \bar{H}_0(x, (I_{B_2}-F_2)y)$.

Let $f : B \to \boldsymbol{R}$ be a bounded Borel function and $g : B \to \boldsymbol{R}$ be an $\mathcal{F}_2$-measurable bounded function. Then it follows from Propositions 3.1, 4.3 and (4.10) that

$$\int_B f(z) g(z) \nu(dz) = \int_{B_1 \oplus B_2} f(x+y) g(x+y) G_2^{-1}\mu_1 \otimes \mu_2(dx \times dy)$$
$$= \int_{B_1 \oplus B_2} f(x+y) g(y) \rho(x, y) \mu_1(dx) \otimes (I_{B_2}-F_2)^{-1}\mu_2(dy)$$
$$= \int_{B_1 \oplus B_2} g(y) \frac{\int_{B_1} f(\tilde{x}+y) \rho(\tilde{x}, y) \mu_1(d\tilde{x})}{\int_{B_1} \rho(\tilde{\tilde{x}}, y) \mu_1(d\tilde{\tilde{x}})} \rho(x, y) \mu_1(dx) \otimes (I_{B_2}-F_2)^{-1}\mu_2(dy)$$
$$= \int_B g(\pi_2 z) \tilde{f}(z) \nu(dz),$$

where

$$\tilde{f}(z) = \frac{\int_B f(\pi_1\tilde{z} + \pi_2 z) \rho(\pi_1\tilde{z}, \pi_2 z) \mu(d\tilde{z})}{\int_B \rho(\pi_1\tilde{\tilde{z}}, \pi_2 z) \mu(d\tilde{\tilde{z}})}.$$

Since $g(\pi_2 z) = g(z)$ for ν-a.e. z and $\tilde{f}$ is $\mathcal{F}_2 \vee \mathcal{N}_\nu$-measurable by Propositions 3.2 and 3.3, we have got $E[f | \mathcal{F}_2](z) = \tilde{f}(z)$ for ν-a.e. z. This completes the proof.

PROPOSITION 4.4. *Suppose that there exists a constant C, $0<C<1$, such that $\|DF(z)\|_{\mathcal{L}^\infty(H,H)} \leq C$ for any $z \in B$. Then* (F-1) *and* (F-2) *lead to* (F-4).

PROOF. Since $\|P_2DF(z)\|_{\mathcal{L}^\infty(H,H)}\leq C$ for any $z\in B$, $I_H-P_2DF(z):H\to H$ is invertible for any $z\in B$. Therefore it suffices to prove that $I_B-F_2:B\to B$ is bijective under the assumptions (F-1) and (F-2). It is easy to see that

$$\|F(z+h)-F(z)\|_H=\left\|\int_0^1 DF(z+th)h\,dt\right\|_H\leq C\|h\|_H,$$

and

$$\|F_2(z+h)-F_2(z)\|_H=\|P_2(F(z+h)-F(z))\|_H\leq C\|h\|_H$$

for any $z\in B$ and $h\in H$. Therefore $I_H-(F(z+\cdot)-F(z)):H\to H$ and $I_H-(F_2(z+\cdot)-F_2(z)):H\to H$ are bijective for any $z\in B$ by virtue of the fixed point theorem for contraction map.

Now let us prove the injectivity of $I_B-F_2:B\to B$. Suppose that $(I_B-F_2)z_1=(I_B-F_2)z_2$ for some $z_1,z_2\in B$. Then we get $(I_B-F)z_2=(I_B-F)z_1+k$, where $k=\bar{P}_1F(z_1)-\bar{P}_1F(z_2)\in H$. Since $I_H-(F(z_1+\cdot)-F(z_1)):H\to H$ is bijective, there exists some $h\in H$ such that $h-(F(z_1+h)-F(z_1))=k$. Thus $(z_1+h)-F(z_1+h)=z_1-F(z_1)+k$. Since $I_B-F:B\to B$ is bijective by (F-1), we get $z_2=z_1+h$. Hence $h-(F_2(z_1+h)-F_2(z_1))=(I_B-F_2)z_2-z_1+F_2(z_1)=0$. The injectivity of $I_H-(F_2(z_1+\cdot)-F_2(z_1)):H\to H$ implies $h=0$, and accordingly we have got $z_1=z_2$. This shows the injectivity of $I_B-F_2:B\to B$.

Let w be an arbitrary element of B. Let $z=(I_B-F)^{-1}w$. Since $\bar{P}_1F(z)\in H$, there exists some $h\in H$ such that $h-(F_2(z+h)-F_2(z))=-\bar{P}_1F(z)$. Then we obtain

$$(I_B-F_2)(z+h)=z-F_2z-\bar{P}_1F(z)=(I_B-F)z=w.$$

This shows the surjectivity of $I_B-F_2:B\to B$. This completes the proof.

By using Schwarts [6, Theorem 1.22], we can also prove the following similarly to Proposition 4.4.

PROPOSITION 4.5. *Suppose that $I_H-DF(z):H\to H$ and $I_H-P_2DF(z):H\to H$ are invertible for any $z\in B$ and that there exists a constant $K>0$ such that*

$$\|(I_H-DF(z))^{-1}\|_{\mathcal{L}^\infty(H,H)}\leq K \quad \textit{and} \quad \|(I_H-P_2DF(z))^{-1}\|_{\mathcal{L}^\infty(H,H)}\leq K$$

for any $z\in B$. Then (F-1) *and* (F-2) *lead to* (F-4).

5. Application.

In this section we will consider the solution of the stochastic pseudo-differential equation treated in [5, Section 5]. We will use the notation introduced in [5] sometimes without explanation.

Let $p(\xi)\in\tilde{\mathcal{S}}^m$, $m\in\boldsymbol{R}$, such that $p(\xi)\neq 0$ for any $\xi\in\boldsymbol{R}^d$ and $p(\xi)^{-1}\in\tilde{\mathcal{S}}^{-m}$, and let $q_j(\xi)\in\tilde{\mathcal{S}}^r$, $j=1,\cdots,n$ and $r\in\boldsymbol{R}$. Moreover let $b:\boldsymbol{R}^n\to\boldsymbol{R}$ be a bounded smooth function such that

$$\|\partial_j b\|_\infty=\sup\left\{\left|\frac{\partial b}{\partial y_j}(y)\right|;\ y\in\boldsymbol{R}^n\right\}<\infty,$$

and

$$\|\partial_{ij} b\|_\infty=\sup\left\{\left|\frac{\partial^2 b}{\partial y_i\partial y_j}(y)\right|;\ y\in\boldsymbol{R}^n\right\}<\infty$$

for any $i,j=1,\cdots,n$. Now let us consider the following stochastic pseudo-differential equation

$$p(D_x)X-b(q_1(D_x)X,\cdots,q_n(D_x)X)=W, \tag{5.1}$$

where W is a Gaussian white noise with d-dimensional parameter. Let $Y=p(D_x)^{-1}W$. Then we get

$$X-p(D_x)^{-1}b(q_1(D_x)X,\cdots,q_n(D_x)X)=Y. \tag{5.2}$$

Assume that $m>r+\dfrac{d}{2}$ and $\sum_{j=1}^n\|\partial_j b\|_\infty\cdot\|q_jp^{-1}\|_{L^\infty}<1$. Then according to [5, Theorem 3], there exists the unique solution X of the equation (5.1). Let D be a bounded domain in $\boldsymbol{R}^d$ with smooth boundary. Let us make some preparation to study about the σ-fields $\mathcal{F}_D$ and $\mathcal{G}_{D^e}$ as in Introduction.

Let $\sigma^t(x)=\langle x\rangle^t$ and $\rho^s(x)=\langle x\rangle^s$, $x\in\boldsymbol{R}^d$, for each $t,s\in\boldsymbol{R}$. Let $W_2^{\sigma^t,\rho^s}$ be a Banach space with a norm $\|\ \|_{W_2^{\sigma^t,\rho^s}}$, the same as in [5], given by

$$W_2^{\sigma^t,\rho^s}=\{u\in\mathcal{S}'(\boldsymbol{R}^d);\ \rho^s(X)\sigma^t(D_x)u\in L^2(\boldsymbol{R}^d)\},$$

and

$$\|u\|_{W_2^{\sigma^t,\rho^s}}=\|\rho^s(X)\sigma^t(D_x)u\|_{L^2}\qquad\text{for each } u\in W_2^{\sigma^t,\rho^s}.$$

The following has been shown in [5, Theorem 2].

PROPOSITION 5.1. *For any $s,t\in\boldsymbol{R}$ and any pseudo-differential operator P belonging to $\tilde{S}^0$, there exists a constant $C>0$ such that*

$$\|Pu\|_{W_2^{\sigma^t,\rho^s}}\leq C\|u\|_{W_2^{\sigma^t,\rho^s}}\qquad\textit{for any } u\in W_2^{\sigma^t,\rho^s}.$$

Therefore P can be considered a bounded linear operator in $W_2^{\sigma^t,\sigma^s}$.

Let $\sigma^t{}_\eta(x)=\langle\eta\cdot x\rangle^t$ and $\rho^s{}_\lambda(x)=\langle\lambda\cdot x\rangle^s$, $x\in\boldsymbol{R}^d$, for each $t,s\in\boldsymbol{R}$ and $\eta,\lambda\in(0,1]$, and let $W_2^{\sigma^t{}_\eta,\rho^s{}_\lambda}$ be a Banach space with a norm $\|\ \|_{W_2^{\sigma^s{}_\eta,\rho^t{}_\lambda}}$ given by

$$W_2^{\sigma^t{}_\eta,\rho^s{}_\lambda}=\{u\in\mathcal{S}'(\boldsymbol{R}^d);\ \rho^s{}_\lambda(X)\sigma^t{}_\eta(D_x)u\in L^2(\boldsymbol{R}^d)\},$$

and

$$\|u\|_{W_2^{\sigma^t_\eta,\rho^s_\lambda}}=\|\rho^s_\lambda(X)\sigma^t_\eta(D_x)u\|_{L^2} \qquad \text{for each } u\in W_2^{\sigma^t_\eta,\rho^s_\lambda}.$$

Then we get the following.

PROPOSITION 5.2. *$W_2^{\sigma^t_\eta,\rho^s_\lambda}=W_2^{\sigma^t,\rho^s}$ as a set and the norms $\|\ \ \|_{W_2^{\sigma^t_\eta,\rho^s_\lambda}}$* and *$\|\ \ \|_{W_2^{\sigma^t,\rho^s}}$ are equivalent for any $s, t\in \boldsymbol{R}$ and $\eta, \lambda\in(0, 1]$.*

PROOF. It is obvious that

$$\|u\|_{W_2^{\sigma^t_\eta,\rho^s_\lambda}}=\|\rho^s_\lambda(X)\rho^s(X)^{-1}(\rho^s(X)\sigma^t_\eta(D_x)\sigma^t(D_x)^{-1}\rho^s(X))\rho^s(X)\sigma^t(D_x)u\|_{L^2}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^d)$. Since $\rho^s_\lambda(X)\rho^s(X)^{-1}$ and $\rho^s(X)\sigma^t_\eta(D_x)\sigma^t(D_x)^{-1}\rho^s(X)^{-1}$ are pseudo-differential operators belonging to $\tilde{\mathcal{S}}^0$ by virtue of [5, Corollary to Lemma 4.1], it follows from Proposition 5.1 that there exists a constant $C>0$ such that

$$\|u\|_{W_2^{\sigma^t_\eta,\rho^s_\lambda}}\leq C\|\rho^s(X)\sigma^t(D_x)u\|_{L^2}=C\|u\|_{W_2^{\sigma^t,\rho^s}} \qquad \text{for any } u\in\mathcal{S}(\boldsymbol{R}^d).$$

Similarly we see that there exists a constant $C'>0$ such that

$$\|u\|_{W_2^{\sigma^t,\rho^s}}\leq C'\|u\|_{W_2^{\sigma^t_\eta,\rho^s_\lambda}} \quad \text{for any } u\in\mathcal{S}(\boldsymbol{R}^d).$$

This proves our assertion.

Let $t_0=-\frac{1}{2}\left(m-r+\frac{d}{2}\right)$ and $s_0=-\frac{d}{2}-1$. Then it is obvious that $\sigma^{t_0}, \rho^{s_0}\in L^2(\boldsymbol{R}^d)$. Let U_1 and U_2 be bounded domains in $\boldsymbol{R}^d$ such that $\bar{D}\subset U_1\subset\bar{U}_1\subset U_2$, and let $g:\boldsymbol{R}^d\to\boldsymbol{R}$ be a smooth function such that $g(x)>0, x\in\boldsymbol{R}^d, g(x)=1$ for any $x\in U_1$ and $g(x)=\rho^{s_0}(x)=\langle x\rangle^{s_0}$ for any $x\in U_2^c$, where $\bar{D}$ and $\bar{U}_1$ denote the closure of D and U_1 and U_2^c denotes the complement of U_2 in $\boldsymbol{R}^d$. Note that $g\in L^2(\boldsymbol{R}^d)$. From now on we denote $p(\xi)p(-\xi)$ by $r(\xi), \xi\in\boldsymbol{R}^d$. Let

$$A_1=g(X)^{-1}r(D_x)^{-1}g(X)r(D_x),\quad A_2=p(D_x)g(X)p(D_x)^{-1}g(x)^{-1}$$
$$\text{and } A_3=g(X)^{-1}p(D_x)g(X)p(D_x)^{-1}.$$

Then we get the following.

PROPOSITION 5.3. (1) *A_1, A_2 and A_3 are pseudo-differential operators belonging to $\tilde{\mathcal{S}}^0$.*
(2) *$g(X)$ can be considered a continuous linear map from $W_2^{\sigma^t,\rho^{s0}}$ into $W_2^{\sigma^t,\rho^0}$ for any $t\in\boldsymbol{R}$.*

PROOF. Our assertion (1) is an immediate consequence of [5, Corollary to Lemma 4.1]. It is obvious that

$$\|g(X)u\|_{W_2^{\sigma^t,\rho^0}}=\|\sigma^t(D_x)g(X)u\|_{L^2}$$
$$=\|(\sigma^t(D_x)g(X)\sigma^t(D_x)^{-1}g(X)^{-1})(g(X)\rho^{s_0}(X)^{-1})\rho^{s_0}(X)\sigma^t(D_x)u\|_{L^2}$$

for any $u\in\mathcal{S}(\boldsymbol{R}^d)$. Therefore the proof of our assertion (2) goes similarly to that of Proposition 5.2.

Now let $G: W_2^{\sigma^{t_0},\rho^{s_0}}\to W_2^{\sigma^0,\rho^{s_0}}$ be a continuous map given by

$$Gu(x)=b(q_1(D_x)p(D_x)^{-1}u(x),\cdots,q_n(D_x)p(D_x)^{-1}u(x)),$$

$x\in\boldsymbol{R}^d$, for each $u\in W_2^{\sigma^{t_0},\rho^{s_0}}$. Then it follows from the proof of [5, Theorem 3] and Proposition 5.2 that $I_{W_2^{\sigma^{t_0},\rho^{s_0}}}-G: W_2^{\sigma^{t_0},\rho^{s_0}}\to W_2^{\sigma^{t_0},\rho^{s_0}}$ is bijective.

Let $t_1=t_0+m=\frac{1}{2}\left(m+r-\frac{d}{2}\right)$, and let B denote $W_2^{\sigma^{t_1},\rho^{s_0}}$ and B_0 denote $W_2^{\sigma^m,\rho^{s_0}}$. By virtue of [5, Theorem 2], $p(D_x)$ can be considered a bijective bicontinuous linear map from B onto $W_2^{\sigma^{t_0},\rho^{s_0}}$ and also considered a bijective bicontinuous linear map from B_0 onto $W_2^{\sigma^0,\rho^{s_0}}$. Therefore we can define a continuous linear map $F: B\to B_0$ by $Fu=p(D_x)^{-1}Gp(D_x)u$ for each $u\in B$, and we see that $I_B-F: B\to B$ is bijective.

Let μ be a probability measure on $\mathcal{S}'(\boldsymbol{R}^d)$ such that

$$\int_{S'(\boldsymbol{R}^d)}\exp(\sqrt{-1}_S\langle f, w\rangle_{S'})\mu(dw)=\exp\left(-\frac{1}{2}\|p(-D_x)^{-1}f\|_{L^2}^2\right)$$

for any $f\in\mathcal{S}(\boldsymbol{R}^d)$. Then μ is the probability law of Y. It follows from [5, Theorem 1] that $\mu(B)=1$. Thus by (5.2), we see that $\nu=(I_B-F)^{-1}\mu$ is the probability law of X. Let H be a Hilbert space with an inner product $(\ ,\ )_H$ given by $H=\{u\in\mathcal{S}'(\boldsymbol{R}^d);\ p(D_x)u\in L^2(\boldsymbol{R}^d)\}$, and $(u,v)_H=(p(D_x)u,p(D_x)v)_{L^2}$ for each $u,v\in H$. Then it is easy to see that $H=W_2^{\sigma^m,\rho^0}$ as a set.

Let us identify the dual space H^* with H. Then it is easy to see that $\mathcal{S}(\boldsymbol{R}^d)\subset B^*\subset H\subset B_0\subset B$ and

$$(u,v)_H={}_B\langle u,v\rangle_{B^*}=(u,r(D_x)v)_{L^2} \tag{5.3}$$

for any $u,v\in\mathcal{S}(\boldsymbol{R}^d)$. Therefore for any $u\in\mathcal{S}(\boldsymbol{R}^d)$, we obtain

$$\int_B\exp(\sqrt{-1}_{B^*}\langle u,w\rangle_B)\mu(dw)=\int_{S'(\boldsymbol{R}^d)}\exp(\sqrt{-1}_S\langle r(D_x)u,w\rangle_{S'})\mu(dw)$$
$$=\exp\left(-\frac{1}{2}\|p(-D_x)^{-1}r(D_x)u\|_{L^2}^2\right)=\exp\left(-\frac{1}{2}\|u\|_H^2\right).$$

Therefore (μ,H,B) is an abstract Wiener space.

Recall that D is a bounded domain in $\boldsymbol{R}^d$ with smooth boundary, and let H_1 and H_2 be closed linear subspaces of H given by

(5.4) $$H_1=\{u\in H\subset\mathcal{S}'(\boldsymbol{R}^d);\ \text{the support of } r(D_x)u \text{ is contained in the closure } \bar{D} \text{ of } D\},$$

and

(5.5) $$H_2=\{u\in H\subset\mathcal{S}'(\boldsymbol{R}^d);\ \text{the support of } u \text{ is contained in the complement } D^c \text{ of } D\}.$$

Then it is obvious that H_1 and H_2 are orthogonal and $H=H_1\oplus H_2$. Let B_1 and B_2 be the closure of H_1 and H_2 in B respectively. Then it is easy to see that

(5.6) $$B_1=\{u\in B\subset\mathcal{S}'(\boldsymbol{R}^d);\ \text{the support of } r(D_x)u \text{ is contained in } \bar{D}\},$$

and

(5.7) $$B_2=\{u\in B\subset\mathcal{S}'(\boldsymbol{R}^d);\ \text{the support of } u \text{ is contained in } D^c\}.$$

Now we get the following.

PROPOSITION 5.4. *The assumptions* (A-1) *and* (A-2) *hold. That is,*
(1) $B_0\cap B_1\cap B_2=\{0\}$, *and*
(2) *the orthogonal projection* $P_1: H\to H_1$ *is extensible to a bounded linear map* $\bar{P}_1: B_0\to H_1$.

PROOF. Since $g(x)^{-1}=1$ around D, we get

(5.8) $$g(X)^{-1}u=g(X)^{-1}r(D_x)^{-1}g(X)g(X)^{-1}r(D_x)u=A_1u$$

for any $u\in B_1$.

Suppose that $u\in B_0\cap B_1\cap B_2$. Then Propositions 5.1 and 5.3 (1) show that $g(X)^{-1}u=A_1u\in B_0$. Thus by Proposition 5.3 (2), we see that $u=g(X)g(X)^{-1}u\in H$. However, it is obvious that $H\cap B_1=H_1$ and $H\cap B_2=H_2$. Therefore $u\in H_1\cap H_2=\{0\}$. This proves (A-1).

Now let us prove (A-2). By (5.3), we see that for any $u\in\mathcal{S}(\boldsymbol{R}^d)$ and $v\in H$,

$$(P_1u, v)_H={}_S\langle u, r(D_x)P_1v\rangle_{S'}={}_S\langle g(X)u, r(D_x)P_1v\rangle_{S'}=(P_1g(X)u, v)_H.$$

Therefore we get

(5.9) $$P_1u=P_1g(X)u \qquad \text{for any } u\in\mathcal{S}(\boldsymbol{R}^d).$$

Hence due to Proposition 5.3 (2), we obtain (A-2). This completes the proof.

Since B_0 is reflexive, we see that B_0, H_1 and H_2 satisfy all the assumptions in Section 2. Now let us study about the property of the Borel map $F: B\to B_0$. For each $w\in B$, let $f(x;w)=b(q_1(D_x)w(x),\cdots,q_n(D_x)w(x)), x\in \boldsymbol{R}^d$, and

$$f_j(x;w)=\frac{\partial b}{\partial y_j}(q_1(D_x)w(x),\cdots,q_n(D_x)w(x)),\ j=1,\cdots,n$$

and $x\in\boldsymbol{R}^d$, and let $T_j(w): L^2(\boldsymbol{R}^d)\to L^2(\boldsymbol{R}^d), j=1,\cdots,n,$ be bounded linear operators given by

$$T_j(w)u(x)=f_j(x;w)u(x),\qquad x\in\boldsymbol{R}^d,\quad \text{for each } u\in L^2(\boldsymbol{R}^d).$$

Note that $p(D_x)$ can be considered an isometry from H into $L^2(\boldsymbol{R}^d)$. Now let $DF(w): H\to H$ be a bounded linear operator given by

$$DF(w)h=\sum_{j=1}^{n} p(D_x)^{-1}T_j(w)q_j(D_x)p(D_x)^{-1}p(D_x)h, \tag{5.10}$$

$h\in H$, for each $w\in B$. It is obvious that DF is well-defined. It is easy to see that for any $w\in B$,

$$\begin{aligned}(5.11)\qquad &\|DF(w)\|_{\mathcal{L}^\infty(H,H)}\\ &=\|p(D_x)DF(w)p(D_x)^{-1}\|_{\mathcal{L}^\infty(L^2(\boldsymbol{R}^d),L^2(\boldsymbol{R}^d))}\\ &\leq\sum_{j=1}^{n}\|T_j(w)\|_{\mathcal{L}^\infty(L^2(\boldsymbol{R}^d),L^2(\boldsymbol{R}^d))}\|q_j(D_x)p(D_x)^{-1}\|_{\mathcal{L}^\infty(L^2(\boldsymbol{R}^d),L^2(\boldsymbol{R}^d))}\\ &\leq\sum_{j=1}^{n}\|\partial_j b\|_\infty\|q_j\cdot p^{-1}\|_{L^\infty}<1.\end{aligned}$$

Since $m-r>\dfrac{d}{2}$, by virtue of Sobolev's lemma there exists a constant $C>0$ such that $\|q_j(D_x)p(D_x)^{-1}u\|_{L^\infty}\leq C\|u\|_{L^2}$, $j=1,\cdots,n$, for any $u\in L^2(\boldsymbol{R}^d)$. Thus we get for any $w\in B$ and $h\in H$,

$$\begin{aligned}(5.12)\qquad &\|DF(w+h)-DF(w)\|_{\mathcal{L}^\infty(H,H)}\leq\sum_{j=1}^{n}\|f_j(\cdot\,;w+h)-f_j(\cdot\,;w)\|_{L^\infty}\|q_j\cdot p^{-1}\|_{L^\infty}\\ &\leq\sum_{i,j=1}^{n}\|\partial_{ij}b\|_\infty\|q_i(D_x)p(D_x)^{-1}p(D_x)h\|_{L^\infty}\|q_j\cdot p^{-1}\|_{L^\infty}\\ &\leq C\Big(\sum_{i,j=1}^{n}\|\partial_{ij}b\|_\infty\|q_j\cdot p^{-1}\|_{L^\infty}\Big)\|h\|_H.\end{aligned}$$

Therefore $DF(w+\cdot): H\to\mathcal{L}^\infty(H,H)$ is continuous for any $w\in B$. It is obvious that for any $w\in B$ and $h\in H$,

$$F(w+h)-F(w)=\int_0^1 DF(w+th)h\,dt,$$

which implies that $F(w+h)-F(w)\in H$ and

$$\|F(w+h)-F(w)-DF(w)h\|_H=o(\|h\|_H),\quad \|h\|_H\to 0.$$

Thus by (5.11) and Proposition 4.4, we get the following.

PROPOSITION 5.5. *The Borel map F satisfies the assumptions* (F-1), (F-2) *and* (F-4).

Now let us prove the following.

PROPOSITION 5.6. *The Borel map F satisfies the assumption* (F-3).

PROOF. It follows from (5.8) and (5.9) that

$$P_1 DF(w)=\sum_{j=1}^{n} P_1 g(X)p(D_x)^{-1}T_j(w)q_j(D_x)p(D_x)^{-1}p(D_x) \tag{5.13}$$

$$=\sum_{j=1}^{n} P_1 p(D_x)^{-1}A_2 T_j(w)(g(X)q_j(D_x)p(D_x)^{-1})p(D_x)$$

and

$$DF(w)P_1=\sum_{j=1}^{n} p(D_x)^{-1}T_j(w)q_j(D_x)p(D_x)^{-1}p(D_x)g(X)g(X)^{-1}P_1 \tag{5.14}$$

$$=\sum_{j=1}^{n} p(D_x)^{-1}T_j(w)(q_j(D_x)p(D_x)^{-1}g(X))A_3 p(D_x)A_1 P_1.$$

Note that A_1 can be considered a bounded linear operator in H and that A_2 and A_3 can be considered bounded linear operators in $L^2(\boldsymbol{R}^d)$, due to Propositions 5.1 and 5.3. Since g and $q_j\cdot p^{-1}, j=1,\cdots,n$, belong to $L^2(\boldsymbol{R}^d)$, we see that

$$g(X)q_j(D_x)p(D_x)^{-1},\quad q_j(D_x)p(D_x)^{-1}g(X),\quad j=1,\cdots,n,$$

can be considered Hilbert-Schmidt operators in $L^2(\boldsymbol{R}^d)$.

Therefore $P_1DF(w): H\to H$ and $DF(w)P_1: H\to H$ are Hilbert-Schmidt operators for each $w\in B$. Similarly to (5.12), we can see that

$$T_j(w+\cdot): H\longrightarrow \mathcal{L}^\infty(L^2(\boldsymbol{R}^d), L^2(\boldsymbol{R}^d)),\quad j=1,\cdots,n,$$

are continuous for any $w\in B$. Thus $P_1DF(w+\cdot): H\to \mathcal{L}^2(H,H)$ and $DF(w+\cdot)P_1: H\to \mathcal{L}^2(H,H)$ are continuous for any $w\in B$. This completes the proof.

PROPOSITION 5.7. *$DF(w+u)-DF(w): H\to H$ is a Hilbert-Schmidt operator for any $w\in B$ and $u\in B_1$. Furthermore, $DF(\cdot+u)-DF(\cdot): B\to\mathcal{L}^2(H,H)$ is continuous for any $u\in B_1$, and there exists a constant $C>0$ such that*

$$\|DF(w+u)-DF(w)\|_{\mathcal{L}^2(H,H)}\leq C\|u\|_B$$

for any $w\in B$ and $u\in B_1$. Therefore the map from $B\times B_1$ into $\mathcal{L}^2(H,H)$ under which (w,u) corresponds to $DF(w+u)-DF(w)$ is continuous. In particular, the Borel map F satisfies (F-5).

PROOF. It is obvious that

$$(5.15)\qquad DF(w+u)-DF(w)=p(D_x)^{-1}\sum_{j=1}^{n}(T_j(w+u)-T_j(w))q_j(D_x)p(D_x)^{-1}p(D_x)$$

for any $w\in B$ and $u\in B_1$. It is easy to see that

$$(5.16)\qquad |f_j(x;w+u)-f_j(x;w)|\leq \sum_{j=1}^{n}\|\partial_{ij}b\|_\infty|q_i(D_x)u(x)|,$$

$x\in \boldsymbol{R}^d$ and $j=1,\cdots,n$, for each $w\in B$ and $u\in B_1$. It follows from (5.8) that

$$(5.17)\qquad q_i(D_x)u=q_i(D_x)g(X)A_1u=g(X)g(X)^{-1}q_i(D_x)g(X)A_1u$$

for any $u\in B_1$ and $i=1,\cdots,n$. By virtue of [5, Corollary to Lemma 4.1 and Theorem 2], $g(X)^{-1}q_i(D_x)g(X), i=1,\cdots,n$, can be considered a continuous linear map from B into $W_2^{\sigma_0,\rho^{s_0}}$. Proposition 5.3 (2) shows that $g(X)$ can be considered a continuous linear map from $W_2^{\sigma_0,\rho^{s_0}}$ into $L^2(\boldsymbol{R}^d)$, and Propositions 5.1 and 5.3 (1) show that A_1 can be considered a bounded linear operator in B. Therefore by (5.17) we see that there exists a constant $C''>0$ such that

$$(5.18)\qquad \|q_i(D_x)u\|_{L^2}\leq C''\|u\|_B,\quad i=1,\cdots,n,$$

for any $u\in B_1$. Thus by virtue of Lebesgue's convergence theorem, (5.16) and (5.18), we get

$$(5.19)\qquad \int_{\boldsymbol{R}^d}|(f_j(x;w'+u)-f_j(x;w'))-(f_j(x;w+u)-f_j(x;w))|^2dx\longrightarrow 0,\quad w'\longrightarrow w \text{ in } B,$$

for any $u\in B_1$. Moreover (5.16) and (5.18) imply that there exists a constant $C'>0$ such that

$$(5.20)\qquad \left\{\int_{\boldsymbol{R}^d}|f_j(x;w+u)-f_j(x;w)|^2dx\right\}^{1/2}\leq C'\|u\|_B$$

for any $u\in B_1$. Since $q_j\cdot p^{-1}\in L^2(\boldsymbol{R}^d), j=1,\cdots,n$, we see by (5.15), (5.19) and (5.20) that $DF(w+u)-DF(w): H\to H$ is a Hilbert-Schmidt operator for any $w\in B$ and $u\in B_1$, $\|(DF(w'+u)-DF(w'))-(DF(w+u)-DF(w))\|_{\mathcal{L}^2(H,H)}\to 0$ as $w'\to w$ in B for any $u\in B_1$, and that there exists a constant $C>0$ such that

$$\|DF(w+u)-DF(w)\|_{\mathcal{L}^2(H,H)}\leq C\|u\|_B \qquad \text{for any } u\in B_1.$$

This proves the first part of our assertion. The latter part is obvious. This completes the proof.

Let $\mathcal{F}_D$ and $\mathcal{J}_{D^e}$ be σ-fields as in Introduction. By ignoring $\mathcal{S}'(\boldsymbol{R}^d)\setminus B$, we obtain $\mathcal{F}_1\vee\mathcal{N}_\mu=\mathcal{F}_D\vee\mathcal{N}_\mu$, $\mathcal{F}_1\vee\mathcal{N}_\nu=\mathcal{F}_D\vee\mathcal{N}_\nu$, $\mathcal{F}_2\vee\mathcal{N}_\mu=\mathcal{J}_{D^e}\vee\mathcal{N}_\mu$ and $\mathcal{F}_2\vee\mathcal{N}_\nu=$

$\mathcal{J}_{D^c}\vee\mathcal{N}_\nu$. Thus according to Theorem 1, Propositions 3.3, 5.5, 5.6 and 5.7, we get the following by letting $\tilde{H}(\tilde{w}, w)=H(\pi_1\tilde{w}, \pi_2 w)$ as in Theorem 1.

THEOREM 2. *Let μ and ν be the probability laws of Y and X respectively, and let D be a bounded domain with smooth boundary. Moreover let $\nu(\cdot|\mathcal{J}_{D^c})$ denote the conditional probability measure relative to the σ-field $\mathcal{J}_{D^c}$ under ν. Then*

(1) *the restricted measures $\mu|_{\mathcal{F}_D}$ and $\nu|_{\mathcal{F}_D}$ relative to the σ-field $\mathcal{F}_D$ are mutually absolutely continuous, and*

(2) *there exists an $\mathcal{F}_D\times\mathcal{J}_{D^c}$-measurable function $\tilde{H}: \mathcal{S}'(\boldsymbol{R}^d)\times\mathcal{S}'(\boldsymbol{R}^d)\to\boldsymbol{R}$ such that for any $E\in\mathcal{F}_D$,*

$$\nu(E|\mathcal{J}_{D^c})(w)=\frac{\int_E \exp \tilde{H}(\tilde{w}, w)\mu(d\tilde{w})}{\int_{\mathcal{S}'} \exp \tilde{H}(\tilde{w}, w)\mu(d\tilde{w})} \quad \textit{for } \nu\text{-a.e. } w.$$

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