Gromov invariant and S^1-actions

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Introduction.

Recently, Gromov introduced the notion of so called the Gromov invariant and proved that it is positive for any hyperbolic manifold. (See [1] and Section 6 of [7].) Thurston, on the other hand, showed that Gromov invariants of three dimensional manifolds with non-trivial S^1-actions are zero (Proposition 6.5.2 and Corollary 6.5.3 of [7]). The purpose of this paper is to generalize Thurston's result to higher dimensions.

After the first draft of this paper, the author received a preprint [1] from Professor M. Gromov and found that the same result was also obtained in it. The methods, however, are quite different.

THEOREM. *Let M be a closed connected smooth manifold which admits a non-trivial smooth S^1-action. Then the Gromov invariant of M is equal to zero.*

COROLLARY. *Let M be an orientable closed connected smooth manifold of dimension grater than one. Suppose that there exists a continuous map f from M to some closed Riemannian manifold N of negative sectional curvature such that $f_*[M]\neq 0$ in $H_*(N; \boldsymbol{R})$. Here $[M]$ denotes the fundamental homology class of M. Then M does not admit non-trivial smooth S^1-actions.*

The corollary is a special case of Corollary 5 of Schoen-Yau [6], while our result reveals an obstruction to the existence of S^1-actions in such a manifold.

The plan is as follows: In Section 1, we give necessary definitions and lemmas. In Section 2, we introduce the notion of hollowings and apply this to manifolds with S^1-actions. Finally we prove the theorem and the corollary in Section 3.

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1. Preliminaries.

Let X be a metric space and Y a subspace of X. We put $S_*(X)=\bigoplus_{k=0}^{\infty} S_k(X)$ to be the real coefficient singular chain complex of X i.e., an element $c\in S_k(X)$

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is a finite sum $c=\sum_i a_i\sigma_i$, where $a_i\in \boldsymbol{R}$ and $\sigma_i\colon \Delta^k\to X$ is a continuous map. We define the *norm*, the *diameter* and the *restriction* to Y of c, denoted by $\|c\|$, $\operatorname{diam} c$ and $c|_Y$ respectively as follows:

$$\|c\|=\sum_i |a_i|,$$

$$\operatorname{diam} c=\sup_i \operatorname{diam} \sigma_i(\Delta^k),$$

$$c|_Y=\sum_{\sigma_i(\Delta^k)\subset Y} a_i\sigma_i.$$

Now recall the definition of the Gromov invariant. Let M be a closed connected manifold. When M is orientable, the *Gromov invariant* of M is defined by

$$\Gamma(M)=\inf\{\|c\|;\ c\in S_*(M) \text{ represents } [M]\}$$

where $[M]\in H_*(M;\boldsymbol{R})$ is the fundamental homology class of M. When M is non-orientable,

$$\Gamma(M)=\frac{1}{2}\Gamma(\tilde{M})$$

where $\tilde{M}$ is the orientable double covering of M.

The following two lemmas are based on Proposition 6.5.1 of Thurston [7].

LEMMA 1. *Let X be a compact metrized polyhedron, m a non-negative integer and ε a positive number. Then there exist positive constants δ and C satisfying the following condition* (*).

(*) *If a cycle $z\in S_m(X)$ is homologous to zero and satisfies* $\operatorname{diam} z\leqq\delta$, *then there exists a chain $w\in S_{m+1}(X)$ such that $\partial w=z$,* $\operatorname{diam} w\leqq\varepsilon$ *and* $\|w\|\leqq C\|z\|$.

PROOF. Let K be a subdivision of X with mesh $K\leqq\varepsilon/2$ and $\{v_j\}$ be the set of vertices of K. For $m+1$ vertices $v_{j_0},\cdots,v_{j_m}$ satisfying $\langle v_{j_0},\cdots,v_{j_m}\rangle\in K$ (may be degenerate), $\sigma_{j_0\cdots j_m}$ denotes the affine singular simplex with $\sigma_{j_0\cdots j_m}(0,\cdots,\overset{s}{\breve{1}},\cdots,0)=v_{j_s}$ and $K_m(X)$ denotes the linear subspace of $S_m(X)$ spaned by these singular simplices. Since $K_m(X)$ is finite dimensional and consists of chains of diameter not greater than $\varepsilon/2$, there is a positive constant C' such that if a cycle $z'\in K_m(X)$ is homologous to zero, there exists a chain $w'\in S_{m+1}(X)$ satisfying $\partial w'=z'$, $\operatorname{diam} w'\leqq\varepsilon$ and $\|w'\|\leqq C'\|z'\|$.

Let K' be the first barycentric subdivision of K and $\{U_j\}$ be a family of mutually disjoint subsets of X such that $\operatorname{Int}\operatorname{Star}(v_j, K')\subset U_j\subset \operatorname{Star}(v_j, K')$ and $X=\cup U_j$. Let δ be a Lebesgue number with respect to the open covering $\{\operatorname{Int}\operatorname{Star}(v'_j, K');\ v'_j \text{ is a vertex of } K'\}$ of X. For any singular m-simplex σ with $\operatorname{diam}\sigma\leqq\delta$, we let $\bar{\sigma}$ denote $\sigma_{j_0\cdots j_m}\in K_m(X)$, where j_s is determined by $\sigma(0,\cdots,\overset{s}{\breve{1}},\cdots,0)\in U_{j_s}$. The assumption that $\operatorname{diam}\sigma\leqq\delta$ guarantees the existence of $\bar{\sigma}$ and implies that for any $x\in\Delta^m$, corresponding points $\sigma(x)$ and $\bar{\sigma}(x)$ lie in a same simplex of K. Thus there is a homotopy $\hat{\sigma}: \Delta^m\times[0, 1]\to X$ connecting σ and $\bar{\sigma}$, defined by $\hat{\sigma}(x, t)=t\sigma(x)+(1-t)\bar{\sigma}(x)$. By using the prism decomposition of $\Delta^m\times[0, 1]$, we regard $\hat{\sigma}$ as an element of $S_{m+1}(X)$ with $\|\hat{\sigma}\|=m+1$ and $\operatorname{diam}\hat{\sigma}\leqq\operatorname{diam}\operatorname{Star}(v'_j, K')\leqq\varepsilon$. Now let $z=\sum_i a_i\sigma_i\in S_m(X)$ be a homologically trivial cycle with $\operatorname{diam} z\leqq\delta$. Put $\bar{z}=\sum_i a_i\bar{\sigma}_i$ and $\hat{w}=\sum_i a_i\hat{\sigma}_i$, where $\bar{\sigma}_i$ and $\hat{\sigma}_i$ are as above. Then we have $\partial\hat{w}=z-\bar{z}$, $\operatorname{diam}\hat{w}\leqq\varepsilon$, $\|\hat{w}\|\leqq(m+1)\|z\|$ and $\|\bar{z}\|\leqq\|z\|$. Since $\bar{z}$ belongs to $K_m(X)$ and is a homologically trivial cycle, this, together with the observation in the first half of this proof, implies the required result. □

LEMMA 2. *Let X, m and ε be as in Lemma* 1. *Then there exist positive constants δ and C satisfying the following condition* (**).

(**) *If a cycle $z\in S_m(X\times S^1)$ is homologous to zero and satisfies* $\operatorname{diam}\pi_*z\leqq\delta$, *then there exists a chain $w\in S_{m+1}(X\times S^1)$ such that $\partial w=z$,* $\operatorname{diam}\pi_*w\leqq\varepsilon$ *and $\|w\|\leqq C\|z\|$. Here $\pi: X\times S^1\to X$ denotes the projection to the first factor.*

PROOF. For any positive integer N, there is an N-fold covering $\rho_N: X\times S^1\to X\times S^1$ defined by $\rho_N(x, t)=(x, Nt)$, where S^1 is regarded as $\boldsymbol{R}/\boldsymbol{Z}$. Let τ_N be the linear map from $S_*(X\times S^1)$ to itself, defined by

$$\tau_N\sigma=\frac{1}{N}\sum_{\rho_N\circ\tilde{\sigma}=\sigma}\tilde{\sigma}$$

for each singular simplex σ. Then τ_N satisfies $\rho_{N*}\circ\tau_N=\mathrm{id}_{S_*(X\times S^1)}$, $\partial\circ\tau_N=\tau_N\circ\partial$ and $\|\tau_N z\|=\|z\|$. Now consider a metric d on $X\times S^1$ defined by $d((x_1, t_1), (x_2, t_2))=\max\{d_X(x_1, x_2), d_{S^1}(t_1, t_2)\}$ for $x_i\in X$ and $t_i\in S^1$, where d_X and d_{S^1} are the metrics on X and S^1, respectively. Take positive numbers δ and C as in Lemma 1 for $X\times S^1$. Suppose that a cycle $z\in S_m(X\times S^1)$ is homologous to zero and satisfies $\operatorname{diam}\pi_*z\leqq\delta$. For a large integer N, we have $\operatorname{diam}(\tau_N z)\leqq\delta$, hence, by Lemma 1, there exists a chain $\tilde{w}$ with $\partial\tilde{w}=\tau_N z$, $\operatorname{diam}\tilde{w}\leqq\varepsilon$ and $\|\tilde{w}\|\leqq C\|\tau_N z\|$. It is easy to see that the chain $w=\rho_{N*}\tilde{w}$ satisfies the required conditions. □

2. Hollowings and Manifolds with S^1-actions.

We use the following notation.

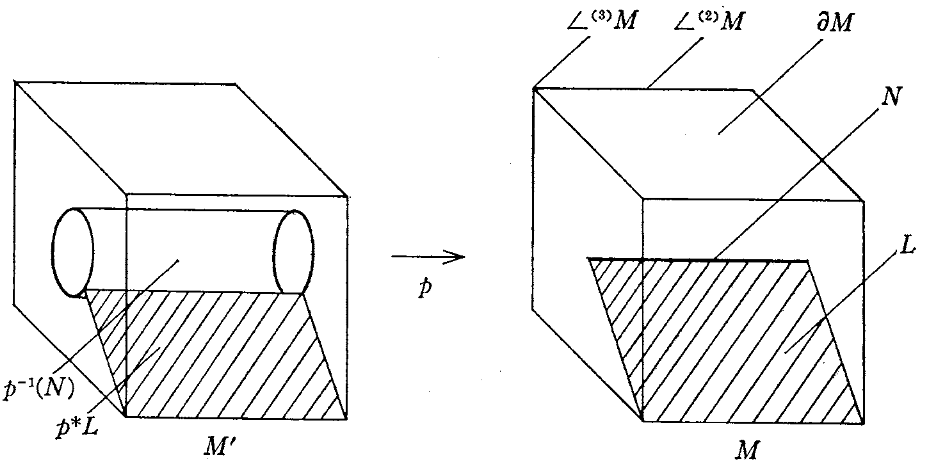
$$\boldsymbol{R}^n_+=\{(x_1, \cdots, x_n);\ x_i\geqq 0\}$$

$$\partial\boldsymbol{R}^n_+=\angle^{(1)}\boldsymbol{R}^n_+=\{(x_i)\in\boldsymbol{R}^n_+;\ {}^\exists j,\ x_j=0\}$$

$$\angle^{(k)}\boldsymbol{R}^n_+=\{(x_i)\in\boldsymbol{R}^n_+;\ {}^\exists j_1<\cdots<j_k,\ x_{j_1}=\cdots=x_{j_k}=0\}.$$

We say that a space M is an *n-dimensional manifold with corner* or simply a *manifold* in case of no confusion, if for each point of M, there is a neighborhood homeomorphic to some open set in $\boldsymbol{R}^n_+$ and associated coordinate transformations are smooth. For such an M, we define the subsets $\partial M=\angle^{(1)}M\supset\cdots\supset\angle^{(n)}M$ by an obvious way.

Let M be a manifold with corner and N a submanifold with corner of M such that N is transverse to each $\angle^{(k)}M-\angle^{(k+1)}M$ and $\angle^{(k)}N=N\cap\angle^{(k)}M$. Then the tubular neighborhood $\nu(N)$ of N in M has a structure as a disk bundle over N. Let $\phi:\nu_S(N)\times[0, 1]\to\nu(N)$ be the polar coordinate i.e., $\nu_S(N)$ is the total space of the associated sphere bundle, $\phi|_{\nu_S(N)\times\{1\}}=\mathrm{id}_{\nu_S(N)}$ and $\phi|_{\nu_S(N)\times\{0\}}$ is the projection of the bundle. Let M' be the space (Closure of $(M-\nu(N))$) $\cup_{\phi|_{\nu_S(N)\times\{1\}}}\nu_S(N)\times[0, 1]$. Then there is a natural map $p: M'\to M$ defined by $p|_{M-\nu(N)}=\mathrm{id}_{M-\nu(N)}$ and $p|_{\nu_S(N)\times[0,1]}=\phi$. It is clear that M' has a canonical structure as a manifold with corner which makes p differentiable. We call this map p the *hollowing* of M at N. (When $N=\phi$, we define $M'=M$ and $p=\mathrm{id}_M$.) The submanifolds N of M and $p^{-1}(N)$ of M' are called the *trace* and the *hollow wall* of p respectively and for a subspace L of M, the closure of $p^{-1}(L-(L\cap N))$ in M' is denoted by p^*L.



The following lemma is immediate and the proof is omitted.

LEMMA 3. *Let M be a manifold with a smooth S^1-action, N an invariant submanifold and $p: M'\to M$ the hollowing at N. Then M' has a smooth S^1-action such that p is equivariant.*

We are now in a position to dissect manifolds with S^1-actions. Hereafter, for a space X with an S^1-action, $\bar{X}$ denotes the orbit space and $\pi: X\to\bar{X}$ the natural projection. Suppose that M is an n-dimensional closed connected smooth manifold with an effective smooth S^1-action. Let F be the fixed point set, L_r be the set of points whose isotropy groups contain $\{0, 1/r, \cdots, (r-1)/r\}\subset S^1\cong \boldsymbol{R}/\boldsymbol{Z}$ for $r=2, 3, \cdots$, and $L=\bigcup_{r=2}^{\infty}L_r$. Note that since M is compact, L_r-F is empty for large r. Then there exists a triangulation of the orbit space $\bar{M}$ compatible to its structure. (See, for example, Matumoto [3] or Verona [8].) More precisely,

(i) $\bar{F}$ and each $\bar{L}_r$ are subpolyhedra, and

(ii) for any l-dimensional simplex Δ of this triangulation, there is a smooth embedding $\sigma: \Delta^l \to \pi^{-1}(\Delta) \subset M$ such that $\pi \circ \sigma: \Delta^l \to \Delta$ is a simplicial isomorphism.

We fix such a triangulation and define the sequence of hollowings

$$M_{2n-3} \xrightarrow{p_{2n-4}} M_{2n-4} \to \cdots \to M_1 \xrightarrow{p_0} M_0 = M$$

as follows. Let $\bar{M}^{(l)}$ denote the l-skeleton of this triangulation. For $k=0, \cdots, n-2$, we define inductively that $p_k: M_{k+1} \to M_k$ is the hollowing at $p_{k-1}^* \cdots p_0^*(\pi^{-1}(\bar{M}^{(k)} \cap \bar{F}))$ and, for $k=n-1, \cdots, 2n-4$, also inductively that $p_k: M_{k+1} \to M_k$ is the hollowing at $p_{k-1}^* \cdots p_0^*(\pi^{-1}(\bar{M}^{(k-n+1)}))$. Then, by Lemma 3, M_k's have smooth S^1-actions such that p_k's are equivariant.

In the rest of this section, we give information about this sequence. For simplicity, we put $p_{j,j'} = p_{j'} \circ \cdots \circ p_{j-1}: M_j \to M_{j'}$ and $p_{j,j'}^* = p_{j-1}^* \cdots p_{j'}^*$. Then $p_{j,0}^* F$ and $p_{j,0}^* L$ are the set of fixed points and the set of points with non-trivial isotropy groups, respectively, of the S^1-action on M_j. Let X_j denote the trace of p_j i. e., $X_j = p_{j,0}^*(\pi^{-1}(\bar{M}^{(j)} \cap \bar{F}))$ when $0 \leqq j \leqq n-2$ and $X_j = p_{j,0}^*(\pi^{-1}(\bar{M}^{(j-n+1)}))$ when $n-1 \leqq j \leqq 2n-4$. Let N_j be the hollow wall of p_j, $\tilde{N}_j = p_{2n-3,j+1}^* N_j$, $\tilde{N}_{j_1 \cdots j_k} = \tilde{N}_{j_1} \cap \cdots \cap \tilde{N}_{j_k}$ and $X_{j_1 \cdots j_k} = p_{2n-3,j_1}(\tilde{N}_{j_1 \cdots j_k}) \subset X_{j_1}$ for mutually distinct $j_1, \cdots, j_k$. Then we have $\angle^{(k)} M_{2n-3} = \cup \tilde{N}_{j_1 \cdots j_k}$ and in particular $\partial M_{2n-3} = \cup \tilde{N}_j$.

LEMMA 4. *Each connected component of $\bar{X}_{j_1 \cdots j_k}$ is contractible.*

PROOF. Let Δ^l be an l-simplex with the standard triangulation and let

$$\Delta_l^l \xrightarrow{q_{l-1}} \Delta_{l-1}^l \to \cdots \to \Delta_1^l \xrightarrow{q_0} \Delta_0^l = \Delta^l$$

be the sequence of hollowings defined inductively by the following: $q_k: \Delta_{k+1}^l \to \Delta_k^l$ is the hollowing at $q_{k-1}^* \cdots q_0^* \Delta^{(k)}$, where $\Delta^{(k)}$ is the k-skeleton of Δ^l. Then, by the construction, each connected component of $\bar{X}_{j_1}$ is diffeomorphic to Δ_l^l for $l=j_1$ when $0 \leqq j_1 \leqq n-2$ and for $l=j_1-n+1$ when $n-1 \leqq j_1 \leqq 2n-4$. Under this identification, every connected component of $\bar{X}_{j_1 \cdots j_k}$ for $k \geqq 2$ is one of the closures of connected components of $\angle^{(k-1)} \Delta_l^l - \angle^{(k)} \Delta_l^l$, and thus it is diffeomorphic to Δ_{l-k+1}^{l-k+1}. This proves Lemma 4. □

LEMMA 5. $X_{j_1 \cdots j_k} \cong \bar{X}_{j_1 \cdots j_k}$ *for* $0 \leqq j_1 \leqq n-2$.

PROOF. If $0 \leqq j_1 \leqq n-2$, we have $X_{j_1 \cdots j_k} \subset X_{j_1} \subset p_{j_1,0}^* F$. Thus the result follows. □

LEMMA 6. *$X_{j_1\cdots j_k}\cong\bar{X}_{j_1\cdots j_k}\times S^1$ for $n-1\leqq j_1\leqq 2n-4$.*

PROOF. In the case that $n-1\leqq j_1\leqq 2n-4$, $X_{j_1\cdots j_k}$ is a total space of a principal S^1-bundle over $\bar{X}_{j_1\cdots j_k}$. Thus Lemma 4 implies the required result. □

LEMMA 7. *When M is orientable, $M_{2n-3}\cong\bar{M}_{2n-3}\times S^1$.*

PROOF. Since $\dim\bar{F}\leqq n-2$ and $\dim(\bar{L}-\bar{F})\leqq n-3$, we have $p_{2n-3,0}{}^*F=p_{2n-3,0}{}^*L=\phi$ and thus the S^1-action on M_{2n-3} is free. On the other hand, since $\bar{M}_{2n-3}$ is homotopy equivalent to $\bar{M}-(\bar{M}^{(n-3)}\cup\bar{F})$, it has a homotopy type of a 1-complex. Therefore the result follows from the fact that every principal S^1-bundle over a 1-complex is trivial. □

3. Proofs of Theorem and Corollary.

It suffices to prove the theorem in the case that the manifold is orientable and the S^1-action is effective, and thus we assume that. We use the notation in the previous section.

Metrize $\bar{M}_j$ $j=0, \cdots, 2n-3$ such that each $\bar{p}_j: \bar{M}_{j+1}\to\bar{M}_j$ does not increase the distance, i.e. $\bar{p}_j$ is Lipschitz with the Lipschitz constant$\leqq 1$. First, we define positive numbers δ_i $i=1, \cdots, n$ and C as follows: We put $\delta_1=1$. When δ_k is defined, for $j_1>\cdots>j_k$, we let $\delta_{(j_1\cdots j_k)}$ and $C_{(j_1\cdots j_k)}$ be as in Lemma 1 if $0\leqq j_1\leqq n-2$ and as in Lemma 2 if $n-1\leqq j_1\leqq 2n-4$, for $X=\bar{X}_{j_1\cdots j_k}$, $m=n-k$ and $\varepsilon=\delta_k$. We put δ_{k+1} to be a positive number which is not greater than δ_k and any $\delta_{(j_1\cdots j_k)}$. When all of δ_i's are defined, we put C to be greater than the maximum of $C_{(j_1\cdots j_k)}$'s. Now let $\bar{K}$ be a triangulation of $\bar{M}_{2n-3}$ such that mesh $\bar{K}\leqq\delta_n$ and each $\angle^{(k)}\bar{M}_{2n-3}$ is a subpolyhedron. Then we have a triangulation K of $M_{2n-3}\cong\bar{M}_{2n-3}\times S^1$ cannonically induced from $\bar{K}$ with each $\angle^{(k)}M_{2n-3}$ a subpolyhedron. Take a positive integer N and let z denote the singular chain $\rho_{N*}K\in S_n(M_{2n-3})$, where $\rho_N: M_{2n-3}\cong\bar{M}_{2n-3}\times S^1\to M_{2n-3}$ is defined by $\rho_N(x, t)=(x, Nt)$ and K is naturally considered as an element of $S_n(M_{2n-3})$ which represents the fundamental homology class $[M_{2n-3}, \partial M_{2n-3}]$. Then z represents the N times multiple of $[M_{2n-3}, \partial M_{2n-3}]$ and $\operatorname{diam}\pi_*z\leqq\delta_n$. We define a chain $z_{j_1\cdots j_k}\in S_{n-k}(\tilde{N}_{j_1\cdots j_k})$ for $j_i=0, \cdots, 2n-4$ as follows. We put $z_j=(\partial z)|_{\tilde{N}_j}$ and inductively $z_{j_1\cdots j_k}=(\partial z_{j_1\cdots j_{k-1}})|_{\tilde{N}_{j_k}}=(\partial z_{j_1\cdots j_{k-1}})|_{\tilde{N}_{j_1\cdots j_k}}$ for mutually distinct $j_1, \cdots, j_k$. If there are $l\neq l'$ with $j_l=j_{l'}$, we put $z_{j_1\cdots j_k}=0$. Then, since each $\angle^{(k)}M_{2n-3}=\cup\tilde{N}_{j_1\cdots j_k}$ is a subpolyhedron of K and $\partial\tilde{N}_{j_1\cdots j_k}=\cup_{j\neq j_1,\cdots,j_k}\tilde{N}_{j_1\cdots j_k j}$, we have $\partial z_{j_1\cdots j_k}=\sum_j z_{j_1\cdots j_k j}$. The following lemma is immediate and the proof is omitted.

LEMMA 8. *The chain $z_{j_1\cdots j_k}$ is alternating with respect to the suffix i.e., if*

τ is a permutation of $\{1, \cdots, k\}$, *then*

$$z_{j_{\tau(1)}\cdots j_{\tau(k)}}=\text{sign}(\tau)z_{j_1\cdots j_k}.$$

Let $A=(n+1)!\|K\|=(n+1)!\|z\|$ and $B=1+(2n-3)C$. Then A and B do not depend on N, and $\|z_{j_1\cdots j_k}\|\leqq A$ for every $j_1, \cdots, j_k$.

LEMMA 9. *There exists a family of chains* $\{w_{j_1\cdots j_k}\}$ $j_i=0, \cdots, 2n-4$, $k=1, \cdots, n-1$, *satisfying the following conditions.*

(1) $w_{j_1\cdots j_k}\in S_{n-k+1}(X_{j_1\cdots j_k})$.

(2) $w_{j_1\cdots j_k}$ *is alternating with respect to the suffix i.e., if τ is a permutation of* $\{1, \cdots, k\}$, *then*

$w_{j_{\tau(1)}\cdots j_{\tau(k)}}=\text{sign}(\tau)p_{j_1, j_{\tau(1)}*}w_{j_1\cdots j_k}$ *when* $j_{\tau(1)}\leqq j_1$, *and*

$p_{j_{\tau(1)}, j_1*}w_{j_{\tau(1)}\cdots j_{\tau(k)}}=\text{sign}(\tau)w_{j_1\cdots j_k}$ *when* $j_{\tau(1)}\geqq j_1$.

(3) $\text{diam}\,\pi_*w_{j_1\cdots j_k}\leqq\delta_k$.

(4) $\|w_{j_1\cdots j_k}\|\leqq ACB^{n-k-1}$.

(5) $\partial w_{j_1\cdots j_{n-1}}=p_{2n-3, j_1*}z_{j_1\cdots j_{n-1}}$, *and*

$\partial w_{j_1\cdots j_k}=p_{2n-3, j_1*}z_{j_1\cdots j_k}-\sum_j w_{j_1\cdots j_k j}$ *for* $k=1, \cdots, n-2$.

PROOF. We prove this lemma by downward induction on the length k of the suffix. Let $k=n-1$. First, we construct $w_{n-2\cdots 0}$ as follows. Since $\angle^{(n)}M_{2n-3}=\phi$ by Lemma 7, the 1-chain $z_{n-2\cdots 0}$ is a cycle and since $H_1(X_{n-2\cdots 0}; \boldsymbol{R})=0$ by Lemmas 4 and 5, it is homologous to zero. Thus, by the definition of δ_n, we can apply Lemma 1 to $z_{n-2\cdots 0}$ and get a chain $w_{n-2\cdots 0}\in S_2(X_{n-2\cdots 0})$ satisfying $\partial w_{n-2\cdots 0}=z_{n-2\cdots 0}$, $\text{diam}\,\pi_*w_{n-2\cdots 0}\leqq\delta_{n-1}$ and $\|w_{n-2\cdots 0}\|\leqq C\|z_{n-2\cdots 0}\|\leqq AC$. Now, for a permutation τ of $\{0, \cdots, n-2\}$, we put $w_{\tau(n-2)\cdots\tau(0)}=\text{sign}(\tau)p_{n-2, \tau(n-2)*}w_{n-2\cdots 0}$ and put $w_{j_1\cdots j_{n-1}}=0$ if $\{j_1, \cdots, j_{n-1}\}\neq\{0, \cdots, n-2\}$. Since $X_j\cap\angle^{(j-n+2)}M_j=\phi$ for $j=n-1, \cdots, 2n-4$, we have $\angle^{(n-1)}M_{2n-3}=\tilde{N}_{0\cdots n-2}$ i. e. $\tilde{N}_{j_1\cdots j_{n-1}}=\phi$ for $\{j_1, \cdots, j_{n-1}\}\neq\{0, \cdots, n-2\}$, and thus $z_{j_1\cdots j_{n-1}}=0$ if $\{j_1, \cdots, j_{n-1}\}\neq\{0, \cdots, n-2\}$. Therefore, by Lemma 8, the $w_{j_1\cdots j_{n-1}}$'s above satisfy the required conditions.

Suppose that there has already been such a family for suffixes of length greater than k with $k\leqq n-2$. For a suffix $j_1\cdots j_k$ satisfying $j_1>\cdots>j_k$, consider the $(n-k)$-chain $\hat{z}_{j_1\cdots j_k}=p_{2n-3, j_1*}z_{j_1\cdots j_k}-\sum_j w_{j_1\cdots j_k j}\in S_{n-k}(X_{j_1\cdots j_k})$. Then, by the induction assumption, we have $\partial\hat{z}_{j_1\cdots j_k}=0$. Recall that, by Lemmas 5 and 6, $X_{j_1\cdots j_k}\cong\bar{X}_{j_1\cdots j_k}$ when $0\leqq j_1\leqq n-2$ and $X_{j_1\cdots j_k}\cong\bar{X}_{j_1\cdots j_k}\times S^1$ when $n-1\leqq j_1\leqq 2n-4$. Hence, by Lemma 4, $H_l(X_{j_1\cdots j_k}; \boldsymbol{R})=0$ for $l\geqq 2$ and thus the cycle $\hat{z}_{j_1\cdots j_k}$ is homologous to zero. Therefore, by the definition of δ_{k+1}, we can apply Lemma 1 when $0\leqq j_1\leqq n-2$ and Lemma 2 when $n-1\leqq j_1\leqq 2n-4$ to $\hat{z}_{j_1\cdots j_k}$ and get a chain $w_{j_1\cdots j_k}\in S_{n-k+1}(X_{j_1\cdots j_k})$ satisfying $\partial w_{j_1\cdots j_k}=\hat{z}_{j_1\cdots j_k}$, $\text{diam}\,\pi_*w_{j_1\cdots j_k}\leqq\delta_k$ and $\|w_{j_1\cdots j_k}\|\leqq C\|\hat{z}_{j_1\cdots j_k}\|\leqq ACB^{n-k-1}$. For a general suffix $j_1\cdots j_k$, $w_{j_1\cdots j_k}$ is defined as follows. If there are $l\neq l'$ with $j_l=j_{l'}$, we put $w_{j_1\cdots j_k}=0$. For mutually distinct $j_1, \cdots, j_k$, we let τ be a permutation of $\{1, \cdots, k\}$ with $j_{\tau(1)}>\cdots>j_{\tau(k)}$ and put $w_{j_1\cdots j_k}=$

$\operatorname{sign}(\tau)p_{j_{\tau(1)},j_1*}w_{j_{\tau(1)}\cdots j_{\tau(k)}}$. Then these $w_{j_1\cdots j_k}$'s satisfy the conditions (1), (2), (3) and (4), and Lemma 8 implies that they also satisfy (5). □

PROOF OF THEOREM. Let $\hat{z}=p_{2n-3,0*}z-\sum_j p_{j,0*}w_j\in S_n(M)$ where w_j is as in Lemma 9. Then the chain $\hat{z}$ is closed and covers a general point of M exactly N times and thus it represents the N times multiple of the fundamental homology class of M. On the other hand, we have $\|\hat{z}\|\leqq\|z\|+\sum_j\|w_j\|\leqq AB^{n-1}$. Therefore the Gromov invariant of M is estimated as follows:

$$\Gamma(M)\leqq\frac{1}{N}AB^{n-1}.$$

Since N is arbitrary and A and B do not depend on N, we have proved the theorem. □

REMARK 1. If one considers the relative Gromov invariant, the theorem is valid also for manifolds with boundaries by a slightly modified proof. (See Section 6.5 of Thurston [7] for such a notion.)

REMARK 2. The converse of the theorem does not hold even in the three dimensional case. The following example was informed by A. Hattori. Let $\alpha: T^2\to T^2$ be a hyperbolic toral automorphism $\left(\text{for example, } \alpha=\begin{pmatrix}2&1\\1&1\end{pmatrix}\right)$ and M_α the suspension of α i.e., $M_\alpha=T^2\times[0, 1]/\sim$, where $(x, 0)\sim(\alpha x, 1)$. Then, since M_α is a total space of a T^2-bundle over S^1, Proposition 6.5.2 of Thurston [7] implies $\Gamma(M_\alpha)=0$. (This is also obtained in Gromov [1] and Morita [4].) On the other hand, since M_α is aspherical and the center of $\pi_1(M_\alpha)$ is trivial, a result of Orlik-Raymond [5] says that there are no non-trivial S^1-actions on M_α.

PROOF OF COROLLARY. In fact, it is proved in Inoue-Yano [2] (see also Gromov [1]) that the Gromov invariant of a manifold as in this corollary is positive and thus the result follows. □

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