On the topology of non-complete algebraic surfaces

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Introduction.

According to the theory of logarithmic Kodaira dimension $\bar{\kappa}$ of algebraic varieties (cf. [I]), (open) surfaces are classified into four classes by $\bar{\kappa}$, which takes the values 2, 1, 0 or $-\infty$. In case $\bar{\kappa}=-\infty$, the ruling theorem of Castelnuovo-Miyanishi-Sugie (cf. [MS]) gives us a powerful tool to study the structure of such surfaces. If $\bar{\kappa}=0$ or 1, the theory of Kawamata [Kw 3] is very useful. As for the case $\bar{\kappa}=2$, little is known. However, in this case, many objects defined by transcendental methods are of algebraic nature (cf. [Sa]).

Recently Miyanishi [My 3] began a more precise study of the structure of surfaces, especially when they admit an $\boldsymbol{A}^1$-ruling or $\boldsymbol{A}^1_*$-ruling. Inspired by his work, we will first study here similar problems from topological viewpoints. Although many of our results thus obtained are not more than copies of Miyanishi's results, we hope that our method helps to understand the meaning of these results. Thus, among others, we give a topological characterization of $\boldsymbol{A}^2$ which led to an affirmative answer to the following cancellation problem: $S\times V\cong\boldsymbol{A}^2\times V$ implies $S\cong\boldsymbol{A}^2$. We give also a classification of surfaces dominated by $\boldsymbol{A}^2$, in terms of their fundamental groups.

We will further proceed to study the cases $\bar{\kappa}=0$ or 1. For this purpose we develop a computational theory of Zariski decomposition of $K+D$, where D is an effective reduced divisor on a complete surface with canonical bundle K. Combined with Kawamata's theory and with a precise study of $\boldsymbol{A}^1_*$-rulings, this method yields various results. Among others we establish a classification theory of affine surfaces with $\bar{\kappa}=0$ and with finite Picard group. In particular, all the possible fundamental groups of such surfaces can be completely determined (cf. (8.64) and (8.65-69)). Another sample of results obtained by this method is: Any algebraic surface S with $\bar{\kappa}(S)=1$, $H^2(S;\boldsymbol{Z})=0$ is rational and $\pi_1(S)\neq 1$ (cf. (7.13) and (7.15)). As an application we get a new proof of the following theorem (Morrow): All the compactifications of $\boldsymbol{C}^2$ are bimeromorphically equivalent to the standard one. Our proof of this fact is by no means simpler than his original one, but our method works for many other surfaces with $H^2(S;\boldsymbol{Z})=0$ in the same way.

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some help in order to show the significance of the classification theory originated and developed by them.

Convention.

Surface means usually a smooth surface. A variety is said to be complete if it is compact with respect to the Euclidian topology. An exceptional curve on a complete surface means a rational normal curve E with $E^2=-1$. "Component" of a divisor means its irreducible (or prime) component, and is never an abbreviation of "connected component".

§1. Quasi-complete invariants.

(1.1) DEFINITION. An open embedding $i: S\to\bar{S}$ of an analytic space S in a compact analytic space $\bar{S}$ is called a *completion* of S if S is dense in $\bar{S}$ and if $X=\bar{S}-S$ is analytically closed, i.e., its germ at each point x on X is the common zero-set of finitely many holomorphic functions in a neighborhood of x in $\bar{S}$.

Two completions $i_1: S\to\bar{S}_1$ and $i_2: S\to\bar{S}_2$ are said to be *equivalent* if there exist another completion $i: S\to\bar{S}$ and morphisms $f_j: \bar{S}\to\bar{S}_j$ for $j=1, 2$ such that $i_j=f_j\circ i$. An equivalence class of completions of S is called a quasi-complete structure (abbr.: *q.c. structure*) of S. An analytic space given a q.c. structure is called a quasi-complete space (abbr.: *q.c. space*). A completion of a q.c. space means a completion of the space which defines the given q.c. structure.

A divisor on a manifold is called an NC-*divisor* if all the irreducible components of it are smooth and if they intersect normally with each other. A completion of a (q.c.) manifold M is called an NC-*completion* if $\bar{M}-M$ is an NC-divisor on $\bar{M}$.

(1.2) THEOREM. *Any q.c. manifold has an* NC-*completion of it.*

This is a consequence of the desingularization theory of Hironaka. In this paper we are chiefly concerned with q.c. manifolds and their NC-completions.

(1.3) DEFINITION. Let S_1 and S_2 be q.c. spaces and let $i_j: S_j\to\bar{S}_j$ $(j=1, 2)$ be their completions. A meromorphic mapping $f: S_1\to S_2$ is said to be *rational* if it can be extended to a meromorphic mapping $\bar{f}: \bar{S}_1\to\bar{S}_2$. Clearly this notion is independent of the choice of completions of $\bar{S}_j$. A morphism $S_1\to S_2$ is called a *q.c. morphism* if it is rational. Q.c. spaces together with q.c. morphisms

REMARK. Our q.c. manifold is the same as Kawamata's compactifiable manifold. See [Kw 1].

form a category, which will be denoted by *QCS*. In particular, q. c. isomorphisms mean biholomorphisms which can be extended to bimeromorphic mappings of the completions. Sometimes we mean q. c. morphisms by saying "morphisms of q. c. spaces".

(1.4) THEOREM. *The category of algebraic spaces of finite type over* $\boldsymbol{C}$ *is naturally isomorphic to a full subcategory of QCS.*

This is a consequence of Nagata's completion theorem (cf. [Ng]).

(1.5) A q. c. morphism from a q. c. space S into $\boldsymbol{A}^1$ (together with the standard q. c. structure) is called a *regular function* on S. They form a $\boldsymbol{C}$-algebra, which will be denoted by $A(S)$.

$A(S)$ is not finitely generated in general. Suppose that $(\overline{M}, D)$ is an NC-completion of a q. c. manifold $M=\overline{M}-D$. Let $\delta\in H^0(\overline{M}, D)$ be an element defining the divisor D and let $\delta_j^i: H^0(\overline{M}, jD)\to H^0(\overline{M}, iD)$ be the mapping defined by the multiplication of δ^{i-j} for each $i\geqq j$. Then $\{H^0(\overline{M}, jD)\}_{j\geq 0}$ together with $\{\delta_j^i\}$ form an inductive system. The limit of this system is isomorphic to $A(M)$. From this we infer that $A(M)$ is finitely generated if and only if the graded $\boldsymbol{C}$-algebra $\bigoplus_{j\geq 0}H^0(\overline{M}, jD)$ is finitely generated.

(1.6) A rational mapping (may not be holomorphic) from a q. c. space S to $\boldsymbol{A}^1$ is called a *rational function* on S. If S is irreducible and reduced, rational functions on S form a field, which is denoted by $\boldsymbol{C}(S)$. Of course $\boldsymbol{C}(\bar{S})=\boldsymbol{C}(S)$ for any completion $\bar{S}$ of S.

(1.7) For any bimeromorphic invariant i, $i(\overline{M})$ is an invariant of a q. c. manifold M, where $\overline{M}$ is a completion of M. For example we have:

1) $\boldsymbol{C}(\overline{M})$, and its transcendental degree over $\boldsymbol{C}$, the algebraic dimension of M.

2) $H^q(\overline{M}, \mathcal{O}_{\overline{M}})$.

3) $H^0(\overline{M}, \Omega^\rho)$, where ρ is any polynomial representation of $\mathrm{GL}(\dim M, \boldsymbol{C})$ and Ω^ρ is the vector bundle induced by ρ from the cotangent bundle of $\overline{M}$. If in particular ρ is $(\det)^m$ with $m>0$, then $\Omega^\rho=mK$ where K is the canonical bundle of $\overline{M}$.

4) The canonical ring $\bigoplus_{t\geq 0}H^0(\overline{M}, tK)$ and the Kodaira dimension $\kappa(\overline{M})$.

5) The fundamental group $\pi_1(\overline{M})$, the homology group $H_1(\overline{M}; \boldsymbol{Z})$ and the torsion part of $H^2(\overline{M}; \boldsymbol{Z})$.

6) The Picard variety $\mathrm{Pic}_0(\overline{M})$, the torsion parts of the Picard group $\mathrm{Pic}(\overline{M})$ and the Neron-Severi group $\mathrm{NS}(\overline{M})$.

7) The Albanese torus $\mathrm{Alb}(\overline{M})$.

(1.8) Let $(\overline{M}, D)$ be an NC-completion of $M=\overline{M}-D$. Let $\Omega(\log D)$ be the sheaf of meromorphic 1-forms on $\overline{M}$ which have only logarithmic poles along D (hence, in particular, holomorphic on M). Then, for any polynomial representation ρ of $\mathrm{GL}(\dim M, \boldsymbol{C})$, $H^0(\overline{M}, \Omega(\log D)^{\rho})$ turns to be independent of the choice of the NC-completion $(\overline{M}, D)$ of M (see [I]). Thus, we get various invariants of the q.c. manifold M. Example: $\bar{q}(M)=h^0(\overline{M}, \Omega(\log D))$, called the *logarithmic irregularity* of M, the logarithmic canonical ring $\bigoplus_{t\geq 0}H^0(\overline{M}, t(K+D))$, and the *logarithmic Kodaira dimension* $\bar{\kappa}(M)=\kappa(K+D, \overline{M})$.

More generally, $H^0(\overline{M}, \Omega^{\rho}\otimes\Omega(\log D)^{\rho'})$ is an invariant of M for any polynomial representations ρ and ρ' of $\mathrm{GL}(\dim M, \boldsymbol{C})$. In particular, $\bigoplus_{t\geq 0}H^0(\overline{M}, tK+(t-1)D)$, which can be identified with an ideal of the logarithmic canonical ring, is an invariant of M. Furthermore, Sakai [Sa] showed that this ideal is actually an invariant of the complex structure of M although the logarithmic canonical ring itself depends on the q.c. structure (cf. (1.24) below). Thus, $\kappa(K+D, \mathcal{O}(K))$ is an invariant of the complex manifold M, which is denoted by $\kappa_c(M)$.

(1.9) THEOREM. *If $\bar{\kappa}(M)=n=\dim M$, then $\kappa_c(M)=n$.*

The proof is easy (see [Sa]).

(1.10) *Notation.* Let G be an abelian group and let T be a closed subspace of a topological space S. Then, by $\tilde{H}_p(S, T; G)$ and $\hat{H}_p(S, T; G)$ we denote $\mathrm{Coker}(H_p(T; G)\to H_p(S; G))$ and $\mathrm{Ker}(H_{p-1}(T; G)\to H_{p-1}(S; G))$ respectively. Note that the homology exact sequence yields a natural exact sequence $0\to\tilde{H}_p(S, T; G)\to H_p(S, T; G)\to\hat{H}_p(S, T; G)\to 0$.

(1.11) LEMMA. *Let $f: M_1\to M_2$ be a bimeromorphic morphism of compact complex manifolds M_1 and M_2. Then $f_*: H_p(M_1; G)\to H_p(M_2; G)$ is surjective, where G is an abelian group.*

PROOF. We may assume G to be finitely generated because G is an inductive limit of such groups and because the inductive limit is an exact functor. Then G is a direct sum of infinite and finite cyclic groups. So it suffices to consider the case in which $G\cong\boldsymbol{Z}$ or G is a finite field $\boldsymbol{Z}/p\boldsymbol{Z}$, p being prime.

Using Poincaré duality we get $H_p(M_2; G)\cong H^{2n-p}(M_2; G)\to H^{2n-p}(M_1; G)\cong H_p(M_1; G)\to H_p(M_2; G)$, which gives the identity of $H_p(M_2; G)$ since f is bimeromorphic. This implies the surjectivity of f_*. Q.E.D.

(1.12) Let $(\overline{M}, D)$ be a completion of a q.c. manifold $M=\overline{M}-D$ and suppose that G is a field or $\boldsymbol{Z}$. Then $H_q(\overline{M}, D; G)\cong H^{2n-q}(M; G)$ by Lefschetz duality, where $n=\dim M$. So $\tilde{H}_q(\overline{M}, D; G)$ and $\hat{H}_q(\overline{M}, D; G)$ define a subgroup and a quotient group of $H^{2n-q}(M; G)$ respectively.

THEOREM. *These sub- and quotient groups are independent of the choice of a completion of M.*

PROOF. We should show that $H_q(\overline{M}_1, D_1; G)$ and $H_q(\overline{M}_2, D_2; G)$ define the same subgroup of $H^{2n-q}(M; G)$ for any equivalent completions $(\overline{M}_1, D_1)$ and $(\overline{M}_2, D_2)$ of M. Let $(\overline{M}, D)$ be a completion of M such that there are morphisms $f_i: \overline{M} \to \overline{M}_i$ for $i=1, 2$ which induce the identity map of M. Then f_i define mappings $\tilde{H}_q(\overline{M}, D; G) \to \tilde{H}_q(\overline{M}_i, D_i; G)$ for $i=1, 2$. They are injective because $\tilde{H}_q$'s are identified with subgroups of $H^{2n-q}(M; G)$ in a natural way. On the other hand, they are surjective by (1.11). Thus they are bijective. Our assertion follows easily from this.

(1.13) The subgroup (resp. quotient group) of $H^p(M; G)$ corresponding to $\tilde{H}_{2n-p}(\overline{M}, D; G)$ (resp. $\hat{H}_{2n-p}(\overline{M}, D; G)$ will be denoted by $\hat{H}^p(\overline{M}; G)$ (resp. $\tilde{H}^p(M; G)$). The rank of the free part of $\hat{H}^p(M; \boldsymbol{Z})$ (resp. $\tilde{H}^p(M; \boldsymbol{Z})$) will be denoted by $\hat{b}_p(M)$ (resp. $\tilde{b}_p(M)$). Of course we have $b_p(M)=\hat{b}_p(M)+\tilde{b}_p(M)$.

(1.14) REMARK. Even if G is an arbitrary abelian group, $\tilde{H}_q(\overline{M}, D; G)$ is independent of the choice of a completion $(\overline{M}, D)$ of M. However, this does not define a subgroup of $H^{2n-q}(M; G)$ in general, because G is not necessarily a ring. Similar remark applies also to $\hat{H}_q(\overline{M}, D; G)$.

(1.15) PROPOSITION. *Let $(\overline{M}, D)$ be a completion of $M=\overline{M}-D$, where $\overline{M}$ is smooth. Then*

1) $\tilde{b}_0(M)=0$ *and* $\hat{b}_0(M)=1$. $\hat{b}_1(M)=b_1(\overline{M})$.

2) $\tilde{b}_1(M)=b_1(M)-b_1(\overline{M})=\bar{q}(M)-q(\overline{M})$.

3) $\tilde{b}_{2n-1}(M)=$(*the number of connected components of* D)-1.

4) $\hat{b}_{2n}(M)=0$ *unless* $D=\emptyset$. $\tilde{b}_{2n}(M)=0$ *always.*

Proofs are straightforward from the definition. As for 2), see also [Kw 1; p. 264].

(1.16) DEFINITION. Let D be a reduced divisor on a complete manifold $\overline{M}$ and let M be the q.c. manifold $\overline{M}-D$. Let $\mathfrak{L}(D)$ be the free abelian group with basis consisting of the prime components of D and let $p: \mathfrak{L}(D) \to \mathrm{Pic}(\overline{M})$ be the natural mapping. Then we define $\mathrm{Pic}(M)$ to be $\mathrm{Coker}(p)$.

Let $c_1: \mathrm{Pic}(\overline{M}) \to H^2(\overline{M}; \boldsymbol{Z})$ be the Chern mapping. Then $\mathrm{Pic}_0(\overline{M})=\mathrm{Ker}(c_1)$

and $NS(\overline{M})=\mathrm{Im}(c_1)$. We define the Neron-Severi group $NS(M)$ (resp. $\mathrm{Pic}_0(M)$) of M to be the cokernel (resp. kernel) of the natural homomorphism $\mathfrak{L}(D)\to NS(\overline{M})$ (resp. $\mathrm{Pic}(M)\to NS(M)$).

Apparently these definitions of $\mathrm{Pic}(M)$, $NS(M)$ and $\mathrm{Pic}_0(M)$ depends on the completion $(\overline{M}, D)$. However, as a matter of fact, they are invariants of the q.c.-structure (cf. (1.19)).

(1.17) PROPOSITION. *Let $\overline{M}$, D and M be as in* (1.16), *and let $A(M)^\times$ be the multiplicative group of invertible elements of $A(M)$ (cf.* (1.5)). *Then*

1) $A(M)^\times/\boldsymbol{C}^\times$ *is isomorphic to* $\mathrm{Ker}(p)$.
2) $A(M)$ *is* UFD *if* $\mathrm{Pic}(M)=0$. *The converse is true if M is affine.*

PROOF. Let $f\in A(M)^\times$. Then f is a meromorphic function on $\overline{M}$, and its poles lie in D. Its zeros also lie in D, because $f^{-1}\in A(M)$. So the divisor (f) is in $\mathfrak{L}(D)$. Thus we get a group homomorphism $\delta\colon A(M)^\times\to\mathfrak{L}(D)$. Clearly $\mathrm{Im}(\delta)=\mathrm{Ker}(p)$ and $\mathrm{Ker}(\delta)=\boldsymbol{C}^\times$. Hence 1) follows.

Suppose that $\mathrm{Pic}(M)=0$. For a meromorphic function f on $\overline{M}$, we denote its pole divisor (resp. zero divisor) by $(f)_\infty$ (resp. $(f)_0$), and we set $(f)=(f)_0-(f)_\infty$. Now, let $\varphi\in A(M)$ and let $(\varphi)_0=\sum\mu_iZ_i$ be the prime decomposition of $(\varphi)_0$. Since p is surjective, there is a function φ_i on $\overline{M}$ for each i such that $(\varphi_i)-Z_i\in\mathfrak{L}(D)$. Then $\varphi_i\in A(M)$ and $\varphi=c\prod\varphi_i^{\mu_i}$ for $c=\varphi\prod\varphi_i^{-\mu_i}$. $c\in A(M)^\times$ because $(c)\in\mathfrak{L}(D)$. This observation implies that, if φ is prime in $A(M)$, then $(\varphi)_0$ contains at most one prime component off D. So we infer that any other irreducible decomposition of φ is equivalent to the above one. Thus we see that $A(M)$ is UFD.

The converse of 2) in the affine case is well-known and easily proved.

Q.E.D.

(1.18) PROPOSITION. *Let $\overline{M}$, D and M be as above. Then we have the following natural exact sequences.*

1) $0\to A(M)^\times/\boldsymbol{C}^\times\to\mathfrak{L}(D)\to\mathrm{Pic}(\overline{M})\to\mathrm{Pic}(M)\to 0$.

2) $0\to\tilde{H}^1(M;\boldsymbol{Z})\to\mathfrak{L}(D)\to NS(\overline{M})\to NS(M)\to 0$ *and* $0\to NS(M)\to\hat{H}^2(M;\boldsymbol{Z})$.

3) $0\to A(M)^\times/\boldsymbol{C}^\times\to\tilde{H}^1(M;\boldsymbol{Z})\to\mathrm{Pic}_0(\overline{M})\to\mathrm{Pic}_0(M)\to 0$.

PROOF. 1) is straightforward from the definition (1.16) and (1.17; 1). 2) follows from the exact sequence $0\to\tilde{H}^1(M;\boldsymbol{Z})\to H_{2n-2}(D;\boldsymbol{Z})\to H_{2n-2}(\overline{M};\boldsymbol{Z})\to\hat{H}^2(M;\boldsymbol{Z})\to 0$ and the natural isomorphisms $H_{2n-2}(D;\boldsymbol{Z})\cong\mathfrak{L}(D)$ and $H_{2n-2}(\overline{M};\boldsymbol{Z})\cong H^2(\overline{M};\boldsymbol{Z})$. To obtain 3) we use $\mathrm{Ker}(\mathrm{Pic}_0(\overline{M})\to\mathrm{Pic}_0(M))\cong p(\mathfrak{L}(D))\cap\mathrm{Pic}_0(\overline{M})=p(\mathrm{Ker}(\mathfrak{L}(D)\to H^2(\overline{M};\boldsymbol{Z})))$ and the above long exact sequence.

(1.19) COROLLARY. *$\mathrm{Pic}(M)$, $\mathrm{NS}(M)$ and $\mathrm{Pic}_0(M)$ are independent of the choice of a completion of the q.c. manifold M.*

PROOF. By a similar argument as in (1.12), we prove the invariance of $\mathrm{NS}(M)$ as a subgroup of $H^2(M;\boldsymbol{Z})$. The invariance of $\mathrm{Pic}_0(M)$ follows from (1.18; 3). Combining these one gets the invariance of $\mathrm{Pic}(M)$.

(1.20) COROLLARY. 1) $A(M)^\times=\boldsymbol{C}^\times$ *if* $\widetilde{H}^1(M;\boldsymbol{Z})=0$.

1') $\widetilde{H}^1(M;\boldsymbol{Z})=0$ *if* $A(M)^\times=\boldsymbol{C}^\times$ *and* $\mathrm{Pic}_0(\overline{M})=0$.

2) $A(M)$ *is* UFD *if* $\hat{H}^2(M;\boldsymbol{Z})=0$ *and* $\mathrm{Pic}_0(\overline{M})=0$.

PROOF. 1) and 1') follow from (1.18; 3). $\hat{H}^2(M;\boldsymbol{Z})=0$ implies $\mathrm{NS}(M)=0$ by (1.18; 2) and $\mathrm{Pic}_0(M)=0$ follows from $\mathrm{Pic}_0(\overline{M})=0$ by 3). So the assertion 2) follows from (1.17; 2).

(1.21) DEFINITION. Let x be a point on a topological space X. A fundamental system of neighborhoods $\{U_j\}_{j=1,2,\dots}$ of x in X will be called excellent if $U_i\subset U_j$ for every $i>j$ and the inclusion $U_i^*\subset U_j^*$ is a homotopy equivalence, where U_j^* denotes $U_j-\{x\}$. If such a system exists, then U_j^* is homotopic to V_k^* for any other excellent system $\{V_k\}$. So the homotopy type of U_j^* is independent of the choice of the excellent system $\{U_j\}$. This will be called the punctured local homotopy type of X at x.

Let X be a non-compact topological space. If the punctured local homotopy type of its one-point-compactification at the infinity is well-defined, then it will be called the homotopy type of X at the infinity, and is denoted by $\infty(X)$. For any homotopical functor F, $F(\infty(X))$ is well-defined too.

Let $(\overline{M}, D)$ be an NC-completion of a q.c. manifold $M=\overline{M}-D$. Following the recipe of Ramanujam [Ra; p. 72], we can find a distance function ρ on $\overline{M}$ such that $D=\{x\in\overline{M}\mid\rho(x)=0\}$, D is a deformation retract of $U_\delta=\{x\in\overline{M}\mid\rho(x)<\delta\}$ for any sufficiently small $\delta>0$ and $S_\delta=\partial U_\delta=\{x\in\overline{M}\mid\rho(x)=\delta\}$ is a deformation retract of $V_\delta^*=\overline{U}_\delta-D=\{x\in\overline{M}\mid 0<\rho(x)<\delta\}$ for any small δ. Then $\{\infty\cup V_\delta^*\}$'s give an excellent system of M at the infinity. Thus $\infty(M)$ is well-defined, and homotopic to S_δ. Obviously this is a homeomorphic invariant of M.

(1.22) *Example.* $\infty(\boldsymbol{C}^p\times(\boldsymbol{C}^\times)^q)=\infty(\boldsymbol{R}^{2p+q}\times(S^1)^q)\sim S^{2p+q-1}\times T^q$.

(1.23) Before closing this section, we describe several different q.c. structures of $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$. Incidentally, we will see that many invariants of q.c. structures are not invariants of the complex analytic structures.

1) Let D_1, D_2 and D_3 be three lines on $\boldsymbol{P}^2$ which do not meet at a common point. Then one sees easily that $\boldsymbol{P}^2-D_1\cup D_2\cup D_3\cong\boldsymbol{C}^\times\times\boldsymbol{C}^\times$. On the other hand,

let $Q=\boldsymbol{P}^1\times\boldsymbol{P}^1$ and let $D=p_1^{-1}(0)\cup p_1^{-1}(\infty)\cup p_2^{-1}(0)\cup p_2^{-1}(\infty)\subset Q$, where p_j is the projection onto the j-th factor. Then $Q-D\cong\boldsymbol{C}^\times\times\boldsymbol{C}^\times$. It is easy to see that these two completions define the same q.c. structure of $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$. By the latter completion we see that this is the direct product of $(\boldsymbol{A}^1)^*$'s in the category QCS. This q.c. structure will be called the standard q.c. structure of $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$.

2) Let C be an elliptic curve and let $0\to[0]\to E\to[0]\to 0$ be a non-trivial extension of vector bundles on C. So, E is an indecomposable vector bundle of rank two. The sub-bundle [0] defines a divisor D on $\bar{S}=\boldsymbol{P}(E)$, and D is a section of $\bar{S}\to C$. We claim that $S=\bar{S}-D$ is biholomorphic to $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$.

To prove this, we note that S is an $\boldsymbol{A}^1$-bundle over C with structure group $\boldsymbol{G}_a$, by construction. Hence $\tilde{S}=S\times_C\tilde{C}$ is an $\boldsymbol{A}^1$-bundle over the universal covering $\tilde{C}\cong\boldsymbol{C}$ of C. Let t be a uniformizing parameter on $\tilde{C}$ such that $C\cong\tilde{C}/(\boldsymbol{Z}\omega_1+\boldsymbol{Z}\omega_2)$ for some $\boldsymbol{R}$-linearly independent scalars ω_1 and ω_2, and let z be a parameter of $\tilde{S}$ along the fiber. Then (t, z) gives a coordinate system of $\tilde{S}\cong\boldsymbol{C}^2$ such that $S\cong\tilde{S}/(\boldsymbol{Z}(\omega_1, a_1)+\boldsymbol{Z}(\omega_2, a_2))$, where a_i's are some scalars. Since E is defined by a non-trivial extension, we infer that (ω_1, a_1) and (ω_2, a_2) are $\boldsymbol{C}$-linearly independent in $\boldsymbol{C}^2$. This implies $S\cong\boldsymbol{C}^\times\times\boldsymbol{C}^\times$.

A q.c. structure of this type will be called an elliptic q.c. structure of $(\boldsymbol{C}^\times)^2$. They form one dimensional moduli, corresponding to that of elliptic curves. Moreover, such a q.c. manifold can be specialized to $C\times\boldsymbol{A}^1$, because the vector bundle E as above can be specialized to $[0]\oplus[0]$.

3) Let G be the subgroup of GL(2; $\boldsymbol{C}$) generated by $g=\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$, where a is a scalar with $|a|<1$. G acts on $\boldsymbol{C}^2-\{(0, 0)\}$ freely, and the quotient $\bar{S}=\boldsymbol{C}^2-\{(0, 0)\}/G$ is a compact complex manifold. $\bar{S}$ is known as a primary Hopf surface (cf. [Ko 2]). Let (x, y) be a coordinate of $\boldsymbol{C}^2$ such that $g(x, y)=(ax, x+ay)$. Then the line $\{x=0\}$ is G-stable, so this defines a divisor D on $\bar{S}$. D is isomorphic to the elliptic curve $\boldsymbol{C}^\times/G'$, where G' is the group generated by $g': y\to ay$. We claim that $S=\bar{S}-D$ is biholomorphic to $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$.

To see this, take a coordinate system (z, y) of the universal covering $\tilde{S}\cong\boldsymbol{C}^2$ of S such that $x=e(z)$, where $e(\cdot)$ is the function $\exp(2\pi i\cdot)$. Take a scalar α such that $a=e(\alpha)$. Then $S\cong\tilde{S}/\langle g_0, g_1\rangle$, where g_i's are the covering transformations defined by $g_0(z, y)=(z+1, y)$ and $g_1(z, y)=(z+\alpha, ay+e(z))$. Introduce another coordinate system (z, u) by setting $u=e(-z)y$. With respect to this system g_i's look like $g_0(z, u)=(z+1, u)$ and $g_1(z, u)=(z+\alpha, u+a^{-1})$. Since (1, 0) and (α, a^{-1}) generate a lattice Λ, we infer that $S\cong\boldsymbol{C}^2/\Lambda\cong\boldsymbol{C}^\times\times\boldsymbol{C}^\times$.

A q.c. structure of this type will be called a Hopf q.c. structure of $(\boldsymbol{C}^\times)^2$. Varying the scalar a, we see that these structures form a one-dimensional family.

4) $\boldsymbol{C}^\times\times\boldsymbol{C}^\times$ has the following biholomorphic invariants: $H^1(S)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$ (for the moment, cohomologies are defined in coefficients in $\boldsymbol{Z}$), $H^2(S)\cong\boldsymbol{Z}$, $H^3(S)=0$,

$\pi_1(S)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}$, $\infty(S)\sim S^1\times S^1\times S^1$, $\kappa_c(S)=-\infty$. $S=(\boldsymbol{C}^\times)^2$ is Stein, of course.

Q.c. invariants of the preceding q.c. structures of S are calculated as follows:

structures	alg. dim	$\pi_1(\bar{S})$	$\bar{q}$	$\bar{\kappa}$	$\hat{H}^1$	$\tilde{H}^1$
standard	2	0	2	0	0	$\boldsymbol{Z}\oplus\boldsymbol{Z}$
elliptic	2	$\boldsymbol{Z}\oplus\boldsymbol{Z}$	1	$-\infty$	$\boldsymbol{Z}\oplus\boldsymbol{Z}$	0
Hopf	0	$\boldsymbol{Z}$	1	$-\infty$	$\boldsymbol{Z}$	$\boldsymbol{Z}$

$\hat{H}^2$	$\tilde{H}^2$	$A(S)^\times/\boldsymbol{C}^\times$	NS(S)	Pic$_0$	others
0	$\boldsymbol{Z}$	$\boldsymbol{Z}\oplus\boldsymbol{Z}$	0	0	affine
$\boldsymbol{Z}$	0	0	$\boldsymbol{Z}$	D	non-affine
0	$\boldsymbol{Z}$	0	0	D	non-Kähler

(1.24) REMARK. If $\kappa_c(M)=\dim M$, then M admits only one q.c. structure. This follows from Sakai's theory ([Sa], cf. (1.8) too).

§2. Vanishing of higher Betti numbers.

From now on we are mainly concerned with surfaces. But at first we prove the following

(2.1) THEOREM. *Let $f: M\to N$ be a proper finite morphism of complex manifolds. Then $f^*: H^p(N;\boldsymbol{C})\to H^p(M;\boldsymbol{C})$ is injective.*

PROOF. In general, the hypercohomology of the de Rham complex $\{\Omega_X^\cdot, d_X^\cdot\}$ of any complex manifold X is canonically isomorphic to $H^\cdot(X;\boldsymbol{C})$. In our situation, $\boldsymbol{H}^\cdot(\Omega_M^\cdot, d_M^\cdot)\cong\boldsymbol{H}^\cdot(f_*\Omega_M^\cdot, f_*d_M^\cdot)$ because f is finite. Therefore, the theorem follows from the result below.

(2.2) PROPOSITION. *Let f, M, N be as in* (2.1) *and let $\Omega^p(f)$ be the natural homomorphism $\Omega_N^p\to f_*\Omega_M^p$. Then there exists a homomorphism $\{\sigma^\cdot\}:\{f_*\Omega_M^\cdot\}\to\{\Omega_N^\cdot\}$ of complexes such that $\sigma^\cdot\circ\Omega^\cdot(f)$ is the identity of $\{\Omega_N^\cdot\}$.*

PROOF. σ^p is constructed in the following way. Let B be the branch locus of f and suppose x to be a point on N off B. Take a neighborhood U of x such that all the connected components $U_1,\cdots,U_k$ of $f^{-1}(U)$ are mapped isomorphically onto U by f. For $\varphi\in f_*\Omega_M^p(U)\cong\oplus_j\Omega_M^p(U_j)\cong\oplus_j\Omega^p(U)$, set

$\sigma^p(\varphi)=k^{-1}\sum_{j=1}^{k}\varphi_j$, where $\varphi_j\in\Omega^p(U)$ is the push down of the j-th component of φ by f. Patching them together, we get a sheaf homomorphism $f_*\Omega_M^p\to\Omega_N^p$ defined off B.

Next consider the restriction of f to $f^{-1}(B)$ with respect to the reduced structure. We have a nowhere dense closed subset S of B such that this restriction is étale over $B-S$ and B is non-singular at any point x on $B-S$. Let $y_1, \cdots, y_r$ be the points on M lying over x. We can find a coordinate neighborhood U (resp. $V_1, \cdots, V_r$) of x (resp. $y_1, \cdots, y_r$) with coordinate system $(u^1, \cdots, u^n)$ (resp. $(v_j^1, \cdots, v_j^n)$) on U (resp. V_j for each $j=1, \cdots, r$) such that the restriction f_j of f to V_j is a surjective morphism onto U and $f_j(v_j^1, \cdots, v_j^n)=((v_j^1)^{\mu_j}, v_j^2, \cdots, v_j^n)$ in terms of the above coordinates for every $j=1, \cdots, r$. Here $\mu_1, \cdots, \mu_r$ are positive integers such that $\mu_1+\cdots+\mu_r=k$, the mapping degree of f. Of course $B\cap U$ is the divisor defined by $u^1=0$. Let $g_j: V_j\to V_j$ be the automorphism defined by $g_j(v_j^1, \cdots, v_j^n)=(\rho_j v_j^1, v_j^2, \cdots, v_j^n)$, where $\rho_j=\exp(2\pi i/\mu_j)$. Then U is the quotient of V_j with respect to g_j. In particular, a holomorphic p-form on V_j comes from U if and only if it is g_j-invariant. Now, let $\varphi\in f_*\Omega_M^p(U)=\bigoplus_{j=1}^{r}\Omega_M^p(V_j)$ and let φ_j be the j-th component of φ. Then $\sum_{\nu=1}^{\mu_j}(g_j^\nu)^*\varphi_j$ is g_j-invariant and hence comes from a p-form ϕ_j on U. Setting $\sigma^p(\varphi)=(\phi_1+\cdots+\phi_r)/k$, we obtain a sheaf homomorphism $f_*\Omega_M^p\to\Omega_N^p$ defined on $N-S$ as an extension of that on $N-B$ defined in the preceding paragraph.

Since $\mathrm{codim}(S)\geqq 2$, one can extend the above homomorphism to the whole space N using the theorem of Hartogs. It is easy to see that this has the desired property. Q.E.D.

(2.3) REMARK. (2.2) gives a mapping $\sigma: H^p(M; \boldsymbol{C})\to H^p(N; \boldsymbol{C})$ such that $\sigma\circ f^*$ is the identity. If M and N are compact, one can verify that σ is k^{-1}-times of the Poincaré dual of $H_{2n-p}(M)\to H_{2n-p}(N)$. In case of $\boldsymbol{Z}$-valued cohomologies, f^* need not be injective. The kernel may have k-torsion.

(2.4) From now on, throughout in this section, $(\bar{S}, D)$ is an NC-completion of a smooth q.c. surface $S=\bar{S}-D$.

THEOREM. 1) $\hat{b}_2(S)\geqq 2p_g(\bar{S})$.
2) *If* $\hat{b}_2(S)=0$ *and* $\hat{b}_1(S)$ *is even, then* $\bar{S}$ *is projectively algebraic.*
3) *If in addition* $\tilde{b}_3(S)=0$, *then* S *is affine.*

PROOF. 1) Let $\nu=c_1\circ p: \mathfrak{L}(D)\to H^2(\bar{S}; \boldsymbol{Z})$ be as in (1.16). Then $\mathrm{Im}(\nu)$ lies in the (1, 1)-part with respect to the Hodge decomposition of $H^2(\bar{S}; \boldsymbol{C})$. Therefore $\hat{b}_2(S)=\mathrm{rank}(\mathrm{Coker}(\nu))\geqq h^{2,0}+h^{0,2}=2p_g(\bar{S})$.

2) $p_g(\bar{S})=0$ by 1) and $b_1(\bar{S})=\hat{b}_1(S)$ by (1.15; 1). So the criterion [Ko 2; p. 758, Theorem 10] applies.

3) Let $\mathfrak{A}$ be the set consisting of all the effective divisors A such that $\mathrm{Supp}(A)\subset D$ and $AC>0$ for any prime component C of A. We first show that $\mathfrak{A}$ contains a non-zero element. $\mathrm{Coker}(\nu)$ is a torsion module since $\hat{b}_2(S)=0$. Hence there exists an ample divisor X on $\bar{S}$ such that $\mathrm{Supp}(X)\subset D$. Write $X=Y-Z$ such that Y and Z are effective and have no common component. Then, for any component C of Y, we have $YC=(X+Z)C\geqq XC>0$. So $Y\in\mathfrak{A}$.

Take $A\in\mathfrak{A}$ such that the number of irreducible components of A is the maximum among elements of $\mathfrak{A}$. We claim that $\mathrm{Supp}(A)=D$. Indeed, otherwise, there is a component C of D such that $C\not\subset\mathrm{Supp}(A)$ and $AC>0$, because D is connected by (1.15; 3). Then $(tA+C)C>0$ for $t>-C^2$, and hence $A'=tA+C\in\mathfrak{A}$. The number of components of A' is greater than that of A, contradicting the hypothesis. Thus we prove $\mathrm{Supp}(A)=D$.

It is enough to show that A is ample. $A^2>0$ follows from the definition of A. So, thanks to Nakai's criterion (cf. [Nk]), it suffices to show $AC>0$ for any curve C in $\bar{S}$. By definition of A we may assume $C\not\subset D$. Then $YC\geqq XC>0$. So $DC>0$. This implies $AC>0$ since $\mathrm{Supp}(A)=D$. Q. E. D.

REMARK. As for the non-algebraic case, see (9.5).

(2.5) COROLLARY (almost due to Ramanujam [Ra]). *The following two conditions are equivalent to each other.*

a) $b_j(S)=0$ *for any* $j>0$.

b) S *is affine*, $A(S)^\times=\boldsymbol{C}^\times$, $\mathrm{Pic}(S)$ *is a torsion group*, $p_g(\bar{S})=q(\bar{S})=0$ *and* $b_1(D)=0$.

PROOF. a)$\Rightarrow$b). $b_1(\bar{S})=0$ by (1.15; 1). So $q(\bar{S})=0$ and $\mathrm{Pic}_0(\bar{S})=0$. In view of (1.18; 2), we infer that $\mathrm{Pic}(S)=\mathrm{NS}(S)\subset H^2(S;\boldsymbol{Z})$ is a torsion group. $H^1(S;\boldsymbol{Z})=0$ because this is torsion-free by the universal coefficient theorem. Hence $A(S)^\times=\boldsymbol{C}^\times$ by (1.18; 3). $p_g(\bar{S})=0$ by (2.4; 1). So S is affine by (2.4; 3). $b_1(D)=0$ because $b_1(\bar{S})=\bar{b}_2(S)=0$ (see the definition of $\bar{b}_2$).

b)$\Rightarrow$a). $H^p(S;\boldsymbol{Z})=0$ for $p>2$ since S is affine. $\hat{b}_1(S)=0$ by (1.15; 1). We have $\mathrm{Pic}_0(\bar{S})=0$ since $q(\bar{S})=0$. So $A(S)^\times=\boldsymbol{C}^\times$ implies $\bar{b}_1=0$ by (1.18; 3). $p_g(\bar{S})=0$ implies $\mathrm{NS}(\bar{S})=H^2(\bar{S};\boldsymbol{Z})$. Hence $\hat{b}_2(S)=\mathrm{rank}\,\mathrm{NS}(S)=\mathrm{rank}\,\mathrm{Pic}(S)=0$. $\bar{b}_2(S)=0$ follows from the definition and $b_1(D)=0$.

REMARK. Actually, we may omit the condition $q(\bar{S})=0$ from b). Indeed, $b_3(S)=0$ since S is affine. So $b_1(D)=0$ implies $b_1(\bar{S})=0$ by definition of $\hat{b}_3$.

(2.6) COROLLARY. *Suppose that the conditions in* (2.5) *are satisfied. Then* S *is rational unless* $\bar{\kappa}(S)=2$.

PROOF. Suppose that S is not rational. Then, by Castelnuovo's criterion, we infer $\kappa(\bar{S})\geqq 0$. Hence $\bar{\kappa}(S)=\kappa(K+D, \bar{S})\geqq\kappa(D, \bar{S})=2$, where K denotes the canonical bundle of $\bar{S}$.

(2.7) DEFINITION. A connected NC-divisor D on a surface will be called a *tree* if: 1) there is no pair (C_i, C_j) of components of D such that $C_iC_j\geqq 2$, and 2) there are no components $C_1, C_2, \cdots, C_r$ of D with $r\geqq 3$ such that $C_iC_j>0$ for every (i, j) with $i-j\equiv 1$ modulo r.

If in addition every component of D is a rational curve, D will be called a *rational tree*.

(2.8) THEOREM. *A connected* NC-*divisor D on a surface is a rational tree if and only if $b_1(D)=0$. Moreover, in this case $\pi_1(D)=H_1(D; \boldsymbol{Z})=0$.*

The proof is easy and well-known. See e.g. [Ra; p. 70].

(2.9) COROLLARY. *Suppose that the conditions in* (2.5) *are satisfied. Then* $\mathrm{Pic}(S)\cong H^2(S; \boldsymbol{Z})$ *and $\bar{S}$ is simply connected.*

PROOF. D is a rational tree by (2.8). So $H_1(D; \boldsymbol{Z})=0$ which implies $\tilde{H}^2(S; \boldsymbol{Z})=0$. Hence $\mathrm{Pic}(S)\cong \mathrm{NS}(S)\cong\hat{H}^2(S; \boldsymbol{Z})=H^2(S; \boldsymbol{Z})$. D is the support of an ample divisor on $\bar{S}$ as we saw in (2.4). Therefore $\pi_1(D)\to\pi_1(\bar{S})$ is surjective by virtue of Lefschetz theorem. So $\bar{S}$ is simply connected as well as D.

(2.10) *Question.* Let S be as above. Then, is S necessarily rational?

The answer is YES if $\kappa(\bar{S})=-\infty$. $\kappa(\bar{S})\neq 0$ since $p_g(\bar{S})=0$ and $\pi_1(\bar{S})=0$. It seems to be improbable that $\kappa(\bar{S})=1$, because of the structure theory of elliptic surfaces (cf. [Ko 1]). It is a long-standing conjecture that $\kappa(\bar{S})=2$ and $\pi_1(\bar{S})=0$ implies $p_g(\bar{S})>0$.

(2.11) REMARK. In the situation (2.5), S itself may not be simply connected. Far from that, $\pi_1(S)$ may be an infinite non-abelian group (see (5.12)).

(2.12) THEOREM. *Let $f: M\to N$ be a dominant morphism of q.c. manifolds. Then* $\mathrm{Im}(\pi_1(M))$ *is a subgroup of $\pi_1(N)$ of finite index.*

PROOF. There exists a proper mapping $\bar{f}: \bar{M}\to N$ and an open embedding $M\subset\bar{M}$ such that f is the restriction of $\bar{f}$. Since $\pi_1(M)\to\pi_1(\bar{M})$ is surjective, it suffices to show the assertion for $\bar{f}$. So we may assume that f is proper.

Let U be an open dense subset of N such that f is smooth over U. Since $\pi_1(U)\to\pi_1(N)$ is surjective, it suffices to show the assertion for $f_U: f^{-1}(U)\to U$.

So we may assume that f is smooth.

Let $M \to W \to N$ be the Stein factorization of f. So any general fiber of $f': M \to W$ is connected. This implies that $\pi_1(f')$ is surjective. Hence, we may further assume that $\dim M = \dim N$.

Thus we reduce the problem to the case in which f is proper, smooth, finite, that means, étale. Here the assertion is obvious. Q.E.D.

(2.13) COROLLARY. *If in addition M is simply connected, then $\pi_1(N)$ is finite.*

(2.14) COROLLARY. *Let f, M and N be as in* (2.13) *and further suppose that f is a finite morphism and that $b_j(M)=0$ for any $j>0$. Then $\pi_1(N)=0$.*

PROOF. Let $\pi: \tilde{N} \to N$ be the universal covering of N. f can be lifted to $\tilde{f}: M \to \tilde{N}$ since M is simply connected. We have $b_j(\tilde{N}) = b_j(N) = 0$ by (2.1). Hence $\chi(\tilde{N}) = \chi(N) = 1$. This implies $\deg(\pi) = 1$, proving the assertion.

§ 3. Weighted dual graphs of NC-divisors on surfaces.

In this section we will fix our terminology and notation.

(3.1) Let D be an NC-divisor on a surface. We define a graph Γ in the following way. The prime components $D_1, \cdots, D_r$ are in one-to-one correspondence with the vertices $v_1, \cdots, v_r$ of Γ. The nodes on $D_i \cap D_j$ are in one-to-one correspondence with the segments connecting v_i and v_j. Furthermore, the *weight* of v_j is defined to be the self intersection number D_j^2. Such a graph Γ will be called the *weighted dual graph* of D. In the sequel we sometimes use terminologies concerning NC-divisors and weighted dual graphs interchangeably.

Example. D is a tree in the sense (2.7) if and only if Γ is a tree.

(3.2) Let v be a vertex of a graph Γ. The number of segments connecting v and other vertices is called the *branch number* of Γ at v and is denoted by $\beta_\Gamma(v)$, or $\beta(v)$ when there is no danger of confusion. Obviously $\beta(v)=0$ if and only if v is isolated. If $\beta(v)=1$, v is called a *tip* of Γ. v is called a *branching* if $\beta(v) \geqq 3$.

Suppose that Γ is connected. Then Γ is linear if and only if Γ has no branching. A *twig* is a linear graph Γ together with a total ordering $v_1 > \cdots > v_r$ among its vertices such that v_j and v_{j-1} are connected by a segment for each j. The highest vertex v_1 is called the *tip* of it. Such a twig is denoted by $[-w_1, \cdots, -w_r]$, where w_j is the weight of v_j.

Let v be a tip of Γ. Then, unless v is in a linear connected component of

Γ, there exist vertices $v_1=v, v_2, \cdots, v_r, v_{r+1}$ of Γ such that v_j and v_{j+1} is connected by precisely one segment for each $j=1, \cdots, r$ and that $\beta(v_2)=\cdots=\beta(v_r)=2$ and $\beta(v_{r+1})\geqq 3$. This linear subgraph consisting of $v_1, \cdots, v_r$ (not including v_{r+1}) is called the *twig* of Γ with the tip v. v_{r+1} is called the *branching* of this twig.

(3.3) $\boldsymbol{Q}(D)$ (resp. $\boldsymbol{Q}(\Gamma)$) denotes the r-dimensional $\boldsymbol{Q}$-vector space of formal linear combinations of prime components of D (resp. vertices of Γ) with coefficients being rational numbers. The intersection pairing induces a $\boldsymbol{Q}$-valued symmetric bilinear form I on $\boldsymbol{Q}(\Gamma)$ such that $I(v_i, v_j)=D_iD_j$. Γ is said to be *contractible* if I is negative definite. By $d(\Gamma)$ we denote the determinant of the (r, r)-matrix with (i, j)-ingredients $-I(v_i, v_j)$.

(3.4) Let q be a point on D. Blow up the surface at q and let D' be the inverse image of D. Let Γ' be the dual graph of D' and let v be the vertex corresponding to the exceptional divisor lying over q.

$\beta(v)=2$ if and only if q is a node of D. In this case the blowing-up is said to be *subdivisional*. $\beta(v)=1$ if and only if q is a smooth point on D. In this case the blowing-up is called *sprouting*. In either case, the weighted dual graph Γ' is obtained from Γ by the obvious combinatorial process.

It is easy to see that $d(\Gamma')=d(\Gamma)$ and Γ' is contractible if and only if Γ is so.

A weighted graph Γ is said to be *minimal* if it does not contain a vertex with weight -1 and $\beta=1$ or 2. If Γ contains such a vertex, then it is obtained from another graph by one of the processes corresponding to the two types of blowing-up described above.

(3.5) Let A be the twig $[a_1, \cdots, a_r]$. We call the twig $[a_r, \cdots, a_1]$ the *transposal* of A and denote it by ${}^tA$. We define also $\bar{A}=[a_2, \cdots, a_r]$ and $\underline{A}={}^t\overline{({}^tA)}=[a_1, \cdots, a_{r-1}]$. $e(A)=d(\bar{A})/d(A)$ is called the *inductance* of A.

A is said to be *integral* if a_j's are integers. If in addition $a_j\geqq 2$ for every j, A is said to be *admissible*. It is easy to see that a twig is admissible if and only if it is integral, contractible and minimal.

(3.6) LEMMA. *Let A be an admissible twig. Then*

1) $d(A)=a_1d(\bar{A})-d(\bar{\bar{A}})=d({}^tA)=a_rd(\underline{A})-d(\underline{\underline{A}})$, *where* $d(\emptyset)=1$ *by convention.*

2) $d(\bar{A})d(\underline{A})-d(A)d(\underline{\bar{A}})=1$.

3) $d(A)$ *and* $d(\bar{A})$ *are coprime integers such that* $d(A)>d(\bar{A})$.

PROOF. 1) follows from the definition of $d(A)$ and by elementary calculations in linear algebra. Applying 1) to $d(A)$ and $d(\underline{A})$, we obtain $d(\bar{A})d(\underline{A})$

$-d(A)d(\underline{\bar{A}})=d(\underline{\bar{A}})d(\bar{\bar{A}})-d(\bar{A})d(\underline{\bar{\bar{A}}})$. So 2) is proved by induction on r. 3) follows easily from 2).

(3.7) COROLLARY. *Let A and B be admissible twigs such that $e(A)+e(B)=1$. Then $d(A)=d(B)$ and $e({}^tA)+e({}^tB)=1$.*

PROOF. $d(A)=d(B)$ is clear by definition of e and (3.6; 3). The assumption implies that $d(\bar{A})\equiv -d(\bar{B}) \bmod d(A)$. (3.6; 2) implies that $d(\bar{A})$ and $d(\underline{A})$ are multiplicatively inverse to each other in $\boldsymbol{Z}/d(A)\boldsymbol{Z}$. Therefore $d(\bar{B})$ and $-d(\underline{A})$ are multiplicatively inverse to each other modulo $d(B)$. So $d(\underline{B})+d(\underline{A})\equiv 0 \bmod d(B)$. From this we infer that $e({}^tB)+e({}^tA)=1$.

(3.8) COROLLARY. *e defines a one-to-one correspondence from the set of all the admissible twigs to the set of rational numbers in the interval $0<q<1$.*

PROOF. (3.6; 1) implies that $e(A)^{-1}=a_1-e(\bar{A})$. Since a_1 is an integer and $0<e(\bar{A})<1$, $e(A)$ determines both a_1 and $e(\bar{A})$. Note also that $e(A)^{-1}$ is an integer if and only if $r=1$, i.e., $\bar{A}=\varnothing$. Thus $A=[a_1, \cdots, a_r]$ can be recovered by the method of continued fraction. This method shows also the existence of A with $e(A)=q$ for every rational number q such that $0<q<1$.

(3.9) The admissible twig of inductance e is denoted by $\Gamma(e)$. For an admissible twig A, $\Gamma(1-e({}^tA))$ is the transposal of $\Gamma(1-e(A))$ by (3.7). This is called the *adjoint* of A and is denoted by A^*. So $e({}^tA)+e(A^*)=1$.

Example. $\Gamma(1/r)=[r]$. $\Gamma(r/r+1)=[r\times 2]$, where $r\times 2$ stands for the r-times repetition of 2. $\Gamma(2/5)=[3, 2]$. $\Gamma(3/5)=[2, 3]$. $\Gamma(2/7)=[4, 2]$. $\Gamma(3/7)=[3, 2, 2]$. $\Gamma(4/7)=[2, 4]$. $\Gamma(5/7)=[2, 2, 3]$. $\Gamma(3/8)=[3, 3]$. $\Gamma(5/8)=[2, 3, 2]$. etc.

§ 4. Geometry on ruled surfaces.

(4.1) DEFINITION. A surjective morphism $f: \bar{S}\to C$ from a smooth complete surface $\bar{S}$ onto a smooth curve C is called a *ruling* (or *$\boldsymbol{P}^1$-ruling*) if any general fiber of f is $\boldsymbol{P}^1$.

(4.2) THEOREM. *Any singular fiber of a ruling $f: \bar{S}\to C$ contains an exceptional curve (of the first kind).*

Proof is easy and well-known.

(4.3) COROLLARY. *f is obtained from a $\boldsymbol{P}^1$-bundle over C by a finite number of successive blowing-ups.*

(4.4) COROLLARY. *Any fiber F of f is set-theoretically a rational tree.*

(4.5) COROLLARY. *The dual graph of F does not have a branching with weight -1.*

(4.6) COROLLARY. *F does not contain two exceptional curves C_i, C_j with $C_iC_j>0$, unless F is the twig $[1, 1]$.*

(4.7) PROPOSITION. *Let F be a rational tree on a smooth complete surface $\bar{S}$. Suppose that the dual graph of F is the twig $[a_1, \cdots, a_r, 1, b_1, \cdots, b_s]$ such that both $A=[a_1, \cdots, a_r]$ and $B=[b_1, \cdots, b_s]$ are admissible. Then, A and B are adjoint to each other if and only if $\bar{S}$ admits a ruling such that F is a fiber of it.*

PROOF. Note first that F is a fiber of a ruling if and only if F can be blown down to a single smooth rational curve with self-intersection number zero. We prove the proposition by induction on $r+s$.

F is blown down to the twig $[a_1, \cdots, a_{r-1}, a_r-1, b_1-1, b_2, \cdots, b_s]=F'$. Suppose that F is a fiber of a ruling. Then F' must contain an exceptional curve, hence $a_r=2$ or $b_1=2$. We may assume $a_r=2$ by symmetry. Then $d(A)=2d(\underline{A})-d(\underline{\underline{A}})$ by (3.6; 1). So $e({}^tA)^{-1}=2-e({}^t\underline{A})$. On the other hand, setting $B'=[b_1-1, b_2, \cdots, b_s]$, we have $d(B')=d(B)-d(\bar{B})$ and $e(B')=e(B)/(1-e(B))$. By the induction hypothesis we have $e({}^t\underline{A})+e(B')=1$. So $e({}^tA)^{-1}=1+e(B')=(1-e(B))^{-1}$, which implies $A=B^*$.

Conversely, suppose that $e({}^tA)+e(B)=1$. In view of (3.6; 3), we infer that $d(A)=d(B)=d(\underline{A})+d(\bar{B})$. By symmetry we may assume $d(\underline{A})\geqq d(\bar{B})$. Then, as we saw in the proof of (3.8), $a_r=2$. So F' is of the form $[\underline{A}, 1, B']$. Reversing the preceding calculation, we obtain $e({}^t\underline{A})+e(B')=1$ from $e({}^tA)+e(B)=1$. Hence F' is obtained from $[0]$ by successive blowing-ups by virtue of the induction hypothesis. Hence so is F. Q.E.D.

(4.8) PROPOSITION. *Let F, A and B be as above. Regarding F to be the blowing-up of the twig $[1, 1]$, let α and β be the total transforms in $\boldsymbol{Q}(F)$ of the upper and lower vertices of $[1, 1]$ respectively. Then the coefficients of the exceptional curve E of F in α and β are $d(\underline{B})$ and $d(\bar{A})$ respectively. Hence, the coefficient of E in the Cartier divisor F is $d(\underline{B})+d(\bar{A})=d(A)=d(B)$.*

PROOF. By symmetry it suffices to prove the assertion for β. Let $u_1, \cdots, u_r$ be the vertices of A with weights $-a_1, \cdots, -a_r$, let x_j be the coefficient of u_j in β and let x be the coefficient of E in β. Note that $x_1=0$, because u_1 is the proper transform of the upper component of $[1, 1]$. Set $\beta'=\sum_{j=1}^{r} x_ju_j$. Then $I(\beta', u_1)=1$, $I(\beta', u_j)=0$ for $j=2, \cdots, r-1$, and $I(\beta', u_r)=-x$, since $I(\beta, u_1)=1$

and $I(\beta, u_j)=0$ for $j\geqq 2$. Therefore $0=x_1=P_{1,1}-xP_{1,r}$, where $(P_{i,j})$ is the inverse (r, r)-matrix of $(-I(u_i, u_j))$. Calculating the cofactors we obtain $d(\bar{A})=x$ from this, as required.

(4.9) DEFINITION. Let $f: \bar{S}\to C$ be a ruling as in (4.1), let D be a reduced effective divisor on $\bar{S}$ and set $S=\bar{S}-D$. A component Y of a fiber F over $x\in C$ is called a *D-component* if $Y\subset D$, or *S-component* if $Y\not\subset D$. The number of S-components of F is denoted by $\sigma_D(x)$, or $\sigma(F)$, $\sigma(x)$ when there is no danger of confusion. The multiplicity of Y is defined to be the coefficient of Y in f^*x, which is a Cartier divisor with support F. The *S-multiplicity* (or multiplicity, when there is no danger of confusion) of F is defined to be the greatest common divisor of the multiplicities of S-components of F, and is denoted by $\mu_S(x)$ or $\mu(F)$, $\mu(x)$. When $\sigma(x)=0$, $\mu(x)$ is defined to be ∞ by convention. Obviously $F\cong \boldsymbol{P}^1$ and $\sigma(x)=\mu(x)=1$ for all but finite points on C. These exceptional fibers are called *D-singular fibers* of f.

F is said to be *D-connected* if $F\cap D$ is connected. A connected component of $F\cap D$ is called a *D-connected component* of F.

F is said to be *D-minimal* if every D-component of F is not an exceptional curve (of the first kind). f is said to be *D-minimal* if every fiber of it is D-minimal.

(4.10) PROPOSITION. *Let $f: \bar{S}\to C$, D and S be as in* (4.9). *Then there exist a birational morphism $p: \bar{S}\to \bar{S}'$, a ruling $f': \bar{S}'\to C$, an effective reduced divisor D' on $\bar{S}'$ and a finite subset Y of $S'=\bar{S}'-D'$ such that $f'\circ p=f$, $p(D)=D'\cup Y$, $D=p^{-1}(D'\cup Y)$, $p_S: S\to S'-Y$ is an isomorphism and f' is D'-minimal. In particular, if D is connected, then $Y=\emptyset$ or $D'=\emptyset$.*

PROOF. If f itself is D-minimal, nothing is to prove. So suppose that we have an exceptional curve E which is a D-component of a fiber of f. Let $p_1: \bar{S}\to \bar{S}_1$ be the blowing down of E to a point y on $\bar{S}_1$ and let D_1 be the image on $\bar{S}_1$ of the union of the components of D other than E. Then $D=p_1^{-1}(D_1\cup y)$ and $S\cong \bar{S}_1-D_1-y$. Moreover f induces a ruling $f_1: \bar{S}_1\to C$ such that $f=f_1\circ p_1$. Repeating such processes one proves the proposition by induction on the number of components of D.

REMARK. $\hat{H}^q(S; \boldsymbol{Z})\cong \hat{H}^q(S'; \boldsymbol{Z})$ for $q\leqq 3$ and $\tilde{H}^q(S; \boldsymbol{Z})\cong \tilde{H}^q(S'; \boldsymbol{Z})$ for $q\leqq 2$. Moreover, S and S' have the same Pic, NS, Pic_0, π_1 and $A(S)\cong A(S')$.

(4.11) REMARK. Let f, $\bar{S}$, D, S, p, $\bar{S}'$, f', D', Y and C be as above and let x be a point on C and let $F=f^{-1}(x)$ and $F'=f'^{-1}(x)$. Then $\sigma(F)=\sigma(F')$ and $\mu(F)=\mu(F')$.

(4.12) LEMMA. *Let f, $\bar{S}$, D be as in* (4.9) *and let F be a D-minimal fiber with $\sigma(F)=0$. Then $F\cong \boldsymbol{P}^1$.*

Obvious by definition and by (4.2).

(4.13) LEMMA. *Let f, $\bar{S}$ and D be as in* (4.9) *and let F be a D-minimal fiber with $\sigma(F)=1$. Then $F\cong \boldsymbol{P}^1$ or $\mu(F)\geqq 2$. Moreover, in the latter case, the S-component of F is the unique exceptional curve in F.*

PROOF. Suppose that F is singular. By (4.2) F contains an exceptional curve E. E must be the unique S-component of F by minimality. For any other component F_j of F we have $F_j^2\leqq -2$ and hence $KF_j\geqq 0$, where K is the canonical bundle of $\bar{S}$. So, $-2=KF\geqq \mu(F)KE=-\mu(F)$ since $KE=E^2=-1$. Thus $\mu(F)\geqq 2$.

(4.14) How is a singular fiber as in (4.13) obtained by successive blowing-ups ?

Let q be a point on a fiber and consider the blowing-up with center q. By (4.6), the number of exceptional curves in the fiber does not decrease except the case in which we get the twig [2, 1, 2] from [1, 1]. Moreover, this number increases unless q lies on some exceptional curve.

Therefore, a fiber as in (4.13) must be obtained from [2, 1, 2] by successive blowing-ups whose centers being on the exceptional curves at each stage. Moreover, such a fiber is D-connected if and only if the final exceptional curve is a tip of the fiber, or equivalently, the final blowing-up is sprouting.

Taking a sprouting blowing-up of a fiber as in (4.7), we can find a D-connected D-minimal singular fiber F as in (4.13) for any positive integer $\mu\geqq 2$. Of course, there are many other types of such fibers, but they are obtained from a fiber of the above type by successive blowing-ups.

(4.15) DEFINITION. Let things be as in (4.9). A component Y of D is said to be horizontal if $f(Y)=C$. Let h be the number of horizontal components of D, let Σ be the sum of $(\sigma(F)-1)$ of all the fibers F with $\sigma(F)>1$, let ν be the number of fibers with $\sigma=0$ and set $\varepsilon=1$ if $\nu>0$, or $\varepsilon=0$ if $\nu=0$. Then:

(4.16; 1) $h-\Sigma+\nu-2=\tilde{b}_1(S)-\hat{b}_2(S)$ and

(4.16; 2) $\tilde{b}_1(S)\geqq \nu-\varepsilon$ and $\hat{b}_2(S)\geqq \Sigma-h-\varepsilon+2$.

PROOF. We have an exact sequence $0\to\tilde{H}^1(S)\to H_2(D)\to H_2(\bar{S})\to\hat{H}^2(S)\to 0$. So $\tilde{b}_1-\hat{b}_2=\{$the number of components of $D\}-b_2(\bar{S})=h-2-(\sum_{x\in C}(\sigma(x)-1))$ $=h-2-\Sigma+\nu$. This proves 1). Clearly the fibers with $\sigma=0$ are algebraically equivalent to each other. Hence $\tilde{b}_1(S)\geqq \nu-\varepsilon$. Combining this with 1), we obtain $\hat{b}_2(S) \geqq \Sigma - h - \varepsilon + 2$.

(4.17) To study the fundamental group of S, we make the following

DEFINITION. Let X be a prime divisor on a manifold M (both M and X are not necessarily complete). Let x be a general point on X and let $(z_1, \cdots, z_n)$ be a system of coordinates of M at x such that X is defined by the equation $z_1 = 0$ in a neighborhood U of x isomorphic to the polydisc. Let $\Delta = \{(z_1, \cdots, z_n) \in U \mid z_2 = \cdots = z_n = 0\}$ and take a small circle γ around the origin of Δ with the counter-clockwise direction. Then the image of γ in $\pi_1(M-D)$ is determined by D uniquely up to conjugacy. This will be called the *vanishing loop* of D. The normal subgroup generated by γ will be called the *vanishing subgroup* of D. Clearly it is contained in the kernel of the homomorphism $\pi_1(M-D) \to \pi_1(M)$.

(4.18) LEMMA. *Let D be a reduced divisor on a smooth complete surface $\bar{S}$ and let C be a smooth irreducible curve on $\bar{S}$ intersecting with D normally. Set $S = \bar{S} - D$ and $S_0 = S - C$. Then $\pi_1(S_0) \to \pi_1(S)$ is surjective and the kernel coincides with the vanishing subgroup of C.*

For a proof, take a tubular neighborhood U of C and apply Van Kampfen's theorem to $S = S_0 \cup (U-D)$. Note that $U-D-C$ has the homotopy type of an S^1-bundle over $C-D$, since C and D intersect normally.

(4.19) PROPOSITION. *Let things be as in* (4.9). *Suppose that D has no horizontal component. Let $F_1, \cdots, F_k$ be the fibers with $\mu > 1$, let $x_j = f(F_j)$, set $\mu_j = \mu(x_j)$, let $C_0 = C - \cup_{j=1}^{k} x_j$ and let $\sigma_j \in \pi_1(C_0)$ be the vanishing loop of x_j. Then $\pi_1(S)$ is the quotient group of $\pi_1(C_0)$ by the relation $\sigma_1^{\mu_1} = \cdots = \sigma_k^{\mu_k} = 1$, where $\sigma_j^{\mu_j} = 1$ means nothing if $\mu_j = \infty$.*

PROOF. If we add all the D-singular fibers to F_j's allowing $\mu(F_j) = 1$, the meaning of the assertion does not change. Hence we suppose so. Then $f_0 \colon S_0 = f^{-1}(C_0) \to C_0$ is a $\boldsymbol{P}^1$-bundle and $\pi_1(S_0) \cong \pi_1(C_0)$. By virtue of (4.18), we infer that $\pi_1(S)$ is the quotient of $\pi_1(S_0)$ by the normal subgroup generated by the vanishing loops of the S-component of F_j's. It is easy to see that the vanishing loop of an S-component Y of F_j of multiplicity m is (conjugate to) σ_j^m. So, the S-components of F_j altogether yield the relation $\sigma_j^{\mu_j} = 1$. Our assertion follows from these observations.

(4.20) COROLLARY. *Let things be as in* (4.19). *Then $\pi_1(S)$ is finite if and only if $C \cong \boldsymbol{P}^1$ and μ_j's are one of the following.*

1) $k \leqq 1$. *In this case $\pi_1(S)$ is trivial.*
2) $k = 2$ *and* $(\mu_1, \mu_2) \neq (\infty, \infty)$. $\pi_1(S) \cong \boldsymbol{Z}/m\boldsymbol{Z}$ *for* $m = g.c.d.(\mu_1, \mu_2)$.

3) *$k=3$ and $(\mu_1, \mu_2, \mu_3)=(2, 2, m)$ for $m<\infty$ (modulo permutation). In this case $\pi_1(S)$ is isomorphic to the dihedral group D_m with $\#=2m$.*

4) *$k=3$ and $(\mu_1, \mu_2, \mu_3)=(2, 3, 3)$. $\pi_1(S)$ is the 4-th antisymmetric group A_4.*

5) *$k=3$ and $(\mu_1, \mu_2, \mu_3)=(2, 3, 4)$. $\pi_1(S)$ is the 4-th symmetric group S_4.*

6) *$k=3$ and $(\mu_1, \mu_2, \mu_3)=(2, 3, 5)$. $\pi_1(S)$ is the 5-th antisymmetric group A_5.*

Since we have established an explicit description of $\pi_1(S)$ in (4.19), the above fact is reduced to a famous result in the group theory. In particular, the 'if' part is an elementary exercise. We present here an algebro-geometric proof of the 'only if' part.

The finiteness of $\pi_1(S)$ implies that $\pi_1(\bar{S})\cong\pi_1(C)$ is finite. So $C\cong\boldsymbol{P}^1$. The assertion is easy to verify if $k\leqq 2$ or if $\mu_j=\infty$ for some j. Hence we assume that $k\geqq 3$ and $\mu_j<\infty$. We are assuming that $G=\pi_1(S)$, which is a group generated by $\sigma_1, \cdots, \sigma_k$ with the relation $\sigma_1\cdots\sigma_k=\sigma_1^{\mu_1}=\cdots=\sigma_k^{\mu_k}=1$, is a finite group. Correspondingly we have a Galois covering $p: C'\to C$ with $\mathrm{Gal}(C'/C)=G$ which is unramified over C_0.

For any given integers a, b and c with $a\geqq b\geqq c\geqq 2$, it is not difficult to find permutations ρ, σ and τ exactly of order a, b and c respectively in some symmetric group S_n such that $\rho\sigma\tau=1$. From this we infer that σ_j is exactly of order μ_j in G. So, any point on C' lying over x_j is a ramification point of order μ_j.

Let $C''\to C'$ be any unramified Galois covering. Then $C''\to C$ is also Galois and σ_j is of order μ_j in $\mathrm{Gal}(C''/C)$ because C'' is ramified over x_j as C'. By definition of G this implies that $\pi_1(C_0)\to\mathrm{Gal}(C''/C)$ factors through G. On the other hand, $G=\mathrm{Gal}(C'/C)$ is a quotient of $\mathrm{Gal}(C''/C)$ by $\mathrm{Gal}(C''/C')$. Therefore we infer that $\mathrm{Gal}(C''/C')=1$ and $C''\cong C'$. Thus we conclude that $C'\cong\boldsymbol{P}^1$.

Now, counting the order of the canonical line bundles and setting $d=\deg(C'/C)$, we obtain $-2=-2d+\sum_j(\mu_j-1)(d/\mu_j)=-d(2-\sum_j(1-\mu_j^{-1}))$. Hence $\sum_j(1-\mu_j^{-1})<2$. From this we infer that there are only the four possibilities 3), 4), 5) and 6). Q.E.D.

REMARK. The 'only if' part of (4.20) is valid even if D has horizontal components.

(4.21) COROLLARY. *Let things be as in* (4.19). *Then $\pi_1(S)$ is abelian if and only if one of the following conditions is satisfied.*

1) *$k=0$ and C is an elliptic curve.*

2) *$C\cong\boldsymbol{P}^1$ and $k\leqq 2$.*

3) *$C\cong\boldsymbol{P}^1$, $k=3$ and $\mu_1=\mu_2=\mu_3=2$. In this case $\pi_1(S)\cong\boldsymbol{Z}/2\boldsymbol{Z}\oplus\boldsymbol{Z}/2\boldsymbol{Z}$.*

The proof is easy and is left to the reader. As well as (4.20), the 'only if' part is valid even if D has horizontal components.

§ 5. A^1-ruled surfaces.

(5.1) Let $f:\bar{S}\to C$, D and $S=\bar{S}-D$ be as in (4.9). Then, f is called an *A^1-ruling* if $DF=1$ for any general fiber F of f. A quasi-complete surface S is said to be *A^1-ruled* if there exists a completion of it admitting an A^1-ruling.

(5.2) Suppose that f is an A^1-ruling. Then D has a unique horizontal component D_∞ which defines a section of f. In particular, for every fiber F of f, D_∞ meets F at exactly one point transversally. So using (4.4) we infer:

a) D is an NC-divisor.

b) D is connected if and only if every fiber of f is D-connected.

c) Any connected component of D is a tree.

(5.3) THEOREM. *$\bar{\kappa}(S)=-\infty$ if S is A^1-ruled. Conversely, if $\bar{\kappa}(S)=-\infty$, $\tilde{b}_3(S)=0$ and if S is algebraic, then S is A^1-ruled.*

For a proof, see [MS], [Sg], [Ru], [My 3]. Note that $\tilde{b}_3(S)=0$ if and only if D is connected (see (1.15; 3)). This condition is satisfied if S is Stein.

(5.4) THEOREM. *Let $f:\bar{S}\to C$ be an A^1-ruling of $S=\bar{S}-D$ and let the notations be as in* (4.15). *Then $\tilde{b}_1(S)=\nu-\varepsilon$ and $\hat{b}_2(S)=\Sigma-\varepsilon+1$.*

PROOF. D_∞ is the unique component of D meeting a general fiber of f. Hence it is numerically independent of the other components of D. Now it is easy to see that all the numerical equivalence relations among the components of D are derived from the equivalences of fibers of f. So $\tilde{b}_1(S)=\nu-\varepsilon$ by (1.18; 2). $\hat{b}_2(S)$ is calculated by this and (4.16; 1).

(5.5) COROLLARY. *$\hat{b}_2(S)=0$ if and only if $\Sigma=0$ and $\nu>0$, i.e., $\sigma(F)\leqq 1$ for every fiber F of f and $\sigma(F)=0$ for some fiber F of f. Moreover, in this case,* $\mathrm{NS}(S)\cong\hat{H}^2(S;\boldsymbol{Z})\cong H^2(S;\boldsymbol{Z})\cong\bigoplus_x\boldsymbol{Z}/\mu(x)\boldsymbol{Z}$, *where x runs through the points on C with $\sigma(x)\geqq 1$.*

PROOF. The first assertion is clear because $\Sigma\geqq 0$ and $\varepsilon\leqq 1$ by definition (4.15). By the observation (5.2) we infer $H_1(D)\cong H_1(D_\infty)\cong H_1(C)\cong H_1(\bar{S})$. Hence $\tilde{H}^2(S)=0$ and $H^2(S;\boldsymbol{Z})\cong\hat{H}^2(S;\boldsymbol{Z})\cong\mathrm{NS}(S)$. On the other hand, $\mathrm{NS}(S)$ is generated by the S-components of the D-singular fibers of f. Since $\sigma(F)\leqq 1$ for every fiber F and $\nu>0$, the second assertion follows now straightforwards.

(5.6) COROLLARY. *Suppose further that* $\mathrm{NS}(S)=0$. *Then, S is isomorphic to an A^1-bundle minus $b_3(S)$ points over an open subset of C.*

PROOF. By virtue of (4.10) we may assume that f is D-minimal. Then (4.13) applies. Note also that $H_1(D)\cong H_1(\bar{S})$ (see (5.5)) implies $\hat{b}_3(S)=0$ and $b_3(S)=\bar{b}_3(S)$.

(5.7) COROLLARY. *Let S be a quasi-complete surface such that $\bar{b}_3(S)=0=\hat{b}_1(S)$, $\bar{\kappa}(S)=-\infty$ and $H^2(S;\boldsymbol{Z})=0$. Then $S\cong\boldsymbol{A}^1\times(\boldsymbol{A}^1$-minus $b_1(S)$ points). In particular, $S\cong\boldsymbol{A}^2$ if in addition $b_1(S)=0$.*

PROOF. S is affine by (2.4). Hence $\boldsymbol{A}^1$-ruled by (5.3). So (5.6) applies. Note that $2g(C)=2h^1(\bar{S},\mathcal{O})=\hat{b}_1(S)=0$. Q.E.D.

(5.8) COROLLARY. *Let $f:\boldsymbol{A}^2\to S$ be a proper finite morphism of quasi-complete surfaces. Then $S\cong\boldsymbol{A}^2$.*

PROOF. $b_j(S)=0$ for $j>0$ by (2.1), and we have $\pi_1(S)=1$ by (2.14). So $H^2(S;\boldsymbol{Z})$ has no torsion as well as $H_1(S;\boldsymbol{Z})$. We have also $\bar{\kappa}(S)\leqq\bar{\kappa}(\boldsymbol{A}^2)=-\infty$. Hence (5.7) applies.

(5.9) PROPOSITION. *Let things be as in (5.2), let D' be the union of components of D other than D_∞ and let $S'=\bar{S}-D'$. Then $\pi_1(S)\cong\pi_1(S')$.*

PROOF. We can take a vanishing loop of D_∞ in a general fiber F of f. Then it vanishes in $\pi_1(S)$ since $F\cap S\cong\boldsymbol{A}^1$ is simply connected. So (4.18) proves our assertion.

(5.10) REMARK. $\mu_S(x)=\mu_{S'}(x)$ for every $x\in C$. Therefore (4.19) applies also to the $\boldsymbol{A}^1$-ruled case. In particular, (4.20) and (4.21) are valid in this context.

(5.11) LEMMA. *Let F be a line bundle on a manifold M and let $B\in|kF|$ for some positive integer k. Suppose that there do not exist integers m, k' and $B'\in|k'F|$ such that $B=mB'$ and $m>1$. Then, there is a manifold $\tilde{M}$ together with a proper morphism $\pi:\tilde{M}\to M$ such that the restriction of π to $\tilde{M}-\pi^{-1}(B)$ is a finite unramified cyclic covering of degree k.*

Several proofs are well-known. For example, let H be the tautological line bundle on $P=\boldsymbol{P}_M(F\oplus\mathcal{O})$ and let V be the member of $|kH|$ on P corresponding to $\beta\in H^0(M, kF)\subset H^0(M, S^k(F\oplus\mathcal{O}))=H^0(P, kH)$, which defines the divisor B on M. Then V is irreducible by the assumption. A non-singular model M of V has the required property.

(5.12) THEOREM. *Let things be as in (5.2) and suppose that $b_1(S)=0$ and $\sigma(x)=0$ for a point x on C. Then the following conditions are equivalent to each* *other.*

a) *There exists at most one fiber with* $1<\mu<\infty$.

b) $\pi_1(S)$ *is abelian.*

c) $\pi_1(S)$ *is finite.*

c′) *Any element of* $\pi_1(S)$ *is of finite order.*

d) $b_1(\tilde{S})=0$ *for any finite unramified covering* $\tilde{S}$ *of* S.

PROOF. By (5.4), $\sigma(x)=0$ for exactly one point. Moreover, $b_1(S)=0$ implies that $C\cong \boldsymbol{P}^1$. So, (5.9) and (4.21; 2) prove a)$\Rightarrow$b). b)$\Rightarrow$c) follows from $b_1(S)=0$. c)$\Rightarrow$c′) is obvious. c′) implies d) because $\pi_1(\tilde{S})$ is a subgroup of $\pi_1(S)$. Thus it suffices to show d)$\Rightarrow$a).

Let o be the point with $\sigma(o)=0$ and suppose that there are two points x, y on C such that $1<\mu(y)\leqq\mu(x)<\infty$. Let $C'\to C$ be the cyclic $\mu(x)$-sheeted branched covering with branch locus precisely o and x. Let S' be the normalization of $S\times_C C'$. Then, S' is unramified over S because every S-component of $F_x=f^{-1}(x)$ has a multiplicity divisible by $\mu(x)$. Clearly $S'\to C'$ can be completed to an $\boldsymbol{A}^1$-ruling $f'\colon \bar{S}'\to C'$. All the $\mu(x)$ points on C' lying over y are with $\mu=\mu(y)$. Using (5.11), take a $\mu(y)$-sheeted cyclic branched covering $C''\to C'$ with branch locus $\mu(y)$ points among these points over y. Let S'' be the normalization of $S'\times_{C'}C''$. Then, similarly as above, $S''\to S'$ is unramified. Moreover, $S''\to C''$ can be completed to an $\boldsymbol{A}^1$-ruling which has $\mu(y)$ fibers with $\sigma=0$ lying over $F_0=f^{-1}(o)$. So $b_1(S'')=\mu(y)-1>0$ by (5.4). Thus we prove d)$\Rightarrow$a).

(5.13) PROPOSITION. *Let things be as in* (5.12). *The above conditions are satisfied if there exists a dominant morphism* $\boldsymbol{A}^2-Y\to S$, *with* Y *being a finite set. Conversely, such a morphism exists if* a) *is satisfied and if every fiber* $F_x=f^{-1}(x)$ *has an S-component of multiplicity* $\mu(x)$ *except the case* $\sigma(x)=0$.

PROOF. The first part follows from (2.13). To show the converse, we may assume that $\sigma(x)=1$ for every x on C except o. If in addition $\pi_1(S)=1$, then we infer that S itself is of the form $\boldsymbol{A}^2-Y$, by virtue of (5.9), (4.20; 2), (5.5) and (5.6). Otherwise we have a unique point x on C with $\mu(x)>1$. Let $C'\to C$ be the $\mu(x)$-sheeted cyclic branched covering with branch locus $o\cup x$, and let S' be the normalization of $S\times_C C'$. Then, as in (5.12), S' is unramified over S. Moreover, S' has $\mu(x)$ components of multiplicity one lying over F_x. Hence, by the first step, we find an open dense subset of S' isomorphic to $\boldsymbol{A}^2-Y$. The assertion follows from this.

REMARK. If $\hat{b}_2(S)=0$, the above hypothesis for the converse is automatically satisfied because of (5.5). Furthermore, Y can be taken to be empty if D is connected.

(5.14) *Example.* Let Σ be $C\times \boldsymbol{P}^1$ with $C\cong \boldsymbol{P}^1$. Let D_0 be the fiber over $o\in C$ of the first projection $p:\Sigma\to C$, and let D_∞ be a fiber of the second projection. Let x be a point on C other than o and let y be the intersection point of D_∞ and $p^{-1}(x)$. Performing successive blowing-ups over y we get a fiber over x looking like the twig $[(m-1)\times 2, 1, m]$, where $(m-1)\times 2$ stands for the $(m-1)$ times repetition of 2. The proper transform D_h of D_∞ meets the highest component of this twig, which is of weight -2. Let $\bar{S}$ be the ruled surface obtained by the sprouting blowing-up at a point on the exceptional component of this twig. Let E be the final exceptional curve on $\bar{S}$ and let F be the fiber of $\bar{S}$ over x. Set $D=D_0\cup D_h\cup\overline{(F-E)}$ and $S=\bar{S}-D$. Then D looks like the twig $[0, 1, m\times 2, m]$. By (5.4) we see also $b_j(S)=0$ for any $j>0$. Moreover, $\mu(x)=m$ and $\mathrm{NS}(S)\cong\mathrm{Pic}(S)\cong \boldsymbol{Z}/m\boldsymbol{Z}\cong\pi_1(S)$.

D is not a minimal twig. Indeed, we can blow it down successively to obtain the twig $[-m-1, m-1]$. Correspondingly, S has an NC-completion $(\bar{S}', D')$ such that D' consists of two prime components D_1 and D_2 with $D_1^2=m+1$ and $D_2^2=1-m$. Since $b_1(S)=b_2(S)=0$, we have $b_2(\bar{S}')=2$. Hence $\bar{S}'$ must be isomorphic to the Hirzebruch surface $\Sigma_{m-1}\cong \boldsymbol{P}(\mathcal{O}(m-1)\oplus\mathcal{O})$ which is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$, and D_1 and D_2 are sections of this bundle map. However, this bundle map has nothing to do with the ruling $\bar{S}\to C$.

In the special case $m=2$, D' can be blown down to $[-4]$, and S is the complement of a smooth plane quadric curve. Its universal covering is isomorphic to $\boldsymbol{P}^1\times\boldsymbol{P}^1$ minus the diagonal, and the covering transformation is the factor changing involution.

(5.15) THEOREM. *Let $f:\bar{S}\to C$ be an $\boldsymbol{A}^1$-ruling of $S=\bar{S}-D$ and let $F_1,\cdots,F_k$ be the fibers with $\mu\geqq 2$. Suppose that there exists a dominant morphism $\boldsymbol{A}^2-Y\to S$, Y being a finite subset of $\boldsymbol{A}^2$. Then $C\cong\boldsymbol{P}^1$ and $k\leqq 3$. Moreover, if $k=3$, the multiplicities $\mu(F_1)$, $\mu(F_2)$, $\mu(F_3)$ are one of the following triplet up to permutation*: (2, 3, 5), (2, 3, 4), (2, 3, 3) or (2, 2, m) *for some* $m<\infty$.

For a proof, combine (2.13), (5.9) and (4.20). As for the converse, we have:

(5.16) THEOREM. *Let $f:\bar{S}\to C$ be an $\boldsymbol{A}^1$-ruling of $S=\bar{S}-D$. For any point x on C, let $\mu'(x)$ be the minimum of the multiplicities of S-components of $F_x=f^{-1}(x)$ which are tips of the tree F_x. If every tip of F_x is a D-component, then $\mu'(x)=\infty$. Suppose that $\sum_{x\in C}(1-(\mu'(x))^{-1})<2$. Then there exists a dominant quasi-finite morphism $\boldsymbol{A}^2-Y\to S$, where Y is a finite subset of $\boldsymbol{A}^2$. Moreover, Y can be taken to be empty if D is connected.*

PROOF. Let E_x be a tip of F_x which is an S-component with multiplicity $\mu'(x)$. Addling the other components to D, we may assume that E_x is the unique S-component of F_x. In particular, $\mu(F_x)=\mu'(F_x)$. Moreover, by virtue of (4.10) and (5.2; a), we may assume that f is D-minimal. Then $\mu\geqq 2$ for every D-singular fiber by (4.13).

Let $F_1, \cdots, F_k$ be the fibers with $\mu\geqq 2$. By (5.13), we may assume that $\mu(F_j)$ is finite for every j. Then our hypothesis implies that $k=3$ and the multiplicities are one of the triplets in (4.20; 3), 4), 5) and 6). In any case $G=\pi_1(S)$ is finite, and we have a Galois covering $p: C'\to C$ with branch locus x_1, x_2 and x_3 such that $\mathrm{Gal}(C'/C)=G$. Let S' be the normalization of $S\times_C C'$. Then, similarly as in (5.12) and (5.13), S' is completed to an A^1-ruling $f': \bar{S}'\to C'$ with $\mu(x)=1$ for every $x\in C$. Moreover, the S'-components are tips of the rational trees F_x. Now, with the help of (4.10) and (4.13), we infer that S' contains an open subset which is isomorphic to A^2-Y. Similarly as in (5.6), Y can be taken to be empty if D is connected. Thus we prove the theorem.

(5.17) Many results in this section are translations in our topological terminology of results of Miyanishi. For example, the characterization of $\boldsymbol{A}^2$ given by (5.7) corresponds to the following criterion (cf. [My 1]): *An affine surface* $S=\mathrm{Spec}(A)$ *is isomorphic to* $\boldsymbol{A}^2$ *if and only if* $A^\times=\boldsymbol{C}^\times$, A *is* UFD *and* S *is* $\boldsymbol{A}^1$*-ruled.*

Furthermore, (5.13) is a weaker version of [My 3; Theorem 4.7]. (5.15) is almost the same with [My 3; Theorem 4.11.1]. (5.16) is slightly better than [My 3; Theorem 4.11.3], where the case (2, 3, 5) was not settled. Why this case was difficult is perhaps explained by the fact that $\pi_1(S)$ is simple and non-solvable unlike the other cases.

§6. Zariski decomposition of effective divisors.

(6.1) DEFINITION. A divisor is a $\boldsymbol{Z}$-linear combination of its prime components. Similarly, a $\boldsymbol{Q}$*-divisor* is defined to be a linear combination of its prime components with coefficients being rational numbers. We use terminologies concerning usual divisors for $\boldsymbol{Q}$-divisors too. For example, a $\boldsymbol{Q}$-divisor is said to be *effective* if every coefficient of its prime components is non-negative.

We define the D-dimension of a $\boldsymbol{Q}$-divisor D to be $\kappa(tD)$, where t is a positive integer such that tD is a usual divisor. $\kappa(tD)$ is independent of the choice of t, hence $\kappa(D)$ is well-defined.

$\boldsymbol{Q}$-valued intersection numbers of $\boldsymbol{Q}$-divisors are defined in the obvious way. A $\boldsymbol{Q}$-divisor D is said to be *semipositive* if $DC\geqq 0$ for any irreducible curve C. A $\boldsymbol{Q}$-divisor D on a surface is said to be *pseudo-effective* if $DH\geqq 0$ for any semipositive $\boldsymbol{Q}$-divisor H. Of course D is pseudo-effective if $\kappa(D)\geqq 0$, e.g., D is effective.

For a $\boldsymbol{Q}$-divisor D on a surface, $\boldsymbol{Q}(D)$ is the $\boldsymbol{Q}$-vector space generated by the prime components of D (see (3.3)). D is said to be *contractible* if the intersection form I defined on $\boldsymbol{Q}(D)$ is negative definite.

(6.2) LEMMA. *Let D be a contractible $\boldsymbol{Q}$-divisor. Let $X \in \boldsymbol{Q}(D)$ and suppose that $XD_j \leqq 0$ for any prime component D_j of D. Then X is effective.*

Proof is easy and is found in [Z; p. 588].

(6.3) THEOREM. *Let D be a pseudo-effective $\boldsymbol{Q}$-divisor on a smooth complete surface $\bar{S}$. Then there exists an effective $\boldsymbol{Q}$-divisor N satisfying the following properties.*

a) *$N=0$ or N is contractible.*

b) *$H=D-N$ is semipositive.*

c) *$HC=0$ for any prime component C of N.*

For a proof, see [F]. The case where D is effective was treated by [Z].

(6.4) LEMMA. *N is determined by the numerical equivalence class of D.*

PROOF. Let $D_1=N_1+H_1$ and $D_2=N_2+H_2$ be decompositions as in (6.3) of two $\boldsymbol{Q}$-divisors D_1 and D_2 which are numerically equivalent to each other. Assume that $N_1 \neq N_2$ and write $N_1-N_2=E_1-E_2$, where E_1 and E_2 are effective and have no common component. By symmetry we may assume $E_1 \neq 0$. Then $E_1^2<0$ because $E_1 \in \boldsymbol{Q}(N_1)$. So $E_1C<0$ for some component C of E_1. Then $C(N_1-N_2)=C(E_1-E_2)<0$. Hence $CN_2>CN_1=CD_1=CD_2 \geqq CN_2$. This contradiction proves the lemma.

(6.5) DEFINITION. N is denoted by D^- and is called the *negative part* of D. $H=D-N$ is denoted by D^+ and is called the *semipositive part* of D. $D=N+H$ is called the *Zariski decomposition* of D. The two facts below are obvious by definition.

(6.6; 1) $(mD)^-=m(D^-)$ and $(mD)^+=m(D^+)$ for any $m>0$.

(6.6; 2) $(D+E)^+=D^+$ for any effective divisor E in $\boldsymbol{Q}(D^-)$.

(6.7) LEMMA. *Let D, N and H be as in (6.3) and let E be an effective $\boldsymbol{Q}$-divisor such that $D-E$ is semipositive. Then $E-N$ is effective.*

PROOF. Assume that $E-N$ is not effective and write $E-N=Y_1-Y_2$, where Y_1 and Y_2 are effective and have no common component. $Y_2^2<0$ since $Y_2 \in \boldsymbol{Q}(N)$. So there is a component C of Y_2 such that $CY_2<0$. Then $CE>CN=CD \geqq CE$. This contradiction proves the lemma.

(6.8) COROLLARY. *Let D be a usual divisor and let F be the fixed part of $|D|$. Then $F-D^-$ is effective.*

(6.9) PROPOSITION. *Let D, N and H be as in* (6.3) *and assume that D is effective. Then H is also effective.*

PROOF. Write $H=H_1-H_2$, where H_1 and H_2 are effective and have no common component. We have $H_2\in \boldsymbol{Q}(N)$ since D is effective. So $0=HH_2=(H_1-H_2)H_2\geqq -H_2^2$. Hence $H_2=0$ because N is contractible. Q.E.D.

(6.10) COROLLARY. $\kappa(D)=\kappa(D^+)$.

(6.11) THEOREM (Kawamata, see [Kw 3] or [My 3]). *Let $(\bar{S}, D)$ be an* NC-*completion of a surface $S=\bar{S}-D$ with $\bar{\kappa}(S)\geqq 0$. Let K be a canonical divisor of $\bar{S}$ and set $H=(K+D)^+$. Then*

1) $\bar{\kappa}(S)=0$ *if and only if $H\sim 0$, where $\sim$ denotes the numerical equivalence.*

2) $\bar{\kappa}(S)=1$ *if and only if $H\not\sim 0$ and $H^2=0$. Moreover, in this case, $Bs|tH|=\emptyset$ for some positive integer t. The rational mapping defined by $|tH|$ makes $\bar{S}$ a fiber space over a curve C (which may have singular fibers). A general fiber F of $\bar{S}\to C$ is either an elliptic curve with $FD=0$ or a rational curve with $FD=2$.*

3) $\bar{\kappa}(S)=2$ *if and only if $H^2>0$.*

REMARK. The 'if' part of 1) is trivial and 3) is easy. The fibration in 2) is a special case of Iitaka's theory. The 'only if' part of 1) is the most essential contribution of Kawamata.

(6.12) In order to study the Zariski decomposition of $K+D$ more precisely, we make several definitions. For the sake of later convenience, we consider also the case in which D has bad singularities.

Let $\bar{S}$ be a smooth complete surface and let D be an effective reduced divisor on it. For any component Y of D, we denote $Y(D-Y)$ by $\beta(Y)$, or by $\beta_D(Y)$ when confusion is possible. This is called the *branching number* of Y in D. Compare (3.2).

Y is called a *tip* of D if $\beta(Y)=1$. It is called a *rational tip* if $Y\cong \boldsymbol{P}^1$. A sequence $C_1, \cdots, C_r$ of components of D is called a *rational twig* of D if each C_i is a rational normal curve, $\beta(C_1)=1$, $\beta(C_j)=2$ and $C_{j-1}C_j=1$ for $2\leqq j\leqq r$. C_1 is called the *tip* of this twig T.

Since $\beta(C_r)=2$, there is a component C of D not in T such that $C_rC=1$. If C is a rational tip of D, then $T'=T+C$ is a connected component of D and will be called a *rational club* of D. A component Y of D such that $Y\cong \boldsymbol{P}^1$, $\beta(Y)=0$ also will be called a rational club of D.

When the above C is isomorphic to $\boldsymbol{P}^1$ and $\beta(C)=2$, T' is a rational twig of D. Otherwise, T is called a *maximal rational twig* of D and C is called the *branching component* of T. Obviously any rational tip of D is contained in a rational club or a maximal rational twig of D.

If T is a contractible rational twig of D, the element $N\in\boldsymbol{Q}(T)$ such that $NC_1=-1$ and $NC_j=0$ for $j\geqq 2$ is called the *bark* of T. If $T'=C_1+\cdots+C_r+C$ is a contractible rational club of D, the bark of T' is defined to be the $\boldsymbol{Q}$-divisor N' in $\boldsymbol{Q}(T')$ such that $N'C_1=N'C=-1$ and $N'C_j=0$ for $2\leqq j\leqq r$. For an isolated rational normal curve Y, its bark is defined to be $2(-Y^2)^{-1}Y$. In any case we have $NZ=(K+D)Z$ for any component Z of T (or T').

If all the rational clubs and maximal twigs of D are contractible, then the sum of their barks are denoted by $\mathrm{Bk}(D)$ and is called the *bark* of D.

(6.13) LEMMA. *Let things be as above and suppose that $K+D$ is pseudo-effective. Then any component of any rational twig of D is a component of $N=(K+D)^-$. Hence all the rational clubs and maximal twigs of D are contractible.*

PROOF. Let $T=C_1+\cdots+C_r$ be a rational twig of D as in (6.12). $C_1\in\boldsymbol{Q}(N)$ since $(K+D)C_1=-1$. Suppose that $C_{j-1}\in\boldsymbol{Q}(N)$ for some $j\geqq 2$. Then, if C_j were not a component of N, we would have $0=(K+D)C_j\geqq NC_j>0$. So $C_j\in\boldsymbol{Q}(N)$. Thus we prove $C_i\in\boldsymbol{Q}(N)$ by induction on i.

(6.14) COROLLARY. *Let things be as in* (6.12) *and suppose that D has a rational twig or club which is not contractible. Then $\kappa(K+D, \bar{S})=\bar{\kappa}(S)=-\infty$.*

Indeed, we have $\bar{\kappa}(S)\leqq\kappa(K+D, \bar{S})$ in general, where $S=\bar{S}-D$. See [I].

(6.15) LEMMA. *Let things be as in* (6.13). *Then $N-\mathrm{Bk}(D)$ is effective.*

PROOF. Write $N-\mathrm{Bk}(D)=E_1-E_2$, where E_1 and E_2 are effective and have no common component. If $E_2\neq 0$, then $E_2^2<0$ since $E_2\in\boldsymbol{Q}(\mathrm{Bk}(D))$. So $E_2Y<0$ for some component Y of E_2. Then $NY>\mathrm{Bk}(D)Y=(K+D)Y\geqq NY$ by definition of $\mathrm{Bk}(D)$. This contradiction proves the lemma.

(6.16) LEMMA. a) *Let $T=C_1+\cdots+C_r$ be a rational twig of D as in* (6.12). *Suppose that the dual graph $\Gamma=[-C_1^2, \cdots, -C_r^2]$ is admissible, that means, contractible and minimal* (recall (3.5)). *Let $N=\sum n_iC_i\in\boldsymbol{Q}(T)$ be the bark of T. Then $n_1=e(\Gamma)$, $n_r=d(\Gamma)^{-1}$, $N^2=-n_1=-e(\Gamma)$ and $0<n_i<1$ for $1\leqq i\leqq r$.*

b) *Let $T'=C_1'+\cdots+C_s'$ be a rational club of D and suppose that the dual graph Γ' of T' is an admissible twig. Let $N'=\sum n_j'C_j'$ be the bark of T'. Then $0<n_j'<1$ for $1\leqq j\leqq s$, except the case in which $(C_j')^2=-2$ for every j and $N'=T'$.*

PROOF. By (6.2) and a similar argument as in (6.13), we infer $n_i>0$ for every i. By definition of N, $-n_j$ is the (i, j)-entry of the inverse matrix of the (r, r)-matrix (C_iC_j). So we get $n_1=e(\Gamma)$ and $n_r=d(\Gamma)^{-1}$ by elementary computation. $N^2=-n_1$ is clear by definition of N. Note that $n_i<1$ for $i=1, r$. Now assume that $M=\mathrm{Max}(n_i)\geqq 1$. Take the least i such that $n_i=M$. Since $1<i<r$, we have $0=NC_i=n_{i-1}+n_iC_i^2+n_{i+1}<M(2+C_i^2)\leqq 0$ because of the minimality of T. This contradiction shows that $M<1$. Thus we prove a).

b) is proved similarly. In particular, we have $n_1'=e(\Gamma')+d(\Gamma')^{-1}$ and $n_s'=e({}^t\Gamma')+d(\Gamma')^{-1}$. Both are less than one except the case $\Gamma'=[s\times 2]$ (see (3.9)). So the above argument works.

(6.17) LEMMA. *Let things be as in* (6.13) *and suppose that (the dual graph of) any rational twig of D is admissible. This means, in particular, D has no isolated exceptional component. Suppose in addition that any component of $N=(K+D)^-$ is a component of D. Let $D_1, \cdots, D_k$ be the connected components of D and write $N=\sum N_j$, where $N_j\in \boldsymbol{Q}(D_j)$ for $j=1, \cdots, k$. Then $N_j=\mathrm{Bk}(D_j)$ unless D_j consists of three maximal rational twigs T_1, T_2, T_3 and their common branching component B such that $B\cong \boldsymbol{P}^1$, $\beta(B)=3$ and $d(T_1)^{-1}+d(T_2)^{-1}+d(T_3)^{-1}>1$.*

PROOF. Set $H_j=K+D-\mathrm{Bk}(D_j)$. If $H_jY\geqq 0$ for any component Y of D_j, we see easily $N_j=\mathrm{Bk}(D_j)$ by (6.7) and (6.15). So we assume $H_jB<0$ for some component B of D_j. Since $(K+D)Y=\mathrm{Bk}(D_j)Y$ for any component Y of $\mathrm{Bk}(D_j)$, B cannot be a component of $\mathrm{Bk}(D_j)$. Let $T_1, \cdots, T_q$ be the rational twigs of D whose branching components is B. Then, using (6.16), we obtain $H_jB=2g(B)-2+\beta(B)-\sum_{i=1}^q d(T_i)^{-1}$, where $g(B)$ is the arithmetic genus of B. Since $\beta(B)\geqq q$ and $d(T_i)\geqq 2$, this can be negative only when $g(B)=0$, $\beta(B)=q=3$ and $\sum d(T_i)^{-1}>1$. Thus $B+T_1+T_2+T_3$ is a connected component of D, which is D_j of course.

(6.18) DEFINITION. A connected component $D_j=B+T_1+T_2+T_3$ of D of the above type will be called an *abnormal rational club* of D. The $\boldsymbol{Q}$-divisor $N\in\boldsymbol{Q}(D_j)$ such that $NY=(K+D)Y$ for every component Y of D_j will be called the *thicker bark* of D_j and is denoted by $\mathrm{Bk}^*(D_j)$. For a connected component D_i which is not an abnormal rational club, we set $\mathrm{Bk}^*(D_i)=\mathrm{Bk}(D_i)$. We define also $\mathrm{Bk}^*(D)=\sum_{j=1}^k \mathrm{Bk}^*(D_j)$, where $D_1, \cdots, D_k$ are the connected components of D.

It is clear that $(K+D)^-=\mathrm{Bk}^*(D)$ in the situation (6.17).

(6.19) LEMMA. *Let $D_j=B+T_1+T_2+T_3$ be an abnormal rational club as above. Set $B^2=-b$, let N_i' be the bark of the transposal of T_i, set $B'=B+N_1'+N_2'+N_3'$, $\delta=\sum_i d(T_i)^{-1}$ and $e=\sum_i e({}^tT_i)$. Then*

1) *$(d(T_i))$ is one of the following triplet modulo permutation:* (2, 3, 3),

(2, 3, 4), (2, 3, 5) *or* (2, 2, m) *for some* $m \geqq 2$.

2) $b>e$.

3) $\mathrm{Bk}^*(D_j)=(\delta-1)(b-e)^{-1}B'+\mathrm{Bk}(D)$.

4) *Both* $\mathrm{Bk}^*(D_j)$ *and* $D_j-\mathrm{Bk}^*(D_j)$ *are effective.*

PROOF. 1) follows from an elementary argument. By (6.16), we have $BN_i'=e({}^tT_i)$. We have also $B'Y=0$ for any component Y of T_i. Hence $0>(B')^2=B'B=-b+e$. Thus we prove 2). 3) is easily checked by definition of Bk^* and by $\mathrm{Bk}(D)B=\delta$. To show 4), note that $b>e\geqq\delta>1$. So $B^2\leqq-2$ and $KB\geqq 0$. Hence $KY\geqq 0$ for any component Y of D_j. Write $D_j-\mathrm{Bk}^*(D_j)=E_1-E_2$, where E_1 and E_2 are effective and have no common component. If $E_2\neq 0$, then $E_2^2<0$ since $E_2\in\boldsymbol{Q}(D_j)$. So $E_2Y<0$ for some component Y of E_2. Then $D_jY>\mathrm{Bk}^*(D_j)Y=(K+D)Y\geqq DY=D_jY$. This contradiction proves that $D_j-\mathrm{Bk}^*(D_j)$ is effective. The effectivity of $\mathrm{Bk}^*(D_j)$ is clear by 3). Q.E.D.

REMARK. By the above argument we can show the effectivity of $D-\mathrm{Bk}(D)$ for a usual rational club too. Compare (6.16). Moreover, similarly as in (6.16; b), the coefficient of any component of D_j in $D_j-\mathrm{Bk}^*(D_j)$ is positive unless $Y^2=-2$ and $KY=0$ for any component Y of D_j, in which case we have $D_j=\mathrm{Bk}^*(D_j)$.

(6.20) LEMMA. *Let things be as in* (6.13) *and suppose that any rational twig of D is admissible. Then, if* $N\neq\mathrm{Bk}^*(D)$, *there exists a component E of N which is an exceptional curve not in D and satisfies one of the following conditions.*

1) $DE=0$, *i.e.*, $D\cap E=\emptyset$.

2) $DE=1$ *and E meets a component of* $\mathrm{Bk}^*(D)$.

3) $DE>1$ *and E meets two components of D, one of which is a tip of a rational club of D.*

PROOF. Set $L=K+D-\mathrm{Bk}^*(D)$. If this is semipositive, then $N=\mathrm{Bk}^*(D)$ by definition of Zariski decomposition. We have $LY\geqq 0$ for any component Y of D by definition of thicker bark. So we may assume that $LE<0$ for some curve E not in D. Any such curve must be a component of N. Hence $E^2<0$. Moreover, $KE\leqq LE<0$ since $D-\mathrm{Bk}^*(D)$ is effective by (6.16) and (6.19). Therefore E must be an exceptional curve. Let $E_1, \cdots, E_t$ be all such curves with $LE_i<0$, and set $D'=D+E_1+\cdots+E_t$. By (6.6; 2) we have $(K+D')^-=N+\sum E_i$.

Suppose that $DE_i=1$ for some E_i. Then E_i meets $\mathrm{Bk}^*(D)$ since $LE_i=0$ otherwise. So it suffices to consider the case in which $DE_i\geqq 2$ for every E_i. Then any rational tip of D' is a rational tip of D. Assume that none of the E_i's satisfies the condition 3). Then E_i is not contained in any rational twig of D'. Therefore any rational twig of D' is a subset of a rational twig of D.

Hence $(\mathrm{Bk}^*(D)-\mathrm{Bk}^*(D'))Y\leqq 0$ for any component Y of $\mathrm{Bk}^*(D)$. So $\mathrm{Bk}^*(D)-\mathrm{Bk}^*(D')$ is effective by (6.2). Since $(K+D')^-\neq \mathrm{Bk}^*(D')$, there is a curve C not in D' such that $(K+D'-\mathrm{Bk}^*(D'))C<0$ by (6.17). We have $LC<0$, because $\mathrm{Bk}^*(D)-\mathrm{Bk}^*(D')+D'-D$ is effective. So C must be one of the curves E_i's, contradicting $C\not\subset D'$. Thus we prove the lemma (6.20).

(6.21) Let things be as in (6.13) and let E be an exceptional curve on $\bar{S}$. Let $\pi:\bar{S}\to\bar{S}'$ be the blowing-down of E to a point p. We consider the following five cases.

1) $E\subset D$ and $\beta(E)=1$ or 2. π is called D-blowing-down. We define $D'=\pi(D)$ and $S'=\bar{S}'-D'\cong S$.

2) $E\subset D$ and $\beta(E)=0$. $D'=\pi(D)-\{p\}$ and $S'=\bar{S}'-D'\cong S\cup\{p\}$. S' (resp. S) is called a one point attachment (resp. detachment) of S (resp. S').

3) $E\not\subset D$ and $DE=0$. π is called S-blowing-down. $D'=\pi(D)\cong D$ and $S'=\bar{S}'-D'$.

4) $E\not\subset D$ and $DE=1$. $D'=\pi(D)$ looks like D, and $S'=\bar{S}'-D'\cong S-E$. S' (resp. S) is called a half point detachment (resp. attachment) of S (resp. S').

5) $E\not\subset D$ and E is a component of $N=(K+D)^-$ satisfying the condition (6.20; 3).

(6.22) Remark. In the processes in (6.21), $\hat{b}_i$'s and $\bar{b}_j$'s do not change except

a) $\bar{b}_3(S')=\bar{b}(S)-1$ in case 2) and 5).

b) $\bar{b}_1(S')=\bar{b}_1(S)+1$ in case 4) and 5), provided that E is numerically equivalent to a $\boldsymbol{Q}$-divisor in $\boldsymbol{Q}(D)$.

c) $\hat{b}_2(S')=\hat{b}_2(S)-1$ in case 3), 4) and 5) unless b) is the case. Any way we have $\hat{b}_2(S')-\bar{b}_1(S')=\hat{b}_2(S)-\bar{b}_1(S)-1$ in case 3), 4) and 5) in which $E\not\subset D$.

The proof is easy. As a consequence, we see that the blowing-downs of type (6.21; 2) and 5) (resp. 3)) are impossible if $\bar{b}_3(S)=0$ (resp. $\hat{b}_2(S)=0$).

(6.23) Remark. In each case in (6.21), $H=(K+D)^+$ is the pull-back of $H'=(K'+D')^+$, where K' is the canonical bundle of $\bar{S}'$. Consequently $H^0(\bar{S}', m(K'+D'))\cong H^0(\bar{S}, m(K+D))$ for every positive integer m by (6.9).

To prove $H=\pi^*H'$, we note that $K+D=\pi^*(K'+D')+\mu E$ for some $\mu\geqq 0$ in case (6.21; 1-4). Then $K+D=\pi^*H'+(\pi^*(K'+D')^-+\mu E)$ satisfies the condition in (6.3). So $H=\pi^*H'$ by definition of Zariski decomposition.

In case (6.21; 5), we have $H=(K+D+E)^+$ since E is a component of N. So $H=\pi^*H'$ because $K+D+E=\pi^*(K'+D')$.

(6.24) From (6.20) we obtain the following

Theorem. *Let D be a reduced effective divisor on a smooth complete surface $\bar{S}$. Suppose that $K+D$ is pseudo-effective for the canonical bundle K of $\bar{S}$ and* *that any blowing-down of the five types described in* (6.21) *is impossible. Then* $(K+D)^{-}=\mathrm{Bk}^{*}(D)$. (*Note that every rational twig of* D *is admissible since* (6.21; 1) *is impossible*).

Thus, after several blowing-downs, we know the Zariski decomposition of $K+D$ fairly explicitly by virtue of (6.16) and (6.19).

(6.25) As an application of the above method, we obtain the following

THEOREM. *Let* D *be an effective reduced divisor on a smooth complete surface* $\bar{S}$. *Then the graded algebra* $\bigoplus_{t\geq 0}H^{0}(\bar{S}, t(K+D))$ *is finitely generated.*

In fact, this was proved by Kawamata [Kw 3] when D has no singularity other than nodes. We sketch here how to modify his argument.

We may assume $\kappa(K+D, \bar{S})=2$ by virtue of [Z; Proposition 11.5]. Set $H=(K+D)^{+}$ and let ε be the union of all the curves C such that $HC=0$. Then $H^{2}>0$ since $\kappa(H)=\kappa(K+D)=2$.

Suppose that ε contains an exceptional curve E. Since ε is contractible by the index theorem, $(K+D+\mu E)^{+}=H$ for any $\mu\geqq 0$. Therefore, if $\pi: \bar{S}\rightarrow\bar{S}'$ is the blowing-down of E and if D' is the image divisor of D, we have $H=\pi^{*}((K'+D')^{+})$ similarly as in (6.23).

Repeating similarly we reduce the problem to the case in which ε contains no exceptional curve. In particular the processes in (6.21) is impossible and $(K+D)^{-}=\mathrm{Bk}^{*}(D)$.

Suppose that $C\not\subset D$ for a curve C in ε. Then $C^{2}<0$ and $KC\leqq(K+D-\mathrm{Bk}^{*}(D))C$ $=HC=0$. So $KC=0$ since C is not exceptional. Hence $(D-\mathrm{Bk}^{*}(D))C=0$. So C can meet only those connected components D_{j} of D such that $\mathrm{Bk}^{*}(D_{j})=D_{j}$. By the remark to (6.19), we have $Y^{2}=-2$ and $KY=0$ for any component Y of such D_{j}. From these observations we infer that any connected component of ε not contained in D can be contracted to a rational double point.

As for the connected components contained in D, we see that they are of the types considered by Kawamata [Kw 3; (2.22)-(2.24)] by a similar argument as that in (6.17) (see also (8.7) below). Thereafter we proceed in the same way as Kawamata.

REMARK. There are many counter-examples to the above statement if we do not assume that D is reduced.

§ 7. A_{*}^{1}-ruled surfaces.

(7.1) Let $f: \bar{S}\rightarrow C$ be a ruling and let D be a reduced effective divisor on $\bar{S}$. f is called an A_{*}^{1}-ruling of $S=\bar{S}-D$ if $DF=2$ for any general fiber F of f.

We will study the structure of such rulings in this section. (6.11; 2) is an important motive of our study.

(7.2) DEFINITION. We are chiefly interested in the case where D is an NC-divisor. So we call a fiber F of f to be *almost D-minimal* if F has no exceptional D-component E with $\beta_D(E)\leqq 2$. Similarly as in (4.10), we can reduce the problem to the case in which f is almost D-minimal, that means, every fiber F of f is almost D-minimal. Under the process of blowing-down D remains to be an NC-divisor.

A fiber F of f is said to be *virtually D-connected* if every connected component of $F\cap D$ meets a horizontal component of D. A connected component of $F\cap D$ is called a *rivet* if it meets the horizontal component(s) of D at more than one points, or if it is a node of D_h, the union of horizontal components of D.

In case of A^1_*-rulings there are at most two horizontal components of D. If there are two, both must be a section of f. In this case the ruling is called a *sandwich*. If there is only one, then it is a two-sheeted (branched) covering of C. In this case the ruling is called a *gyoza* (=餃子, in Japanese). In either case if a virtually D-connected fiber contains a rivet, then it is D-connected.

(7.3) LEMMA. *Let F be a fiber of a ruling $f:\bar{S}\to C$. Suppose that F contains an exceptional curve E of multiplicity one* (*cf.* (4.9)). *Then E is a tip of the rational tree F. Moreover, F contains another exceptional curve.*

PROOF. If E were not a tip, it would be the exceptional divisor of a subdivisional blowing-up. So its multiplicity cannot be one. If there were no other exceptional curve in F, then $KF\geqq KE=-1$ as in (4.13). This contradiction completes the proof.

(7.4) From now on, we study the structure of singular fibers of A^1_*-rulings. We assume that D is an NC-divisor and f is almost D-minimal until (7.7).

(7.5) LEMMA. *Suppose that $\sigma(F)=0$, i.e., $F\subset D$. Then either*

1) *$F\cong \boldsymbol{P}^1$ and F is a rivet, or*

2) *F is the twig* [2, 1, 2], *f is a gyoza, $f(F)$ is a branch point of $D_h\to C$ where D_h is the horizontal component of D and D_h meets the exceptional component of F transversally.*

PROOF. Suppose that F is a rivet. Assume that F is singular. Then there is an exceptional curve E in F. By almost D-minimality we have $\beta_D(E)\geqq 3$. Hence E must meet D_h by (4.5), where D_h is the union of horizontal components of D. Since D_h meets F at two points by definition of rivet, E is of multiplicity one, because otherwise $D_hF>2$. So E is a tip of F by (7.3). Therefore $\beta_D(E)\geqq 3$ implies $D_hE\geqq 2$. On the other hand, we have another exceptional component in F by (7.3). By the same reasoning as above, it also must meet D_h. Then $D_hF>2$, a contradiction. Thus we prove $F\cong \boldsymbol{P}^1$.

Suppose that F is not a rivet. Then $D_h\cap F$ must be a point, which is a node of D. So D_h consists of one component and there is only one component C of F that meets D_h. Moreover, C is of multiplicity two. Thus F is singular, and it contains an exceptional component. By almost D-minimality and (4.5) this must meet D_h. Hence C is the unique exceptional component of F. $-2=KF=2KC$ implies $KY=0$ and $Y^2=-2$ for any other component Y of F. Note also that C is not a tip of F by almost D-minimality. Now, by the observation in (4.14), we infer that F is a twig [2, 1, 2].

(7.6) LEMMA. *Suppose that $\sigma(F)=1$ and that F does not contain a rivet. Then*

1) *$F\cong \boldsymbol{P}^1$ and F meets D_h at two different points, or*

2) *F looks like a twig $[A, 1, B]$ as in (4.7), the S-component of F is the unique exceptional component of F, and D_h meets the highest and the lowest components of F, or*

3) *f is a gyoza and $f(F)$ is a branch point of $D_h\to C$.*

PROOF. It suffices to show 2) assuming that F is singular and that 3) is not the case. By (4.2) F contains an exceptional curve E. If E were a D-component, then $\beta(E)\geqq 3$ and $D_hE>0$ by almost D-minimality and (4.5). So the multiplicity of E would be one because otherwise $D_hF>2$. Hence E would be a tip of F by (7.3). But then $D_hE=2$ by $\beta(E)\geqq 3$. So F would contain a rivet. Thus we conclude that E is the S-component of F. In particular, F contains only one exceptional curve. Now, by the observation (4.14), we see that F is obtained by succesive blowing-ups from a twig $[A, 1, B]$ as in (4.7). So, the proper transforms of the highest and the lowest components of this twig are the only components of multiplicity one. D_h meets F at two different points since 3) is not the case. These points must lie on two different components of F of multiplicity one. Hence D_h meets the above two proper transforms. They would become connected in $F\cap D$ if once we perform a sprouting blowing-up. So F is obtained only by subdivisional blowing-ups, and hence looks like a twig $[A, 1, B]$ as in (4.7). In particular, $\mu(F)\geqq 2$.

(7.7) LEMMA. *Suppose that $\sigma(F)=1$ and that F contains a rivet. Then*

1) *$F\cong \boldsymbol{P}^1$, f is a sandwich, $D_1\cap F=D_2\cap F$ where D_j are the horizontal components of D, or*

2) *F is D-minimal and meets D_h at two different points, or*

3) *there is precisely one exceptional D-component E of multiplicity one, E is a tip of F and meets D_h at two different points.*

PROOF. Suppose that F contains a node of D_h. Since D is an NC-divisor, this node must be on the S-component of F. Moreover, f is a sandwich. Now, we infer that this S-component must be of multiplicity one, because otherwise $D_hF>2$. Moreover, the almost D-minimality implies the D-minimality of F, since D_h does not meet D-components of F. So, using (4.13), we see that 1) is satisfied.

Second suppose that D_h meets F at two different points. If F is not D-minimal, we infer that an exceptional D-component must meet D_h, be of multiplicity one, be a tip of F and meet D_h at two points, by the same argument as in (7.6). Thus 3) is the case. Q.E.D.

REMARK. In both cases 2) and 3), the S-component of F is an exceptional curve.

(7.8) Now we will calculate numerical invariants of S. Let ν be the number of fibers with $\sigma=0$, let Σ be the sum of $(\sigma(F)-1)$ where F runs through all the fibers of f with $\sigma>1$, let ρ be the numbers of rivets contained in fibers of f, and let ε be the function defined by $\varepsilon(0)=0$ and $\varepsilon(t)=1$ for $t>0$.

(7.9) LEMMA. *Let $f:\bar{S}\to C$ be an A^1_*-ruling of $S=\bar{S}-D$, where D is an NC-divisor. Suppose that f is a gyoza. Then $\tilde{b}_1(S)=\nu-\varepsilon(\nu)$, $\hat{b}_2(S)=\Sigma+1-\varepsilon(\nu)$ and $\bar{b}_2(S)=\rho+2(g(D_h)-g(C))$, where D_h is the horizontal component of D and g denote the genus of curves.*

PROOF. $\tilde{b}_1(S)$ and $\hat{b}_2(S)$ are calculated in the same way as in (5.4). $\bar{b}_2(S)=b_1(D)-b_1(\bar{S})$ by definition of $\bar{b}_2$ and by the surjectivity of $H_1(D_h)\to H_1(D)\to H_1(\bar{S})\to H_1(C)$. It is easy to see $b_1(\bar{S})=2g(C)$ and $b_1(D)=2g(D_h)+\rho$. Thus we obtain the desired formula.

(7.10) LEMMA. *Let things be as in (7.9), but this time suppose that f is a sandwich. Then $\tilde{b}_1(S)-\hat{b}_2(S)=\nu-\Sigma$, $\tilde{b}_1(S)=\nu-\varepsilon(\nu)$ or $\nu-\varepsilon(\nu)+1$, $\hat{b}_2(S)=\Sigma-\varepsilon(\nu)$ or $\Sigma-\varepsilon(\nu)+1$, $\bar{b}_2(S)=2g(C)+\rho-\varepsilon(\rho)$.*

PROOF. The first equality is a special case of (4.16; 1). Let D_1 and D_2 be the horizontal components of D. If D_2 is linearly $\boldsymbol{Q}$-independent of the other components of D, then $\tilde{b}_1(S)=\nu-\varepsilon(\nu)$ by a similar reasoning as in (5.4). If D_2 is linearly dependent on the other components, or equivalently, D_1-D_2 is linearly dependent over $\boldsymbol{Q}$ on the non-horizontal components of D, then we have one more relation among the components of D and $\tilde{b}_1(S)=\nu-\varepsilon(\nu)+1$. Thus we prove the assertion for $\tilde{b}_1(S)$. The claim for $\hat{b}_2(S)$ follows from this. As for

$\bar{b}_2(S)$, we have $b_1(D)=2g(D_1)+2g(D_2)+\rho-\varepsilon(\rho)$. So the formula is proved similarly as in (7.9).

(7.11) LEMMA. *Let $\gamma(F)$ be the number of connected components of $F\cap D$ which do not meet the horizontal component(s) of D, and let Γ be the sum of $\gamma(F)$ of all the fibers F of f. Then, $\bar{b}_3(S)=\Gamma$ if f is a gyoza, and $\bar{b}_3(S)=\Gamma+1-\varepsilon(\rho)$ if f is a sandwich.*

This is proved easily by (1.15; 3).

(7.12) LEMMA. *Suppose that $b_2(S)=0$. Then $g(C)=0$ and $\bar{S}$ is rational.*

PROOF. If f is a sandwich, $\bar{b}_2(S)=0$ implies $g(C)=0$ by the formula (7.10). So we assume f to be a gyoza. Then, by (7.9) and $\bar{b}_2(S)=0$, we obtain $\rho=0$ and $g(D_h)=g(C)$. Since D_h is a covering of C, $g(C)$ can be positive only when C is an elliptic curve and $D_h\to C$ is unramified. We may assume that f is almost D-minimal. Therefore (7.5) applies to the effect $\nu=0$. Then $\hat{b}_2(S)=\Sigma+1>0$ by (7.9). This contradiction to $b_2(S)=0$ proves $g(C)=0$.

(7.13) COROLLARY. *Let S be an algebraic surface such that $b_2(S)=0$ and $\bar{\kappa}(S)=1$. Then S is rational.*

For a proof, use (6.11; 2). Note that $FD=0$ is impossible since $\hat{b}_2(S)=0$ implies that every curve on $\bar{S}$ meets D.

(7.14) LEMMA. *Let $f:\bar{S}\to C$ be an A^1_*-ruling of $S=\bar{S}-D$, where D is an effective reduced divisor. Then* NS$(S)\neq 0$ *in any of the following cases.*

1) *f is a gyoza.*
2) *There are two fibers F_1, F_2 with $\mu(F_j)\geqq 2$, and $\sigma(F)>0$ for every fiber F.*
3) *There are fibers F_1, F_2 such that $\sigma(F_1)\geqq 2$ and $1<\mu(F_2)<\infty$.*

PROOF. In case 1), the class of a section of f is not zero in NS(S). Actually, we have a surjective homomorphism NS$(S)\to \boldsymbol{Z}/2\boldsymbol{Z}$.

In case 2), adding several components to D if necessary, we may assume $\sigma(x)=1$ for every $x\in C$. Let E_x be the S-component of $F_x=f^{-1}(x)$. Then $F=\mu(x)E_x$ in NS(S) for any x. Letting D_1 and D_2 be the horizontal components of D (we may assume that f is a sandwich by 1), we have $0=D_1-D_2=\sum_y d_y E_y+tF$ in NS(S), where y runs through the singular locus of f and d_y's and t are integers. In view of $\mu_1E_1=F=\mu_2E_2$, we may assume that $d_1>0$ and $d_2>0$. Adding further relations $E_y=0$ for $y\neq 1, 2$, we obtain a quotient group G of NS(S) generated by the classes e_1, e_2 of E_1 and E_2 with the relations $\mu_1e_1=\mu_2e_2=d_1e_1+d_2e_2=0$ where $\mu_x=\mu(x)$. Setting $\varphi(e_1)=\mu_2$ and $\varphi(e_2)=\mu_1$, we obtain a well-defined non-trivial homomorphism $\varphi: G\to \boldsymbol{Z}/(d_1\mu_2+d_2\mu_1)\boldsymbol{Z}$. Thus $\mathrm{NS}(S)\neq 0$ is proved.

The case 3) is treated similarly as 2). We sketch the method. We may assume f to be a sandwich. We may assume $\sigma(F_1)=2$, because otherwise $\hat{b}_2(S)=\mathrm{rank}\,\mathrm{NS}(S)>0$ by (7.10). Furthermore, adding several components to D if necessary, we may assume that $\sigma(F)=0$ for any singular fiber F other than F_1 and F_2. Let A_1 and A_2 be the S-components of F_1 and let m_1, m_2 be their multiplicities. Let μ be the multiplicity of the S-component E of F_2 and write $D_1-D_2=\nu E+n_1A_1+n_2A_2+tF$ in $\mathrm{NS}(S)$ as in 2). Since $m_1A_1+m_2A_2=\mu E=F$, we may assume that $n_1>0$, $n_2>0$ and $0\leqq\nu<\mu$. Then, $\mathrm{NS}(S)$ is isomorphic to the abelian group G generated by the classes a_1, a_2 and e of A_1, A_2 and E under the relation $m_1a_1+m_2a_2=\mu e=n_1a_1+n_2a_2+\nu e=0$. We will show $G\neq 0$.

If m_1 and m_2 are not coprime, then (m_1, m_2) and (n_1, n_2) cannot generate $\boldsymbol{Z}\oplus\boldsymbol{Z}$. Therefore $G/\{e=0\}$ is non-trivial. So we may assume $\mathrm{g.c.d.}(m_1, m_2)=1$. Let c be the greatest common divisor of μ and ν and take integers x and y such that $\mu x+\nu y=c$. Set $\varphi(a_1)=-cm_2$, $\varphi(a_2)=cm_1$, $\varphi(e)=y(n_1m_2-n_2m_1)$. It is easy to see that φ gives rise to a homomorphism $G\to\boldsymbol{Z}/\mu\boldsymbol{Z}$. If this were trivial, then μ would divide both cm_1 and cm_2, hence $\mathrm{g.c.d.}(cm_1, cm_2)=c$. But this is impossible by definition of c and $\nu<\mu$.

(7.15) PROPOSITION. *Let $f: \bar{S}\to C$ be an A^1_*-ruling of $S=\bar{S}-D$, where D is an* NC-*divisor. Suppose that $H^2(S;\boldsymbol{Z})=0$ and $\bar{\kappa}(S)\geqq 0$. Then f is a sandwich, and $\bar{\kappa}(S)=b_1(S)=1$, and $\pi_1(S)$ is non-abelian.*

PROOF. f is a sandwich by (7.14; 1). Let D_1 and D_2 be the horizontal components of D and let Σ, ν, ρ be as in (7.10) and (7.8). Then $\tilde{b}_2(S)=0$ implies $g(C)=0$ and $\nu\leqq\rho\leqq 1$. So $\hat{b}_1(S)=0$ and $b_1(S)=\tilde{b}_1(S)=\nu-\Sigma$, since $\hat{b}_2(S)=0$. We may assume that f is almost D-minimal.

Assume that $\nu=0$. Then, by (7.14; 2), there is at most one fiber with $\mu\geqq 2$. By virtue of (4.13) we infer that there are at most two singular fibers, and if there are two, one of them must contain a rivet and with $\mu=1$. The other singular fiber is of type (7.6; 2). From these observations we see easily that both D_1 and D_2 are contained in some rational twigs of D. Hence $(K+D)^+F<0$ for a general fiber F by (6.13). This is absurd.

Thus we prove $\nu>0$, and hence $\rho=\nu=1$. Let F_∞ be the fiber with $\sigma(F_\infty)=0$. Assume that there is a fiber F_0 such that $\sigma(F_0)\geqq 2$. Then $\sigma(F_0)=2$ since $\tilde{b}_1(S)=1-\Sigma$. Moreover, by (7.14; 3), there is no other fiber with $\mu>1$, nor $\sigma>1$. So, by (7.6), every other fiber is non-singular.

It is easy to see that we can find a linear subgraph T of the dual graph of F_0 such that T consists of components $Y_1, \cdots, Y_t$ of F_0 in this order where and $Y_1D_1=Y_tD_2=1$. Since F does not contain a rivet, one of Y_i's must be an

S-component. Let Y' be the other S-component, set $D'=D+Y'$, and let F_j be the connected component of $F_0\cap D'$ containing the point $D_j\cap F$ for each $j=1, 2$. By symmetry we may assume $Y'\subset F_1$. Then we claim that F_2 is a rational twig of D (or possibly consists of only the point $D_2\cap F_0$). Indeed, otherwise, F_2 would contain an exceptional curve by a reasoning as in (4.14). But this is impossible by the almost D-minimality, (4.5) and (7.3). Thus we infer that F_2 is a rational twig of D, and hence so is F_2+D_2. Then $(K+D)^+F<0$ by (6.13), which is absurd.

Thus we prove $\sigma(F)\leqq 1$ for every fiber F. Hence $b_1(S)=1$, and every singular fiber is of type (7.6; 2) except for F_∞. Let $F_1, \cdots, F_k$ be the singular fibers and set $\mu_j=\mu(F_j)$. We may assume $k\geqq 2$, because otherwise D_j would be contained in a rational twig of D and this would lead to a contradiction as above. So $\pi_1(S)$ is non-abelian by (4.21).

Moreover, by the same reasoning, we infer that $(K+D)^-$ contains no horizontal component. Therefore we see that $(K+D)^-=\mathrm{Bk}(D)$ by (6.17). Furthermore, $(K+D)^+D_j=-1+\sum_{j=1}^{k}(1-\mu_j^{-1})=0$ if and only if $\bar{\kappa}(S)=0$. This is possible only when $k=2$ and $\mu_1=\mu_2=2$. But then we have $\mathrm{NS}(S)\cong \boldsymbol{Z}/2\boldsymbol{Z}$. Thus we conclude that $\bar{\kappa}(S)=1$, completing the proof of (7.15).

REMARK. In the above argument, we just used that $(K+D)$ is pseudo-effective. However, on rational surfaces, this apparently weaker condition is actually equivalent to $\bar{\kappa}(S)\geqq 0$ (cf. [F ; (2.8)]).

(7.16) COROLLARY. *Let S be an algebraic surface such that $b_1(S)=0$ and $H^2(S ; \boldsymbol{Z})=0$. Then $\bar{\kappa}(S)\neq 1$.*

(7.17) In order to calculate the fundamental group, the following lemma is useful.

LEMMA. *Let x be a node of an effective reduced divisor D on a smooth surface $\bar{S}$ and let $S=\bar{S}-D$. Let $\bar{S}'$ be the blowing-up of $\bar{S}$ at x and let E be the exceptional curve over x. Then a vanishing loop γ of E in $\pi_1(S)$ is of the form $\gamma_1\gamma_2=\gamma_2\gamma_1$, where γ_1 and γ_2 are vanishing loops of the two analytic branches of D at x.* (Recall that a vanishing loop is determined up to conjugacy, (4.17)).

PROOF. Take a coordinate neighborhood $U\cong\{(z_1, z_2)\in \boldsymbol{C}^2 \mid |z_j|<1\}$ of x such that $D\cap U=\{z_1z_2=0\}$. Define a path γ in $U-D$ by $\gamma(t)=(\varepsilon\exp(2\pi it), \varepsilon\exp(2\pi it))$ for $0\leqq t\leqq 1$, where ε is a small positive number. It is easy to see that γ lifts to a vanishing loop of E, and γ is of the desired form. Any other vanishing loop of E is conjugate to γ, and hence of the desired form as well as γ.

(7.18) LEMMA. *Let $\bar{S}$, D and S be as above and let x be a smooth point (resp. node) of D. Let $p:\bar{S}'\to\bar{S}$ be a successive blowing-ups with centers lying over x and let $D'=p^{-1}(D)$ and $S'=\bar{S}'-D'\cong S$. Let Y be a component of D' over x and let m (resp. m_1 and m_2) be the coefficient(s) of Y in the total transform(s) of the component(s) D_0 (resp. D_1 and D_2) of D passing x. Here, m_1 and m_2 should be counted separatedly, when two analytic branches at x happen to belong to a same irreducible component of D. Then, a vanishing loop of Y is of the form γ^m (resp. $\gamma_1^{m_1}\gamma_2^{m_2}$ with $\gamma_1\gamma_2=\gamma_2\gamma_1$), where γ (resp. each γ_j) is a vanishing loop of D_0 (resp. D_j).*

PROOF. Take a coordinate neighborhood U of x as in (7.17) and we consider everything over U. $U\cap S\cong(\varDelta^*)^2$ if x is a node, $U\cap S\cong\varDelta\times\varDelta^*$ if x is a smooth point on D. In either case $\pi_1(U\cap S)$ is abelian and we get rid of the troubles coming from the non-commutativity. In particular, vanishing loops are determined uniquely in π_1. So, we just apply (7.17) successively to obtain the desired result.

REMARK. When the two analytic branches of D at x belong to a same component of D, their vanishing loops γ_1 and γ_2 are conjugate in $\pi_1(S)$, but may not be the same.

(7.19) COROLLARY. *Suppose that there is an exceptional curve E not in D with $DE=1$. Let Y be the component of D meeting E and let D' be the divisor $D-Y$. Then $\pi_1(S)\cong\pi_1(S')$ where $S'=\bar{S}-D'$.*

PROOF. Let $p:\bar{S}\to\bar{S}_1$ be the blowing-down of E to a point. Set $D_1=p(D)$, $D_1'=p(D')$. The vanishing loop of E in $\pi_1(\bar{S}-(D+E))\cong\pi_1(\bar{S}_1-D_1)$ is that of $p(Y)$ by (7.18). Hence $\pi_1(S)\cong\pi_1(\bar{S}_1-D_1')\cong\pi_1(S')$ by (4.18).

(7.20) We give now a recipe to calculate $\pi_1(S)$ of an A^1_*-ruled surface S.

First we consider the case of sandwich. Let $F_1,\cdots,F_k$ be all the singular fibers of $f:\bar{S}\to C$ with respect to D and let $S_0=\bar{S}-(D\cup F_1\cup\cdots\cup F_k)$. Then S_0 is a $\boldsymbol{C}^*$-bundle over $C_0=f(S_0)$, which is topologically trivial unless $C_0=C$. If $C_0=C$, then $D=D_1+D_2$ and these two horizontal components are disjoint. In this case $\pi_1(S)\to\pi_1(C)$ is surjective and its kernel is a cyclic group of order $|D_1^2|=|D_2^2|$. So we consider the case $C_0\neq C$. In this case $\pi_1(S_0)\cong\pi_1(C_0)\times\pi_1(F)$, where $F\cong\boldsymbol{C}^*$ is a general fiber and $\pi_1(F)\cong\boldsymbol{Z}$ is in the center of $\pi_1(S_0)$. $\pi_1(S)$ is the quotient group of $\pi_1(S_0)$ by the relations coming from vanishing loops of S-components of F_j (recall (4.18)). By virtue of (7.18), the vanishing loop of an S-component Y of F_j is of the form $\gamma_j^m t^n$ for some integers m, n, where γ_j is a vanishing loop of $x_j=f(F_j)$ in $\pi_1(C_0)$ and t is a generator of $\pi_1(F)\cong\boldsymbol{Z}$. Thus we can calculate $\pi_1(S)$.

When $f: \bar{S} \to C$ is a gyoza, the business becomes a little involved. Let D_h be the horizontal component of D and let B be the branch locus of $D_h \to C$. Set $S_0 = S - f^{-1}(B)$, and $S'_0 = S_0 \times_C D_h$. Then S'_0 is an unramified double covering of S_0, and we easily see that S'_0 can be completed to an A^1_*-ruled surface over D_h, which is a sandwich. By the preceding method we calculate $\pi_1(S'_0)$, which is a subgroup of $\pi_1(S_0)$ of index two. Thus we describe the structure of $\pi_1(S_0)$, and our $\pi_1(S)$ is a quotient group of $\pi_1(S_0)$ by relations given by (7.18). See also (7.24).

(7.21) *Example.* Suppose that $f: \bar{S} \to C$ has a fiber F of type (7.7; 1). Then, by the following process in order to calculate $\pi_1(S)$, we can transform F to a fiber of type (7.6; 1).

Let x be the intersection point of D_1 and D_2 on F. Let $p: \bar{S}' \to \bar{S}$ be the blowing-up at x, $D' = p^{-1}(D)$, E be the exceptional curve over x and let E' be the proper transform of F. Then, thanks to (7.19), we have $\pi_1(S) \cong \pi_1(\bar{S}' - D')$ $\cong \pi_1(\bar{S}' - (D' - E))$ since E' is an exceptional curve on $\bar{S}'$ with $D'E' = 1$. Blowing down E' we get a fiber of type (7.6; 1) with respect to $D' - E$.

(7.22) *Example.* Suppose that there is a fiber F consisting of two exceptional curves E_1 and E_2, each of which is an S-component of F and meets a horizontal component of D. Then $\pi_1(S) \cong \pi_1(\bar{S} - D_*)$, where D_* is the divisor D minus its horizontal component(s). Indeed, this is a special case of (7.19). $\pi_1(\bar{S} - D_*)$ is described by (4.19).

(7.23) *Example.* Suppose that there is a fiber F of type (7.6; 2). Then the S-component of F gives a relation of the form $\gamma^m t^n = 1$, where γ is a vanishing loop of $f(F)$ and t is a generator of π_1 (general fiber of $f) \cong \boldsymbol{Z}$ (see (7.20)), and $m = \mu(F)$ and n is an integer calculated with the help of (4.8). In particular, n is coprime to m.

(7.24) *Example.* Suppose that f is a gyoza and has a fiber F of type (7.5; 2). If we blow down two times to make the fiber Y isomorphic to $\boldsymbol{P}^1$, then Y and the horizontal component D_h of D has a contact of order two. Let y be the vanishing loop of Y and let t be the vanishing loop of D_h. Then we have $yty^{-1} = t^{-1}$.

To see this, take a coordinate neighborhood $U \cong \{(x, s) \in \boldsymbol{C}^2 \mid \mathrm{Max}(|x|, |s|) < 1\}$ of the contact point such that $U \cap Y = \{s = 0\}$ and $U \cap D_h = \{s = x^2\}$. Take vanishing loops in an s-disc in U where x is a fixed small positive number ε. Then move this s-disc along a path in the x-disc to another s-disc over $-\varepsilon$, where the path is homotopic to a half-circle. By this process yty^{-1} is transformed to a vanishing loop of D_h in the s-disc over $-\varepsilon$. The vanishing loops of D_h

over ε and $-\varepsilon$ are the same in π_1 as the two vanishing loops of D_h in the x-disc where $s=\varepsilon^2$. So they are inverse to each other. Thus we obtain $yty^{-1}=t^{-1}$.

Note that this relation implies $yt^{-1}y^{-1}=t$, and hence t and y^2 commute with each other.

§ 8. The case $\bar{\kappa}=0$.

(8.1) We recall the following important result.

THEOREM (Kawamata, [Kw 2]). *Let $f: M \to T$ be a dominant morphism of algebraic varieties such that any general fiber F is an irreducible curve (M, T and F may not be complete). Then $\bar{\kappa}(M) \geqq \bar{\kappa}(T)+\bar{\kappa}(F)$.*

(8.2) THEOREM. *Let S be an algebraic surface such that $\bar{\kappa}(S)=0$ and $\hat{b}_2(S)=0$. Then S is rational except the following case: $S=\bar{S}-D-Y$, $\bar{S}=\boldsymbol{P}_C(L\oplus\mathcal{O})$, L is an ample line bundle on an elliptic curve C, $D=D_1+D_2$, D_1 and D_2 are the sections of $f: \bar{S}\to C$ corresponding to the direct sum factors L and $\mathcal{O}$, and Y is a finite subset of $\bar{S}$.*

PROOF. Take an NC-completion $(\bar{S}, D)$ of S. We claim $\kappa(\bar{S})=-\infty$. Indeed, if not, we have an effective $\boldsymbol{Q}$-divisor K representing the canonical bundle of $\bar{S}$. $\bar{\kappa}(S)=0$ implies that $H=(K+D)^+$ is numerically equivalent to zero, while (6.9) says that H is effective. So $H=0$ and $K+D=(K+D)^-$. Hence $D\in\boldsymbol{Q}(N)$ is contractible. But this is impossible since $\hat{b}_2(S)=0$ and S is algebraic.

Thus $\kappa(\bar{S})=-\infty$ and hence $\bar{S}$ is ruled. We analyze the case in which $\bar{S}$ is not rational. We have a ruling $f: \bar{S}\to C$ over a curve C with genus $g>0$. $DF\geqq 2$ for a general fiber F, because otherwise $\bar{\kappa}(S)=-\infty$. On the other hand $DF\leqq 2$ because otherwise $\bar{\kappa}(S)\geqq\bar{\kappa}(F\cap S)=1$ by (8.1). So f is an A^1_*-ruling of S. From (8.1) we infer also that there is no fiber with $\sigma=0$, which means $\nu=0$ (see (7.8)).

If f were a gyoza, we would have $\hat{b}_2(S)=1+\Sigma>0$ by (7.9). So f is a sandwich. Using (7.10) and $\hat{b}_2(S)=0$, we obtain $\Sigma=0$ and hence $\sigma(F)=1$ for every fiber F. Adding several points to S if necessary, we may assume that f is almost D-minimal. Then every fiber F is either of type (7.6; 1), (7.6; 2) or (7.7).

Assume that $(K+D)Z<0$ for some curve Z not in D. Then Z is a component of $(K+D)^-$. Z cannot be horizontal, because otherwise $(K+D)^+F<0$ for a general fiber F of f. So Z must be the S-component of some fiber F_0. By (6.6; 2) we have $\bar{\kappa}(S)=\kappa(K+D, \bar{S})=\kappa(K+D+Z, \bar{S})=\bar{\kappa}(S-Z)$. But $\bar{\kappa}(S-Z)\geqq\bar{\kappa}(C-f(F_0))=1$ by (8.1). This contradiction proves that $(K+D)Z\geqq 0$ for any curve Z not in D. Hence $(K+D)^-=\mathrm{Bk}(D)$ by virtue of (6.17).

In view of (6.16), we infer that $(K+D)^+D_i>0$ for each horizontal component

D_i of D, unless $g(C)=1$ and all the fibers are of type (7.6; 1). Therefore we conclude that D is the union of its horizontal components, which are disjoint to each other. So $\bar{S}=\boldsymbol{P}_C(L\oplus\mathcal{O})$ for some line bundle L on C. Moreover, $\deg(L)\neq 0$ because otherwise D_1 and D_2 would be numerically equivalent to each other and $\hat{b}_2(S)=1$. Q.E.D.

REMARK. $\sharp(Y)=\bar{b}_3(S)-1$.

(8.3) COROLLARY. *Let S be an algebraic surface with $\bar{\kappa}(S)=\hat{b}_2(S)=0$. Then S is rational under any of the following additional hypotheses.* 1) $\bar{b}_1(S)>0$. 2) $\bar{b}_2(S)=0$. 3) $\bar{b}_3(S)=0$. 4) *S is affine.* 5) $\overline{p_g}(S)=h^0(\bar{S}, K+D)=0$.

(8.4) REMARK. Let S be a rational surface with $\bar{\kappa}(S)=0$ and let $(\bar{S}, D)$ be its NC-completion. Then the logarithmic m-genus $\bar{P}_m(S)=h^0(\bar{S}, m(K+D))$ is positive if and only if $m(K+D)^-$ is a usual divisor with integral coefficients.

Indeed, $(K+D)^+$ is linearly equivalent to zero by (6.11). So 'if' part is obvious. The 'only if' part follows from (6.9).

(8.5) THEOREM. *Let S be a rational surface such that $\bar{\kappa}(S)=0$, $b_2(S)=0$ and $\bar{b}_1(S)>0$. Then there is a ruling $f: \bar{S}\to C\cong\boldsymbol{P}^1$ together with an NC-divisor D on $\bar{S}$ and a finite subset Y of $\bar{S}$, such that $S\cong\bar{S}-D-Y$, f is a gyoza, there are precisely two branch points of $D_h\to C$ where D_h is the horizontal component of D, the fibers over the branch points are of type* (7.5; 2) *and all the other fibers are non-singular, that means, of type* (7.6; 1).

PROOF. By (1.18; 3) we have a non-constant invertible rational function on S. This gives a dominant morphism $\varphi: S\to\boldsymbol{A}^1_*=\boldsymbol{A}^1-\{0\}$. Let $S\to C_0\to\boldsymbol{A}^1_*$ be its Stein factorization. By $\bar{\kappa}(S)=0$ and (8.1) we infer that $\bar{\kappa}(C_0)=0$ and $\bar{\kappa}(F_0)=0$ for any general fiber F_0 of $f_0: S\to C_0$. The former implies that $C_0\cong\boldsymbol{A}^1_*$. The latter implies that f_0 can be completed to an $\boldsymbol{A}^1_*$-ruling $f: \bar{S}\to C\cong\boldsymbol{P}^1$ of S. By a similar reasoning as in (4.10), we reduce the problem to the case in which f is almost D-minimal, where $D=\bar{S}-S$ is an NC-divisor on $\bar{S}$.

Thus, f has exactly two fibers F_1 and F_2 with $\sigma=0$. So $\nu\geqq 2$, where the notation is as in (7.8). If f were a sandwich, then $\bar{b}_2(S)=\rho-1>0$ by (7.10). So f is a gyoza. Then $\bar{b}_2(S)=0$ implies $\rho=g(D_h)=0$. Hence $D_h\cong\boldsymbol{P}^1$ and $D_h\to C$ has two branch point, while F_1 and F_2 must be of type (7.5; 2). Thus $f(F_1)$ and $f(F_2)$ are exactly the branch locus. Moreover, from $\hat{b}_2(S)=0$, we infer that $\Sigma=0$ and $\sigma(F)\leqq 1$ for every fiber F. Hence F is either of type (7.6; 1) or (7.6; 2).

Now, calculating $(K+D)^-$ similarly as in (8.2), we infer that there is no fiber of type (7.6; 2). This completes the proof.

REMARK. $Y=\emptyset$ if $\bar{b}_3(S)=0$.

(8.6) COROLLARY. *Let S be a surface as in* (8.5). *Then* $\bar{b}_1(S)=1$, $\mathrm{NS}(S)\cong\mathrm{Pic}(S)\cong \boldsymbol{Z}/2\boldsymbol{Z}$, $\bar{P}_m(S)=h^0(\bar{S}, m(K+D))=1$ *if and only if m is a non-negative even integer, $\pi_1(S)$ is the group generated by t and y under the relation* $yty^{-1}=t^{-1}$.

PROOF. It is easy to calculate $\bar{b}_1(S)$ and $\mathrm{NS}(S)=\mathrm{Pic}(S)$. $\bar{P}_m$ is calculated by (8.4). As for $\pi_1(S)$, use (7.20) and (7.24).

(8.7) LEMMA. *Let D be a reduced effective divisor as in* (6.13). *Suppose that every rational twig of D is admissible (hence* $\mathrm{Bk}^*(D)$ *is well-defined). Let C be a component of D such that* $(K+D)C=\mathrm{Bk}^*(D)C$. *Then*

1) *C is a component of* $\mathrm{Bk}^*(D)$, *or*

2) $C\cong \boldsymbol{P}^1$, $\beta(C)=2$ *and* $\mathrm{Bk}^*(D)C=0$, *or*

3) $C\cong \boldsymbol{P}^1$, $\beta(C)=3$ *and there are two rational tips T_1, T_2 such that* $T_j^2=-2$ *and* $T_jC=1$ *for each* $j=1, 2$, *or*

4) $C\cong \boldsymbol{P}^1$, $\beta(C)=3$ *and there are three rational twigs T_1, T_2 and T_3 with* $\sum d(T_j)^{-1}=1$ *and with common branching component C, or*

5) $C\cong \boldsymbol{P}^1$, $\beta(C)=4$ *and there are four rational tips* $T_1, \cdots, T_4$ *with* $T_j^2=-2$ *and* $T_jC=1$ *for each j, or*

6) $\beta(C)=0$ *and the arithmetic genus of C is one.*

PROOF. Let D_j be the connected component of D containing C. We may assume $C\not\subset\mathrm{Bk}^*(D)$ and hence D_j is neither a (usual) rational club nor an abnormal rational club. So $D_j-\mathrm{Bk}^*(D_j)$ is an effective $\boldsymbol{Q}$-divisor with the same components as D_j (see (6.16)). Therefore $(D-\mathrm{Bk}^*(D)-C)C>0$ unless $\beta(C)=0$. When $(K+C)C\geqq 0$, we have $(D-\mathrm{Bk}^*(D)-C)C\leqq 0$ by the assumption $(K+D)C=\mathrm{Bk}^*(D)C$. This implies $\beta(C)=0$. Then $(K+C)C=0$ and 6) is the case. When $(K+C)C<0$, we have $C\cong\boldsymbol{P}^1$ and $2=(D-\mathrm{Bk}^*(D)-C)C=\beta(C)-\mathrm{Bk}^*(D)C$. Using (6.16), we infer that 2), 3), 4) or 5) must be the case.

(8.8) COROLLARY. *Let D be as above and let D_j be a connected component of D such that* $(K+D)C=\mathrm{Bk}^*(D)C$ *for every component C of D_j. Then D_j is*

(*I*) *a (usual or abnormal) rational club, or*

(*O*) *a rational cycle such that $\beta(C)=2$ for every component C of D_j, or*

(*H*) *a rational tree with precisely two branching components, both of which are of type* (8.7; 3), *or*

(*Y*) *a rational tree with three twigs and their common branching component of type* (8.7; 4), *or*

(*X*) *a rational tree consisting of four tips and a branching component of type* (8.7; 5), *or*

(*) *an isolated prime component of arithmetic genus one.*

PROOF. If D_j contains a component of type (8.7; 4) (resp. 5) or 6)), then D_j is of the above type (Y) (resp. (X) or $(*)$). If all the components of D_j are of type (8.7; 2), then D_j is of type (O). Otherwise, D_j has a component with $\beta(C)\geqq 3$ unless D_j is a rational club. If this C is of type (8.7; 1), D_j is an abnormal rational club. Thus, we may assume that C is of type (8.7; 3). Then D_j must have another branching component, which also is of type (8.7; 3). So D_j is of type (H).

(8.9) Now we will study the structure of a smooth algebraic surface S such that $\hat{b}_2(S)=\bar{b}_3(S)=\bar{\kappa}(S)=0$. S is affine by (2.4; 3) and is rational by (8.3). An NC-completion $(\bar{S}, D)$ of S is said to be NC-*minimal* if none of the five types of blowing-downs as in (6.21) is possible. We study first the cases in which there exists an NC-minimal completion, and then proceed to the general case.

(8.10) Suppose that $(\bar{S}, D)$ is NC-minimal (until (8.64)). Then $(K+D)^-=\mathrm{Bk}^*(D)$ by (6.24). We have also $(K+D)^+=0$ by (6.11; 1). Hence (8.8) applies to D. Note that D is connected and is not contractible since S is affine. So D is either of type $(*)$, (O), (H), (X) or (Y).

(8.11) Type $(*)$. We have $b_2(\bar{S})\leqq b_2(D)+\hat{b}_2(S)=1$. Hence $\bar{S}\cong \boldsymbol{P}^2$. We have also $K+D=\mathrm{Bk}^*(D)=0$. So $\deg(D)=3$. Now it is easy to see $\mathrm{NS}(S)=\mathrm{Pic}(S)\cong \boldsymbol{Z}/3\boldsymbol{Z}\cong\pi_1(S)$, $b_1(S)=0$ and $\bar{P}_m(S)=1$ for any integer $m>0$. D is a non-singular elliptic curve by definition of NC-divisor. However, the case in which D is a rational curve with one node can be treated similarly. In particular, we have $\pi_1(S)\cong\boldsymbol{Z}/3\boldsymbol{Z}$. See (8.17).

(8.12) Type (O). By definition of NC-minimality, D has no exceptional component. Moreover, in fact, $\bar{S}$ is relatively minimal. To see this, assume that there is an exceptional curve E on $\bar{S}$. Then $ED=1$, since $K+D=\mathrm{Bk}^*(D)=0$. Hence (6.21; 4) would be possible, contradicting the NC-minimality. Thus $\bar{S}\cong\boldsymbol{P}^2$, or $\bar{S}$ is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$. Note that $K+D=0$ in any case.

(8.13) Suppose in addition that $\bar{S}\cong\boldsymbol{P}^2$. Then $\deg(D)=3$ and hence D is either a union of three lines in a general position or a union of a smooth quadric and a line intersecting normally with each other. In the former case $S\cong\boldsymbol{A}^2_*$ and the completion is denoted by $O(1, 1, 1)$. The latter type is denoted by $O(4, 1)$. In this case $b_1(S)=1$. We have $\mathrm{Pic}(S)=\mathrm{NS}(S)=0$ in both cases.

(8.14) Second we consider the case $\bar{S}\cong\Sigma_k=\boldsymbol{P}(\mathcal{O}(k)\oplus\mathcal{O})$. Let $f:\bar{S}\to\boldsymbol{P}^1$ be the bundle mapping and let ν be the number of fibers with $\sigma=0$. It is easy to see that $\nu\leqq 2$ and that f is an $\boldsymbol{A}^1_*$-ruling. Moreover, if $\nu=2$, there are exactly two horizontal components of D and $S\cong A_*^2$ in this case.

(8.15.1) Let things be as in (8.14) and suppose in addition $\nu=1$. Let F_0 be the fiber with $\sigma=0$. Here we consider the case in which f is a sandwich.

We see $D=D_1+D_2+F_0$ and $D_1F_0=D_2F_0=D_1D_2=1$. Since $K^2=c_1(\bar{S})^2=8$ and $K+D=0$, we obtain $D_1^2+D_2^2=-4-K(D_1+D_2)=-4-KD+KF_0=2$.

Blow up $\bar{S}$ at the point $q=D_1\cap F_0$. Then the proper transform of F_0 is an exceptional curve on it. By blowing down this curve, we get another NC-completion $\bar{S}'$ of S. $\bar{S}'$ is a $\boldsymbol{P}^1$-bundle over $\boldsymbol{P}^1$ and $D'=\bar{S}'-S$ is of the form $F_0'+D_1'+D_2'$, where F_0' is a fiber of $f':\bar{S}'\to\boldsymbol{P}^1$ and D_j' is a transform of D_j for $j=1, 2$. Note that $(D_1')^2=D_1^2-1$ and $(D_2')^2=D_2^2+1$. Such an NC-completion will be called an *elementary transformation* of $(\bar{S}, D)$.

By successive elementary transformations we can transform D_1 to an exceptional curve. Blowing it down to a point, we get a completion of type $O(4, 1)$.

On the other hand, we can transform the completion so that $(D_1'')^2=(D_2'')^2=1$. Let $\bar{S}''\cong\Sigma_{k''}$ for some $k''\geqq 0$. $(D_j'')^2\equiv k''$ modulo 2. If $k''\geqq 3$, we would have $D''C=-K''C=2-k''<0$ for the section C of $\bar{S}''\to\boldsymbol{P}^1$ such that $C^2=-k''$. This is absurd. Now we conclude $k''=1$. Let E be the exceptional curve on $\bar{S}''\cong\Sigma_1$. Then $EF_0''=1$, $ED_1''=ED_2''=0$. Therefore S is a half-point attachment of A_*^2. Moreover, we have $\pi_1(S)\cong\boldsymbol{Z}$ be virtue of (7.19).

Thus, incidentally, we prove $\pi_1(S)\cong\boldsymbol{Z}$ for a surface S of type $O(4, 1)$.

(8.15.2) Here we consider the case in which f is a gyoza. So $D=D_h+F_0$. Such a completion $(\bar{S}, D)$ (or S) is said to be of type $O(4, 0)$. By elementary transformations we obtain another completion $(\bar{S}'', D'')$ of S such that $\bar{S}''\cong\Sigma_1$. Write $D''=D_h''+F_0''$ and let E be the exceptional curve on $\bar{S}''$. Then $D_h''E=0$ and $F_0''E=1$. Hence S is a half-point attachment of a surface of type $O(4, 1)$. Using (7.19), we obtain $\pi_1(S)\cong\pi_1(\boldsymbol{P}^2-\{\textit{smooth quadric}\})\cong\boldsymbol{Z}/2\boldsymbol{Z}$.

(8.16) Let things be as in (8.14) and suppose in addition that $\nu=0$. Then f is a sandwich and $D=D_1+D_2$. $-KD=K^2=8$ implies $D_1^2+D_2^2=4$.

Let F_x and F_y be the fibers of f containing the two points on $D_1\cap D_2$. It is easy to see that $S_0=S-(F_x\cup F_y)$ is isomorphic to A_*^2. Hence $\pi_1(S)$ is a homomorphic image of $\pi_1(S_0)\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$ and so abelian. Therefore $\pi_1(S)\cong H_1(S;\boldsymbol{Z})$, which is non-canonically isomorphic to the torsion part of $H^2(S;\boldsymbol{Z})$ by the universal coefficient theorem. Thus it is isomorphic to $\hat{H}^2(S;\boldsymbol{Z})\cong\mathrm{NS}(S)\cong\mathrm{Pic}(S)$, because $\tilde{H}^2(S;\boldsymbol{Z})\cong H_1(D;\boldsymbol{Z})\cong\boldsymbol{Z}$.

(8.16.1) Suppose that $k=0$, i.e., $\bar{S}\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$. Then D is of bidegree $(2, 2)$. So the bidegrees of D_j's are $(1, 1)+(1, 1)$ or $(1, 2)+(1, 0)$. In the former case D_1 is numerically equivalent to D_2 and $\hat{b}_2(S)=1$, contradicting the hypothesis.

In the latter case, changing the role of two rulings of $\bar{S}$, we see that $(\bar{S}, D)$ is of type $O(4, 0)$.

(8.16.2) Suppose that $k\geqq 2$. Let C be the curve on $\bar{S}\cong\Sigma_k$ such that $C^2=-k$. Then $DC=-KC=2-k\leqq 0$. So C is a component of D, say D_1. Then $D_2=-K-C\sim C+(k+2)F$, where F is the class of a fiber of f. So $D_2^2=k+4$ and $\mathrm{Pic}(S)\cong\mathrm{NS}(S)\cong \boldsymbol{Z}/(k+2)\boldsymbol{Z}$. This type is called $O(k+4, -k)$.

(8.17) REMARK. In (8.16), the case $k=1$ is ruled out by the assumption on the NC-minimality of $(\bar{S}, D)$. But this case can be treated similarly as above. This type will be called $O(3, 1)$ (resp. $O(5, -1)$) if the exceptional curve C on $\bar{S}\cong\Sigma_1$ is not (resp. lies) in D. In case of $O(3, 1)$, $D_1\in|C+F|$ and $D_2\in|C+2F|$, hence $\mathrm{NS}(S)=0$. S is a half-point attachment of a surface of type $O(4, 1)$, where the half-point lying over the quadric. In case of $O(5, -1)$, $D_1=C$ and $D_2\in|C+3F|$, hence $\mathrm{NS}(S)\cong\boldsymbol{Z}/3\boldsymbol{Z}$. By blowing down C to a point, we see that S is the complement in $\boldsymbol{P}^2$ of a rational curve of degree three with one node. Incidentally we prove $\pi_1(S)\cong\boldsymbol{Z}/3\boldsymbol{Z}$.

(8.18) Type (H). Let B_1 and B_2 be the two branching components of D and let T_1, T_3 (resp. T_2, T_4) be the tips meeting B_1 (resp. B_2). Then $K+D=(K+D)^-=\mathrm{Bk}^*(D)=(1/2)(T_1+T_2+T_3+T_4)$. So $\bar{P}_m(S)=1$ if and only if m is even and non-negative. Note that $(K+D)^2=-2=(K+D)D$. So $(K+D)K=0$.

(8.19) $b_1(D)=0$ because D is a rational tree. So $b_2(S)=\bar{b}_2(S)=0$. Hence (8.5) applies if $b_1(S)>0$. This type will be called $H[-1, 0, -1]$. From now on, we consider the case in which $b_1(S)=0$.

(8.20) Since $b_1(S)=\hat{b}_2(S)=0$, we have $b_2(\bar{S})=b_2(D)\geqq 6$. So there exists an exceptional curve E on $\bar{S}$. In view of $K=(1/2)(\sum_i T_i)-D=-(1/2)(\sum_i T_i)-(D-\sum_i T_i)$, $EK=-1$ and the NC-minimality of $(\bar{S}, D)$, we infer that one of the following conditions are satisfied.

a) $E=B_j$ for $j=1$ or 2.
b) E meets two tips with the same branching component.
c) E meets two tips whose branching components are different.
d) E meets only one tip T_i, and $ET_i=2$.

(8.21) Actually, we have the following stronger result: Any irreducible reduced curve C on $\bar{S}$ with $C^2<0$ and $C\not\subset D$ must be an exceptional curve of the above type b), c) or d).

Indeed, C meets D because S is affine. From this we infer $KC<0$. Hence C must be an exceptional curve.

(8.22) *Claim.* If there is an exceptional curve E of type a), then there is another exceptional curve of type c).

To show this, we may assume $E=B_1$. Then $T_1\cup B_1\cup T_3$ looks like a twig [2, 1, 2]. Hence $h^0(\bar{S}, F_1)=2$ for $F_1=T_1+2B_1+T_3$ and F_1 appears as a fiber of a ruling f of $\bar{S}$. F_1 is of type (7.5; 2) and f is a gyoza. By virtue of (8.21), it suffices to show that there exists a section C of f such that $C^2<0$. By (4.3), $\bar{S}$ is a blowing-up of a $\boldsymbol{P}^1$-bundle$\cong\Sigma_k$. If $k>0$, take C to be the proper transform of the section Δ of Σ_k such that $\Delta^2=-k$. If $k=0$, take C to be the proper transform of a horizontal line passing through a center of the blowing-up $\bar{S}\to\Sigma_k$. In either case C has the required property.

(8.23) *Claim.* If there is an exceptional curve E of type b), then there is another exceptional curve of type c).

To show this, we may assume $ET_2=ET_4=1$. Similarly as above, $F_2=T_2+2E+T_4$ appears as a fiber of a ruling $f:\bar{S}\to\boldsymbol{P}^1$. Again f is a gyoza and B_2 is the horizontal component. From now on the argument is the same as in (8.22).

(8.24) *Claim.* There exists an exceptional curve of type (8.20; c).

Indeed, otherwise, all the curves as in (8.21) must be of type d). Recall that $\bar{S}$ is obtained from some Σ_k by successive blowing-ups. Let $E_1, \cdots, E_r$ be the proper transforms of the exceptional curves of these blowing-ups. Then $E_j^2<0$. By (8.21), E_j is of type d) or a component of D. Moreover, the exceptional curve E_r of the final blowing-up must be of type d), since case a) is ruled out.

We may assume $E_rT_1=2$. Let E'_{r-1} be the exceptional curve of the blowing-up just before the final step. E'_{r-1} cannot be the image of T_1, which is singular. E'_{r-1} is not any other component of the image of D, because otherwise E_{r-1} would be a component of D and of type a). Thus E_{r-1} is not in D and of type d). Moreover, we have $E_{r-1}T_1=2$. In fact, if $E_{r-1}T_i=2$ for some $i\neq1$, then $(T_1+2E_r)^2=(T_i+2E_{r-1})^2=2$ and $(T_1+2E_r)(T_i+2E_{r-1})=0$. This contradicts the index theorem.

By similar reasoning we infer that all the above curves $E_1, \cdots, E_r$ are of type d), with $T_1E_j=2$, and disjoint from each other. Therefore T_2, T_3 and T_4 are mapped isomorphically onto Σ_k. However, Σ_k contains only one curve with negative self-intersection number. This contradiction proves the claim.

(8.25) Now, let E_1 be an exceptional curve of type (8.20; c). W may assume $E_1T_1=E_1T_2=1$ by symmetry. Then $h^0(\bar{S}, F_1)=2$ for $F_1=T_1+2E_1+T_2$ and hence F_1 appears as a fiber of a ruling $f:\bar{S}\to\boldsymbol{P}^1$. Let F_2 be the fiber containing T_3. Then F_2 contains an exceptional curve E_2 by (4.2). This can be neither of type

a) nor b). E_2 is not of type d) because F_2 is a rational tree by (4.4). Thus E_2 is of type c). This implies $E_2T_3=E_2T_4=1$ and $F_2=T_3+2E_2+T_4$. Clearly B_1 and B_2 are horizontal and f is a sandwich.

By the above observation and by (8.20), we infer that any fiber F different from F_1 and F_2 does not contain an exceptional curve. So $F\cong \boldsymbol{P}^1$ by (4.2). Hence $b_2(D)=b_2(\bar{S})=6$. Therefore $D=B_1+B_2+\sum_i T_i$ and $B_1B_2=1$. So there is a fiber F_0 of type (7.7; 1). Any other fiber is of type (7.6; 1). Consequently $\pi_1(S)$ is abelian by virtue of (7.21).

(8.26) $K(K+D)=0$ (cf. (8.18)) and $K^2=10-b_2(\bar{S})=4$ imply $B_1^2+B_2^2=0$. By symmetry we may assume that $B_1^2=-B_2^2=k$ for some $k\geqq 0$. In this case $(\bar{S}, D)$ will be said to be of type $H[k, -k]$.

If $k=0$, set $L=2B_1-2B_2+T_1-T_2+T_3-T_4$. Then $LD_i=0$ for any component D_i of D. So L is numerically equivalent to zero. This implies $b_1(S)>0$, contradicting our hypothesis. Thus, k must be positive.

(8.27) To calculate NS(S), we blow down E_1 and E_2, and then further blow down the images of T_1 and T_3. The result is Σ_k, because B_2 is mapped to a curve C with $C^2=-k<0$. Since $B_1B_2=1$, the image of B_1 is a member of $|C+(k+1)F|$. Therefore $B_1\in|B_2+(k+1)F-(T_1+E_1)-(T_3+E_2)|$. Hence, in NS($S$), we have $0=B_1-B_2=(k+1)F-E_1-E_2=2(k+1)E_1-E_1-E_2$. So $E_2=(2k+1)E_1$ in NS(S). In view of the relation $F=2E_1=2E_2$, we infer that NS(S) is a cyclic group of order $4k$.

By virtue of (8.25) and the universal coefficient theorem, we obtain also $\pi_1(S)\cong H_1(S; \boldsymbol{Z})\cong \boldsymbol{Z}/4k\boldsymbol{Z}$.

(8.28) Type (X). Our conclusion is that this case does not happen.

$D=B+T_1+T_2+T_3+T_4$, where B is the branching component. $K+D=\mathrm{Bk}^*(D)=(1/2)(\sum_i T_i)$. So $K=-B-(1/2)(\sum_i T_i)$. $(K+D)^2=-2=(K+D)D$ as in (8.18). Moreover $b_2(\bar{S})=b_2(D)=5$. Hence $K^2=10-b_2(\bar{S})=5$. So $KB=KD=-5$. This implies $B^2=3$.

By similar arguments as in the case of type (H), we infer that there is an exceptional curve E which meets two tips, say T_1 and T_2. Then $F_1=T_1+2E+T_2$ appears as a fiber of a ruling $f: \bar{S}\to \boldsymbol{P}^1$. There exists another fiber F_2 of the form $T_3+2E_2+T_4$, where E_2 is another exceptional curve. f is a gyoza and B is the horizontal component. Then, by (7.9), we get $\hat{b}_2(S)>0$ since $\nu=0$. This contradiction proves that type (X) cannot exist.

(8.29) Type (Y). Let $D=B+T_1+T_2+T_3$, where B is the branching component and T_i's are the rational twigs of D. They are admissible, and hence their components are $\boldsymbol{Q}$-linearly independent. B is not dependent on them in

$NS(\bar{S})$, because otherwise D would be contractible. Thus the components of D are independent in $NS(\bar{S})\otimes\boldsymbol{Q}$, and so $b_2(\bar{S})=b_2(D)$ and $b_1(S)=0$.

(8.30) By symmetry we may assume $d(T_1)\leqq d(T_2)\leqq d(T_3)$. Since $\sum d(T_i)^{-1}=1$, the triplet $(d(T_1), d(T_2), d(T_3))$ is one of the following: $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$. In these cases $(\bar{S}, D)$ is said to be of type $Y\{3, 3, 3\}$, $Y\{2, 4, 4\}$ or $Y\{2, 3, 6\}$ respectively.

(8.31) Type $Y\{3, 3, 3\}$. Set N_j be the bark of T_j. Then $(K+D)^2=(N_1+N_2+N_3)^2=N_1^2+N_2^2+N_3^2=-e(T_1)-e(T_2)-e(T_3)$ by (6.16). Since $d(T_j)=3$, $e(T_j)=1/3$ or $2/3$. So, in order that $(K+D)^2$ is an integer, we must have either $e(T_1)=e(T_2)=e(T_3)=1/3$ or $e(T_1)=e(T_2)=e(T_3)=2/3$.

(8.32) In the case $e(T_j)=1/3$, T_j is a tip with $T_j^2=-3$ (cf. (3.8) and (3.9)). So $K+D=\mathrm{Bk}(D)=(T_1+T_2+T_3)/3$ and $(K+D)^2=-1$. We have also $(K+D)D=-2$ and $K^2=10-b_2(\bar{S})=6$. Hence $KD=-5$ and $KB=KD-\sum_i KT_i=-8$. So $B^2=6$. We will show that such a case does not occur.

Indeed, for an exceptional curve E on $\bar{S}$, we have $1=-KE=(B+(2/3)\sum_i T_i)E$. By the above observation we see that $E\not\subset D$. So the above equality implies $BE=1$. But this contradicts the NC-minimality of $(\bar{S}, D)$. See (6.21; 4).

(8.33) Thus $e(T_i)=2/3$ for each i, and T_i is a twig $[2, 2]$. Write $T_i=T_{i1}+T_{i2}$, where T_{i1} is the tip of T_i. Then $N_i=\mathrm{Bk}(T_i)=(2T_{i1}+T_{i2})/3$. $(K+D)^2=-2=(K+D)D$. $K^2=10-b_2(\bar{S})=3$. So $KB=KD=-3$ and $B^2=1$.

(8.34) $B\cong\boldsymbol{P}^1$ and $0\to H^0(\bar{S}, \mathcal{O})\to H^0(\bar{S}, \mathcal{O}[B])\to H^0(B, \mathcal{O}_B[B])\to H^1(\bar{S}, \mathcal{O})=0$ is exact. From this we infer $h^0(\bar{S}, B)=3$, $Bs|B|=\varnothing$ and $|B|$ defines a birational morphism $p:\bar{S}\to\boldsymbol{P}^2$. Moreover, for each i, $L_i=p(T_{i2})$ is a line on $\boldsymbol{P}^2$.

(8.35) Assume that L_1, L_2 and L_3 contain a common point x on $\boldsymbol{P}^2$. Clearly p is not étale over x. So p is factored as $\bar{S}\to\tilde{P}\to\boldsymbol{P}^2$, where $\tilde{P}$ is the blowing-up of $\boldsymbol{P}^2$ at x. Let E be the exceptional divisor on $\tilde{P}$ over x, and let C be the proper transform of E on $\bar{S}$. Then $C\cong\boldsymbol{P}^1$ and $C^2<0$. By a similar argument as in (8.21), we infer that $C\subset D$ unless C is exceptional. If $C^2=-1$, then $\bar{S}\to\tilde{P}$ is étale over E, which implies $CT_{i2}=1$ for each i. Then $-KC\geqq C(\sum_i T_{i2})(2/3)=2$, which is absurd.

Thus we conclude $C\subset D$. So $C=T_{i1}$ for some i, and $C^2=-2$. We may assume $i=1$ without loss of generality. Since $C\cap T_{22}=C\cap T_{32}=\varnothing$, the morphism $\bar{S}\to\tilde{P}$ is not étale at the two points $E\cap\tilde{L}_2$ and $E\cap\tilde{L}_3$, where $\tilde{L}_i$ is the proper transform of L_i on $\tilde{P}$. Therefore $C^2\leqq E^2-2=-3$.

This contradiction proves $L_1\cap L_2\cap L_3=\varnothing$.

(8.36) Set $x_1=L_2\cap L_3$, $x_2=L_3\cap L_1$ and $x_3=L_1\cap L_2$. p is not étale over these points, and is lifted to a morphism $p':\bar{S}\to P'$, where P' is the blowing-up of $\boldsymbol{P}^2$ at x_1, x_2 and x_3. Let E_i be the exceptional curve on P' over x_i, let L_i' be the proper transform of L_i on P' and let C_i be the proper transform of E_i on $\bar{S}$. Similarly as in (8.35), we infer that C_i cannot be exceptional on $\bar{S}$, and must be a component of D. This implies $C_i^2=-2$ and there exists exactly one point y_i on E_i such that p' is not étale over y_i.

(8.37) Since $b_2(\bar{S})=7=b_2(P')+3$, $\bar{S}$ is the blowing-up of P' at y_1, y_2 and y_3. Since $(L_i')^2=-1=T_{i2}^2+1$, each L_i' contains exactly one point among y_j's. So there are two possibilities: (1) $y_1=E_1\cap L_2'$, $y_2=E_2\cap L_3'$ and $y_3=E_3\cap L_1'$, or (2) $y_1=E_1\cap L_3'$, $y_2=E_2\cap L_1'$ and $y_3=E_3\cap L_2'$. Both are the same except the difference between the ways of numbering. We have $C_1=T_{31}$, $C_2=T_{11}$ and $C_3=T_{21}$ in case (1), or $C_1=T_{21}$, $C_2=T_{31}$ and $C_3=T_{11}$ in case (2). Note that $p(B)$ is a line on $\boldsymbol{P}^2$ not passing the points x_1, x_2 and x_3. Thus we describe the structure $(\bar{S}, D)$ completely.

(8.38) To calculate $\mathrm{NS}(S)\cong\mathrm{Pic}(S)$, let z_i be the class in $\mathrm{NS}(S)$ of the exceptional curves on $\bar{S}$ lying over y_i. Clearly $\mathrm{NS}(S)$ is generated by z_1, z_2 and z_3. $E_i=z_i$ in $\mathrm{NS}(S)$ since $C_i\subset D$. To proceed further, we assume that (1) is the case. Then, in $\mathrm{NS}(S)$, $0=T_{12}=B-E_2-E_3-z_3$, so $z_2+2z_3=0$. Similarly $z_3+2z_1=0=z_1+2z_2$. These give $z_3=-2z_1$, $z_2=4z_1$ and $9z_1=0$. Hence $\mathrm{NS}(S)\cong\boldsymbol{Z}/9\boldsymbol{Z}$. This implies $H_1(S;\boldsymbol{Z})\cong\boldsymbol{Z}/9\boldsymbol{Z}$ by the universal coefficient theorem, since $H^2(S;\boldsymbol{Z})\cong\mathrm{NS}(S)$.

(8.39) We will prove that $\pi_1(S)$ is abelian. To show this, let $D''=D-B$ and $S''=\bar{S}-D''$. By a similar argument as in (8.38), we can show that $\mathrm{NS}(S'')\cong\boldsymbol{Z}$. Hence $H_1(S'';\boldsymbol{Z})$ is torsion free. So $b_1(S'')\leqq b_1(S)=0$ implies $H_1(S'';\boldsymbol{Z})=0$. On the other hand, S'' contains an open dense subset isomorphic to $\boldsymbol{P}^2-(L_1\cup L_2\cup L_3)\cong\boldsymbol{A}_*^2$. Therefore $\pi_1(S'')$ is abelian, so we have $\pi_1(S'')\cong H_1(S'';\boldsymbol{Z})=\{1\}$. By (4.18), we infer that the vanishing subgroup of B coincides with $\pi_1(S)$. Hence it is enough to show that a vanishing loop of B is in the center of $\pi_1(S)$.

(8.40) In order to make use of the techniques in [Ra], we fix our notation. For each prime component D_i of D, take a sufficiently small tubular neighborhood U_i and let $U=\cup_i U_i$. Let $W=\partial U$ be the boundary of U and set $W_i=W\cap\bar{U}_i$. Then, as we saw in (1.21), W has the homotopy type $\infty(S)$. W_i will be called a *wrap* of D_i.

(8.41) Using Morse theory as in the proof of the Lefschetz theorem (cf. [M1; pp. 42]), we can prove that $\pi_1(U\cap S)\to\pi_1(S)$ is surjective, since S is affine. Hence $\pi_1(W)\to\pi_1(S)$ is surjective.

(8.42) Let $n_1, \cdots, n_q$ be the nodes of D lying on D_i. Then W_i has the structure of an S^1-bundle over $D_i - \bigcup_{j=1}^q \Delta_j$, where Δ_j is a small disc on D_i with center n_j. Moreover, each fiber ($\cong S^1$) equipped with the orientation coming from the complex structure represents a vanishing loop of D_i in $\pi_1(S)$.

Thus, W_i is a real three dimensional manifold with boundary $\bigcup_{j=1}^q W_{ij}$, where W_{ij} is an S^1-bundle over $\partial\Delta_j \cong S^1$. Let D_k be the other component of D passing n_j. Then W_{kj} is an S^1-bundle over $\partial\Delta'_j$, where Δ'_j is a small disc on D_k with center n_j. W_{ij} and W_{kj} are identified in W in such a way that the two projections $W_{ij} \to \partial\Delta_j$ and $W_{kj} \to \partial\Delta'_j$ define an isomorphism $W_{ij} = W_{kj} \cong \partial\Delta_j \times \partial\Delta'_j \cong S^1 \times S^1$.

(8.43) Suppose that $D_i \cong \boldsymbol{P}^1$. Then, as was explained in [Ra; p. 76], $\pi_1(W_i)$ is described in the following way.

Let α_j (resp. β_j) $\in \pi_1(W_{ij})$ be the element defined by a fiber of the projection $W_{ij} \to \partial\Delta_j$ (resp. $\partial\Delta'_j$). Be careful to define mappings $\iota_j : \pi_1(W_{ij}) \to \pi_1(W_i)$, because there is no canonical way to define them simultaneously. Following the recipe in [Ra], we do this by choosing a base point o on W_i, and a base point o_j on W_{ij} together with a path in W_i connecting o and o_j for each $j=1, \cdots, q$, so that $\pi_1(W_i)$ is generated by the images of α_j's and β_j's (which will be denoted by α_j or β_j by abuse of notation) under the relation $\alpha_1 = \alpha_2 = \cdots = \alpha_q$ and $\beta_1\beta_2 \cdots \beta_q = (\alpha_j)^a$, where $a = -D_i^2$.

Theoretically, we can calculate $\pi_1(W)$ by combining the above descriptions of each $\pi_1(W_i)$ with the help of van Kampfen's theorem.

(8.44) Let $T = T_1 + \cdots + T_r$ be a rational twig of D whose dual graph $[a_1, \cdots, a_r]$ is admissible. Let W be as in (8.40) and let W_i be the wrap of T_i. Let B be the component with $T_r B = 1$ and not in T, and let n be the node $T_r \cap B$. Set $W(T) = \bigcup_i W_i$. Then the boundary $\partial W(T)$ is a trivial S^1-bundle over $\partial\Delta \cong S^1$, where Δ is a small disc on T_r with center n. Let $\alpha(T)$ and $\beta(T)$ be the generator system of $\pi_1(\partial W(T)) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}$ as in (8.43), so that $\alpha(T)$ (resp. $\beta(T)$) represents a vanishing loop of T_r (resp. B) in $\pi_1(S)$. Then we have:

(8.45) LEMMA. *There is an isomorphism $\pi_1(W(T)) \cong \boldsymbol{Z}$ such that $\beta(T)$ and $\alpha(T)$ are mapped to $d[a_1, \cdots, a_r]$ and $d[a_1, \cdots, a_{r-1}]$ respectively.*

PROOF. We use the induction on r. If $r=1$, by the observation (8.43) we infer that $\pi_1(W(T))$ is generated by $\alpha(T)$ and $\beta(T)$ under the relation $\beta(T) = \alpha(T)^{a_1}$. So our assertion is true ($d[\varnothing]=1$ by convention). If $r \geqq 2$, let $T' = T_1 + \cdots + T_{r-1}$ and take $\alpha(T')$ and $\beta(T')$ as before. By the induction hypothesis $\pi_1(W(T')) \cong \boldsymbol{Z}$ and we have a generator γ of it such that $\alpha(T') = \gamma^{d''}$ and $\beta(T') = \gamma^{d'}$, where $d'' = d[a_1, \cdots, a_{r-2}]$ and $d' = d[a_1, \cdots, a_{r-1}]$. By (8.43), $\pi_1(W_r)$

is the group generated by $\alpha(T')$, $\beta(T')$, $\alpha(T)$ and $\beta(T)$ under the relation $\alpha(T)=\beta(T')$ and $\beta(T)\alpha(T')=\beta(T')^{a_r}$. So, in $\pi_1(W(T))$, we have $\alpha(T)=\gamma^{d'}$ and $\beta(T)=\gamma^{d}$, where $d=a_r d'-d''=d[a_1, \cdots, a_r]$ by (3.6; 1). This proves the lemma.

(8.46) PROPOSITION. *Let D be a rational tree on a smooth complete surface $\bar{S}$ consisting of several admissible rational twigs $T_1, T_2, \cdots, T_q$ and their common branching component B. Let W be as in* (8.40). *Then the vanishing loop of B represented by a fiber of the wrap of B lies in the center of $\pi_1(W)$.*

PROOF. By (8.45) and by van Kampfen's theorem, we infer that $\pi_1(W_B)\to\pi_1(W)$ is surjective, where W_B is the wrap of B. Moreover, there is a generator γ_j of $\pi_1(W(T_j))\cong \boldsymbol{Z}$ such that $\beta(T_j)=\gamma_j^{d(T_j)}$. Hence $\pi_1(W)$ is generated by $\gamma_1, \cdots, \gamma_q$. The vanishing loop of B is $\beta(T_1)=\cdots=\beta(T_q)$, so it commutes with any γ_j, and hence lies in the center of $\pi_1(W)$. Q.E.D.

(8.47) Now, combining (8.41) and (8.46), we prove that a vanishing loop of B is in the center of $\pi_1(S)$. Thus we obtain $\pi_1(S)\cong H_1(S;\boldsymbol{Z})\cong \boldsymbol{Z}/9\boldsymbol{Z}$ for a surface of type $Y\{3, 3, 3\}$.

(8.48) Type $Y\{2, 4, 4\}$. Similar arguments as in the case $Y\{3, 3, 3\}$ will apply in this case too.

As in (8.31), we must have $(e(T_1), e(T_2), e(T_3))=(1/2, 1/4, 1/4)$ or $(1/2, 3/4, 3/4)$ in order that $(K+D)^2=\Sigma_i-e(T_i)$ is an integer. In the former case, each T_i is a tip of D and $T_1^2=-2$, $T_2^2=-4=T_3^2$. By a similar calculation as in (8.32), we obtain $B^2=7$. Since $K=-B-(2T_1+3T_2+3T_3)/4$ in this case, by similar arguments as in (8.21) and (8.32) we prove the following claim: Any curve C on $\bar{S}$ with $C^2<0$ is a component of D or an exceptional curve with $CT_1=2$.

$\bar{S}$ is a successive blowing-up of a $\boldsymbol{P}^1$-bundle Σ_k over $\boldsymbol{P}^1$. By a similar argument as in (8.24) using the above claim, we infer that both T_2 and T_3 are mapped isomorphically onto their images on Σ_k. This is impossible because Σ_k contains at most one curve with negative self-intersection number.

Thus we conclude: $(e(T_1), e(T_2), e(T_3))=(1/2, 3/4, 3/4)$.

(8.49) For $i=2$ and 3, we have $T_i=T_{i1}+T_{i2}+T_{i3}$ with $T_{ij}^2=-2$. T_{i1} is the tip and $N_i=(3T_{i1}+2T_{i2}+T_{i3})/4$ is the bark of T_i. Hence we have $-K=B+T_1/2+\sum_{i=2,3;\, j=1,2,3} jT_{ij}/4$. Similarly as in (8.33), we obtain $B^2=0$.

(8.50) LEMMA. *Let C be an irreducible reduced curve on $\bar{S}$ such that $C^2<0$. Then C is an exceptional curve unless $C\subset D$.*

Proof is similar as in (8.31).

(8.51) LEMMA. *Let F be a singular fiber of a ruling of $\bar{S}$. Then any S-component of F is an exceptional curve. Moreover, the sum of the multiplicities of the S-components of F is not greater than two.*

PROOF. The first assertion follows straightforward from (8.50). The second assertion follows from $KF=-2$.

(8.52) Since $B^2=0$, there is a ruling $f: \bar{S}\to \boldsymbol{P}^1$ such that B is a fiber of f. T_1, T_{23} and T_{33} are sections of f. Let F_2 and F_3 be the fibers containing $T_{21}+T_{22}$ and $T_{31}+T_{32}$ respectively. We first claim that $F_2\neq F_3$.

Assume that $F_2=F_3$. Let X be the component of this fiber F such that $XT_1>0$. Since $FT_1=1$, the multiplicity of X is one. X is an S-component and hence exceptional. By (7.3), F contains another exceptional curve Y. In view of (8.51), we infer that Y is of multiplicity one and there is no other S-component of F. (7.3) says that X and Y are tips of F. Therefore $F-X-Y=T_{21}+T_{22}+T_{31}+T_{32}$ should be connected. But this is not the case. Thus we prove $F_2\neq F_3$.

(8.53) Let E_i be the component of F_i meeting T_1. Since $F_iT_1=1$, E_i is of multiplicity one in F_i and is exceptional by (8.51). By (7.3) F_i has another exceptional component E_i'. This is of multiplicity one by (8.51) and both E_i and E_i' are tips of F_i. Moreover, by (8.51), we infer that there is no other component of F_i. Thus we obtain $F_i=E_i+T_{i2}+T_{i1}+E_i'$ for $i=2, 3$. Using the formula for K in (8.49), we see $E_iT_{i2}=T_{i1}E_i'=T_{23}E_3'=T_{33}E_2'=1$.

In view of $b_2(\bar{S})=b_2(D)=8$, we infer that any other fiber of f is isomorphic to $\boldsymbol{P}^1$. Moreover, any exceptional curve E on $\bar{S}$ is one of E_2, E_3, E_2', E_3'. Indeed, by NC-minimality we have $EB=0$. This implies that E is contained in some fiber of f. So E must be a component of either F_2 or F_3.

(8.54) $T_1\cup E_3\cup T_{32}$ looks like a twig [2, 1, 2]. Hence $G_3=T_1+2E_3+T_{32}$ is a fiber of a ruling $g: \bar{S}\to \boldsymbol{P}^1$. Then B, T_{31} and T_{33} are sections of g. There exists another singular fiber $G_2=E_2'+T_{21}+T_{22}+T_{23}+E_3'$ of g. Counting $b_2(\bar{S})$, we infer that there is no other singular fiber of g.

We blow down first E_3, E_2' and E_3' and then further the images of T_1, T_{21} and T_{23}. Then we get a $\boldsymbol{P}^1$-bundle $\bar{S}'$ over $\boldsymbol{P}^1$, with $g': \bar{S}'\to \boldsymbol{P}^1$ induced by g. The images T_{31}' and T_{33}' of T_{31} and T_{33} on $\bar{S}'$ are sections of g', disjoint with each other, and with self-intersection number zero. Therefore $\bar{S}'\cong \boldsymbol{P}^1\times \boldsymbol{P}^1$, the second projection being given by $|T_{31}'|=|T_{33}'|$, which will be denoted by $|Z|$, with $Z\in \mathrm{Pic}(\bar{S}')$.

Thus, the observations (8.53) and (8.54) describe the structure $(\bar{S}, D)$ fairly explicitly.

(8.55) Clearly $\mathrm{Pic}(S)\cong\mathrm{NS}(S)$ is generated by E_3, E'_2 and E'_3 because of (8.54). Let Z be as above and let G be the class of a fiber of g or g'. Then we have $T_{31}=Z-(T_{23}+E'_3)-E'_3$, $T_{33}=Z-(T_{21}+E'_2)-E'_2$, $B=(Z+G)-(T_1+E_3)-(T_{23}+E'_3)$ and $T_1+2E_3+T_{32}=G=E'_2+T_{21}+T_{22}+T_{23}+E'_3$. From this we obtain $E'_2=3E_3$, $E'_3=-E_3$ and $8E_3=0$ in $\mathrm{NS}(S)$. Thus we prove $\mathrm{NS}(S)\cong\boldsymbol{Z}/8\boldsymbol{Z}$.

This implies $H_1(S;\boldsymbol{Z})\cong\boldsymbol{Z}/8\boldsymbol{Z}$ by the universal coefficient theorem.

(8.56) To calculate $\pi_1(S)$, let $D''=D-B$ and $S''=\bar{S}-D''$. By (8.54), we infer that S'' contains a Zariski open subset which is isomorphic to $\bar{S}'-(T'_{31}\cup T'_{33}\cup G'_2\cup G'_3)\cong A^2_*$. Therefore $\pi_1(S'')$ is abelian. So $\pi_1(S'')\cong H_1(S'';\boldsymbol{Z})$, which is a homomorphic image of $H_1(S;\boldsymbol{Z})$. Hence $\pi_1(S'')$ is cyclic.

On the other hand, combining (4.18), (8.46) and a similar argument as in (8.41), we infer that $\mathrm{Ker}(\pi_1(S)\to\pi_1(S''))$ is a cyclic group lying in the center of $\pi_1(S)$. Now we see easily that $\pi_1(S)$ is abelian.

Thus we obtain $\pi_1(S)\cong H_1(S;\boldsymbol{Z})\cong\boldsymbol{Z}/8\boldsymbol{Z}$.

(8.57) The case of type $Y\{2, 3, 6\}$.

We obtain $(e(T_1), e(T_2), e(T_3))=(1/2, 2/3, 5/6)$ by a similar method as in (8.48). Hence $T_2=T_{21}+T_{22}$, $T_3=T_{31}+T_{32}+T_{33}+T_{34}+T_{35}$, $K=-B-(1/2)T_1-(T_{21}+2T_{22})/3-(T_{31}+2T_{32}+3T_{33}+4T_{34}+5T_{35})/6$, $b_2(\bar{S})=b_2(D)=9$ and $B^2=-1$.

(8.58) The Lemmas (8.50) and (8.51) are valid in this context too.

(8.59) $T_{22}\cup B\cup T_{35}$ looks like a twig $[2, 1, 2]$. So $F=T_{22}+2B+T_{35}$ is a fiber of a ruling $f:\bar{S}\to\boldsymbol{P}^1$. T_{21} and T_{34} are sections of f. T_1 is a double section, that means, $T_1F_x=2$ for any fiber F_x of f.

Let F_3 be the fiber containing $T_{31}+T_{32}+T_{33}$. Let B^* be the component of F_3 meeting T_{21}. Since $F_3T_{21}=1$, B^* is of multiplicity one in F_3. By (7.3) we infer that there is another exceptional component E of F_3. Thanks to (8.58), we see that E is of multiplicity one and F_3 has no other component. Thus $F_3=B^*+T_{31}+T_{32}+T_{33}+E$ and both B^* and E are tips of F_3 by (7.3). So $B^*E=0$. If $T_1E=2$, then $KE=-1$ would imply $T_{3j}E=0$ by the formula of K in (8.57). This is impossible because F_3 is connected. So $T_1E<2$, and similarly we have $B^*T_1<2$. Hence $T_1F_3=2$ implies $T_1E=T_1B^*=1$. Now, from $KE=KB^*=-1$ and from the formula of K in (8.57), we obtain $B^*T_{31}=ET_{33}=1$. Thus the structure of F_3 is described completely.

Since $b_2(\bar{S})=9$, there is still another singular fiber F_0 of f. F_0 has no D-component, so $F_0=E_1+E_2$ for some exceptional curves E_1 and E_2. By symmetry we may assume $E_2T_{34}=1$. Then $E_2T_1=0$ by $E_2K=-2$ and the formula of K. So $F_0T_1=2$ implies $E_1T_1=2$. Hence $E_1T_{21}=0$ by a similar reason as above. So $E_2T_{21}=1$ by $F_0T_{21}=1$.

From $b_2(\bar{S})=9$, we infer that f has no other singular fiber.

(8.60) Studing the ruling f^* given by $F^*=T_{21}+2B^*+T_{31}$, we find a fiber F_0^* of the form $E_1+E_2^*$ for an exceptional curve E_2^* with $E_2^*T_{22}=E_2^*T_{32}=1$. The situation is symmetric to (8.59).

(8.61) $T_1\cup E\cup T_{33}$ looks like a tree [2, 1, 2]. So $G=T_1+2E+T_{33}$ is a fiber of a ruling $g: \bar{S}\to \boldsymbol{P}^1$. T_{32}, T_{34}, B (and also B^*) are sections of g. Let G_2 be the fiber containing $T_{21}\cup T_{22}$. In view of (8.60) we see $G_2=E_2+T_{21}+T_{22}+E_2^*$. Let G_3 be the fiber containing T_{35}. It must contain an S-component E_3 meeting T_{35}. E_3 is exceptional by (8.58). From $E_3K=-1$ and from the formula of K in (8.57) we infer $E_3T_{31}=1$. Thus we see $G_3=T_{31}+2E_3+T_{35}$.

Since $b_2(\bar{S})=9$, any other fiber of g is smooth.

REMARK. Although we don't use the fact, $\bar{S}$ contains no other exceptional curve except B, B^*, E, E_1, E_2, E_2^* and E_3.

(8.62) Using (8.61), we calculate $\mathrm{Pic}(S)\cong \mathrm{NS}(S)$ in the following way.

We blow down E, E_3, E_2 and E_2^* first, and then the images of T_1, T_{21} and T_{31}. Then we get a $\boldsymbol{P}^1$-bundle $g': \bar{S}'\to \boldsymbol{P}^1$, where g' is induced by g. The images of B, T_{32} and T_{34} on $\bar{S}'$ is a section of g', disjoint with each other and with self-intersection number zero. Therefore they define the same line bundle Z on $\bar{S}'$, and $\bar{S}'\cong \boldsymbol{P}^1\times\boldsymbol{P}^1$ with the second projection given by $|Z|$.

Clearly $\mathrm{NS}(S)$ is generated by E, E_2, E_2^* and E_3. In $\mathrm{NS}(\bar{S})$ we have $B=Z-(T_1+E)$, $T_{32}=Z-(T_{31}+E_3)-E_2^*$, $T_{34}=Z-(T_{21}+E_2)-E_2$ and $G=T_1+2E+T_{33}=E_2+T_{21}+T_{22}+E_2^*=T_{31}+2E_3+T_{35}$. From them we obtain $E=2E_2$, $E_2^*=3E_2$, $E_3=-E_2$ and $6E_2=0$ in $\mathrm{NS}(S)$. Thus we see $\mathrm{NS}(S)\cong \boldsymbol{Z}/6\boldsymbol{Z}$.

(8.63) We will prove $\pi_1(S)\cong H_1(S;\boldsymbol{Z})\cong \boldsymbol{Z}/6\boldsymbol{Z}$. Let $D''=D-B$ and $S''=\bar{S}-D''$. Similarly as in (8.56), it suffices to show that $\pi_1(S'')$ is cyclic.

We use (8.61) and (8.62). Let T_{32}', T_{34}', G', G_2' and G_3' be the images of T_{32}, T_{34}, G, G_2 and G_3 on $\bar{S}'$. Then S'' contains a Zariski open subset $S_0''=S''-(E\cup E_2\cup E_2^*\cup E_3)\cong \bar{S}'-(T_{32}'\cup T_{34}'\cup G'\cup G_2'\cup G_3')\cong \boldsymbol{A}_*\times(\boldsymbol{P}^1-\{\textit{three points}\})$. So $\pi_1(S_0'')$ is isomorphic to the group generated by three elements t, γ_2 and γ_3 under the relation $t\gamma_j=\gamma_j t$ for $j=2, 3$. Here we can take t, γ_2 and γ_3 in such a way that γ_j is a vanishing loop of G_j' for each j and t (resp. t^{-1}) is a vanishing loop of T_{32}' (resp. T_{34}').

$\pi_1(S'')$ is a quotient of this group, and the kernel is generated by the vanishing subgroups of E, E_2, E_2^* and E_3, which are described by (7.18). Thus, in $\pi_1(S'')$, we have the relation $1=\gamma_2 t^{-2}=\gamma_2 t=\gamma_3^2 t$. From this we obtain $t=\gamma_3^{-2}$, $\gamma_2=\gamma_3^2$ and $\gamma_3^6=1$. So $\pi_1(S'')$ is generated by γ_3, as desired.

In fact, we can prove $\pi_1(S'')\cong\pi_1(S)\cong Z/6Z$.

(8.64) Summarizing the preceding arguments (8.10)$\sim$(8.63), we obtain the following list of NC-minimal completions $(\bar{S}, D)$ of S such that $\bar{\kappa}(S)=\hat{b}_2(S)=\bar{b}_3(S)=0$. Type $O(3, 1)$ and $O(5, -1)$ are included in the list, but they are not NC-minimal in the strict sense. See (8.17).

Table to (8.64).

Type of $(\bar{S}, D)$	$b_1(S)$	$b_2(S)$	$m(S)$	Pic(S)	$\pi_1(S)$	for details, see:
$(*)$	0	2	1	$Z/3Z$	$Z/3Z$	(8.11)
$O(1, 1, 1)$	2	1	1	0	$Z\oplus Z$	(8.13); $S\cong A^2_*$
$O(4, 1)$	1	1	1	0	Z	(8.13)
$O(k+4, -k)$; $k\geqq -1$	0	1	1	$Z/(k+2)Z$	$Z/(k+2)Z$	(8.16) & (8.17)
$H[-1, 0, -1]$	1	0	2	$Z/2Z$	$\langle y, t\rangle/yty^{-1}t=1$	(8.5)
$H[k, -k]$; $k\geqq 1$	0	0	2	$Z/4kZ$	$Z/4kZ$	(8.26)
$Y\{3, 3, 3\}$	0	0	3	$Z/9Z$	$Z/9Z$	(8.37)
$Y\{2, 4, 4\}$	0	0	4	$Z/8Z$	$Z/8Z$	(8.53) & (8.54)
$Y\{2, 3, 6\}$	0	0	6	$Z/6Z$	$Z/6Z$	(8.59) & (8.61)

Here, $m(S)$ is the least positive integer such that $\bar{P}_m(S)>0$.

(8.65) Now we consider the case in which $(\bar{S}, D)$ is not NC-minimal. However the blowing-downs of the types (6.21; 2), 3) and 5) are impossible by (6.22) because $\bar{b}_3(S)=\hat{b}_2(S)=0$. If a D-blowing-down (6.21; 1) is possible, we do it. The result D' is an NC-divisor, unless E meets a single component Y of D at different two points. In any case D' has no singularities other than nodes, and S itself does not change. When we do half point detachment as in (6.21; 4), then $b_1(S)$ increases because $\hat{b}_2(S)=0$.

Thus, after several blowing-downs, we reach a situation $(\bar{S}', D')$ where any blowing-down as in (6.21) is impossible. $S'=\bar{S}'-D'$ is obtained from S by detaching $b_1(S')-b_1(S)$ half-points. Since D' has no singularities other than nodes, we can apply (6.11; 1) to the effect $K+D'=(K+D')^{-}$. This is Bk*(D')

by (6.24). So (8.8) applies. Since D' is not contractible, it is either of type (O), (H), (X), (Y) or $(*)$. Hence $(\bar{S}', D')$ is an NC-minimal NC-completion unless D' is an irreducible rational curve with one node. In this case S' is isomorphic to a surface of type $O(5, -1)$ (see (8.17)). So $b_1(S')=0$ and hence $S\cong S'$.

Of course, here we are interested in the case where $S'\neq S$. So $b_1(S') > b_1(S)\geqq 0$. Hence $(\bar{S}', D')$ is one of the types $O(1, 1, 1)$, $O(4, 1)$ or $H[-1, 0, -1]$. S is obtained from S' by attaching one or two half points. Let us examine these three cases.

(8.66) The case of type $O(1, 1, 1)$. Obviously we have $b_2(S)=b_2(S')=1$ and $\bar{P}_m(S)=\bar{P}_m(S')$. $\pi_1(S')\rightarrow\pi_1(S)$ is surjective and hence $\pi_1(S)\cong H_1(S; \boldsymbol{Z})$. This is generated by at most two elements. However, we cannot say very much about the structure of $(\bar{S}, D)$ in detail, because there are too many ways to attach half-point(s) to S'. Here we just present a couple of examples.

(8.67) Let $\bar{S}'=\boldsymbol{P}^2$ and $D'=L_1+L_2+L_3$, where L_i are lines on $\boldsymbol{P}^2$ having no common point. $S'=\bar{S}'-D'\cong \boldsymbol{A}^2_*$. Let x_i be the vanishing loop of L_i in $\pi_1(S')\cong \boldsymbol{Z}\oplus\boldsymbol{Z}$. Then $\pi_1(S')$ is generated by them under the relation $x_1+x_2+x_3=0$.

1) Take a point q on $L_1-(L_2\cup L_3)$. Blow up $\bar{S}'$ m times with each center being the point on the proper transform of L_1 and over q, where m is a positive number. Take a general point y on the final exceptional divisor and let $\bar{S}$ be the blowing-up at y. Let $D=p^{-1}(D')-E$, where $p: \bar{S}\rightarrow\bar{S}'$ is the natural morphism and E is the exceptional divisor over y. Then $S=\bar{S}-D$ is a half-point attachment of $\bar{S}-p^{-1}(D)\cong S'$. E is of multiplicity m in p^*D. Hence, by (7.18), $\pi_1(S)\cong\{x_1, x_2, x_3/x_1+x_2+x_3=mx_1=0\}\cong \boldsymbol{Z}\oplus(\boldsymbol{Z}/m\boldsymbol{Z})$. So $\mathrm{Pic}(S)\cong \mathrm{NS}(S)\cong \hat{H}^2(S; \boldsymbol{Z})\cong \boldsymbol{Z}/m\boldsymbol{Z}$, because $\tilde{H}^2(S; \boldsymbol{Z})\cong\boldsymbol{Z}$.

This type of half point will be said to be of multiplicity m over $q\in L_1$.

2) Take a point q on $L_1-(L_2\cup L_3)$ and let n be a positive integer. Blow up $\bar{S}'$ n-times over q with each center being a general point on the exceptional curve at each stage. Let $p: \bar{S}\rightarrow\bar{S}'$ be the natural morphism and let $D=p^{-1}(D')-E$, where E is the exceptional divisor of the final blowing-up. Then $S=\bar{S}-D$ is a half-point attachment of S'. It is easy to check $\hat{b}_2(S)=b_3(S)=0$. E is of multiplicity one in p^*D. Hence, using (7.18), we see $\pi_1(S)\cong H_1(S; \boldsymbol{Z})\cong\boldsymbol{Z}$ and $\mathrm{Pic}(S)\cong\mathrm{NS}(S)\cong 0$.

This type of half point will be called a *simple half point* with moment n over q.

3) Let q_1 be the point $L_2\cap L_3$ and let m_2 and m_3 be positive integers coprime with each other. As we saw in (4.7), we can obtain a twig $[\Gamma(m_2/(m_2+m_3)), 1, {}^t\Gamma(m_3/(m_2+m_3))]$ from $[1, 1]$ by successive subdivisional blowing-ups. Doing blow-ups over q_1 in the same way, we obtain $p: \bar{S}''\rightarrow\bar{S}'$ such that the multiplicities of the exceptional curve E'' on $\bar{S}''$ of the final

blowing-up in p^*L_2 and p^*L_3 are m_2 and m_3 respectively. Let S be $\bar{S}''-p^{-1}(D')$ plus a simple half-point with moment one over a general point on E''. Then we have $\pi_1(S)\cong\{x_1, x_2, x_3/x_1+x_2+x_3=m_2x_2+m_3x_3=0\}\cong\boldsymbol{Z}$. If we attach a half-point of multiplicity m over a general point on E'', then we would have $\pi_1(S)\cong\{x_1, x_2, x_3/x_1+x_2+x_3=m(m_2x_2+m_3x_3)=0\}\cong\boldsymbol{Z}\oplus(\boldsymbol{Z}/m\boldsymbol{Z})$.

A half-point of the former type will be said to be of bi-multiplicity (m_2, m_3) over q_1.

4) In general, let $p:\bar{S}\rightarrow\bar{S}'$ be any successive blowing-ups over a point q on D'. Suppose that the final blowing-up is sprouting, or equivalently, the final exceptional curve E is of type (6.21; 4), that means, $ED=1$ for $D=p^{-1}(D')-E$. Then D is connected and hence $\bar{b}_3(S)=0$ for $S=\bar{S}-D$. We can easily verify that $b_1(S)=1$ and $\hat{b}_2(S)=0$. Thus we get many examples of such surfaces.

5) There are also lots of ways to attach two half-points to S'. But we must be a little careful, because sometimes we get a surface with $b_1=\hat{b}_2=1$, not with the desired property $b_1=\hat{b}_2=0$. For example, if the both half-points lie over general points on L_1, then we have $b_1(S)=1$. But if they lie over general points on different lines, say L_2 and L_3, then we have $b_1=\hat{b}_2=0$ as desired.

6) For any abelian group generated by two elements, one can find a surface S of type (8.66) whose fundamental group is the given group.

(8.68) The case of type $O(4, 1)$. Roughly speaking, what is valid in case of type $O(1, 1, 1)$ is valid in case of type $O(4, 1)$ too. Actually, a surface of type $O(4, 1)$ itself can be viewed as a half-point attachment of $\boldsymbol{A}_*^2$. See (8.15.1).

(8.69) The case of type $H[-1, 0, -1]$. We have $b_2(S)=b_2(S')=0$ in this case. As described in (8.5), $\bar{S}'$ admits an $\boldsymbol{A}_*^1$-ruling, which is a gyoza. Hence so does $\bar{S}$. Therefore there is a non-trivial surjective homomorphism $\mathrm{Pic}(S)\cong\mathrm{NS}(S)\rightarrow\boldsymbol{Z}/2\boldsymbol{Z}$. $\pi_1(S)$ may and may not be abelian, infinite, according to the nature of the attached half-point.

Let $D'=T_{11}+B_1+T_{12}+D'_h+B_2+T_{21}+T_{22}$, where D'_h is the horizontal component with respect to the $\boldsymbol{A}_*^1$-ruling f and $F_i=T_{i1}+2B_i+T_{i2}$ $(i=1, 2)$ are the singular fibers of f. In order that $b_1(S)=\hat{b}_2(S)=0$, the attached half-point must lie on F_1 or F_2. By symmetry, we may assume that it lies over a point q on F_1.

We can take vanishing loops x, y, z and t of T_{11}, T_{12}, B_1 and D'_h respectively such that $\pi_1(S')$ is the group generated by them under the relation $x=yt$, $y=tx$, $z=x^2=y^2$. Note that the subgroup generated by z and t is of index two and is isomorphic to $\boldsymbol{Z}\oplus\boldsymbol{Z}$. z is in the center of $\pi_1(S')$.

In $\pi_1(S)$, we have an additional relation described in (7.18). We examine several cases.

(1) The case in which $y^m=1$, m being even. In this case $\pi_1(S)$ is non-abelian and infinite. Indeed, even if $y^2=1$, $\pi_1(S)$ is the group generated by x and y under the relation $x^2=y^2=1$, the "infinite dihedral group".

(2) $y^m=1$, m being odd. In this case y is a multiple of z, and hence in the center of $\pi_1(S)$. So $\pi_1(S)$ is abelian, because it is generated by t and y. Then we have $t^2=1$ also. Thus $\pi_1(S)\cong(\boldsymbol{Z}/m\boldsymbol{Z})\oplus(\boldsymbol{Z}/2\boldsymbol{Z})\cong\boldsymbol{Z}/2m\boldsymbol{Z}$.

(3) $t^m z^n=1$, $m>0$. In this case $\pi_1(S)$ is finite. Indeed, let H be the subgroup generated by t and z. Then H is abelian and of index at most two. Using additive notation, we have $mt+nz=0$ in H. Considering the inner automorphism induced by y, we get $-mt+nz=0$. So H is finite by $m>0$. Thus $\pi_1(S)$ is finite.

$\pi_1(S)$ is abelian if and only if $t=-t$ in H. This is equivalent to $m=1$. If so, $\pi_1(S)\cong\boldsymbol{Z}/4n\boldsymbol{Z}$.

(3) corresponds to the case in which $q\in B_1\cap D'_h$. Otherwise, we get a relation of type (1) or (2) unless $q\in T_{11}$. In case $q\in T_{11}$, we get a relation of type $x^m=1$, and this case can be treated similarly as in (1) and (2).

(8.70) Summarizing we obtain the following

THEOREM. *Let S be an algebraic surface with $\bar{\kappa}(S)=\hat{b}_2(S)=\tilde{b}_3(S)=0$. Then*

1) *if $\bar{P}_2(S)=0$, then S is one of the types $Y\{3, 3, 3\}$, $Y\{2, 4, 4\}$, $Y\{2, 3, 6\}$.*

2) *$\pi_1(S)$ is abelian unless S is of type $H[-1, 0, -1]$ or half-point attachment of it, where S admits an A^1_*-ruling which is a gyoza.*

3) *$\mathrm{Pic}(S)\cong\mathrm{NS}(S)\neq 0$ unless S is of type (O), or half-point(s) attachment of it. In particular, $b_2(S)=1$ in this case. So $H^2(S;\boldsymbol{Z})\neq 0$ in any case.*

4) *$\bar{P}_1(S)=1$ if and only if $b_2(S)>0$.*

5) *$b_2(S)\leqq 2$. The equality holds if and only if S is of type $(*)$.*

6) *$\pi_1(S)$ is generated by suitable two elements.*

7) *$H_1(S;\boldsymbol{Z})$ is cyclic unless S is (a half-point attachment of) a surface of type $O(1, 1, 1)$ or $H[-1, 0, -1]$.*

8) *$b_1(S)=0$ unless S is either of type $H[-1, 0, -1]$ or (a half-point attachment of) A^2_*.*

§ 9. Applications and comments.

(9.1) THEOREM. *Let S be a quasi-complete surface such that $b_1(S)=\tilde{b}_3(S)=0$ and $H^2(S;\boldsymbol{Z})=0$. Then $\bar{\kappa}(S)=2$ unless $S\cong\boldsymbol{A}^2$.*

PROOF. S is algebraic by (2.4; 2). $\bar{\kappa}(S)\neq 1$ by (7.15). $\bar{\kappa}(S)\neq 0$ by (8.70; 3). If $\bar{\kappa}(S)=-\infty$, then $S\cong\boldsymbol{A}^2$ by (5.7).

(9.2) COROLLARY. *C^2 admits only one quasi-complete structure.*

For a proof, use (1.9) to infer $\bar{\kappa}\neq 2$. This fact itself was proved by Ramanujam by a different method. See [Ra].

(9.3) THEOREM. *Let S be a quasi-complete surface such that $b_2(S)=\breve{b}_3(S)=0$, and $\pi_1(S)\cong \boldsymbol{Z}$. Then one of the following conditions is satisfied.*

1) $\bar{\kappa}(S)=2$.

2) $S\cong \boldsymbol{A}^1\times \boldsymbol{A}^1_*$.

3) *The one point compactification of S admits a structure of a normal analytic space. In particular, there is no non-constant holomorphic functions on S.*

PROOF. We have $b_1(S)=\hat{b}_1(S)+\breve{b}_1(S)=1$. When $\hat{b}_1(S)=b_1(\bar{S})=0$, S is algebraic by (2.4; 2). Actually S is affine. $\bar{\kappa}(S)\neq 1$ by (7.15). $\bar{\kappa}(S)\neq 0$ by (8.70; 3). If $\bar{\kappa}(S)=-\infty$, then (5.7) proves the condition 2). Thus 1) or 2) is the case.

So consider the case $\hat{b}_1(S)=1=b_1(\bar{S})$, where $(\bar{S}, D)$ is an NC-completion of S. So, following [Ko 2], we see $h^{1,0}(\bar{S})=0$ and $h^{0,1}(\bar{S})=1$. $b_2(S)=0$ implies that $H^2(\bar{S}; \boldsymbol{Q})$ is generated by the Chern classes of components of D. Hence $p_g(\bar{S})=0$. Therefore $\chi(\bar{S}, \mathcal{O})=0$ and $c_1^2=-c_2=-b_2(\bar{S})$ by Noether's formula. So $b^+-b^-=(c_1^2-2c_2)/3=-b_2(\bar{S})$ by index theorem, which implies that the cup product pairing on $H^2(\bar{S}; \boldsymbol{Q})$ is negative definite. Hence D is connected and contractible in the sense (3.3). By Grauert's criterion [G], D can be actually contracted to a normal point. Thus 3) is valid.

REMARK. We do not know whether a surface of the above type 3) really exists or not. It is not difficult to show that $\bar{S}$ must be of algebraic dimension zero. Furthermore, $\bar{S}$ cannot be a Hopf surface. Indeed, since $b_1(D)\leqq b_1(\bar{S})+\breve{b}_2(S)=1$, any component of D is a rational curve. So we have a non-trivial morphism $\boldsymbol{P}^1\to\bar{S}$, which is lifted to a non-constant morphism $\boldsymbol{P}^1\to U$ to the universal covering U of $\bar{S}$. Thus U contains a compact analytic curve, hence cannot be a domain in any Stein manifold.

(9.4) COROLLARY. *$\boldsymbol{C}\times\boldsymbol{C}^*$ admits only one quasi-complete structure.*

This was proved by Suzuki [Sz] by a similar method as in [Ra].

(9.5) Actually, the above arguments work in more general context too. First we have the following

THEOREM. *Let S be a quasi-complete surface with $\hat{b}_2(S)=0$. Then S is algebraic unless S is compactified to a normal analytic space by adding finite points. Thus, in the non-algebraic case, there is no non-constant holomorphic* *functions on S.*

Outline of proof. Let $(\bar{S}, D)$ be an NC-completion of S. If $b_1(\bar{S})$ is even, then $\bar{S}$ is algebraic by (2.4; 2). We have $p_g(\bar{S})=0$ by (2.4; 1) in any case. So, if $b_1(\bar{S})$ is odd, we infer that D is contractible by a similar argument as in (9.3). Hence, by Grauert's criterion, any connected component of D can be contracted to a normal point. This proves the assertion.

(9.6) THEOREM. *$\boldsymbol{C}\times\{\boldsymbol{C}$-several points$\}$ admits only one quasi-complete structure.*

PROOF. Any q.c. structure is algebraic by (9.5). $\bar{\kappa}\neq 2$ by Nevannlinna-Sakai theory (cf. (1.9)). $\bar{\kappa}\neq 1$ by (7.15). $\bar{\kappa}\neq 0$ by (8.70; 3). Therefore $\bar{\kappa}=-\infty$, and (5.7) applies. Q.E.D.

(9.7) REMARK. In case of $\boldsymbol{C}^*\times\boldsymbol{C}^*$, the problem becomes a little subtler. However, Ueda [U] and Suzuki [Sz] managed to prove that the quasi-complete structures on it are one of the types described in (1.23).

(9.8) Now we consider cancellation problems.

DEFINITION. A quasi-complete variety V is said to be *cancellation stable* if $V\times T\cong W\times T$ implies $V\cong W$. A quasi-complete invariant ι is called a *cancellation invariant* if $V\times T\cong W\times T$ implies $\iota(V)=\iota(W)$. Here, T and W are also quasi-complete varieties.

Of course, a variety is cancellation stable if it is characterized by cancellation invariants.

(9.9) Examples of cancellation invariants.

1) $\dim V$.

2) The number of irreducible components of the singular locus of V. In particular, if $V\times T\cong W\times T$ and if V is smooth, then W is smooth.

3) $H_q(V;\boldsymbol{Z})$ and $H^p(V;\boldsymbol{Z})$. Cancellation invariance follows from the Künneth formula.

4) Logarithmic tensor invariants as in (1.8), including $\bar{\kappa}$. For a proof of their cancellation invariance, see [Km].

5) $A(V)^\times/\boldsymbol{C}^\times$ and $\tilde{H}^1(V;\boldsymbol{Z})$. See [Km; 4.5].

6) Any birational cancellation invariants of $\bar{V}$. Its algebraic dimension. Irregularity $q(\bar{V})=h^{0,1}(\bar{V})$. And so on.

7) $\hat{H}^2(V;\boldsymbol{Z})$ is a cancellation invariant for a smooth manifold V. To see this, we note first that $V\times T\cong W\times T$ implies that W is smooth. Hence, considering the smooth part, we obtain $V\times T_0\cong W\times T_0$ for the smooth part of T. So it is enough to show the cancellation invariance assuming that T is smooth.

Let $\bar{V}$ and $\bar{T}$ be smooth completions of V and T. Then we have $H^2(\bar{V}\times\bar{T})\cong H^2(\bar{V})\oplus(H^1(\bar{V})\otimes H^1(\bar{T}))\oplus H^2(\bar{T})$, where the coefficients of the cohomology groups are $\boldsymbol{Z}$. From this we obtain $\hat{H}^2(V\times T)\cong\hat{H}^2(V)\oplus(H^1(\bar{V})\otimes H^1(\bar{T}))\oplus\hat{H}^2(T)$. We have $H^1(\bar{W})\cong H^1(\bar{V})$ by 6). Applying the above formula to $W\times T$ too, we infer that $\hat{H}^2(V)\cong\hat{H}^2(W)$.

We don't know whether other $\hat{H}^p$'s and $\tilde{H}^q$'s are cancellation invariants or not.

8) If $\bar{V}$ is a manifold with $q(\bar{V})=0$ (e.g. rational), then $\mathrm{NS}(V)\cong\mathrm{Pic}(V)$ is a cancellation invariant of V. To show this, we may assume that T is smooth as in 7). Then, similarly as above, we have $\mathrm{NS}(V)\oplus\mathrm{NS}(T)\cong\mathrm{NS}(V\times T)$, because $q(\bar{V})=0$ implies $H^1(\bar{V})=0$. Since $H^1(\bar{W})=0$ by 6), we have similarly $\mathrm{NS}(W\times T)=\mathrm{NS}(W)\oplus\mathrm{NS}(T)$. Now we infer $\mathrm{NS}(V)\cong\mathrm{NS}(W)$. By virtue of (1.18; 3), $q(\bar{V})=0$ implies $\mathrm{Pic}(V)\cong\mathrm{NS}(V)$. So $\mathrm{Pic}(W)\cong\mathrm{NS}(W)$ too.

(9.10) Examples of cancellation stable varieties.

1) $\boldsymbol{A}^1\times\{\boldsymbol{A}^1-k\ \textit{points}\}$, $k<4$. To prove the cancellation stability, use (5.7). When $k=b_1(V)\geqq 4$, then we can prove $W\cong\boldsymbol{A}^1\times\{\boldsymbol{A}^1-k\ \textit{points}\}$, but the isomorphism classes of $\{\boldsymbol{A}^1-k\ \textit{points}\}$ may depend on the position of these k points.

2) $\boldsymbol{A}^2_*$. For a proof, use (8.70).

3) Any surface of type (Y) in (8.64) is cancellation stable. We give a proof in case of type $Y\{3, 3, 3\}$. Similar arguments work in other cases too.

Using (8.70; 1), we infer that W is also of type $Y\{3, 3, 3\}$. By (8.37), the isomorphism class of a surface of type $Y\{3, 3, 3\}$ is determined by the four lines L_1, L_2, L_3 and $p(B)$ on $\boldsymbol{P}^2$. All such quadruplets are projectively equivalent to each other on $\boldsymbol{P}^2$. Hence all surfaces of type $Y\{3, 3, 3\}$ are isomorphic to each other. In particular, $V\cong W$.

4) Perhaps you can find many other examples for yourself.

(9.11) *Problem.* Is $\pi_1(V)$ a cancellation invariant?

(9.12) Now we discuss positive characteristic versions of our theory. The first trouble is the lack of a desingularization theory. However, as was shown in [Km], many invariants in §1 can be defined for resolvable varieties. In particular, $\mathrm{Pic}(V)$, $\mathrm{NS}(V)$, $A(S)$ and any logarithmic tensor invariants including $\bar{\kappa}$ are well-defined.

$\hat{H}^p(V;\boldsymbol{Z})$ and $\tilde{H}^q(V;\boldsymbol{Z})$ are not defined in the abstract context. However, $\hat{b}_p(V)$ and $\tilde{b}_q(V)$ can be defined by virtue of étale cohomology theory.

(9.13) After necessary modifications, Theorem (2.4) can be proved in positive characteristic cases too. (2.1) will be proved also, if we assume that f is separable. (2.12) is proved for $\pi_1^{\{p\}}$, $p=\mathrm{char}(\mathfrak{K})$. Here, similarly as the algebraic fundamental group π_1, $\pi_1^{(p)}$ is defined by considering all the finite étale Galois coverings whose mapping orders are prime to p.

(9.14) The theory of Zariski decomposition of pseudo-effective divisors are valid in any characteristic cases. However, at present, the author does not know how we can generalize Kawamata's theory as in (6.11), (6.25) and (8.1). Perhaps the statements given in this paper are valid without any change.

(9.15) The analysis of singular fibers of a ruling can be done independently of char($\mathfrak{K}$). So, most results in §4 and many results in §5 and §7 are valid in positive characteristic cases too. However, the author does not know exactly how to generalize the results concerning fundamental groups, especially when char($\mathfrak{K}$)=2, 3 or 5.

(9.16) The A^1-ruling theorem (5.3) is valid in any characteristic (cf. [Km]). So (5.7), and hence (9.10; 1), are true in positive characteristic cases too. The author does not know how we can generalize (5.8), (5.12), (5.13), (5.15) and (5.16), especially in low characteristic cases.

(9.17) At present, the author can prove (6.25) under the additional assumption that $\mathrm{Pic}_0(\bar{S})$ is reduced.

(9.18) The proposition (7.15) is valid in any characteristic, if we assume $NS(S)=0=\bar{b}_2(S)$ instead of $H^2(S; \boldsymbol{Z})=0$. The results in §8, except those concerning π_1, are true in any characteristic, if we can prove (6.11; 1) and (8.1).

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Note added in proof. The author finds an error in the argument (7.14). In fact, the assertion is not true in case 2). So, in (7.15), $b_1(S)=0$ is possible although we can still prove that $\pi_1(S)$ is non-abelian. In (7.16) and (9.1) we need to assume $\pi_1(S)=\{1\}$. Details will appear in future.