On the semi-discrete finite element approximation for the nonstationary Navier-Stokes equation

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1. Introduction.

The purpose of the present paper is to give error estimates for the semi-discrete finite element approximation applied to the nonstationary Stokes or Navier-Stokes equation. Here "semi-discrete" means that we discretize only the space variables, keeping the time variable continuous. In the preceding paper [24] we have shown that the error for the nonstationary Stokes equation is of optimal order when the initial value belongs to the function space H, i.e., if the initial value is square summable and weakly divergence-free.

In [24] we have used Bercovier and Pironneau's finite element spaces. In this paper we again use these spaces and we shall show that the error for the nonstationary Navier-Stokes equation is of optimal order. To this end we study the approximation for the nonstationary Stokes equation in the case where the initial value is smooth, more concretely, where the initial value belongs to the domain of the Stokes operator or the domain of the square root of the Stokes operator. These linear problems are interesting in itself, since the method used here is a real method in its nature and is independent of the complex one, i.e., the "Dunford-integral" method employed previously in Okamoto [24]. In our study, our goal is not merely to show the convergence of the approximate solutions to the original solution, but to obtain the optimal rate of convergence. As the finite element spaces used in this paper are piecewise linear, the "optimal rate" means that

$$\|u(t)-u_h(t)\|_1=O(h) \qquad \text{as } h \longrightarrow 0 \quad (t>0),$$

$$\|u(t)-u_h(t)\|_0=O(h^2) \qquad \text{as } h \longrightarrow 0 \quad (t>0).$$

Here h is the mesh size of the triangulation of the domain Ω occupied with the fluid, and $u(t)$ is the solution of the nonstationary Stokes or Navier-Stokes equation, and $u_h(t)$ is the semi-discretized solution. $\| \ \|_j$ is the usual norm of the Sobolev space $H^j(\Omega)$.

The finite element approximation for the Navier-Stokes equation has been actively investigated by many authors. However, there are not so many papers dealing with the nonstationary problems. Particularly, papers intended to show the optimal rate of convergence are very few. At the present time, the author can refer only to Heywood and Rannacher's one ([22]) by a method quite different from ours.

Our results have the following features:

i) In the two dimensional problems we have obtained the fact that the error are majorized by ch^{2-j} in $H^j(\Omega)$-norm ($j=0, 1$), where c is a positive constant expressed as a polynomial of certain norms of the initial value and does not depend on the time variable (see Theorem (7.1), (7.2) and (8.1)).

ii) From the practical viewpoint, any assumption stated in terms of the solution $u(t)$ would be inappropriate, since $u(t)$ is not known before we carry out the computation. So we have made assumptions only on the initial value and the domain Ω, avoiding those on the solution $u(t)$ itself, except for Theorem 9.1.

The present paper is composed of nine sections. In Section 2 we introduce various function spaces and formulate the linear problem, i.e., the Stokes problem. In Section 3 we derive error estimates by making use of the method used in Fujita [9]. In Sections 4, 5, $\cdots$, 9 we are concerned with the semi-discretization of the nonstationary Navier-Stokes equation. The formulation of the problem in consideration and the semi-discrete scheme for them are given in Section 4. In Section 5 we clarify some crucial properties of the discrete Stokes operator and of the nonlinear term arising in the Navier-Stokes equation. Deriving some a priori estimates for the semi-discrete solutions in Section 6, we give error estimates in Sections 7 and 8 in the case of the two dimensional problem. Actually, Section 7 is devoted to the error estimation in $H^1(\Omega)^2$, while Section 8 is devoted to that in $L^2(\Omega)^2$. Three dimensional problem is considered in Section 9.

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2. Formulation of the Stokes problem.

Let Ω be a convex polygonal domain in $\boldsymbol{R}^2$. As in [24] we use the following function spaces.

$X=H_0^1(\Omega)^2$,

$M=\left\{q\in L^2(\Omega)\,;\ \int_\Omega q(x)dx=0\right\}$,

$V=\{v\in X;\ \operatorname{div} v=0 \text{ in } \Omega\}$,

$H=$the closure of V in $L^2(\Omega)^2$,

$X_h=\{v_h\in X;\ v_h|K \text{ is linear for all } K\in\mathfrak{T}_h\}$,

$M_h=\{q_h\in M\cap H^1(\Omega)\,;\ q_h|K$ is linear for all $K\in\mathcal{T}_h\}$,
$V_h=\{v_h\in X_h\,;\ (v_h, \nabla q_h)=0$ for all $q_h\in M_h\}$,
$Q_h: L^2(\Omega)^2\to V_h \cdots$ the orthogonal projection.

Here $\mathcal{T}_h$ is a triangulation of Ω with the maximum mesh size h. $\tilde{\mathcal{T}}_h$ is a refinement of $\mathcal{T}_h$ such that the element triangles of $\tilde{\mathcal{T}}_h$ are obtained from $K\in\mathcal{T}_h$ by dividing K into four equal triangles. We assume that the family $\{\mathcal{T}_h\}_{h>0}$ is regular (see Ciarlet [7]) and that $\mathcal{T}_h$ satisfies the inverse assumption (see Ciarlet [7]) and the hypothesis of Lemma 2.1 in [24].

Let us recall that the inverse assumption implies the inverse estimate

$$\|v_h\|_s\leqq ch^{-s}\|v_h\|_0 \quad (v_h\in X_h) \qquad \text{for all } s\in[0, 1]. \tag{2.1}$$

Here and in what follows we denote the norm of the Sobolev space $H^s(\Omega)$ by $\|\ \|_s$ and the symbol c stands for positive constants depending only on Ω, which may be different in different contexts.

Since Ω is a convex polygon, we have

$$D(A)=H^2(\Omega)^2\cap V,\ \|Av\|_0\leqq c\|v\|_2,\ \|v\|_2\leqq c\|Av\|_0 \qquad \text{for all } v\in D(A). \tag{2.2}$$

Here A is the Stokes operator in H associated with the quadratic form $(\nabla\cdot, \nabla\cdot)$ on V (see [24]), and $D(A)$ is the domain of A.

Now the nonstationary Stokes equation in the operator theoretical form is formulated as follows:

Find $u\in C([0, \infty[\,; H)\cap C^1(]0, \infty[\,; H)$ *and*
$p\in C(]0, \infty[\,; M)$ *such that*

$$(S)_1\quad\begin{cases} (2.3) & u(t)\in V & (t>0), \\ (2.4) & (du/dt, v)+(\nabla u, \nabla v)-(\operatorname{div} v, p)=0 & (t>0), \\ & \text{for all } v\in X, & \\ (2.5) & u(0)=a. & \end{cases}$$

Here $(\ ,\ )$ means the usual inner-product of $L^2(\Omega)^2$. The initial value a is assumed to belong to H.

The solution $\{u, p\}$ of the problem $(S)_1$ exists uniquely and u is characterized by

$$du/dt+Au(t)=0 \quad \text{and} \quad u(0)=a,$$

where A is the Stokes operator. Hence $u(t)$ is represented by means of the semi-group e^{-tA} as $u(t)=e^{-tA}a$ (see Okamoto [24]).

Now we proceed to the semi-discrete problem.

Find $u_h\in C^1([0, \infty[\,; V_h)$ *and*

$p_h \in C(]0, \infty[; M_h)$ *such that*

$$(S)_1^h \quad \begin{cases} (2.6) & (du_h/dt, v_h)+(\nabla u_h, \nabla v_h)-(\operatorname{div} v_h, p_h)=0 \qquad (t \geqq 0) \\ & \text{for all } v_h \in X_h , \\ (2.7) & u_h(0)=Q_h a . \end{cases}$$

In [24] we have shown the unique existence of the solution of $(S)_1^h$. Furthermore, $u_h(t)$ is characterized by

$$du_h/dt+A_h u_h(t)=0 \quad (t>0) \quad \text{and} \quad u_h(t)=Q_h a ,$$

i.e., $u_h(t)=e^{-tA_h}Q_h a$, where A_h is the discrete Stokes operator in V_h associated with the quadratic form $(\nabla\cdot, \nabla\cdot)$ on V_h.

In [24] we have proved an error estimate

$$\|u(t)-u_h(t)\|_j \leqq ch^{2-j}\|a\|_0/t \quad (t>0) \qquad \text{for } j=0, 1. \tag{2.8}$$

In the next section we modify (2.8) according as $a \in D(A)$ or $a \in D(A^{1/2})=V$, which is necessary for our later consideration of the nonlinear problems.

3. Error estimates for the nonstationary Stokes equation.

The aim of this section is to prove the following

THEOREM 3.1. *There exist positive constants c and δ depending only on Ω such that*

$$\|u(t)-u_h(t)\|_j \leqq ch^{2-j}\|a\|_1 e^{-\delta t}t^{-1/2} \qquad (t>0) \tag{3.1}$$

for all $a \in V$, $j=0, 1$,

$$\|u(t)-u_h(t)\|_j \leqq ch^{2-j}\|a\|_2 e^{-\delta t} \qquad (t>0) \tag{3.2}$$

for all $a \in D(A)$, $j=0, 1$.

PROOF. Firstly we note some easy inequalities;

$$\|A^{1/2}v\|_0 \leqq c\|v\|_1, \quad \|v\|_1 \leqq c\|A^{1/2}v\|_0 \qquad (v \in V) \tag{3.3}$$

$$\|A_h^{1/2}v_h\|_0 \leqq c\|v_h\|_1, \quad \|v_h\|_1 \leqq c\|A_h^{1/2}v_h\|_0 \qquad (v_h \in V_h). \tag{3.4}$$

In the next place we state the definition of the so-called Ritz projection.

DEFINITION. We define *the Ritz projection* $R_h ; V+V_h \to V_h$ by

$$(\nabla R_h w, \nabla v_h)=(\nabla w, \nabla v_h) \qquad (v_h \in V_h, w \in V+V_h).$$

From the definition it follows immediately that

(3.5) $$\|R_h w\|_1 \leqq c\|w\|_1 \qquad (w \in V+V_h).$$

Concerning R_h and Q_h, we need the following

LEMMA 3.1. *There exists a positive constant c depending only on Ω such that for $j=0, 1$ we have*

(3.6) $$\|w-R_h w\|_j \leqq ch^{2-j}\|w\|_2 \qquad (w \in D(A)),$$

(3.7) $$\|w-R_h w\|_0 \leqq ch\|w\|_1 \qquad (w \in V),$$

(3.8) $$\|w-Q_h w\|_j \leqq ch^{2-j}\|w\|_2 \qquad (w \in D(A)),$$

(3.9) $$\|w-Q_h w\|_0 \leqq ch\|w\|_1 \qquad (w \in V),$$

(3.10) $$\|A_h^\alpha Q_h w\|_0 \leqq c\|A^\alpha w\|_0 \qquad (0 \leqq \alpha \leqq 1,\ w \in D(A^\alpha)),$$

(3.11) $$\|Q_h w\|_1 \leqq c\|w\|_1 \qquad (w \in V).$$

The estimates of above type are well-known if A, V and V_h are replaced by $-\Delta$, X and X_h, respectively. The proof of Lemma 3.1 will be given after we have completed the proof of Theorem 3.1.

Following Fujita [9], we put

$$v_h(t)=e^{-tA}a-R_h e^{-tA}a \quad (=(I-R_h)u(t)),$$

$$w_h(t)=R_h e^{-tA}a-e^{-tA_h}Q_h a \quad (=R_h u(t)-u_h(t)).$$

Hence we have $u(t)-u_h(t)=v_h(t)+w_h(t)$. Then we obtain from Lemma 3.1 and (3.3)

(3.12) $$\begin{aligned}\|v_h(t)\|_j &= \|(I-R_h)e^{-tA}a\|_j \\ &\leqq ch^{2-j}\|e^{-tA}a\|_2 \\ &\leqq c'h^{2-j}\|Ae^{-tA}a\|_0 \\ &\leqq \begin{cases} c''h^{2-j}e^{-\delta t}\|a\|_2 & (a \in D(A)) \\ c''h^{2-j}e^{-\delta t}t^{-1/2}\|a\|_1 & (a \in V). \end{cases}\end{aligned}$$

Here use has been made of

(3.13) $$\|A^\alpha e^{-tA}\| \leqq ct^{-\alpha}e^{-\delta t} \qquad (0 \leqq \alpha \leqq 1,\ 0<t)$$

(δ is a domain constant, $\|\ \|$ is the operator norm), which is valid since Ω is a bounded domain.

Therefore, we have only to show

$$\|w_h(t)\|_j \leqq \begin{cases} ch^{2-j}e^{-\delta t}\|a\|_2 & (a \in D(A)) \\ ch^{2-j}e^{-\delta t}t^{-1/2}\|a\|_1 & (a \in V). \end{cases} \tag{3.14}$$

To this end we note firstly that

$$(du/dt, \phi)+(\nabla u, \nabla\phi)-(\operatorname{div}\phi, p)=0 \qquad (\phi \in X),$$

$$(du_h/dt, \phi_h)+(\nabla u_h, \nabla\phi_h)-(\operatorname{div}\phi_h, p_h)=0 \quad (\phi_h \in X_h),$$

and

$$(\nabla R_h u, \nabla\phi_h)=(\nabla u, \nabla\phi_h) \qquad (\phi_h \in V_h).$$

From these identities we easily obtain

$$(dw_h/dt, \phi_h)+(\nabla w_h, \nabla\phi_h)=-(dv_h/dt, \phi_h)-(\phi_h, \nabla p) \qquad (\phi_h \in V_h),$$

or equivalently

$$dw_h/dt+A_h w_h(t)=-Q_h(dv_h/dt+\nabla p(t)) \qquad (t>0).$$

Hence we have by Duhamel's principle

$$\begin{aligned} w_h(t) &= e^{-tA_h}w_h(0)-\int_0^t e^{-(t-s)A_h}Q_h\{dv_h/ds+\nabla p(s)\}\,ds \\ &= e^{-tA_h}w_h(0)-\int_0^t e^{-(t-s)A_h}Q_h\frac{dv_h}{ds}\,ds \\ &\quad -\int_0^t e^{-(t-s)A_h}Q_h\{\nabla p(s)-\nabla p(t)\}\,ds \\ &\quad -\int_0^t e^{-(t-s)A_h}Q_h\nabla p(t)ds \\ &= I_1+I_2+I_3. \end{aligned} \tag{3.15}$$

We are going to prove for $j=0, 1$

$$\|I_1\|_j,\ \|I_2\|_j,\ \|I_3\|_j \leqq \begin{cases} ch^{2-j}\|a\|_2 e^{-\delta t} & (a \in D(A)) \\ ch^{2-j}\|a\|_1 t^{-1/2}e^{-\delta t} & (a \in V), \end{cases} \tag{3.16}$$

which yields (3.14). If $a \in D(A)$, then we have

$$\|I_1\|_j \leqq ch^{2-j}\|a\|_2 e^{-\delta t} \qquad (t>0,\ j=0, 1) \tag{3.17}$$

in the same way as in the proof of Theorem 3.1 of Fujita [9]. If $a \in D(A^{1/2})$, we only make a slight modification of the proof of (3.17) as follows. After integration by parts we have

$$I_1=e^{-tA_h}w_h(0)-\int_0^t e^{-(t-s)A_h}\frac{d}{ds}Q_h\{v_h(s)-v_h(t)\}\,ds$$

$$=e^{-tA_h}w_h(0)+e^{-tA_h}Q_h\{v_h(0)-v_h(t)\}$$
$$+\int_0^t A_h e^{-(t-s)A_h}Q_h\{v_h(s)-v_h(t)\}\,ds\,.$$

Hence we obtain

$$I_1=-e^{-tA_h}Q_h v_h(t)+\int_0^t A_h e^{-(t-s)A_h}Q_h\{v_h(s)-v_h(t)\}\,ds\,. \tag{3.18}$$

The first term in the right side is majorized as

$$\|e^{-tA_h}Q_h v_h(t)\|_j\leqq c\|v_h(t)\|_j e^{-\delta t}\leqq c'h^{2-j}e^{-\delta t}t^{-1/2}\|a\|_1\,. \tag{3.19}$$

The second term is majorized as

$$\left\|\int_0^t A_h e^{-(t-s)A_h}Q_h\{v_h(s)-v_h(t)\}\,ds\right\|_j\leqq c\int_0^t(t-s)^{-1}e^{-\delta(t-s)}\|v_h(s)-v_h(t)\|_j ds$$

by means of the obvious inequality

$$\|A_h^\alpha e^{-tA_h}\|_0\leqq ct^{-\alpha}e^{-\delta t}\qquad(0\leqq\alpha\leqq1)\,.$$

On the other hand, it holds true that for $j=0, 1$ and $0<s<t$

$$\|v_h(s)-v_h(t)\|_j\leqq ch^{2-j}(t-s)^{1/4}s^{-3/4}e^{-\delta s}\|a\|_1\,. \tag{3.20}$$

Let us admit this inequality for the moment. Then we have

$$\left\|\int_0^t A_h e^{-(t-s)A_h}Q_h\{v_h(s)-v_h(t)\}\,ds\right\|_j$$
$$\leqq c\int_0^t(t-s)^{-1}e^{-\delta(t-s)}ch^{2-j}(t-s)^{1/4}s^{-3/4}e^{-\delta s}\|a\|_1 ds$$
$$\leqq c'h^{2-j}e^{-\delta t}t^{-1/2}\|a\|_1\,,$$

which, together with (3.19), yields

$$\|I_1\|_j\leqq ch^{2-j}e^{-\delta t}t^{-1/2}\|a\|_1\,.$$

The inequality (3.20) is shown as follows. Firstly we have by (3.12)

$$\begin{aligned}\|v_h(s)-v_h(t)\|_j&\leqq\|v_h(s)\|_j+\|v_h(t)\|_j\\&\leqq ch^{2-j}s^{-1/2}\|a\|_1e^{-\delta s}+ch^{2-j}t^{-1/2}\|a\|_1e^{-\delta t}\\&\leqq 2ch^{2-j}s^{-1/2}e^{-\delta s}\|a\|_1\,.\end{aligned} \tag{3.21}$$

Secondly we have by (3.6)

$$\|v_h(s)-v_h(t)\|_1=\|(I-R_h)(e^{-sA}-e^{-tA})a\|_j$$
$$\leqq ch^{2-j}\|A(e^{-sA}-e^{-tA})a\|_0$$
$$=ch^{2-j}\|A^{1/2}(e^{-sA}-e^{-tA})\|\|A^{1/2}a\|_0,$$

while

$$\|A^{1/2}(e^{-sA}-e^{-tA})\|=\left\|\int_s^t A^{3/2}e^{-rA}dr\right\|$$
$$\leqq c\int_s^t r^{-3/2}e^{-\delta r}dr$$
$$\leqq cs^{-3/2}(t-s)e^{-\delta s}.$$

Hence we obtain

$$\|v_h(s)-v_h(t)\|_j\leqq ch^{2-j}(t-s)s^{-3/2}e^{-\delta s}\|a\|_1. \tag{3.22}$$

From (3.21) and (3.22) we have

$$\|v_h(s)-v_h(t)\|_j\leqq(ch^{2-j}s^{-1/2}e^{-\delta s}\|a\|_1)^{3/4}(ch^{2-j}(t-s)s^{-3/2}e^{-\delta s}\|a\|_1)^{1/4}$$
$$\leqq ch^{2-j}(t-s)^{1/4}s^{-3/4}\|a\|_1e^{-\delta s}.$$

Next we prove (3.16) for I_2 and I_3. To this end we make use of the following

LEMMA 3.2. *There exists a positive constant c depending only on Ω such that for all $r\in H^1(\Omega)$ we have*

$$\|A_h^{-(2-j)/2}Q_h\nabla r\|_0\leqq ch^{2-j}\|\nabla r\|_0 \qquad (j=0, 1). \tag{3.23}$$

For the moment let us admit this lemma. Then we have

$$\|I_2\|_j\leqq c\int_0^t\|A_he^{-(t-s)A_h}\|\|A_h^{-(2-j)/2}Q_h\nabla\{p(s)-p(t)\}\|_0ds \tag{3.24}$$
$$\leqq c'h^{2-j}\int_0^t(t-s)^{-1}e^{-\delta(t-s)}\|\nabla\{p(s)-p(t)\}\|_0ds.$$

On the other hand, we have from the definition of $p(t)$

$$\|\nabla\{p(s)-p(t)\}\|_0=\|-du/ds+du/dt+\Delta\{u(s)-u(t)\}\|_0 \tag{3.25}$$
$$\leqq c\|A(e^{-sA}-e^{-tA})a\|_0$$
$$\leqq\begin{cases} c'(t-s)^{1/2}s^{-1/2}e^{-\delta s}\|a\|_2 & (a\in D(A)) \\ c'(t-s)^{1/4}s^{-3/4}e^{-\delta s}\|a\|_1 & (a\in V)\end{cases}$$

in a way similar to the proof of (3.20). From (3.24) and (3.25) we easily obtain

$$\|I_2\|_j \leq \begin{cases} ch^{2-j}e^{-\delta t}\|a\|_2 & (a\in D(A)), \\ ch^{2-j}e^{-\delta t}t^{-1/2}\|a\|_1 & (a\in V). \end{cases}$$

Next we note the obvious equality

$$I_3 = -A_h^{-1}\{I-e^{-tA_h}\}Q_h\nabla p(t).$$

From this we have

$$\begin{aligned}\|I_3\|_j &\leq c\|A_h^{-1}Q_h\nabla p(t)\|_j \\ &\leq c'\|A_h^{-(2-j)/2}Q_h\nabla p(t)\|_0 \\ &\leq c''h^{2-j}\|\nabla p(t)\|_0 .\end{aligned}$$

Since it is obvious that

$$\|\nabla p(t)\|_0 \leq c\|Ae^{-tA}\|_0 \leq \begin{cases} c'e^{-\delta t}\|a\|_2 & (a\in D(A)) \\ c'e^{-\delta t}t^{-1/2}\|a\|_1 & (a\in V), \end{cases}$$

we obtain (3.16) for I_3.

Now there remain only the proofs of Lemmas 3.1 and 3.2.

Proof of (3.6) for the case of $j=1$: For any $w\in D(A)$ we put $Aw=g$. By making use of Theorem 3.2 in [24] with $z=0$, we have

$$\|A^{-1}g-A_h^{-1}Q_hg\|_1 \leq ch\|g\|_0, \qquad \text{i.e.,}$$

$$\|w-A_h^{-1}Q_hg\|_1 \leq ch\|g\|_0 \leq c'h\|w\|_2. \tag{3.26}$$

On the other hand, we have

$$\|w-R_hw\|_1 \leq c\|w-A_h^{-1}Q_hg\|_1 \tag{3.27}$$

by the definition of R_h. From (3.26) and (3.27) we arrive at

$$\|w-R_hw\|_1 \leq ch\|w\|_2. \tag{3.28}$$

Proof of (3.6) for the case of $j=0$ and the proof of (3.7): We carry out the proof by means of Nitsche's trick. Let $\{\eta, \pi\}\in X\times M$ be a solution of the problem

$$(\nabla\eta, \nabla v)-(\operatorname{div} v, \pi)=(w-R_hw, v) \qquad (v\in X), \tag{3.29}$$

$$\eta\in V. \tag{3.30}$$

Then it is well-known that

$$\eta\in D(A), \qquad \pi\in H^1(\Omega)\cap M, \tag{3.31}$$

$$\|\eta\|_2+\|\nabla\pi\|_0 \leq c\|w-R_hw\|_0. \tag{3.32}$$

Substituting $w-R_h w$ for v in (3.29), we have

$$\|w-R_h w\|_0^2=(\nabla\eta, \nabla(w-R_h w))-(\mathrm{div}(w-R_h w), \pi)$$
$$=(\nabla(\eta-\eta_h), \nabla(w-R_h w))-(\mathrm{div}(w-R_h w), \pi-q_h)$$

for any $\eta_h\in V_h$ and $q_h\in M_h$. Hence it follows with the aid of (3.28) and (3.32) that

$$\|w-R_h w\|_0^2\leqq\|\nabla(w-R_h w)\|_0 \inf_{\eta_h\in V_h}\|\nabla(\eta-\eta_h)\|_0+c\|w-R_h w\|_1 \inf_{q_h\in M_h}\|\pi-q_h\|_0$$
$$\leqq c'h(\|\eta\|_2+\|\nabla\pi\|_0)\|w-R_h w\|_1$$
$$\leqq c''h\|w-R_h w\|_0\|w-R_h w\|_1$$
$$\leqq\begin{cases} c'''h\|w\|_1\|w-R_h w\|_0 & (w\in V+V_h) \\ c'''h^2\|w\|_2\|w-R_h w\|_0 & (w\in D(A)), \end{cases}$$

which implies (3.6) for the case of $j=0$ and (3.7). Since $\|w-Q_h w\|_0\leqq\|w-R_h w\|_0$, (3.8) and (3.9) are obvious from (3.6) and (3.7), except for $\|w-Q_h w\|_1\leqq ch\|w\|_2$. However, by the inverse estimate (2.1) and (3.6) we have

$$\|w-Q_h w\|_1\leqq\|w-R_h w\|_1+\|R_h w-Q_h w\|_1$$
$$\leqq ch\|w\|_2+ch^{-1}\|R_h w-Q_h w\|_0$$
$$\leqq ch\|w\|_2+ch^{-1}(\|R_h w-w\|_0+\|w-Q_h w\|_0)$$
$$\leqq c'h\|w\|_2 \qquad (w\in D(A)).$$

Proof of (3.10): If we can show

$$\|A_h Q_h w\|_0\leqq c\|Aw\|_0 \qquad (w\in D(A)), \tag{3.33}$$

then (3.10) follows from (3.33) and Heinz-Kato's theorem (see Kato [18]). (3.33) is proved as follows.

For any $v_h\in V_h$ we have

$$(A_h Q_h w, v_h)=(\nabla Q_h w, \nabla v_h)$$
$$=(\nabla(Q_h w-w), \nabla v_h)+(\nabla w, \nabla v_h)$$
$$=(\nabla(Q_h w-w), \nabla v_h)-(\nabla w, v_h).$$

Hence we have

$$|(A_h Q_h w, v_h)|\leqq c\|Q_h w-w\|_1\|v_h\|_1+c\|w\|_2\|v_h\|_0$$
$$\leqq c'h\|w\|_2 c'h^{-1}\|v_h\|_0+c\|w\|_2\|v_h\|_0$$
$$\leqq c''\|w\|_2\|v_h\|_0,$$

which implies (3.33) since $v_h \in V_h$ is arbitrary. (3.11) is a particular case of (3.10). The proof of Lemma 3.1 is now completed.

PROOF of LEMMA 3.2. Firstly we consider the case of $j=1$. For arbitrary $v_h \in V_h$ and $q_h \in M_h$, we have

$$\begin{aligned}(A_h^{-1/2} Q_h \nabla r, v_h) &= (\nabla r, A_h^{-1/2} v_h) \\ &= (\nabla (r-q_h), A_h^{-1/2} v_h) \\ &= -(r-q_h, \operatorname{div}(A_h^{-1/2} v_h)).\end{aligned}$$

Hence we obtain

$$\begin{aligned}|(A_h^{-1/2} Q_h \nabla r, v_h)| &\leqq c \inf_{q_h \in M_h} \|r-q_h\|_0 \|A_h^{-1/2} v_h\|_1 \\ &\leqq c' h \|\nabla r\|_0 \|v_h\|_0,\end{aligned}$$

which implies (3.23) for $j=1$. To deal with the case of $j=0$, we note firstly that for an arbitrary $v_h \in V_h$ there exists $\psi \in D(A)$ satisfying

$$\|A_h^{-1} v_h - \psi\|_1 \leqq c h \|v_h\|_0. \tag{3.34}$$

In fact, as ψ we may take the solution of the problem

$$\begin{cases} (\nabla \psi, \nabla \phi) = (v_h, \phi) & (\phi \in V), \\ \psi \in V. \end{cases}$$

Then we can obtain (3.34) in the same way as in Bercovier-Pironneau [2] or Okamoto [24], since v_h obviously satisfies

$$(\nabla A_h^{-1} v_h, \nabla \varphi_h) = (v_h, \varphi_h) \qquad (\varphi_h \in V_h).$$

For an arbitrary $q_h \in M_h$ we have the identity

$$\begin{aligned}(A_h^{-1} Q_h \nabla r, v_h) &= (\nabla (r-q_h), A_h^{-1} v_h - \psi) \\ &= -(r-q_h, \operatorname{div}(A_h^{-1} v_h - \psi)),\end{aligned}$$

whence follows that

$$\begin{aligned}|(A_h^{-1} Q_h \nabla r, v_h)| &\leqq c \inf_{q_h \in M_h} \|r-q_h\|_0 \|A_h^{-1} v_h - \psi\|_1 \\ &\leqq c' h \|\nabla r\|_0 c' h \|v_h\|_0.\end{aligned}$$

From this we have (3.23) for $j=0$ by virtue of the arbitrariness of $v_h \in V_h$.

The proof of Theorem 3.1 is now completed. Q. E. D.

REMARK 3.1. The estimates (3.1) and (3.2) are valid also for the three-dimensional problem, if we admit that

(3.35) $$D(A)=H^2(\Omega)^3\cap V\,,$$

$$\|Av\|_0\leqq c\|v\|_2\,,\qquad \|v\|_2\leqq c\|Av\|_0\qquad (v\in D(A))\,,$$

which is not explicitly proved yet for polyhedral domain Ω.

4. Formulation of the Navier-Stokes initial value problem.

Henceforth we consider the nonstationary Navier-Stokes equation. Then the two dimensional problems and the three dimensional ones are quite different from viewpoint of mathematical analysis. Actually, in the two dimensional case, the solution exists globally in time, whatever the initial value may be. On the other hand, it is not yet known whether the three dimensional solution exists globally unless the initial value is small enough. Hence we firstly consider the two dimensional problem, while the three dimensional one is discussed in Section 9.

Again, let Ω be a convex polygonal domain in $\boldsymbol{R}^2$. In what follows we employ the same notations as in Sections 2 and 3. Recall that we have

(4.1) $$\begin{cases} D(A)=H^2(\Omega)^2\cap V\,, \\ \|Av\|_0\leqq c\|v\|_2, \quad \|v\|_2\leqq c\|Av\|_0 \qquad (v\in D(A))\,. \end{cases}$$

Let γ be a number satisfying $1/2<\gamma<3/4$ and fix it throughout this paper. Now we define the nonlinear operator F: $D(A^\gamma)\rightarrow H$ by

(4.2) $$(Fw,\, v)=((w\cdot\nabla)w,\, v)\qquad (w\in D(A^\gamma),\, v\in H)\,,$$

i.e., $Fw=P(w\cdot\nabla)w$, where P is the orthogonal projection from $L^2(\Omega)^2$ onto H. By means of the Sobolev imbedding theorems and theorems due to Fujita and Morimoto [13] and Fujiwara [14], we see easily that (4.2) defines $Fw\in H$ uniquely. Furthermore we have the following estimates;

(4.3) $$\|Fw\|_0\leqq c\|A^\gamma w\|_0\|A^{1/2}w\|_0\qquad (w\in D(A^\gamma))\,,$$

(4.4) $$\|Fw-Fv\|_0\leqq c\|A^\gamma w\|_0\|A^{1/2}(w-v)\|_0 + c\|A^{1/2}v\|_0\|A^\gamma(w-v)\|_0\qquad (w,\, v\in D(A^\gamma))\,.$$

The nonstationary Navier-Stokes equation is formulated within the framework of the operator theory as follows:

Find $u\in C([0,\infty[\,;H)\cap C^1(]0,\infty[\,;H)\cap C(]0,\infty[\,;D(A))$ *such that*

(N.S.) $$\begin{cases} (4.5)\quad du/dt+Au(t)+Fu(t)=0 \qquad (t>0)\,, \\ (4.6)\quad u(0)=a\,. \end{cases}$$

Since we are considering the two dimensional problem, it is well-known that for any $a \in H$ there exists a unique solution u on $[0, \infty[$ (see Kato and Fujita [19]).

Nextly, we formulate the semi-discretization of $(N.S.)$ as follows:

Find $u_h \in C^1([0, \infty[; V_h)$ *such that*

$$(N.S.)_h \qquad \begin{cases} (4.7) \qquad du_h/dt + A_h u_h(t) + F_h u_h(t) = 0 \qquad (t>0), \\ (4.8) \qquad u_h(0) = Q_h a, \end{cases}$$

where the nonlinear operator F_h is defined through

$$(4.9) \qquad (F_h w, v_h) = b(w\,; w, v_h) \equiv \frac{1}{2}((w\cdot\nabla)w, v_h) - \frac{1}{2}((w\cdot\nabla)v_h, w),$$

and

$$(4.10) \qquad F_h w \in V_h .$$

We note that F_h is analogous to F in the following manner;

$$(F_h v_h, v_h) = 0 \ (v_h \in V_h) \quad \text{and} \quad (F_h w, v_h) = ((w\cdot\nabla)w, v_h), \qquad \text{if } \operatorname{div} w = 0 .$$

Furthermore $F_h | V_h\,;\ V_h \to V_h$ is a smooth function. Bearing these facts in mind, we prove the following

LEMMA 4.1. *There exists a unique solution* u_h *of* $(N.S.)_h$ *on* $[0, \infty[$.

PROOF. Let $\{w_j\,; j=1, 2, \cdots, N\}$ be a base of V_h. Then we look for a solution of the form $u_h(t) = \sum_{j=1}^{N} \lambda_j(t) w_j$. If we represent $Q_h a$ by $Q_h a = \sum_{j=1}^{N} \mu_j w_j$, then $(N.S.)_h$ is equivalent to the following system of ordinary differential equations;

$$(4.7)' \qquad \sum_{j=1}^{N} \lambda_j'(t)(w_j, w_i) + \sum_{j=1}^{N} \lambda_j(t)(\nabla w_j, \nabla w_i) + \sum_{j,k=1}^{N} \lambda_j(t)\lambda_k(t) b(w_j\,; w_k, w_i) = 0 \qquad (t>0)$$

where $i=1, 2, \cdots, N$ ($'$ means the differentiation), and

$$(4.8)' \qquad \lambda_j(0) = \mu_j \qquad (j=1, 2, \cdots, N).$$

Since $\{w_j\,; j=1, 2, \cdots, N\}$ is a base of V_h, the matrix $[(w_j, w_i)]_{i,j}$ is invertible. Hence the system of ordinary differential equations (4.7)$'$ and (4.8)$'$ is uniquely solvable at least locally. To show that the solution exists on $[0, \infty[$, we need the following

LEMMA 4.2. *For all* $t>0$ *we have*

$$\frac{1}{2}\|u_h(t)\|_0^2+\int_0^t\|\nabla u_h(s)\|_0^2 ds=\frac{1}{2}\|u_h(0)\|_0^2 \le \frac{1}{2}\|a\|_0^2. \tag{4.11}$$

PROOF OF LEMMA 4.2. Taking the inner-product of both sides of (4.7) and $u_h(t)$, we get

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|_0^2+\|\nabla u_h(t)\|_0^2=0 \qquad (t>0), \tag{4.12}$$

since $(F_h u_h, u_h)=0$. Integrating (4.12) on $[0, t]$, we have (4.11).

PROOF OF LEMMA 4.1. By virtue of (4.11), $u_h(t)$ remains in a bounded set of V_h. Hence we can conclude that the solution $u_h(t)$ exists on $[0, \infty[$.

Q.E.D.

REMARK 4.1. Lemmas 4.1 and 4.2 are valid even when Ω is a three dimensional domain, or when Ω is not convex.

5. Some properties of the discrete Stokes operator.

We denote the Laplace operator in $L^2(\Omega)^2$ by $-B$, namely, $Bu=-\Delta u$, $D(B)=H^2(\Omega)^2\cap H_0^1(\Omega)^2$. Then we have the following

LEMMA 5.1. *For any α such that $0\le\alpha\le 1/2$, we have*

$$\|B^\alpha w_h\|_0\le\|A_h^\alpha w_h\|_0 \qquad (w_h\in V_h). \tag{5.1}$$

PROOF. If we have shown

$$\|B^{1/2} w_h\|_0\le\|A_h^{1/2} w_h\|_0 \qquad (w_h\in V_h), \tag{5.2}$$

then (5.1) follows immediately from Heinz-Kato's theorem (see Kato [18]). However, (5.2) holds true since

$$\|B^{1/2}w_h\|_0^2=(\nabla w_h, \nabla w_h)=\|A_h^{1/2}w_h\|_0^2 \qquad (w_h\in V_h).$$

COROLLARY 5.2.

$$\|B^{1/4}A_h^{-1/4}w_h\|_0\le\|w_h\|_0 \qquad (w_h\in V_h). \tag{5.3}$$

Now let us recall that

$$\|A^{-1/4}Fw\|_0\le c\|A^{1/2}w\|_0\|A^{1/4}w\|_0 \qquad (w\in V) \tag{5.4}$$

(see Kato and Fujita [19]). The next lemma is a discrete version of (5.4).

LEMMA 5.3. *There exists a positive constant c depending only on Ω such that for any $w_h \in V_h$ we have*

(5.5) $$\|A_h^{-1/4}F_h w_h\|_0 \leqq c\|A_h^{1/2}w_h\|_0\|A_h^{1/4}w_h\|_0 .$$

PROOF. We carry out the proof by the duality argument. For an arbitrary $v_h \in V_h$ we have

$$(A_h^{-1/4}F_h w_h,\ v_h)=b(w_h\,;\ w_h,\ A_h^{-1/4}v_h)$$
$$=\frac{1}{2}((w_h\cdot\nabla)w_h,\ A_h^{-1/4}v_h)-\frac{1}{2}((w_h\cdot\nabla)A_h^{-1/4}v_h,\ w_h).$$

By means of integration by parts we obtain

(5.6) $$\begin{aligned}(A_h^{-1/4}F_h w_h,\ v_h)&=((w_h\cdot\nabla)w_h,\ A_h^{-1/4}v_h)\\&\quad+\frac{1}{2}((\mathrm{div}\ w_h)A_h^{-1/4}v_h,\ w_h)\\&=(B^{-1/4}(w_h\cdot\nabla)w_h,\ B^{1/4}A_h^{-1/4}v_h)\\&\quad+\frac{1}{2}(B^{-1/4}(\mathrm{div}\ w_h)w_h,\ B^{1/4}A_h^{-1/4}v_h).\end{aligned}$$

From (5.3) and (5.6) it follows that

(5.7) $$\|A_h^{-1/4}F_h w_h\|_0 \leqq \|B^{-1/4}((w_h\cdot\nabla)w_h\|_0+\frac{1}{2}\|B^{-1/4}(\mathrm{div}\ w_h)w_h\|_0 .$$

On the other hand, we can show the inequality

(5.8) $$\|B^{-1/4}(wv)\|_0 \leqq c\|w\|_0\|B^{1/4}v\|_0 \qquad (w\in L^2(\Omega)^2,\ v\in D(B^{1/4})).$$

From Lemma 5.1, (5.7) and (5.8) we easily obtain (5.5).

The inequality (5.8) can be proved by the method used in Appendix of Kato and Fujita [19]. For completeness we give a proof of (5.8) below.

Without loss of generality we may assume that w and v are scalar-valued functions and that B is an operator in $L^2(\Omega)$. Now $B^{-1/4}$ is an integral operator with a kernel having a singulality of the form $|x-y|^{-3/2}$. Hence we have

$$\begin{aligned}\|B^{-1/4}(wv)\|_0^2 &\leqq c\iiint_{\Omega\times\Omega\times\Omega}\frac{|w(y)v(y)\,w(z)v(z)|}{|x-y|^{3/2}|x-z|^{3/2}}\,dy\,dz\,dx\\&\leqq c'\iint_{\Omega\times\Omega}\frac{|w(y)v(y)\,w(z)v(z)|}{|y-z|}\,dy\,dz\\&\leqq c'\left\{\iint_{\Omega\times\Omega}\frac{|w(y)v(z)|^2}{|y-z|}\,dy\,dz\right\}^{1/2}\left\{\iint_{\Omega\times\Omega}\frac{|w(z)v(y)|^2}{|y-z|}\,dy\,dz\right\}^{1/2}\\&=c'\iint_{\Omega\times\Omega}\frac{|w(y)v(z)|^2}{|y-z|}\,dy\,dz .\end{aligned}$$

On the other hand, it is well-known that there exists a positive constant c such that for any $y\in\Omega$ and $v\in D(B^{1/4})$ we have

$$\int_\Omega \frac{|v(z)|^2}{|y-z|}dz \leqq c\|B^{1/4}v\|_0^2$$

(see, e.g., Kato and Fujita [19]).

From this we have

$$\|B^{-1/4}(wv)\|_0^2 \leqq c\|B^{1/4}v\|_0^2 \int_\Omega |w(y)|^2 dy = c\|B^{1/4}v\|_0^2 \|w\|_0^2 . \qquad \text{Q.E.D.}$$

6. A priori estimates.

In this section we derive several a priori estimates for $u(t)$ and $u_h(t)$, which will be used in the error analysis in Section 7 and Section 8. From now on we assume that $a\in V$.

PROPOSITION 6.1. *There exist positive constants c and δ depending only on Ω such that for all $t>0$*

$$\|u(t)\|_0^2 + 2\int_0^t \|\nabla u(s)\|_0^2 ds = \|a\|_0^2 , \tag{6.1}$$

$$\|u(t)\|_0 \leqq c\|a\|_0 e^{-\delta t} , \tag{6.2}$$

$$\|A^{1/4}u(t)\|_0 \leqq \|A^{1/4}a\|_0 \exp\{c\|a\|_0^2 - \delta t\} , \tag{6.3}$$

and

$$\|A^{1/2}u(t)\|_0 \leqq \|A^{1/2}a\|_0 \exp\{c\|a\|_0^4 - \delta t\} . \tag{6.4}$$

PROOF. Taking the inner-product of (4.5) and $u(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_0^2 + \|\nabla u(t)\|_0^2 = 0 \qquad (t>0) . \tag{6.5}$$

Integration of (6.5) on $[0, t]$ yields (6.1). On the other hand, by the Poincaré inequality, (6.5) gives

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_0^2 + c\|u(t)\|_0^2 \leqq 0 \qquad (t>0) ,$$

or equivalently

$$\frac{d}{dt}(e^{2ct}\|u(t)\|_0^2) \leqq 0 \qquad (t>0) ,$$

from which we obtain (6.2). To show (6.3) we take the inner-product of (4.5) and $A^{1/2}u(t)$. Then we have

(6.6) $$\frac{1}{2}\frac{d}{dt}\|A^{1/4}u(t)\|_0^2+\|A^{3/4}u(t)\|_0^2=-(A^{-1/4}Fu(t),\ A^{3/4}u(t))$$

$$\leqq\frac{1}{2}\|A^{3/4}u(t)\|_0^2+\frac{1}{2}\|A^{-1/4}Fu(t)\|_0^2.$$

From (5.4), (6.6) and the Poincaré inequality, we obtain

(6.7) $$\frac{1}{2}\frac{d}{dt}\|A^{1/4}u(t)\|_0^2+c\|A^{1/4}u(t)\|_0^2\leqq c\|A^{1/2}u(t)\|_0^2\|A^{1/4}u(t)\|_0^2.$$

Putting $\phi(t)=e^{2ct}\|A^{1/4}u(t)\|_0^2$ we can rewrite (6.7) as

(6.8) $$\frac{d}{dt}\phi(t)\leqq 2c\|A^{1/2}u(t)\|_0^2\phi(t) \qquad (t>0).$$

Applying Gronwall's lemma to (6.8), we have

$$\|A^{1/4}u(t)\|_0^2e^{2ct}\leqq\phi(0)\exp\left\{\int_0^t 2c\|A^{1/2}u(s)\|_0^2ds\right\}$$

$$\leqq\|A^{1/4}a\|_0^2\exp(c'\|a\|_0^2),$$

which implies (6.3).

To obtain (6.4) we form the inner-product of (4.5) and $Au(t)$. Then we have

(6.9) $$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u(t)\|_0^2+\|Au(t)\|_0^2=-(Au(t),\ Fu(t))$$

$$\leqq\|Au(t)\|_0\|Fu(t)\|_0.$$

On the other hand, by Hölder's inequality and the inequality (1) in page 10 of Ladyzhenskaya [23], we obtain

$$\|Fu(t)\|_0\leqq\|(u\cdot\nabla)u(t)\|_0$$

$$\leqq c\|u(t)\|_{L^4}\|\nabla u(t)\|_{L^4}$$

$$\leqq c'\|u(t)\|_0^{1/2}\|\nabla u(t)\|_0\|Au(t)\|_0^{1/2}.$$

Therefore we have

(6.10) $$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u(t)\|_0^2+\|Au(t)\|_0^2\leqq c\|a\|_0^{1/2}\|A^{1/2}u(t)\|_0\|Au(t)\|_0^{3/2}$$

$$\leqq\frac{1}{2}\|Au(t)\|_0^2+c'\|a\|_0^2\|A^{1/2}u(t)\|_0^4$$

by Young's inequality and (6.1).

Again by means of the Poincaré inequality we have

(6.11) $$\frac{d}{dt}\|A^{1/2}u(t)\|_0^2+c\|A^{1/2}u(t)\|_0^2\leqq c\|a\|_0^2\|A^{1/2}u(t)\|_0^4.$$

Putting $\phi(t)=\|A^{1/2}u(t)\|_0^2 e^{ct}$ we can rewrite (6.11) as

$$\frac{d}{dt}\phi(t)\leqq c\|a\|_0^2\|A^{1/2}u(t)\|_0^2\phi(t) \qquad (t>0). \tag{6.12}$$

Applying Gronwall's lemma to (6.12), we easily obtain (6.4). Q.E.D.

In what follows we denote δ, $(1/2)\delta$, $\min\{1, \delta\}$ simply by δ or δ' which may be different in different contexts. Furthermore we employ the following notation;

$$K(a),\ K'(a)=\text{polynomials of } e^{c\|a\|_0^2},\ e^{c\|a\|_0^4},\ \|A^\alpha a\|_0 \quad (\alpha=0,\ 1/4,\ 1/2,\ 1/2+\theta,\ \gamma+\theta,\ \gamma,\ 1), \tag{6.13}$$

$$L(a),\ L'(a)=\text{polynomials of } e^{c\|a\|_0^2},\ e^{c\|a\|_0^4},\ \|A^\alpha a\|_0 \quad (\alpha=0,\ 1/4,\ 1/2). \tag{6.14}$$

Here θ is a fixed number with $0<\theta<3/4-\gamma$.

PROPOSITION 6.2. *There exists a positive constant δ depending only on Ω such that for all $t>s>0$ we have*

$$\|A^\gamma u(t)\|_0\leqq K(a)e^{-\delta t} \qquad (a\in D(A)), \tag{6.15}$$

$$\|A^\gamma u(t)\|_0\leqq L(a)e^{-\delta t}t^{-(\gamma-1/2)} \qquad (a\in V), \tag{6.16}$$

$$\|A^{1/2}(u(t)-u(s))\|_0\leqq (t-s)^\theta e^{-\delta s}K(a) \qquad (a\in D(A)), \tag{6.17}$$

$$\|A^{1/2}(u(t)-u(s))\|_0\leqq (t-s)^\theta e^{-\delta s}s^{-\theta}L(a) \qquad (a\in V), \tag{6.18}$$

$$\|A^\gamma(u(t)-u(s))\|_0\leqq (t-s)^\theta e^{-\delta s}K(a) \qquad (a\in D(A)), \tag{6.19}$$

and

$$\|A^\gamma(u(t)-u(s))\|_0\leqq (t-s)^\theta e^{-\delta s}s^{-\theta-(\gamma-1/2)}L(a) \qquad (a\in D(A)). \tag{6.20}$$

PROOF. We start with the integral form of (4.5);

$$u(t)=e^{-tA}a-\int_0^t e^{-(t-s)A}Fu(s)ds \qquad (t>0). \tag{6.21}$$

By means of (3.13), (5.4), (6.3) and (6.4) we have

$$\begin{aligned}\|A^\gamma u(t)\|_0 &\leqq \|A^\gamma e^{-tA}a\|_0+c\int_0^t (t-s)^{-(1/4)-\gamma}e^{-\delta(t-s)}\|A^{1/2}u(s)\|_0\|A^{1/4}u(s)\|_0\,ds\\ &\leqq \|A^\gamma e^{-tA}a\|_0+L(a)e^{-\delta t}\end{aligned}$$

$$\leqq\begin{cases} ce^{-\delta t}\|A^{\gamma}a\|_0+L(a)e^{-\delta t} & (a\in D(A)), \\ ce^{-\delta t}t^{-(\gamma-1/2)}\|A^{1/2}a\|_0+L(a)e^{-\delta t} & (a\in V). \end{cases}$$

Thus we obtain (6.15) and (6.16).

In order to show (6.17) and (6.18) we note firstly that

$$A^{1/2}\{u(t)-u(s)\}=A^{1/2}\{e^{-(t-s)A}-I\}e^{-sA}a -\int_s^t A^{1/2}e^{-(t-r)A}Fu(r)dr -\int_0^s A^{1/2}\{e^{-(t-s)A}-I\}e^{-(s-r)A}Fu(r)dr \equiv I_1+I_2+I_3 \qquad (t>s>0). \tag{6.22}$$

Recall the inequality

$$\|\{e^{-tA}-I\}A^{-\theta}\|\leqq\frac{t^{\theta}}{\theta} \qquad (t>0,\ 0<\theta\leqq 1), \tag{6.23}$$

which is shown by means of the spectral resolution of A (see Fujita and Kato [10]). By (6.23) and (3.13) I_1 is estimated as follows;

$$\|I_1\|_0\leqq\frac{(t-s)^{\theta}}{\theta}\|A^{1/2+\theta}e^{-sA}a\|_0 \leqq\begin{cases} \dfrac{(t-s)^{\theta}}{\theta}ce^{-\delta s}\|A^{1/2+\theta}a\|_0 & (a\in D(A)), \\ \dfrac{(t-s)^{\theta}}{\theta}ce^{-\delta s}s^{-\theta}\|A^{1/2}a\|_0 & (a\in V). \end{cases} \tag{6.24}$$

We estimate I_2 and I_3 by making use of (6.3) and (6.4);

$$\|I_2\|_0\leqq c\int_s^t(t-r)^{-3/4}e^{-\delta(t-r)}\|A^{1/2}u(r)\|_0\|A^{1/4}u(r)\|_0dr \leqq(t-s)^{1/4}e^{-\delta t}L(a) \leqq(t-s)^{1/4}e^{-\delta s}L(a), \tag{6.25}$$

$$\|I_3\|_0\leqq c\int_0^s\frac{(t-s)^{\theta}}{\theta}(s-r)^{-3/4-\theta}e^{-\delta(s-r)}\|A^{1/2}u(r)\|_0\|A^{1/4}u(r)\|_0dr \leqq(t-s)^{\theta}e^{-\delta s}s^{1/4-\theta}L(a). \tag{6.26}$$

Now (6.17) and (6.18) follow from (6.24), (6.25) and (6.26).

(6.19) and (6.20) also can be proved in a way similar to the proof of (6.17) and (6.18). So we omit the proof. Q.E.D.

PROPOSITION 6.3. *For all $t>0$ we have*

(6.27) $$\|Au(t)\|_0 \leqq K(a)e^{-\delta t} \qquad (a \in D(A)),$$

and

(6.28) $$\|Au(t)\|_0 \leqq L(a)e^{-\delta t}t^{-1/2} \qquad (a \in V).$$

PROOF. From (6.21) we obtain

(6.29) $$\begin{aligned} Au(t) &= Ae^{-tA}a \\ &\quad -\int_0^t Ae^{-(t-s)A}\{Fu(s)-Fu(t)\}\,ds \\ &\quad -\int_0^t Ae^{-(t-s)A}Fu(t)\,ds \\ &\equiv J_1+J_2+J_3. \end{aligned}$$

J_1 is subject to

(6.30) $$\|J_1\|_0 \leqq \begin{cases} c\|Aa\|_0 e^{-\delta t} & (a \in D(A)) \\ c\|A^{1/2}a\|_0 e^{-\delta t}t^{-1/2} & (a \in V). \end{cases}$$

By means of (4.4), (6.18) and (6.20) we have

(6.31) $$\begin{aligned} \|J_2\|_0 &\leqq \int_0^t \|Ae^{-(t-s)A}\|\,\|Fu(s)-Fu(t)\|_0\,ds \\ &\leqq c\int_0^t (t-s)^{-1}e^{-\delta(t-s)}\{\|A^{1/2}u(s)\|_0\|A^{\gamma}\{u(s)-u(t)\}\|_0 \\ &\qquad +\|A^{\gamma}u(t)\|_0\|A^{1/2}\{u(s)-u(t)\}\|_0\}\,ds \\ &\leqq L(a)e^{-\delta' t}\int_0^t (t-s)^{\theta-1}s^{-\theta-(\gamma-1/2)}\,ds \\ &\leqq L'(a)e^{-\delta' t}t^{-(\gamma-1/2)} \\ &\leqq L''(a)e^{-\delta' t}t^{-1/2}, \end{aligned}$$

if $a \in V$. In a similar manner we obtain

(6.32) $$\|J_2\|_0 \leqq K(a)e^{-\delta t} \qquad (a \in D(A)).$$

Since $J_3=(e^{-tA}-I)Fu(t)$, we have

(6.33) $$\begin{aligned} \|J_3\|_0 &\leqq c\|A^{1/2}u(t)\|_0\|A^{\gamma}u(t)\|_0 \\ &\leqq \begin{cases} K(a)e^{-\delta t} & (a \in D(A)), \\ L(a)e^{-\delta t}t^{-(\gamma-1/2)} & (a \in V) \end{cases} \end{aligned}$$

by means of (4.3), (6.4), (6.15) and (6.16).

Now (6.27) and (6.28) follow from (6.29), (6.30), (6.32) and (6.33). Q.E.D.

PROPOSITION 6.4. *There exist positive constants c and δ depending only on the domain Ω such that for any $t>0$ we have*

$$\|A_h^{1/4}u_h(t)\|_0 \leqq c\|A^{1/4}a\|_0 \exp\{c\|a\|_0^2-\delta t\}. \tag{6.34}$$

PROOF. We take the inner-product of (4.7) and $A_h^{1/2}u_h(t)$, and we carry out the proof similarly to that of (6.3). This is possible since we can use (4.11) and (5.5) instead of (6.1) and (5.4), respectively. We omit the details. Q.E.D.

Now we prepare a basic lemma concerning a certain integral inequality of Volterra type.

LEMMA 6.5. *Let T, α and β be positive constants and let r be a constant with $0<r<1$. Then for any continuous function f; $[0, T]\to[0, \infty[$ satisfying*

$$f(t)\leqq\alpha+\beta\int_0^t(t-s)^{-r}f(s)ds \qquad (0\leqq t\leqq T), \tag{6.35}$$

we have

$$f(t)\leqq c\alpha\exp\{c\beta^{1/(1-r)}t\} \qquad (0\leqq t\leqq T) \tag{6.36}$$

with a positive constant c which depends only on r.

PROOF. If we put $m=\sup_{0\leq t\leq T} f(t)$, then we have by (6.35)

$$\begin{aligned} f(t) &\leqq\alpha+m\beta\int_0^t(t-s)^{-r}ds \\ &=\alpha+\beta m t^{1-r}B(1-r, 1) \qquad (0\leqq t\leqq T), \end{aligned} \tag{6.37}$$

where $B(\,,\,)$ is Euler's beta function. Substitution of (6.37) into the right side of (6.35) yields

$$f(t)\leqq\alpha+\alpha\beta t^{1-r}B(1-r, 1)+\beta^2 m t^{2-2r}B(1-r, 1)B(1-r, 2-r).$$

Repeating this procedure, we have for any natural number l

$$\begin{aligned} f(t)\leqq\alpha\Big\{1+\sum_{k=1}^{l-1}(\beta t^{1-r})^k B(1-r, 1)\cdots B(1-r, 1+(k-1)(1-r))\Big\} \\ +m(\beta t^{1-r})^l B(1-r, 1)B(1-r, 2-r)\cdots B(1-r, 1+(l-1)(1-r)). \end{aligned}$$

On the other hand, we have

$$B(1-r, 1)B(1-r, 2-r)\cdots B(1-r, 1+(l-1)(1-r))=\frac{\{\Gamma(1-r)\}^l}{\Gamma(1+l(1-r))},$$

where $\Gamma(\,)$ is Euler's gamma function.

We choose l large enough so that the inequality

$$\frac{\{\beta T^{1-r}\Gamma(1-r)\}^l}{\Gamma(1+l(1-r))}<1/2$$

holds good. Then we have

$$\frac{1}{2}m\leqq\alpha\Big\{1+\sum_{k=1}^{\infty}(\beta t^{1-r})^k\frac{\Gamma(1-r)^k}{\Gamma(1+k(1-r))}\Big\}$$

$$\leqq c\alpha\exp\{c\beta^{1/(1-r)}t\},$$

which implies (6.36).

The last inequality is shown as follows. Putting

$$g(x)=\{[\beta\Gamma(1-r)]^{1/(1-r)}t\}^x/\Gamma(1+x),$$

we have

$$\sum_{k=1}^{\infty}[\beta t^{1-r}]^k\Gamma(1-r)^k/\Gamma(1+k(1-r))=\sum_{k=1}^{\infty}g(k(1-r)).$$

The function $g(\cdot(1-r))$ is decreasing in $k\geqq k_0$ for some $k_0\in N$. Therefore

$$\sum_{k=1}^{\infty}g(k(1-r))=\sum_{k=1}^{k_0-1}g(k(1-r))+\int_{k_0-1}^{\infty}g(x(1-r))dx$$

$$=\sum_{k=1}^{k_0-1}g(k(1-r))+\frac{1}{1-r}\int_{(k_0-1)(1-r)}^{\infty}g(y)dy$$

$$\leqq\sum_{k=1}^{k_0-1}g(k(1-r))+\frac{1}{1-r}\sum_j\frac{\{[\beta\Gamma(1-r)]^{1/(1-r)}t\}^j}{j!}$$

$$\leqq c\exp(c\beta^{1/(1-r)}t).$$

Q.E.D.

As a first application of this lemma, we prove the following

PROPOSITION 6.6. *For all $t>0$ the inequality*

$$\|A_h^{1/2}u_h(t)\|_0\leqq L(a)e^{L(a)-\delta t}\|a\|_1 \qquad (a\in V) \tag{6.38}$$

holds true.

PROOF. By (4.7) we have

$$u_h(t)=e^{-tA_h}Q_ha-\int_0^t e^{-(t-s)A_h}F_hu_h(s)ds.$$

Therefore we obtain

$$A_h^{1/2}u_h(t)=A_h^{1/2}e^{-tA_h}Q_ha-\int_0^t A_h^{3/4}e^{-(t-s)A_h}A_h^{-1/4}F_hu_h(s)ds.$$

By means of (5.5), (6.34) and the obvious inequality $\|A_h^{\alpha}e^{-tA_h}\|\leqq ct^{-\alpha}e^{-\delta t}$ $(0\leqq\alpha\leqq1)$, we obtain

(6.39)
$$\|A_h^{1/2}u_h(t)\|_0 \leqq ce^{-\delta t}\|A_h^{1/2}Q_h a\|_0 + c\int_0^t (t-s)^{-3/4}e^{-\delta(t-s)}\|A_h^{1/4}u_h(s)\|_0\|A_h^{1/2}u(s)\|_0 ds$$
$$\leqq ce^{-\delta t}\|A_h^{1/2}Q_h a\|_0 + c\int_0^t (t-s)^{-3/4}e^{-\delta(t-s)}\{\|A^{1/4}a\|_0 e^{c\|a\|_0^2}\|A_h^{1/2}u_h(s)\|_0\}\, ds\,.$$

Putting $\phi(t)=e^{\delta t}\|A_h^{1/2}u_h(t)\|_0$, we have by (3.10) and (6.39)

(6.40)
$$\phi(t) \leqq c\|A^{1/2}a\|_0 + L(a)\int_0^t (t-s)^{-3/4}\phi(s)ds\,.$$

Applying Lemma 6.5 to (6.40), we obtain

(6.41)
$$\|A_h^{1/2}u_h(t)\|_0 \leqq c\|A^{1/2}a\|_0 \exp\{(L(a)^4-\delta)t\},$$

where $L(a)=\|A^{1/4}a\|_0 \exp(c\|a\|_0^2)$. In the same way we have for all $t_0 \geqq 0$

(6.42)
$$\|A_h^{1/2}u_h(t)\|_0 \leqq c\|A_h^{1/2}u_h(t_0)\|_0 \exp\{(L(u_h(t_0)^4)-\delta)(t-t_0)\}$$
$$\leqq c\|A^{1/2}a\|_0 \exp\{(L(a)^4+\delta)t_0\}\exp\{(L(u_h(t_0))^4-\delta)t\}\,.$$

From (4.11) and (6.34) it follows that

(6.43)
$$L(u_h(t_0))^4 \leqq c\{\|A^{1/4}a\|_0 \exp(c\|a\|_0^2-\delta t_0)\}^4 e^{c\|a\|_0^2}\,.$$

In view of (6.43) we put

$$t_0=\max\left\{0,\ \frac{1}{4\delta}\log\left(\frac{2c}{\delta}\|A^{1/4}a\|_0^4 e^{5c\|a\|_0^2}\right)\right\}$$

Then it is obvious that

$$L(u_h(t_0))^4 \leqq \delta/2 \quad \text{and} \quad t_0 \leqq c\|A^{1/4}a\|_0^4 e^{c\|a\|_0^2}\,.$$

Therefore we obtain from (6.42)

$$\|A_h^{1/2}u_h(t)\|_0 \leqq c\|A^{1/2}a\|_0 \exp\{L(a)^4 t_0+\delta t_0-\delta t/2\}$$
$$\leqq c\|A^{1/2}a\|_0 \exp\{L'(a)-\delta t/2\} \qquad (t \geqq t_0)\,.$$

On the other hand, we have

(6.44)
$$\|A_h^{1/2}u_h(t)\|_0 \leqq c\|A^{1/2}a\|_0 e^{L(a)^4 t-\delta t}$$
$$\leqq c\|A^{1/2}a\|_0 e^{L'(a)}e^{-\delta t}$$

for $0 \leqq t \leqq t_0$. Thus we have (6.38) for all $t \geqq 0$.

Q.E.D.

7. Error estimates in $H_0^1(\Omega)^2$.

In this section we first prove the following

THEOREM 7.1. *If* $a\in D(A)$, *then we have the estimate*

$$\|u(t)-u_h(t)\|_1\leqq K(a)e^{L(a)}e^{-\delta t}h \qquad (t>0)\,, \tag{7.1}$$

where $K(a)$ *and* $L(a)$ *are such as* (6.13), (6.14).

PROOF. We first recall the equalities

$$u(t)=e^{-tA}a-\int_0^t e^{-(t-s)A}Fu(s)ds\,, \tag{7.2}$$

$$u_h(t)=e^{-tA_h}a-\int_0^t e^{-(t-s)A_h}F_hu_h(s)ds\,. \tag{7.3}$$

From these equalities the error $e(t)=u(t)-u_h(t)$ is represented as

$$\begin{aligned} e(t)=&e^{-tA}a-e^{-tA_h}Q_ha \\ &-\int_0^t\{e^{-(t-s)A}-e^{-(t-s)A_h}Q_h\}\{Fu(s)-Fu(t)\}ds \\ &-\int_0^t\{e^{-(t-s)A}-e^{-(t-s)A_h}Q_h\}Fu(t)ds \\ &-\int_0^t e^{-(t-s)A_h}\{Q_hFu(s)-F_hu(s)\}ds \\ &-\int_0^t e^{-(t-s)A_h}\{F_hu(s)-F_hu_h(s)\}ds \\ =&I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)\,. \end{aligned} \tag{7.4}$$

Then we intend to estimate each of $I_j(t)$ and show

$$\|I_j(t)\|_1\leqq K(a)e^{-\delta t}h \quad (t>0) \qquad \text{for } j=1, 2, 3, 4\,, \tag{7.5}$$

and

$$\|I_5(t)\|_1\leqq L(a)\int_0^t(t-s)^{-3/4}e^{-\delta(t-s)}\|e(s)\|_1ds\,. \tag{7.6}$$

Firstly for $j=1$ the inequality (7.5) is obvious from (3.2). To estimate $I_2(t)$ we note that for any $a\in H$

$$\|e^{-tA}a-e^{-tA_h}Q_ha\|_1\leqq\frac{ch}{t}e^{-\delta t}\|a\|_0 \qquad (t>0)\,. \tag{7.7}$$

This inequality is proved in [24]. Actually in [24] the term $e^{-\delta t}$ does not appear. However, we can obtain (7.7) if we use the path Γ contained in $\{z\in \boldsymbol{C};$ $\operatorname{Re} z\geqq\delta\}$ for sufficiently small $\delta>0$ (see Okamoto [24]).

From (7.7) we have

$$\|I_2(t)\|_1\leqq ch\int_0^t (t-s)^{-1}e^{-\delta(t-s)}\|Fu(s)-Fu(t)\|_0 ds$$

$$\leqq c'h\int_0^t (t-s)^{-1}e^{-\delta(t-s)}\{\|A^{1/2}u(s)\|_0$$

$$\times\|A^{\gamma}(u(s)-u(t))\|_0+\|A^{\gamma}u(t)\|_0\|A^{1/2}(u(s)-u(t))\|_0\}\, ds\,.$$

Hence we obtain

$$\|I_2(t)\|_0\leqq ch\int_0^t (t-s)^{-1}e^{-\delta(t-s)} \tag{7.8}$$

$$\times\{L(a)K(a)(t-s)^{\theta}e^{-\delta s}+K(a)K(a)(t-s)^{\theta}e^{-\delta s}\}\, ds$$

$$\leqq K'(a)he^{-\delta t}\qquad (t>0)$$

by (6.4) and Proposition 6.2.

We proceed to estimation of $I_3(t)$, and notice

$$I_3(t)=\{-A^{-1}(e^{-tA}-I)+A_h^{-1}(e^{-tA_h}-I)Q_h\}\,Fu(t)\,. \tag{7.9}$$

On the other hand, we easily obtain

$$\|\{-A^{-1}(e^{-tA}-I)+A_h^{-1}(e^{-tA_h}-I)Q_h\}\,g\|_j\leqq ch^{2-j}\|g\|\quad (g\in H,\ j=0,\ 1) \tag{7.10}$$

in the same way as in Fujita and Mizutani [11]. (Actually they proved (7.10) for $j=0$ in the case of a conforming finite element space. However, their argument is valid even in our problem without any change.) Therefore we can estimate $I_3(t)$ as follows;

$$\|I_3(t)\|_1\leqq ch\|Fu(t)\|_0 \tag{7.11}$$

$$\leqq c'h\|A^{\gamma}u(t)\|_0\|A^{1/2}u(t)\|_0$$

$$\leqq K(a)he^{-\delta t}\qquad (t>0)\,.$$

To show the estimate of $I_4(t)$, we first prove the following

LEMMA 7.1. *Let* $P;\ L^2(\Omega)^2\to H$ *be the orthogonal projection. Then there exists a positive constant* c *such that for any* $w_h\in V_h$ *we have*

$$\|B^{-1/2}(P-I)w_h\|_0\leqq ch\|w_h\|_0\,, \tag{7.12}$$

and

$$\|B^{-1/2}(P-I)w_h\|_0\leqq ch^2\|w_h\|_1\,. \tag{7.13}$$

(We recall that $-B$ is the Laplace operator in $L^2(\Omega)^2$, i.e., $Bw=-\Delta w$, $D(B)=$

$H^2(\Omega)^2 \cap H_0^1(\Omega)^2$.)

PROOF OF LEMMA 7.1. Let v be an arbitrary element of $L^2(\Omega)^2$, then we put $z=B^{-1/2}v$. By means of what is called the Helmholtz decomposition we can express z as $z=y+\nabla r$ ($y\in H$, $r\in H^1(\Omega)\cap M$). Here we have $y\in H^1(\Omega)^2$ and $r\in H^2(\Omega)$, since $z\in D(B^{1/2})=H_0^1(\Omega)^2$ (see, e.g., Temam [26]). Furthermore we have

$$\|\nabla r\|_1 \leqq c\|z\|_1 \leqq c'\|v\|_0 . \tag{7.14}$$

Now we obtain for any $q_h\in M_h$

$$\begin{aligned}(B^{-1/2}(P-I)w_h, v) &= (w_h, (P-I)z) \\ &= -(w_h, \nabla r) \\ &= -(w_h, \nabla(r-q_h))\end{aligned} \tag{7.15}$$

since w_h is an element of V_h. Hence we have

$$\begin{aligned}|(B^{-1/2}(P-I)w_h, v)| &\leqq \|w_h\|_0 \inf_{q_h\in M_h} \|\nabla(r-q_h)\|_0 \\ &\leqq ch\|w_h\|_0\|\nabla r\|_1 \\ &\leqq ch\|w_h\|_0\|v\|_0\end{aligned}$$

from (7.14). By the arbitrariness of $v\in L^2(\Omega)^2$ we have (7.12).

As for (7.13) we note that

$$(B^{-1/2}(P-I)w_h, v)=(\operatorname{div} w_h, r-q_h) \qquad (q_h\in M_h), \tag{7.16}$$

which follows immediately from (7.15). By (7.16) we have

$$\begin{aligned}|(B^{-1/2}(P-I)w_h, v)| &\leqq c\|w_h\|_1 \inf_{q_h\in M_h} \|r-q_h\|_0 \\ &\leqq c'h^2\|w_h\|_1\|\nabla r\|_1 \\ &\leqq c''h^2\|w_h\|_1\|v\|_0 ,\end{aligned}$$

which implies (7.13). Q.E.D.

We can now estimate $I_4(t)$ as follows;

$$\begin{aligned}\|I_4(t)\|_1 &\leqq c\int_0^t \|A_h^{1/2}e^{-(t-s)A_h}\| \|Q_hFu(s)-F_hu(s)\|_0 ds \\ &\leqq c'\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}\|Q_hFu(s)-F_hu(s)\|_0 ds .\end{aligned} \tag{7.17}$$

On the other hand, we can prove

$$\|Q_hFu(s)-F_hu(s)\|_0 \leqq K(a)e^{-\delta s}h \qquad (s>0). \tag{7.18}$$

In fact, for an arbitrary $v_h\in V_h$ we have

$$
\begin{aligned}
(Q_h Fu(s)-F_h u(s),\ v_h) &= (Fu(s),\ v_h)-(F_h u(s),\ v_h) \\
&= b(u(s)\,;\ u(s),\ (P-I)v_h) \\
&= ((u\cdot\nabla)u,\ (P-I)v_h) \\
&= (B^{1/2}(u\cdot\nabla)u,\ B^{-1/2}(P-I)v_h)\,.
\end{aligned}
$$

Therefore,

$$|(Q_h Fu(s)-F_h u(s),\ v_h)|\leqq \|B^{1/2}(u\cdot\nabla)u\|_0 ch\|v_h\|_0$$

by (7.12). Since the inequality

$$\|(u\cdot\nabla)u\|_1\leqq c\|Au\|_0\|A^{\gamma}u\|_0\leqq K(a)e^{-\delta s} \qquad (s>0)$$

is obvious from Sobolev's imbedding theorems, we obtain (7.18). From (7.17) and (7.18) it follows that

$$
\begin{aligned}
\|I_4(t)\|_1 &\leqq ch\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}K(a)e^{-\delta s}ds \\
&\leqq K'(a)e^{-\delta t}h \qquad (t>0)\,.
\end{aligned}
$$

Thus all the inequalities in (7.5) have been proved.

It remains to show (7.6). We easily obtain

$$
\begin{aligned}
\|I_5(t)\|_1 &\leqq c\int_0^t \|A_h^{1/2}e^{-(t-s)A_h}\{F_h u(s)-F_h u_h(s)\}\|_0 ds \\
&\leqq c\int_0^t \|A_h^{3/4}e^{-(t-s)A_h}\|\,\|A_h^{-1/4}\{F_h u(s)-F_h u_h(s)\}\|_0 ds \\
&\leqq c'\int_0^t (t-s)^{-3/4}e^{-\delta(t-s)}\|A_h^{-1/4}\{F_h u(s)-F_h u_h(s)\}\|_0 ds\,.
\end{aligned}
$$

To obtain (7.6), it is therefore sufficient to show

(7.19) $$\|A_h^{-1/4}\{F_h u(s)-F_h u_h(s)\}\|_0\leqq L(a)\|e(s)\|_0$$

for all $s>0$. We prove (7.19) again by the duality argument. For any $v_h\in V_h$ we have

(7.20)
$$
\begin{aligned}
&(A_h^{-1/4}\{F_h u(s)-F_h u_h(s)\},\ v_h) \\
&= b(u\,;\ u,\ A_h^{-1/4}v_h)-b(u_h\,;\ u_h,\ A_h^{-1/4}v_h) \\
&= b(u\,;\ u-u_h,\ A_h^{-1/4}v_h)+b(u-u_h\,;\ u_h,\ A_h^{-1/4}v_h)\,.
\end{aligned}
$$

Since $\operatorname{div} u=0$, $b(u\,;\ u-u_h,\ A_h^{-1/4}v_h)$ is estimated as

(7.21)
$$
\begin{aligned}
&|b(u\,;\ u-u_h,\ A_h^{-1/4}v_h)|=|((u\cdot\nabla)(u-u_h),\ A_h^{-1/4}v_h)| \\
&\leqq \|B^{-1/4}(u\cdot\nabla)(u-u_h)\|_0\|B^{1/4}A_h^{-1/4}v_h\|_0
\end{aligned}
$$

$$\leqq c\|A^{1/4}u(s)\|_0\|e(s)\|_1\|v_h\|_0 \leqq L(a)\|e(s)\|_1\|v_h\|_0$$

by means of (5.3), (5.8) and (6.3).

On the other hand, integration by parts yields

$$b(u-u_h\,;\,u_h,\,A_h^{-1/4}v_h)=(((u-u_h)\cdot\nabla)u_h,\,A_h^{-1/4}v_h)+\frac{1}{2}(\mathrm{div}(u-u_h)u_h,\,A_h^{-1/4}v_h)\,.$$

Hence we have by (5.3) and (5.8)

$$\begin{aligned}(7.22)\qquad &|b(u-u_h\,;\,u_h,\,A_h^{-1/4}v_h)|\\ &\leqq \{\|B^{-1/4}((u-u_h)\cdot\nabla)(u_h-u)\|_0\\ &\quad+\|B^{-1/4}((u-u_h)\cdot\nabla)u\|_0\}\,\|B^{1/4}A_h^{-1/4}v_h\|_0\\ &\quad+\frac{1}{2}\|B^{-1/4}\,\mathrm{div}(u-u_h)u_h\|_0\|B^{1/4}A_h^{-1/4}v_h\|_0\\ &\leqq c\|v_h\|_0\|e(s)\|_1\{\|B^{1/4}u(s)\|_0+\|B^{1/4}u_h(s)\|_0+\|B^{1/2}u(s)\|_0\}\\ &\leqq L(a)\|e(s)\|_1\|v_h\|_0\,.\end{aligned}$$

Here use has been made of (6.3), (6.4) and (6.34). From (7.20), (7.21) and (7.22) we obtain (7.19).

PROOF OF THEOREM 7.1. From (7.5) and (7.6) it follows that

$$(7.23)\qquad e^{\delta t}\|e(t)\|_1 \leqq K(a)h+L(a)\int_0^t (t-s)^{-3/4}e^{\delta s}\|e(s)\|_1 ds\,.$$

Applying Lemma 6.5 to (7.23), we obtain

$$(7.24)\qquad \|e(t)\|_1 \leqq K(a)h\cdot\exp\{(L(a)^4-\delta)t\}\qquad (t>0)\,.$$

By the same trick as in the proof of Proposition 6.6, we have (7.1) from (7.24).

Q.E.D.

Even when the initial value a belongs only to V, we can employ the same argument as in Theorem 7.1 to obtain the following

THEOREM 7.2. *If $a\in V$, then the estimate*

$$(7.25)\qquad \|u(t)-u_h(t)\|_1 \leqq L(a)e^{L(a)}e^{-\delta t}\frac{h}{t^{1/2}}\qquad (t>0)$$

holds true.

PROOF. We again start with (7.4), and claim

$$(7.26)\qquad \|I_j(t)\|_1 \leqq L(a)e^{-\delta t}\frac{h}{t^{1/2}}\qquad (t>0)$$

for $j=1, 2, 3, 4$. The case of $j=1$ is already known in (3.1). The cases of $j=2, 3, 4$ are proved similarly to (7.5), if we use (6.16), (6.18), (6.20) and (6.28) instead of (6.15), (6.17), (6.19) and (6.27), respectively. For instance, let us write down the case of $j=4$:

$$\begin{aligned}\|I_4(t)\|_1 &\leqq c\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}\|Q_hFu(s)-F_hu(s)\|_0 ds \\ &\leqq c'h\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}\|(u\cdot\nabla)u(s)\|_1 ds \\ &\leqq c''h\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}\|Au(s)\|_0\|A^{\gamma}u(s)\|_0 ds \\ &\leqq L(a)h\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}s^{-1/2}s^{-(\gamma-1/2)}e^{-\delta s}ds \\ &\leqq L'(a)he^{-\delta t}t^{-1/2} \qquad (t>0).\end{aligned}$$

Introducing $\phi(t)=e^{\delta t}\|e(t)\|_1 t^{1/2}$, we have by (7.26) and (7.6)

$$\phi(t)\leqq L(a)h+L(a)t^{1/2}\int_0^t (t-s)^{-3/4}s^{-1/2}\phi(s)ds\,.$$

This inequality can be solved in a way similar to the proof of Lemma 6.5, and leads to

$$\|u(t)-u_h(t)\|_1\leqq L(a)\,e^{L(a)^4 t}e^{-\delta t}\frac{h}{t^{1/2}} \qquad (t>0)\,.$$

In the same way we have

$$\|u(t)-u_h(t)\|_1\leqq L(u(t_1))(t-t_1)^{-1/2}h\exp\{(L(u(t_1))^4-\delta)(t-t_1)\}$$

for all $t>t_1>0$. Then we have by putting $t_1=t/2$

$$\|u(t)-u_h(t)\|_1\leqq L\Big(u\Big(\frac{t}{2}\Big)\Big)t^{-1/2}h\exp\Big\{\Big(L\Big(u\Big(\frac{t}{2}\Big)\Big)-\delta\Big)\frac{t}{2}\Big\}\,. \tag{7.27}$$

Again we choose t_0 so that $L(u(t_0/2))\leqq\delta/2$ and $t_0\leqq L'(a)$. Then we estimate the right side of (7.27) for $t\geqq t_0$ and for $0\leqq t\leqq t_0$, separately. In a way similar to the proof of Proposition 6.6, we obtain (7.25). Q.E.D.

8. Error estimates in $L^2(\Omega)^2$.

We still consider the two-dimensional problem. The aim of this section is to show the following

THEOREM 8.1. *For any $t>0$ the estimates*

(8.1) $$\|u(t)-u_h(t)\|_0 \leqq K(a)e^{K(a)}e^{-\delta t}h^2 \qquad (a\in D(A)),$$

(8.2) $$\|u(t)-u_h(t)\|_0 \leqq L(a)e^{L(a)}e^{-\delta t}\frac{h^2}{t^{1/2}} \qquad (a\in V).$$

hold true.

PROOF. In the first place we prove (8.1). To this end we come back to (7.4), and we show

(8.3) $$\|I_j(t)\|_0 \leqq K(a)e^{-\delta t}h^2 \qquad (t>0)$$

for $j=1, 2, 3, 4$, and

(8.4) $$\|I_5(t)\|_0 \leqq K(a)e^{L(a)}e^{-\delta t}h^2 + K(a)\int_0^t (t-s)^{-1/2}e^{-\delta(t-s)}\|e(s)\|_0 ds.$$

The estimate for $I_1(t)$ is obvious from (3.2). By making use of the inequality

$$\|e^{-tA}a - e^{-tA_h}Q_h a\|_0 \leqq ch^2t^{-1}e^{-\delta t}\|a\|_0 \qquad (a\in H,\ t>0)$$

(see (7.7) and Okamoto [24]), we have

$$\|I_2(t)\|_0 \leqq ch^2\int_0^t (t-s)^{-1}e^{-\delta(t-s)}\|Fu(s)-Fu(t)\|_0 ds$$
$$\leqq K(a)e^{-\delta t}h^2 \qquad (t>0).$$

From (6.4), (6.15) and (7.10) we easily obtain

$$\|I_3(t)\|_0 \leqq K(a)e^{-\delta t}h^2 \qquad (t>0).$$

To estimate $I_4(t)$ we use the inequality

(8.5) $$\|A_h^{-1/2}\{Q_hFu(s)-F_hu(s)\}\|_0 \leqq K(a)e^{-\delta s}h^2,$$

which is proved as follows. For any $v_h\in V_h$ we have

$$(A_h^{-1/2}\{Q_hFu(s)-F_hu(s)\},\ v_h) = b(u(s);\ u(s),\ (P-I)A_h^{-1/2}v_h)$$
$$=((u\cdot\nabla)u,\ (P-I)A_h^{-1/2}v_h)$$
$$=(B^{1/2}(u\cdot\nabla)u,\ B^{-1/2}(P-I)A_h^{-1/2}v_h).$$

Hence we have by (7.13)

$$|(A_h^{-1/2}\{Q_hFu(s)-F_hu(s)\},\ v_h)| \leqq ch^2\|Au(s)\|_0\|A^r u(s)\|_0\|A_h^{-1/2}v_h\|_1$$
$$\leqq K(a)e^{-\delta s}h^2\|v_h\|_0 \qquad (s>0),$$

which implies (8.5) since $v_h\in V_h$ is arbitrary.

Thus it follows immediately that

(8.6)
$$\|I_4(t)\|_0 \leqq \int_0^t \|A_h^{1/2} e^{-(t-s)A_h}\| \|A_h^{-1/2}\{Q_h F u(s) - F_h u(s)\}\|_0 ds$$
$$\leqq K(a) e^{-\delta t} h^2 \qquad (t>0).$$

Next we deal with (8.4). We note that

(8.7)
$$\|I_5(t)\|_0 \leqq \int_0^t \|A_h^{1/2} e^{-(t-s)A_h}\| \|A_h^{-1/2}\{F_h u(s) - F_h u_h(s)\}\|_0 ds$$
$$\leqq \int_0^t (t-s)^{-1/2} e^{-\delta(t-s)} \|A_h^{-1/2}\{F_h u(s) - F_h u_h(s)\}\|_0 ds .$$

Thus it is enough to estimate $\|A_h^{-1/2}\{F_h u(s) - F_h u_h(s)\}\|_0$. This is done by the duality argument below. For any $v_h \in V_h$ we have

(8.8)
$$(A_h^{-1/2}\{F_h u(s) - F_h u_h(s)\}, v_h)$$
$$= b(u; u, A_h^{-1/2} v_h) - b(u_h; u_h, A_h^{-1/2} v_h)$$
$$= b(u; u-u_h, A_h^{-1/2} v_h) + b(u-u_h; u, A_h^{-1/2} v_h) + b(u-u_h; u_h-u, A_h^{-1/2} v_h)$$
$$\equiv b_1(s) + b_2(s) + b_3(s) .$$

Since $\mathrm{div}\, u=0$, we have

(8.9)
$$|b_1(s)| = |((u\cdot\nabla) A_h^{-1/2} v_h, u-u_h)|$$
$$\leqq c\|u(s)\|_{L^\infty} \|v_h\|_0 \|u(s)-u_h(s)\|_0$$
$$\leqq c' \|A^\gamma u(s)\|_0 \|v_h\|_0 \|e(s)\|_0$$
$$\leqq K(a) e^{-\delta s} \|e(s)\|_0 \|v_h\|_0 .$$

On the other hand, we have

(8.10)
$$|b_2(t)| = \left| \frac{1}{2}(((u-u_h)\cdot\nabla)u, A_h^{-1/2} v_h) - \frac{1}{2}(((u-u_h)\cdot\nabla) A_h^{-1/2} v_h, u) \right|$$
$$\leqq c\|u(s)-u_h(s)\|_0 \|A^\gamma u(s)\|_0 \|A_h^{-1/2} v_h\|_1$$
$$\leqq K(a) e^{-\delta s} \|e(s)\|_0 \|v_h\|_0 \qquad (s>0)$$

and

(8.11)
$$|b_3(s)| \leqq c\|u(s)-u_h(s)\|_{1/2} \|u(s)-u_h(s)\|_1 \|A_h^{-1/2} v_h\|_1$$
$$\leqq K(a) e^{L(a)} e^{-\delta s} h^2 \|v_h\|_0 .$$

Here use has been made of Theorem 7.1 and the inequality

$$|((v\cdot\nabla)w, y)| \leqq c\|v\|_r \|w\|_{s+1} \|y\|_t$$
$$(v \in H^r(\Omega)^2,\ w \in H^{s+1}(\Omega)^2,\ y \in H^t(\Omega)^2)$$

for $r \geqq 0$, $s \geqq 0$, $t \geqq 0$, $r+s+t>1$. For the proof of this inequality see, e.g., Foias and Temam [29].

From (8.9), (8.10) and (8.11) we obtain

$$\|A_h^{-1/2}\{F_h u(s)-F_h u_h(s)\}\|_0 \leqq K(a)e^{-\delta s}\|e(s)\|_0+K(a)e^{L(a)}e^{-\delta s}h^2 . \tag{8.12}$$

By this inequality and (8.7) we have (8.4).

Now it follows from (8.3) and (8.4) that

$$e^{\delta t}\|e(t)\|_0 \leqq K(a)e^{L(a)}h^2+K(a)\int_0^t (t-s)^{-1/2}e^{\delta s}\|e(s)\|_0 ds .$$

Applying Lemma 6.5 and the argument used in the proof of Proposition 6.6, we get to (8.1).

In order to prove (8.2) we have only to make a slight modification of the proof above in a way similar to the proof of Theorem 7.2. This modification is easy except for the treatment of $b_3(s)$. To estimate $b_3(s)$ we make use of

$$\|u(s)-u_h(s)\|_{1/2} \leqq L(a)e^{L(a)}e^{-\delta s}s^{-1/4}h \tag{8.13}$$

for $s>0$ and $a \in V$. If we admit this inequality, then we have

$$\begin{aligned} |b_3(s)| &\leqq c\|u(s)-u_h(s)\|_{1/2}\|u(s)-u_h(s)\|_1\|v_h\|_0 \\ &\leqq L(a)e^{L(a)}e^{-\delta s}s^{-3/4}h^2\|v_h\|_0 \qquad (s>0) . \end{aligned} \tag{8.14}$$

By (8.14) and other modified estimates we obtain

$$e^{\delta t}\|e(t)\|_0 \leqq L(a)e^{L(a)}t^{-1/2}h^2+L(a)\int_0^t (t-s)^{-1/2}e^{\delta s}\|e(s)\|_0 s^{-(\gamma-1/2)}ds .$$

Solving this inequality in the same way as in the proof of Theorem 7.2, we get (8.2). Therefore we have only to show (8.13). However, the proof of (8.13) is parallel to that of (7.1), and there a crucial part is to show

$$\|e^{-tA}a-e^{-tA_h}Q_h a\|_{1/2} \leqq ch\|A^{1/2}a\|_0 t^{-1/4} . \tag{8.15}$$

This inequality follows from (3.1),

$$\|v\|_{1/2} \leqq c\|v\|_0^{1/2}\|v\|_1^{1/2} \qquad (v \in V) , \tag{8.16}$$

and

$$\|e^{-tA}a-e^{-tA_h}Q_h a\|_0 \leqq ch\|A^{1/2}a\|_0 , \tag{8.17}$$

while (8.17) is obtained in the same way as in the proof of (3.1). The remaining part of the proof of (8.13) is shown by the arguments used so far. Hence we may omit the proof. Q.E.D.

REMARK 8.1. We can generalize the estimates (7.1), (7.25), (8.1) and (8.2) as follows: Assume that $a\in D(A^{\varepsilon})$ $(0<\varepsilon\leqq 1)$. Then there exists a positive constant $c_{\varepsilon}=c_{\varepsilon}(\Omega,\varepsilon,\|A^{\varepsilon}a\|_0)$ such that for $j=0, 1$

$$\|u(t)-u_h(t)\|_j\leqq c_{\varepsilon}h^{2-j}t^{-(1-\varepsilon)} \qquad (t>0).$$

9. Error estimates for the three dimensional problem.

In this section we consider the three dimensional problem, admitting that

$$D(A)=H^2(\Omega)^3\cap V, \tag{9.1}$$

$$\|Av\|_0\leqq c\|v\|_2, \quad \|v\|_2\leqq c\|Av\|_0 \qquad (v\in D(A)).$$

The state of mathematical analysis of the three dimensional problem is so different from the two dimensional one in that global existence of the solution of the nonstationary Navier-Stokes equation is not known unless the initial value is small enough. Hence we initially assume that there exists $T_0>0$, which may be equal to ∞, such that the strong solution of (N.S.) exists in $[0, T_0[$. Under this assumption our objective is to show the following.

THEOREM 9.1. *We assume that* $a\in D(A)$. *Then,* T_0 *being as above, for any* $T_1<T_0$ *there exist positive constants* K *and* h_0 *such that*

$$\|u(t)-u_h(t)\|_1\leqq Kh \qquad (0\leqq t\leqq T_1), \tag{9.2}$$

$$\|u(t)-u_h(t)\|_0\leqq Kh^2 \qquad (0\leqq t\leqq T_1), \tag{9.3}$$

for h *such that* $0<h\leqq h_0$. *The constants* K *and* h_0 *depend only on* Ω, T_1 *and* N_1, N_2 *defined by* (9.7) *and* (9.8) *below.*

Before entering on the proof we recall several well-known facts;

$$\|Fv\|_0\leqq c\|A^{1/2}v\|_0\|A^{3/4}v\|_0 \qquad (v\in D(A^{3/4})), \tag{9.4}$$

$$\|Fv-Fw\|_0\leqq c\|A^{1/2}v\|_0\|A^{3/4}(v-w)\|_0+c\|A^{3/4}w\|_0\|A^{1/2}(v-w)\|_0 \qquad (v, w\in D(A^{3/4})), \tag{9.5}$$

$$\|A^{-1/4}Fv\|_0\leqq c\|A^{1/2}v\|_0^2 \qquad (v\in V). \tag{9.6}$$

The solution $u(t)$ and the associated pressure $p(t)$ satisfy

$$u\in C^1([0, T_1]; H)\cap C([0, T_1]; D(A)) \quad \text{and}$$

$$p\in C(]0, T_1]; M\cap H^1(\Omega)) \quad \text{and}$$

$$N_1=\sup_{0\leq t\leq T_1}\{\|Au(t)\|_0+\|du/dt\|_0+\|\nabla p(t)\|_0\}<\infty. \tag{9.7}$$

For fixed $\theta\in]0, 1/4[$ there exists $N_2<\infty$ such that for any $0\leqq s\leqq t\leqq T_1$ we have

(9.8) $$\|A^{\alpha}\{u(t)-u(s)\}\|_0 \leqq N_2(t-s)^{\theta} \qquad (\alpha=1/2,\ 3/4).$$

As for the above facts, see Fujita and Kato [10]. The properties (9.7) and (9.8) are not explicitly written there, however, we can understand (9.7) and (9.8) by the argument used in the proof of Proposition 6.3.

REMARK 9.1. We have

$$N_1,\ N_2 \leqq c \sup_{0\leq t\leq T_1} \|Au(t)\|_0$$

with a domain constant $c>0$. Furthermore the right hand side is majorized by a constant depending only on $\|a\|_2$ and $\sup_{0\leq t\leq T_1} \|A^{1/2}u(t)\|$. In fact, by the same technique as in Fujita and Kato [10], we can estimate $\|Au(t)\|$ by means of (7.2) and (9.6).

We note also that the results in Section 3 are valid in the three dimensional case.

PROOF OF THEOREM 9.1. We estimate $I_i(t)$ $(i=1, 2, \cdots, 5)$ in (7.4) under the present circumstances. By means of (3.2), (9.4), $\cdots$, (9.8) we can easily obtain

(9.9) $$\|I_i(t)\|_j \leqq Kh^{2-j} \qquad (0\leqq t\leqq T_1)$$

for $i=1, 2, 3$ and $j=0, 1$, where K is a constant depending only on Ω, T_1, N_1 and N_2. In the same way as in the proof of (7.17) and (7.18) we obtain

(9.10) $$\|I_4(t)\|_j \leqq ch^{2-j}\int_0^t (t-s)^{-1/2}\|(u(s)\cdot\nabla)u(s)\|_1 ds.$$

(We note that Lemma 7.1 is still valid.) On the other hand, we have by Sobolev's imbedding theorems

(9.11) $$\|(u\cdot\nabla)u\|_1 \leqq c\|Au\|_0\|A^{\tilde{\gamma}}u\|_0,$$

where $\tilde{\gamma}$ is a number $>3/4$. Hence we obtain for $j=0, 1$

(9.12) $$\|I_4(t)\|_j \leqq Kh^{2-j} \qquad (0\leqq t\leqq T_1)$$

with a constant $K=K(\Omega, T_1, N_1, N_2)>0$.

If the inequalities

(9.13) $$\|I_5(t)\|_1 \leqq K\int_0^t (t-s)^{-3/4}\|e(s)\|_1 ds,$$

(9.14) $$\|I_5(t)\|_0 \leqq K\int_0^t (t-s)^{-1/2}\|e(s)\|_0 ds + Kh^2$$

are shown, then we have

$$\|e(t)\|_1 \leqq Kh + K\int_0^t (t-s)^{-3/4}\|e(s)\|_1 ds\,, \tag{9.15}$$

$$\|e(t)\|_0 \leqq Kh^2 + K\int_0^t (t-s)^{-1/2}\|e(s)\|_0 ds\,, \tag{9.16}$$

respectively, which yield (9.2) and (9.3) with the aid of Lemma 6.5. We now prove (9.13) below, while the proof of (9.14) is omitted for it is quite similar.

The inequality

$$\|I_5(t)\|_1 \leqq c\int_0^t (t-s)^{-3/4}\|A_h^{-1/4}\{F_h u(s) - F_h u_h(s)\}\|_0 ds \tag{9.17}$$

is obvious. Instead of (5.8) we use the inequality

$$\|B^{-1/4}(vw)\|_0 \leqq c\|v\|_0\|B^{1/2}w\|_0 \qquad (v \in L^2(\Omega)^3,\ w \in D(B^{1/2}))\,. \tag{9.18}$$

This inequality is proved in the same way as the proof of (5.8). Actually, the only difference is the different singulality of the kernel of $B^{-1/4}$. Hence we may omit the proof of (9.18). From the inequality (9.18) we have

$$\|A_h^{-1/4}\{F_h u(s) - F_h u_h(s)\}\|_0 \leqq c\|u(s) - u_h(s)\|_1\|A_h^{1/2}u_h(s)\|_0\|v_h\|_0 \tag{9.19}$$

in a way similar to the proof of (7.18). Hence, in order to obtain (9.13), we have only to show

$$\|A_h^{1/2}u_h(s)\|_0 \leqq K = K(\Omega, T_1, N_1, N_2) \qquad (0 \leqq s \leqq T_1)\,. \tag{9.20}$$

To show (9.20) we proceed as follows.

$$\begin{aligned}\|A_h^{1/2}u_h(s)\|_0 &\leqq c\|u_h(s) - R_h u(s)\|_1 + \|R_h u(s)\|_1 \\ &\leqq c'h^{-1}\|u_h(s) - R_h u(s)\|_0 + c\|u(s)\|_1 \\ &\leqq c'h^{-1}\|u_h(s) - u(s)\|_0 + c'h^{-1}\|u(s) - R_h u(s)\|_0 + c'\|u(s)\|_1 \\ &\leqq c'h^{-1}\|u_h(s) - u(s)\|_0 + c''\|u(s)\|_1\,.\end{aligned}$$

Therefore it suffices to show

$$\|u(s) - u_h(s)\|_0 \leqq Kh \qquad (0 \leqq s \leqq T_1)\,. \tag{9.21}$$

To this end we write the equality

$$\begin{aligned}&(du/dt - du_h/dt, v_h) + (\nabla(u - u_h), \nabla v_h) + (\nabla p, v_h) \\ &\quad + b(u\,;\, u, v_h) - b(u_h\,;\, u_h, v_h) = 0 \qquad (0 \leqq t \leqq T_1)\end{aligned} \tag{9.22}$$

for all $v_h \in V_h$, which follows from the defining equations of u and u_h. We substitute $Q_h u(t) - u_h(t)$ for v_h in (9.22). Then we have

(9.23)
$$\frac{1}{2}\frac{d}{dt}\|u(t)-u_h(t)\|_0^2+\|\nabla(u(t)-u_h(t))\|_0^2$$
$$=-(\nabla p(t),\ Q_hu-u_h)+(du/du-du_h/dt,\ u-Q_hu)$$
$$+(\nabla(u-u_h),\ \nabla(u-Q_hu))-b(u\,;\,u-u_h,\ Q_hu-u_h)$$
$$-b(u-u_h\,;\,u_h-u,\ Q_hu-u_h)-b(u-u_h\,;\,u,\ Q_hu-u_h)$$

by straightforward calculations. We estimate each term of the right side of (9.23). Firstly we have

(9.24)
$$|(\nabla p,\ Q_hu-u_h)|\leqq\varepsilon\|\nabla(u-u_h)\|_0^2+c_\varepsilon N_1^2h^2 \qquad \text{for all } \varepsilon>0\,.$$

In fact we have for any $q_h\in M_h$

$$(\nabla p,\ Q_hu-u_h)=(\nabla p,\ Q_hu-u)+(\nabla p,\ u-u_h)$$
$$=(\nabla p,\ Q_hu-u)-(p-q_h,\ \mathrm{div}(u-u_h))\,.$$

Hence it follows that

$$|(\nabla p,\ Q_hu-u_h)|\leqq\|\nabla p\|_0\|(Q_h-I)u\|_0+\|\mathrm{div}(u-u_h)\|_0 \inf_{q_h\in M_h}\|p-q_h\|_0$$
$$\leqq ch^2\|\nabla p\|_0\|Au\|_0+ch\|\nabla p\|_0\|u-u_h\|_1$$
$$\leqq\varepsilon\|\nabla(u-u_h)\|_0^2+c'N_1^2h^2\,.$$

Secondly we obtain

(9.25)
$$|(du/dt-du_h/dt,\ u-Q_hu)|=|(du/dt,\ u-Q_hu)|$$
$$\leqq\|du/dt\|_0\|(Q_h-I)u\|_0$$
$$\leqq ch^2\|du/dt\|_0\|Au(t)\|_0$$
$$\leqq c'h^2N_1^2.$$

Thirdly we have

(9.26)
$$|(\nabla(u-u_h),\ \nabla(u-Q_hu))|\leqq\varepsilon\|\nabla(u-u_h)\|_0^2+c_\varepsilon\|\nabla(u-Q_hu)\|_0^2$$
$$\leqq\varepsilon\|\nabla(u-u_h)\|_0^2+c'_\varepsilon N_1^2h^2 \qquad (\varepsilon>0)\,.$$

The nonlinear terms are estimated as follows. Since div $u=0$, we have

(9.27)
$$b(u\,;\,u-u_h,\ Q_hu-u_h)=((u\cdot\nabla)(u-u_h),\ Q_hu-u)$$
$$\leqq c\|u(t)\|_1\|\nabla(u-u_h)\|_0\|Q_hu-u\|_1$$
$$\leqq\varepsilon\|\nabla(u-u_h)\|_0^2+c_\varepsilon N_1^4h^2\,.$$

Here use has been made of

(9.28) $|((v\cdot\nabla)w, y)|\leqq\|v\|_r\|w\|_{s+1}\|y\|_t \quad (v\in H^r(\Omega)^3,\ w\in H^{s+1}(\Omega)^3,\ y\in H^t(\Omega)^3)$

for $r+s+t\geqq 3/2$, $r, s, t\geqq 0$, $\neq 3/2$, which was proved in Foias and Temam [29]. By (9.28) we obtain also

$$\begin{aligned} b(u-u_h; u_h-u, Q_hu-u_h) &= b(u-u_h; u_h-u, Q_hu-u) \\ &\leqq c\|u-u_h\|_1^2\|Q_hu-u\|_1 \\ &\leqq c'hN_1\|\nabla(u(t)-u_h(t))\|_0^2 \end{aligned}$$

and

$$\begin{aligned} b(u-u_h; u, Q_hu-u_h) &\leqq c\|u-u_h\|_1\|Au\|_0\|Q_h(u-u_h)\|_0 \\ &\leqq \varepsilon\|\nabla(u-u_h)\|_0^2+c_\varepsilon N_1^2\|u-u_h\|_0^2 \end{aligned}$$

for all $\varepsilon>0$. From above inequalities, it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)-u_h(t)\|_0^2+(1-4\varepsilon-cN_1h)\|\nabla(u(t)-u_h(t))\|_0^2$$
$$\leqq c_\varepsilon N_1^2\|u(t)-u_h(t)\|_0^2+c_\varepsilon h^2(N_1^2+N_1^4) \qquad (\varepsilon>0).$$

We put $\varepsilon=1/16$ and choose h so small that $cN_1h<1/4$. Then we have

(9.29) $$\frac{d}{dt}\|u(t)-u_h(t)\|_0^2\leqq cN_1^2\|u(t)-u_h(t)\|_0^2+ch^2(N_1^2+N_1^4).$$

Applying Gronwall's lemma to (9.29), we have (9.21). The proof of Theorem 9.1 is now complete. Q.E.D.

One might say that Theorem 9.1 is unsatisfactory in that it assumes the existence of the solution $u(t)$, which is not known a priori. Logically, what is known a priori is only the initial value a. So it is desirable to have a result in which the assumption involves only a. In this respect, we can prove the following

THEOREM 9.2. *There exists a positive constant η_0 depending only on Ω such that if a satisfies*

(9.30) $$\|Aa\|_0<\eta_0,$$

then the solution $u(t)$ exists on $[0, \infty[$ and, moreover, the error estimate

(9.31) $$\|u(t)-u_h(t)\|_j\leqq K(a)h^{2-j} \qquad (0<t<\infty)$$

holds true for $j=0, 1$. Here $K(a)$ is a constant such as in (6.13).

This theorem can be proved in the same way as the proof of Theorem 9.1. What we do is to show the following series of propositions

i) Under the assumption of Theorem 9.2 the solution $u(t)$ exists globally.

ii) Under the assumption of Theorem 9.2 the constants N_1 and N_2 are majorized by $K(a)$.

iii) Under the assumption of Theorem 9.2 $\|A_h^{1/2}u_h(t)\|_0$ is majorized by $K(a)$.

i) is a well-known fact (see Fujita and Kato [10]). To show ii) it is sufficient to prove that

$$\|A^{3/4}u(t)\|_0, \qquad \|A^{1/2}u(t)\|_0 \leqq K(a) \qquad (0<t).$$

We can prove this inequality if we re-examine the successive approximation in the existence proof of $u(t)$ (see Fujita and Kato [10]). The proof of iii) is almost the same as that of ii).

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