Ultradistributions, III.

Vector valued ultradistributions and the theory of kernels

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The theory of ultradifferentiable functions and ultradistributions with values in a sequentially complete locally convex space is developed for certain classes (M_p) and $\{M_p\}$ including the Gevrey classes.

It is applied to obtain an analogue of L. Schwartz' theory of kernels for ultradistributions, terminating in a characterization of ultradifferential operators.

As preparations ε tensor products of sequentially complete locally convex spaces and those of locally convex sheaves are discussed.

An infinitely differentiable function φ on an open set Ω in $\boldsymbol{R}^n$ is said to be ultradifferentiable of class (M_p) (resp. of class $\{M_p\}$) if for each compact set K in Ω and $h>0$ there is a constant C (resp. there are constants h and C) such that

(0.1) $$\sup_{x\in K}|D^\alpha\varphi(x)|\leqq Ch^{|\alpha|}M_{|\alpha|}, \qquad |\alpha|=0, 1, 2, \cdots.$$

The sequence M_p of positive numbers is assumed to satisfy the following conditions:

(M. 0) $$M_0=M_1=1;$$

(M. 1) $$M_p^2\leqq M_{p-1}M_{p+1}, \qquad p=1, 2, \cdots;$$

(M. 2) $$\frac{M_{p+q}}{M_pM_q}\leqq AH^{p+q}, \qquad p, q=0, 1, 2, \cdots;$$

(M. 3) $$\sum_{q=p+1}^{\infty}\frac{M_{q-1}}{M_q}\leqq Ap\frac{M_p}{M_{p+1}}, \qquad p=1, 2, \cdots;$$

for some theorems (M. 2) and (M. 3) may be replaced by the following weaker conditions respectively:

(M. 2)′ $$M_{p+1}\leqq AH^{p+1}M_p, \qquad p=0, 1, 2, \cdots;$$

(M. 3)′ $$\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p}<\infty,$$

where A and H are constants. The Gevrey sequences

$$M_p=p!^s, \qquad s>1,$$

satisfy all the conditions.

We denote by $*$ either (M_p) or $\{M_p\}$. The space $\mathcal{E}^*(\Omega)$ of all ultradifferentiable functions of class $*$ on Ω and its subspace $\mathcal{D}^*(\Omega)$ of all functions with compact support have natural locally convex topologies respectively. An ultradistribution of class $*$ on Ω is by definition a continuous linear functional on $\mathcal{D}^*(\Omega)$. We denote by $\mathcal{D}^{*\prime}(\Omega)$ the space of all ultradistributions of class $*$ on Ω which is endowed with the strong topology as the dual of $\mathcal{D}^*(\Omega)$.

In the first two parts [8] and [10] (hereafter quoted as [I] and [II]) of our series we have established the part of the theory of ultradistributions that corresponds to L. Schwartz' book [13] on distributions. This part deals with the same type of problems as Schwartz discussed in his extensive works [14], [15], [16] and [17], Chap. I on vector valued differentiable functions and vector valued distributions.

As in Schwartz' works his theory of ε tensor products $E\varepsilon F$ of locally convex spaces E and F plays an essential role in our study. He assumed that E and F are boundedly complete (=quasi-complete) but actually all results remain valid when E and F are sequentially complete as we see in § 1.

In § 2 we give an abstract formulation of Schwartz' kernel theorem.

Section 3 corresponds to Schwartz [16]. We define and topologize the spaces $\mathcal{E}^*(\Omega; F)$, $\mathcal{D}^*(\Omega; F)$ etc. of ultradifferentiable functions of class $*$ on Ω with values in a sequentially complete locally convex space F so that we have natural isomorphisms $\mathcal{E}^*(\Omega; F)=\mathcal{E}^*(\Omega)\varepsilon F$, $\mathcal{D}^*(\Omega; F)=\mathcal{D}^*(\Omega)\varepsilon F$ etc. of locally convex spaces.

In § 4 we define the space $\mathcal{D}^{*\prime}(\Omega; F)$ of ultradistributions of class $*$ on Ω with values in a sequentially complete locally convex space F by

$$\mathcal{D}^{*\prime}(\Omega; F)=L_\beta(\mathcal{D}^*(\Omega); F) \quad (\cong\mathcal{D}^{*\prime}(\Omega)\varepsilon F).$$

The two structure theorems of scalar valued ultradistributions ([I], Theorems 8.1, 8.7, and 10.3) are extended to locally bounded ultradistributions with values in F. Here an $f\in\mathcal{D}^{*\prime}(\Omega; F)$ is said to be bounded if it maps a neighborhood of 0 in $\mathcal{D}^*(\Omega)$ into a bounded set in F and locally bounded if its restriction to every relatively compact open set is bounded. Actually the vector valued versions are reduced to the scalar case by the nuclearity of $\mathcal{D}^*(\Omega)$.

The structure theorem of ultradistributions with support at the origin ([II], Theorem 3.1) is also extended to the vector valued case.

Section 5 is devoted to the theory of kernels. We have already proved in [II] the kernel theorem asserting that

$$L_\beta(\mathcal{D}^*(\Omega'), \mathcal{D}^{*\prime}(\Omega''))\cong\mathcal{D}^{*\prime}(\Omega'\times\Omega'')$$

under the correspondence

$$T\varphi(y)=\int_{\Omega'}\varphi(x)h(x, y)dx\,.$$

In this section we determine the spaces of kernels $h(x, y)$ corresponding to restricted classes of continuous linear mappings T. For example we have the following isomorphisms of locally convex spaces:

$$L_\beta(\mathcal{E}^{*\prime}(\Omega'), \mathcal{E}^*(\Omega'')) \cong \mathcal{E}^*(\Omega' \times \Omega''),$$

$$L_\beta(\mathcal{D}^*(\Omega'), \mathcal{E}^*(\Omega'')) \cong \mathcal{D}^{*\prime}(\Omega') \varepsilon \mathcal{E}^*(\Omega''),$$

$$L_\beta(\mathcal{E}^{*\prime}(\Omega'), \mathcal{D}^{*\prime}(\Omega'')) \cong \mathcal{E}^*(\Omega') \varepsilon \mathcal{D}^{*\prime}(\Omega''),$$

$$L_\beta(\mathcal{E}^*(\Omega'), \mathcal{E}^{*\prime}(\Omega'')) \cong \mathcal{E}^{*\prime}(\Omega') \varepsilon \mathcal{E}^{*\prime}(\Omega'').$$

The last space coincides with $\mathcal{E}^{*\prime}(\Omega' \times \Omega'')$ as a linear space. If $*=(M_p)$, then they are isomorphic but otherwise they are not.

The spaces $\mathcal{E}^*(\Omega)$, $\Omega \subset \boldsymbol{R}^n$, of ultradifferentiable functions and $\mathcal{D}^{*\prime}(\Omega)$, $\Omega \subset \boldsymbol{R}^n$, of ultradistributions form sheaves ([I], Theorem 5.6). To prove the same for the spaces $\mathcal{D}^{*\prime}(\Omega') \varepsilon \mathcal{E}^*(\Omega'')$, $\Omega' \subset \boldsymbol{R}^{n'}$, $\Omega'' \subset \boldsymbol{R}^{n''}$, we introduce in § 6 the notion of ε tensor products of locally convex sheaves.

$\mathcal{E}^*(\Omega)$ acts as sheaf homomorphisms $\mathcal{E}^* \to \mathcal{E}^*$ and $\mathcal{D}^{*\prime} \to \mathcal{D}^{*\prime}$ over Ω by multiplication. We prove that $\mathcal{E}^*_{x,y}(\Omega)$ acts as a continuous sheaf homomorphism $\mathcal{D}^{*\prime}_x \varepsilon \mathcal{E}^*_y \to \mathcal{D}^{*\prime}_x \varepsilon \mathcal{E}^*_y$ over Ω.

Integrals of ultradistributions in parameters are discussed from the viewpoint of ε tensor products of locally convex sheaves.

In the last Section 7 we characterize continuous sheaf homomorphisms $\mathcal{E}^* \to \mathcal{D}^{*\prime}$, $\mathcal{E}^* \to \mathcal{E}^*$ and $\mathcal{D}^{*\prime} \to \mathcal{D}^{*\prime}$ over open sets Ω in $\boldsymbol{R}^n$ as ultradifferential operators. At an Oberwolfach Conference in April, 1980 E. Albrecht and M. Neumann announced that they obtained the same characterization without continuity.

J.-L. Lions and E. Magenes [12] defined vector valued ultradistributions by duality and gave a structure theorem. The relationship with our results is briefly discussed in § 4.

The results in § 5 have been announced in [11].

Contents

1. ε tensor products of locally convex spaces.

In this section we extend L. Schwartz' theory ([15], [16], [17]) of ε tensor products of locally convex spaces to the case where spaces are sequentially complete.

We will use the following terminology. Let A be a subset of a locally convex space E which we always assume to be Hausdorff. A point f in E is said to be a *sequential limit point* (resp. *a bounding limit point*) of A if there is a sequence (resp. a bounded net) in A which converges to f in E. The set of all sequential limit points (resp. bounding limit points) of A is called the *sequential limit set* (resp. the *bounding limit set*) of A. A is said to be *sequentially closed* (resp. *boundedly closed*) if it coincides with its sequential limit set (resp. its bounding limit set). The intersection of sequentially closed sets (resp. boundedly closed sets) is always sequentially closed (resp. boundedly closed). Hence there is the smallest sequentially closed set (resp. boundedly closed set) including A, which we call the *sequential closure* (resp. the *bounding closure*) of A. The sequential (resp. bounding) closure of A clearly includes the sequential (resp. bounding) limit set of A. The former can often be strictly larger than the latter.

A is said to be *sequentially complete* (resp. *boundedly complete* if every Cauchy sequence (resp. every bounded Cauchy net) in A converges in A. Every sequentially (resp. boundedly) complete subset of E is sequentially (resp. boundedly) closed in E, and if E is sequentially (resp. boundedly) complete, every sequentially (resp. boundedly) closed subset of E is sequentially (resp. boundedly) complete.

We call the sequential closure $\bar{A}$ (resp. the bounding closure $\hat{A}$) of A in the completion $\hat{A}$ of A the *sequential completion* (resp. the *bounding completion*) of A.

Schwartz [16], [17] uses "quasi-fermé" for boundedly closed, "adhérance stricte" for bounding closure, "quasi-complet" for boundedly complete and "quasi-complété" for bounding completion.

Let E and F be locally convex spaces and let $T: E \to F$ be a continuous linear mapping. Then the inverse image $T^{-1}(A)$ of a sequentially (resp. boundedly) closed set A in F is sequentially (resp. boundedly) closed.

The mapping T is uniquely continued to a continuous linear mapping $\bar{T}: \bar{E} \to \bar{F}$ (resp. $\hat{T}: \hat{E} \to \hat{F}$, resp. $\hat{T}: \hat{E} \to \hat{F}$).

We denote by E' the dual, i.e. the space of all continuous linear functionals on E. E'_{σ}, and E'_c stand for the dual E' equipped with the weak* topology $\sigma(E', E)$ and the topology of uniform convergence on the absolutely convex compact sets in E respectively, and E_{σ} the original space E equipped with the weak topology $\sigma(E, E')$.

Schwartz [15], [16], [17] defines the ε tensor product $E\varepsilon F$ of two locally convex spaces E and F to be the space of all bilinear functionals $g(e', f')$ on $E'_c \times F'_c$ which is hypocontinuous with respect to the equicontinuous sets in E' and F'. It is equipped with the topology of uniform convergence on the products of equicontinuous sets in E' and F'.

Each $g \in E\varepsilon F$ is separately continuous on $E'_{\sigma'} \times F'_{\sigma'}$. Since $(E'_{\sigma'})' = E$ and $(F'_{\sigma'})' = F$, there are one-one correspondences among the separately continuous bilinear functionals g on $E'_{\sigma'} \times F'_{\sigma'}$, the continuous linear mappings $L: E'_{\sigma'} \to F_\sigma$ and their duals $L': F'_{\sigma'} \to E_\sigma$ defined by

$$g(e', f') = \langle Le', f'\rangle = \langle e', L'f'\rangle . \tag{1.1}$$

In terms of those linear mappings the elements of the ε tensor product $E\varepsilon F$ are characterized by each of the following properties:

(a) $L: E'_{\sigma'} \to F_\sigma$ is a continuous linear mapping which maps each equicontinuous set into a relatively compact set in F;

(b) $L: E'_{\sigma'} \to F_\sigma$ is a continuous linear mapping whose restriction to each equicontinuous set in E' is continuous with respect to the weak* topology $\sigma(E', E)$ and the original topology of F;

(c) $L: E'_c \to F$ is a continuous linear mapping;

(a′) $L': F'_{\sigma'} \to E_\sigma$ is a continuous linear mapping which maps each equicontinuous set into a relatively compact set in E;

(b′) $L': F'_{\sigma'} \to E_\sigma$ is a continuous linear mapping whose restriction to each equicontinuous set in E' is continuous with respect to the weak* topology $\sigma(F', F)$ and the original topology of E;

(c′) $L': F'_c \to E$ is a continuous linear mapping.

When the ε tensor product $E\varepsilon F$ is identified with the space $L(E'_c, F)$ of all continuous linear mappings $L: E'_c \to F$ by property (c), the topology becomes the topology ε of uniform convergence on the equicontinuous sets in E'. Thus we have the canonical isomorphisms of locally convex spaces:

$$E\varepsilon F \cong L_\varepsilon(E'_c, F) \cong L_\varepsilon(F'_c, E) . \tag{1.2}$$

Schwartz [17] Chap. I, p. 29 shows that $E\varepsilon F$ is boundedly complete (resp. complete) if both E and F are boundedly complete (resp. complete). Similarly we have

PROPOSITION 1.1. *If E and F are sequentially complete, then $E\varepsilon F$ is sequentially complete.*

PROOF. Let $L_n \in L_\varepsilon(E'_c, F)$ be a Cauchy sequence. For each $e' \in E'$ $L_n e'$ is a Cauchy sequence in F and hence converges to an Le'. Similarly for each $f' \in F'$ $L'_n f'$ converges to an Mf'. As the limit of $\langle L_n e', f'\rangle = \langle e, L'_n f'\rangle$ we have

$$\langle Le', f'\rangle = \langle e', Mf'\rangle .$$

This shows that $L: E'_{\sigma'} \to F_\sigma$ is continuous.

On the other hand, since the convergence is uniform on each equicontinuous set, L is continuous on each equicontinuous set with respect to the weak* topology and the original topology. Hence L belongs to $L_\varepsilon(E'_c, F)$ by property (b). Then it is easy to see that L_n converges to L in $L_\varepsilon(E'_c, F)$.

The tensor product $E\otimes F$ is canonically imbedded in $E\varepsilon F$ under

$$(1.3) \qquad (e\otimes f)(e', f') = \langle e, e'\rangle \langle f, f'\rangle .$$

Then the induced topology on $E\otimes F$ is the topology ε of the biequicontinuous convergence of A. Grothendieck [6], Chap. I, p. 89. Thus we have the topological imbedding

$$(1.4) \qquad E\otimes_\varepsilon F \subset E\varepsilon F .$$

Let p (resp. q) be a continuous semi-norm on E (resp. F). We define the semi-norm $p\varepsilon q$ on $E\varepsilon F$ by

$$(1.5) \qquad p\varepsilon q(g) = \sup\{|g(e', f')| \,;\, e'\in A,\ f'\in B\},$$

where A (resp. B) is the equicontinuous set of all $e'\in E'$ such that $|\langle e, e'\rangle| \leqq 1$ for all e with $p(e)\leqq 1$ (resp. of all $f'\in F'$ such that $|\langle f, f'\rangle| \leqq 1$ for all f with $q(f)\leqq 1$).

If $\mathfrak{S}$ (resp. $\mathfrak{T}$) is a family of continuous semi-norms on E (resp. F) which defines the topology of E (resp. F), then $\{p\varepsilon q\,;\, p\in\mathfrak{S}, q\in\mathfrak{T}\}$ is a family of continuous semi-norms on $E\varepsilon F$ which defines the locally convex topology of $E\varepsilon F$. The topology of $E\otimes_\varepsilon F$ is defined by their restrictions to $E\otimes F$.

In order to have the denseness of $E\otimes_\varepsilon F$ in $E\varepsilon F$ we introduce the following approximation properties (cf. Grothendieck [6] Chap. I, p. 167, Schwartz [17] Chap. I, p. 5 and G. Köthe [7], p. 232).

DEFINITION 1.2. A locally convex space is said to have the *sequential approximation property* (resp. the *weak sequential approximation property*) if the identity mapping $1: E\to E$ is in the sequential limit set (resp. the sequential closure) of the subspace $E'\otimes E$ of continuous linear mappings of finite rank in the space $L_c(E, E)$ of all continuous linear mappings from E into E equipped with the topology of uniform convergence on the absolutely convex compact sets in E.

E is said to have the *bounded approximation property* (resp. the *weak bounded approximation property*, resp. the *weak approximation property*) if $1: E\to E$ is in the bounding limit set (resp. the bounding closure, resp. the closure) of $E'\otimes E$ in $L_c(E, E)$.

Every Grothendieck space (=nuclear space) has the weak approximation property but there is an (FG)-space (=nuclear Fréchet space) which does not have the sequential approximation property (see E. Dubinsky [4]).

PROPOSITION 1.3. *If a locally convex space E has the sequential approximation property (resp. the weak sequential approximation, resp. the bounded approximation property, resp. the weak bounded approximation property, resp. the weak approximation property), then every element of $E\varepsilon F$ is in the sequential limit set (resp. the sequential closure, resp. the bounding limit set, resp. the bounding closure, resp. the closure) of $E\otimes F$ in $E\varepsilon F$ for any locally convex space F.*

Conversely if the sequential limit set (resp. the bounding limit set, resp. the closure) of $E\otimes E'_c$ in $E\varepsilon E'_c$ coincides with $E\varepsilon E'_c$, then E has the sequential approximation property (resp. the bounded approximation property, resp. the weak approximation property). If E has the Mackey topology $\tau(E, E')$ or more generally the topology γ of Schwartz [17] and if the sequential closure (resp. the bounding closure) of $E\otimes E'_c$ in $E\varepsilon E'_c$ coincides with $E\varepsilon E'_c$, then E has the weak sequential approximation property (resp. the weak bounded approximation property).

PROOF is similar to Schwartz [17] Chap. I, p. 47. Suppose that $L'\in L_\varepsilon(F'_c, E)$. By property (a′) the linear mapping $T\mapsto TL'$ from $L_c(E, E)$ into $L_\varepsilon(F'_c, E)$ is continuous. If T is in $E'\otimes E$, then TL' is clearly in $E\otimes F$ identified with a linear subspace of $E\varepsilon F\cong L_\varepsilon(F'_c, E)$. Therefore if 1 is in the sequential limit set etc. of $E'\otimes E$ in $L_c(E, E)$, then L' is in the sequential limit set etc. of $E\otimes F$ in $E\varepsilon F$.

Schwartz [17] Chap. I, p. 17 shows that $(E'_c)'_c$ is the vector space E equipped with the topology γ which is stronger than the original topology and weaker than the Mackey topology $\tau(E, E')$. Hence $L_c(E, E)$ is a topological linear subspace of $L_\varepsilon((E'_c)'_c, E)\cong E\varepsilon E'_c$. Therefore if the sequential limit set (resp. the bounding limit set, resp. the closure) of $E\otimes E'_c$ in $E\varepsilon E'_c$ is equal to $E\varepsilon E'_c$, the identity $1\in L_c(E, E)$ is in the sequential limit set (resp. the bounding limit set, resp. the closure) of $E\otimes E'_c$ in $L_c(E, E)$.

When E has the topology γ, $L_c(E, E)$ coincides with $L_\varepsilon((E'_c)'_c, E)$, so that the last part is clear.

We denote by $E\bar{\otimes}_\varepsilon F$ (resp. $E\tilde{\otimes}_\varepsilon F$, resp. $E\hat{\otimes}_\varepsilon F$) the sequential completion (resp. the bounding completion, resp. the completion) of $E\otimes_\varepsilon F$. Then we have the following proposition as a corollary.

PROPOSITION 1.4. *If E and F are sequentially complete (resp. boundedly complete, resp. complete) locally convex spaces and if either E or F has the weak sequential approximation property (resp. the weak bounded approximation property, resp. the weak approximation property), then $E\varepsilon F$ is canonically isomorphic to*

$E\bar{\otimes}_{\varepsilon}F$ (*resp.* $E\hat{\otimes}_{\varepsilon}F$, *resp.* $E\hat{\otimes}_{\varepsilon}F$).

Let E, F, H and K be locally convex spaces and let $S: E\to H$ and $T: F\to K$ be continuous linear mappings. Then the continuous linear mapping $S\varepsilon T$: $E\varepsilon F\to H\varepsilon K$ is defined by

$$((S\varepsilon T)g)(h', k')=g(S'h', T'k'), \tag{1.6}$$

its restriction $S\otimes_{\varepsilon}T: E\otimes_{\varepsilon}F\to H\otimes_{\varepsilon}K$ and the completions $S\bar{\otimes}_{\varepsilon}T: E\bar{\otimes}_{\varepsilon}F\to H\bar{\otimes}_{\varepsilon}K$, $S\hat{\otimes}_{\varepsilon}T: E\hat{\otimes}_{\varepsilon}F\to H\hat{\otimes}_{\varepsilon}K$ and $S\hat{\otimes}_{\varepsilon}T: E\hat{\otimes}_{\varepsilon}F\to H\hat{\otimes}_{\varepsilon}K$ are also continuous.

If S and T are injective homomorphisms, then these mappings are also injective homomorphisms. $S\varepsilon T$ and $S\otimes_{\varepsilon}T$ are injective whenever S and T are injective (Schwartz [17], Chap. I, p. 20).

PROPOSITION 1.5. *Let E be a locally convex space and let F_{α} be a (projective) family of locally convex spaces. Then we have the following canonical isomorphisms of locally convex spaces*:

$$E\varepsilon(\prod_{\alpha} F_{\alpha})=\prod_{\alpha}(E\varepsilon F_{\alpha}); \tag{1.7}$$

$$E\varepsilon(\varprojlim F_{\alpha})=\varprojlim(E\varepsilon F_{\alpha}). \tag{1.8}$$

The same hold if ε is replaced by $\hat{\otimes}_{\varepsilon}$ or $\hat{\otimes}_{\varepsilon}$. If $\{\alpha\}$ is countable, then the same are true with ε replaced by $\bar{\otimes}_{\varepsilon}$.

PROOF. In view of the isomorphism $E\varepsilon F\cong L_{\varepsilon}(E'_c, F)$ we have

$$E\varepsilon(\prod F_{\alpha})\cong L_{\varepsilon}(E'_c, \prod F_{\alpha})\cong\prod L_{\varepsilon}(E'_c, F_{\alpha})\cong\prod(E\varepsilon F_{\alpha}).$$

To prove the other versions, we note the canonical isomorphism of linear spaces

$$E\otimes(\bigoplus F_{\alpha})\cong\bigoplus(E\otimes F_{\alpha}).$$

Each side is imbedded in each side of the completion of (1.7). If we take the sequential closures etc. of both sides, we obtain (1.7) with ε replaced by $\bar{\otimes}_{\varepsilon}$ etc.

(1.8) and its variants are consequences of (1.7) and its variants together with Schwartz' result mentioned before the proposition.

Let E be a locally convex space. As we remarked earlier $(E'_c)'_c$ is the original space equipped with the topology γ of Schwartz [17]. According to the bipolar theorem a set in E is equicontinuous in $(E'_c)'$ if and only if it is included in an absolutely convex compact set in E. Therefore if E has the topology γ, we have the topological isomorphism

$$E'_c\varepsilon F\cong L_{\varepsilon}((E'_c)'_c, F)=L_c(E, F)$$

for any locally convex space F.

Every $L\in L(E, F)$ induces the separately continuous bilinear functional g

on $E_\sigma \times F'_{\sigma'}$ defined by

$$g(e, f')=\langle Le, f'\rangle . \tag{1.9}$$

If E has the Mackey topology, then conversely every separately continuous bilinear functional g on $E_\sigma \times F'_{\sigma'}$ can be represented as (1.9) with a unique $L\in L(E, F)$.

We denote by $B^s(E_\sigma, F'_{\sigma'})$ the linear space of all separately continuous bilinear functionals on $E_\sigma \times F'_{\sigma'}$. The topology on $L(E, F)$ of uniform convergence on a family $\mathfrak{S}$ of bounded sets in E then corresponds to the topology on $B^s(E_\sigma, F'_{\sigma'})$ of uniform convergence on the family of sets of the form $A\times B$, where A is a member of $\mathfrak{S}$ and B is an equicontinuous set in F'. We write $B^s_{\mathfrak{S},\varepsilon}(E_\sigma, F'_{\sigma'})$ for $B^s(E_\sigma, F'_{\sigma'})$ equipped with this topology. Since a locally convex space E with the Mackey topology has the topology γ, we obtain the following.

PROPOSITION 1.6. *If a locally convex space E has the Mackey topology, then we have the topological isomorphisms*

$$E'_c\varepsilon F\cong L_c(E, F)\cong B^s_{c,\varepsilon}(E_\sigma, F'_{\sigma'}) \tag{1.10}$$

for any locally convex space F.

If E is barrelled, then it has the topology γ and the dual E' is by the Banach-Steinhaus theorem boundedly complete in the weak* topology and hence in the topology of uniform convergence on the compact sets in E.

If E is bornologic and sequentially complete, then it has the topology γ and E'_c is complete because the absolutely convex closed hull of every sequence converging to zero is compact (see Proposition 1.11). Hence we have the following.

PROPOSITION 1.7. (i) *If E is a barrelled space, if F is a sequentially complete (resp. boundedly complete) locally convex space and if E'_c or F has the weak sequential (resp. the weak bounded) approximation property, then we have the topological isomorphisms*

$$L_c(E, F)\cong E'_c\varepsilon F\cong E'_c\hat{\otimes}_\varepsilon F \qquad (\textit{resp.}\ \cong E'_c\tilde{\otimes}_\varepsilon F). \tag{1.11}$$

(ii) *If E is a sequentially complete bornologic space, if F is a complete locally convex space and if E'_c or F has the weak approximation property, then we have the topological isomorphisms*

$$L_c(E, F)\cong E'_c\varepsilon F\cong E'_c\hat{\otimes}_\varepsilon F. \tag{1.12}$$

If E is moreover Montel, then we can replace $L_c(E, F)$ in (1.10), (1.11) and (1.12) by $L_\beta(E, F)$ and E'_c there by E'_β each equipped with the topology of uniform convergence on the bounded sets in E.

Let Ω be a locally compact Hausdorff space. We denote by $C(\Omega)$ the space of all continuous functions on Ω with values in $\boldsymbol{C}$ (or in $\boldsymbol{R}$), equipped with the topology of uniform convergence on the compact sets in Ω. $C(\Omega)$ is clearly a complete locally convex space. It is a Fréchet space if and only if Ω is σ compact and is a Banach space if and only if Ω is compact.

PROPOSITION 1.8. *For any locally compact space Ω $C(\Omega)$ has the weak approximation property. If Ω is paracompact, then $C(\Omega)$ has the bounded approximation property. If Ω is σ compact and metrizable, then $C(\Omega)$ has the sequential approximation property.*

PROOF. Let A be an absolutely convex compact set in $C(\Omega)$ and let K be an arbitrary compact set in Ω. Since $f \in A$ are equicontinuous on a compact neighborhood L of K, we can find for each $\varepsilon > 0$ a finite open covering $\mathcal{U} = \{U_1, \cdots, U_m\}$ of K in Ω such that if x and y are in the same U_j then $|f(x) - f(y)| \leqq \varepsilon$ for all $f \in A$. Let χ_j, $j = 1, \cdots, m$, be a partition of unity subordinate to $\mathcal{U}$, that is, continuous functions with values in $[0, 1]$ with compact support in U_j and such that $\sum_{j=1}^{m} \chi_j(x) \leqq 1$ on Ω and $=1$ on K. Let $\delta(x_j)$, $j=1, \cdots, m$, be the point measures at some x_j in U_j. Then $\sum_{j=1}^{m} \delta(x_j) \otimes \chi_j$ belongs to $C(\Omega)' \otimes C(\Omega)$ and we have

$$\sup_{x \in K} \left| \sum_{j=1}^{m} \langle f, \delta(x_j) \rangle \chi_j(x) - f(x) \right| \leqq \sup_{x \in K} \sum_{j=1}^{m} |f(x_j) - f(x)| \chi_j(x) \leqq \varepsilon$$

for all f in A.

When Ω is paracompact, we take a locally finite relatively compact open covering $\mathcal{V}$ of Ω and choose the above open sets U_j so that each U_j is included in an element of $\mathcal{V}$. Then $\{\sum \delta(x_j) \otimes \chi_j\}$ is equicontinuous in $L(C(\Omega), C(\Omega))$ and hence bounded in $L_c(C(\Omega), C(\Omega))$. In fact, for any compact set L in Ω the closure L_1 of the union of all V in $\mathcal{V}$ with $V \cap L \neq \emptyset$ is compact and we have

$$\sup_{x \in L} \left| \sum_{j=1}^{m} f(x_j) \chi_j(x) \right| \leqq \sup_{x \in L_1} |f(x)|.$$

When Ω is σ compact and metrizable, we choose a locally finite relatively compact open covering $\mathcal{V}$ of Ω and a sequence of compact sets $K_1 \Subset K_2 \Subset \cdots \Subset K_i \Subset \cdots$ in Ω such that $\bigcup K_i = \Omega$. Then for each $K = K_i$ we take a finite open covering $\mathcal{U} = \{U_1, \cdots, U_m\}$ of K in Ω so that each U_j is included in some element of $\mathcal{V}$ and has diameter less than $1/i$. Let $\chi_j^{(i)}$, $j=1, \cdots, m$, be a partition of unity as above and let $x_j^{(i)}$ be a point in $\operatorname{supp} \chi_j$. Then $I_i = \sum \delta(x_j^{(i)}) \otimes \chi_j^{(i)}$ is an equicontinuous sequence in $L(C(\Omega), C(\Omega))$ and $I_i f$ converges to f in $C(\Omega)$ for every $f \in C(\Omega)$ because f is uniformly continuous on each compact set in Ω. Since the topology of uniform convergence on the precompact sets coincides with the topology of simple convergence on each equicontinuous set, I_i converges to the identity in $L_c(C(\Omega), C(\Omega))$.

Now let F be a locally convex space. We denote by $C(\Omega; F)$ the space of all continuous functions $\boldsymbol{f}$ on Ω with values in F and endow it with the topology of uniform convergence on the compact sets in Ω. If K is a compact set in Ω and q is a continuous semi-norm on F, q_K defined by

$$q_K(\boldsymbol{f}) = \sup_{x \in K} q(\boldsymbol{f}(x)) \tag{1.13}$$

is a continuous semi-norm on $C(\Omega; F)$. The locally convex topology of $C(\Omega; F)$ is determined by such semi-norms.

It is easily proved that if F is sequentially complete (resp. boundedly complete, resp. complete), then $C(\Omega; F)$ is also sequentially complete (resp. boundedly complete, resp. complete). We also have the topological imbedding $C(\Omega) \otimes_\varepsilon F \subset C(\Omega; F)$. More precisely we have the following.

PROPOSITION 1.9. *The ε tensor product $C(\Omega)\varepsilon F$ is canonically imbedded in $C(\Omega; F)$ for any locally convex space F.*

PROOF. Let $L \in L_\varepsilon(C(\Omega)'_c, F)$. Since the mapping $x \mapsto \delta(x)$ is continuous from Ω into $C(\Omega)'_c$, $\boldsymbol{f}(x) = L\delta(x)$ belongs to $C(\Omega; F)$. The mapping $L \mapsto \boldsymbol{f}$ is injective because $\{\delta(x); x \in \Omega\}$ is total in $C(\Omega)'_c$, whose dual is equal to $C(\Omega)$.

In order to prove that the imbedding is topological we first prove that for any $\mu \in C(\Omega)'$ we have

$$L\mu = \int_\Omega \boldsymbol{f}(x)\mu(dx) \tag{1.14}$$

in the sense of the Dunford-Pettis integral, i.e.

$$\langle L\mu, f' \rangle = \int_\Omega \langle \boldsymbol{f}(x), f' \rangle \mu(dx) \tag{1.15}$$

for all $f' \in F'$.

μ is known to be a measure with compact support (see Bourbaki [2]). Let K be the support. For each open covering $\mathcal{U}$ of K in Ω take a partition of unity $\{\chi_j; j=1, \cdots, m\}$ subordinate to $\mathcal{U}$ and points x_j in $\operatorname{supp}\chi_j$. If φ is in a compact set in $C(\Omega)$, then $\sum_{j=1}^m \varphi(x_j)\chi_j(x)$ converges to φ uniformly on K and uniformly in φ as $\mathcal{U}$ becomes finer. Hence the linear combination of point measures $\sum_{j=1}^m \int \chi_j(x)\mu(dx)\delta(x_j)$ tends to μ in $C(\Omega)'_c$, so that $\sum \int \chi_j(x)\mu(dx)\boldsymbol{f}(x_j)$ converges to $L\mu$ in F. On the other hand, $\sum \int \chi_j(x)\mu(dx)\langle \boldsymbol{f}(x_j), f' \rangle$ converges to $\int \langle \boldsymbol{f}(x), f' \rangle \mu(dx)$ for any $f' \in F'$. Hence we have (1.15).

Let K be a compact set in Ω, let q be a continuous semi-norm on F and consider the equicontinuous sets

$$A=\{\mu\in C(\Omega)';\ \mathrm{supp}\,\mu\subset K,\ \int|\mu(dx)|\leqq 1\},$$

$$B=\{f'\in F';\ |\langle f, f'\rangle|\leqq 1 \text{ for all } f \text{ with } q(f)\leqq 1\}.$$

Then we have

$$\begin{aligned}&\sup\{|\langle L\mu, f'\rangle|;\ \mu\in A,\ f'\in B\}\\ &=\sup\left\{\left|\int\langle \boldsymbol{f}(x), f'\rangle\mu(dx)\right|;\ \mu\in A,\ f'\in B\right\}\\ &=q_K(\boldsymbol{f}).\end{aligned}$$

Since the locally convex topology of $C(\Omega)\varepsilon F$ is determined by these semi-norms, the imbedding is topological.

THEOREM 1.10. (i) *If Ω is a σ compact metrizable locally compact space and if F is a sequentially complete locally convex space, then we have the canonical isomorphisms of locally convex spaces*

$$C(\Omega)\bar{\otimes}_\varepsilon F\cong C(\Omega)\varepsilon F\cong C(\Omega;F). \tag{1.16}$$

(ii) *If Ω is a paracompact locally compact space and if F is a boundedly complete locally convex space, then*

$$C(\Omega)\hat{\otimes}_\varepsilon F\cong C(\Omega)\varepsilon F\cong C(\Omega;F). \tag{1.17}$$

(iii) *If Ω is a locally compact space ana if F is a complete locally convex space, then*

$$C(\Omega)\hat{\hat{\otimes}}_\varepsilon F\cong C(\Omega)\varepsilon F\cong C(\Omega;F). \tag{1.18}$$

PROOF. The isomorphisms on the left follow from Propositions 1.4 and 1.8. In order to prove the isomorphisms on the right it suffices to show that $C(\Omega)\otimes F$ is sequentially dense (resp. boundedly dense, resp. dense) in $C(\Omega;F)$. Let $\boldsymbol{f}$ be an arbitrary element in $C(\Omega;F)$. If $\chi_j\in C(\Omega)$, $j=1,\cdots,m$, are partition of unity and $x_j\in\mathrm{supp}\,\chi_j$ as in the proof of Proposition 1.8, then we can prove in the same way that $\sum_{j=1}^m\chi_j(x)\otimes\boldsymbol{f}(x_j)\in C(\Omega)\otimes F$ converges to $\boldsymbol{f}$ in $C(\Omega;F)$.

Under the assumption of the theorem we can identify $C(\Omega;F)$ with $C(\Omega)\varepsilon F\cong L_\varepsilon(C(\Omega)'_c, F)$. Since $C(\Omega)'$ is the space of all measures with compact support on Ω, this means that the Dunford-Pettis integral $\int\boldsymbol{f}(x)\mu(dx)$ belongs to F for any $\boldsymbol{f}\in C(\Omega;F)$ and $\mu\in C(\Omega)'$ and that the mapping $\mu\mapsto\int\boldsymbol{f}(x)\mu(dx)$ is continuous from $C(\Omega)'_c$ into F.

PROPOSITION 1.11. *Suppose either that F is a sequentially complete locally convex space and Ω is a metrizable compact space or that F is a boundedly complete locally convex space and Ω is a compact space. Then for any continuous* *mapping* $\boldsymbol{f}: \Omega \to F$ *the convex closed hull and the absolute convex closed hull of the image* $\boldsymbol{f}(\Omega)$ *in* F *are compact and are represented as the set of integrals*

$$\left\{\int_\Omega \boldsymbol{f}(x)\mu(dx);\ \mu \in C(\Omega)',\ \mu \geqq 0,\ \mu(\Omega)=1\right\} \tag{1.19}$$

and

$$\left\{\int_\Omega f(x)\mu(dx);\ \mu \in C(\Omega)',\ \|\mu\|_{C(\Omega)'} \leqq 1\right\} \tag{1.20}$$

respectively.

PROOF. The set of measures $\{\mu \in C(\Omega)';\ \mu \geqq 0,\ \mu(\Omega)=1\}$ (resp. $\{\mu \in C(\Omega)';\ \|\mu\|_{C(\Omega)'} \leqq 1\}$) is convex (resp. absolutely convex) and compact in $C(\Omega)'_c$. Hence its continuous linear image (1.19) (resp. (1.20)) is also convex (resp. absolutely convex) and compact in F. If we approximate $\boldsymbol{f}$ by elements in $C(\Omega)' \otimes F$ as in the proof of Theorem 1.10, we find that the convex hull (resp. the absolutely convex hull) of $\boldsymbol{f}(\Omega)$ is dense in the image (1.19) (resp. (1.20)).

We assume from now on that Ω is a σ compact metrizable locally convex space or an open set in $\boldsymbol{R}^n$ and that F is a sequentially complete locally convex space.

Suppose that E is a locally convex space of numerical functions on Ω which is included in $C(\Omega)$ and has a stronger topology than that of $C(\Omega)$. Then $E \varepsilon F$ is continuously included in $C(\Omega; F) = C(\Omega)\varepsilon F$. An $\boldsymbol{f} \in C(\Omega; F)$ belongs to $E \varepsilon F$ if and only if the following conditions are satisfied:

(i) For any $f' \in F'$ $\langle \boldsymbol{f}(x), f' \rangle$ belongs to E;

(ii) The mapping $L': f' \mapsto \langle \boldsymbol{f}(x), f' \rangle$ from F' into E satisfies equivalent conditions (a′), (b′) and (c′) in order that L' belong to $L(F'_c, E)$.

According to (a′) condition (ii) is divided into the following two conditions:

(ii)′ L' maps every equicontinuous set in F' to a relatively compact set in E;

(ii)″ $L': F'_{\sigma'} \to E_\sigma$ is continuous or equivalently for any $e' \in E'$ there is an $f \in F$ such that

$$\langle \langle \boldsymbol{f}(\cdot), f' \rangle, e' \rangle = \langle f, f' \rangle, \quad f' \in F'. \tag{1.21}$$

As Grothendieck [6] Chap. II, p. 78 shows (ii)′ implies (ii)″ if F is complete. In fact, (ii)′ implies that $(L')': E'_c \to (F')^*_\varepsilon$ is continuous, where $(F')^*_\varepsilon$ is the space of all linear functionals on F' equipped with the topology of uniform convergence on the equicontinuous sets in F'. For each $x \in \Omega$ let $\varepsilon(x) \in E'$ be the functional represented by the point measure $\delta(x)$ at x, i.e. $\langle f, \varepsilon(x) \rangle = f(x)$. Then we have

$$\langle \langle \boldsymbol{f}(\cdot), f' \rangle, \varepsilon(x) \rangle = \langle \boldsymbol{f}(x), f' \rangle,$$

so that $(L')'\varepsilon(x) = \boldsymbol{f}(x) \in F$ for all $x \in \Omega$. On the other hand, $\{\varepsilon(x); x \in \Omega\}$ is total in E'_c because for any $f \in E=(E'_c)'$ $\langle f, \varepsilon(x)\rangle = f(x)=0$, $x\in\Omega$, implies $f=0$. Therefore the linear combinations of $\varepsilon(x)$, $x\in\Omega$, are dense in E'_c. Since F is a complete linear subspace of $(F')^*_\varepsilon$, $(L')'$ maps E' into F.

If the sequential closure (resp. the bounding closure) of the set of linear combinations of $\varepsilon(x)$, $x\in\Omega$, in E'_c coincides with E'_c we have the same conclusion for any sequentially complete (resp. boundedly complete) locally convex space F.

More generally if $\nu\in E'$ is the functional represented by a measure $\mu\in C(\Omega)'$ with compact support in Ω, then we have by Theorem 1.10

$$(L')'\nu=\int \boldsymbol{f}(x)\mu(dx)\in F.$$

Hence we obtain the first part of the following.

THEOREM 1.12. *Let Ω be a σ compact metrizable locally compact space, let E be a space of continuous numerical functions on Ω equipped with a locally convex topology stronger than the topology of uniform convergence on the compact sets in Ω and let F be a locally convex space.*

Assume one of the following.

(a) *F is sequentially complete and the sequential closure in E'_c of the functionals represented by measures with compact support on Ω coincides with E'_c.*

(b) *F is boundedly complete and the bounding closure in E'_c of the functionals represented by measures with compact support on Ω coincides with E'_c.*

(c) *F is complete.*

Then $E\varepsilon F=L_\varepsilon(F'_c, E)$ is identified with the space of all $\boldsymbol{f}\in C(\Omega; F)$ which satisfies the following conditions:

(i) *For any $f'\in F'$ the function $\langle \boldsymbol{f}(\cdot), f'\rangle$ belongs to E;*

(ii) *For any equicontinuous set A in F' $\{\langle \boldsymbol{f}(\cdot), f'\rangle; f'\in A\}$ is relatively compact in E.*

Condition (ii) *is not necessary if either E is semi-Montel (i.e. every bounded set is relatively compact) or F is a Schwartz space, and if E is a webbed space of De Wilde* [3].

The last part is proved in the same way as Grothendieck [6] Chap. II, pp. 79-80.

2. Abstract kernel theorems.

Grothendieck [6] introduced two other important locally convex topologies in the tensor product $E\otimes F$ of locally convex spaces E and F.

Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be the set of all continuous semi-norms on E (resp. F) or a family of continuous semi-norms which defines the topology of E (resp. F). Then the projective tensor product topology π on $E\otimes F$ is defined by the family of semi-norms $\{p\otimes_\pi q\,;\ p\in\mathfrak{S},\ q\in\mathfrak{T}\}$, where

$$(2.1)\qquad p\otimes_\pi q(g)=\inf\left\{\sum_{j=1}^{m} p(e_j)q(f_j)\,;\ g=\sum_{j=1}^{m} e_j\otimes f_j\right\}.$$

The topology π is the strongest locally convex topology on $E\otimes F$ such that the canonical bilinear mapping $E\times F\to E\otimes F$ is continuous. The topology π is also a unique locally convex topology on $E\otimes F$ such that for any locally convex space H the canonical bilinear mapping $E\times F\to E\otimes F$ induces an isomorphism from the space $L(E\otimes F, H)$ of continuous linear mappings onto the space $B(E, F; H)$ of continuous bilinear mappings (Grothendieck [6], Chap. I, p. 30).

Consequently, if we denote by $E\otimes_\pi F$ the tensor product $E\otimes F$ equipped with the topology π, the dual $(E\otimes_\pi F)'$ is identified with the space $B(E, F)$ of all continuous bilinear functionals on $E\times F$ and the equicontinuous sets in $(E\otimes_\pi F)'$ with the equicontinuous sets in $B(E, F)$. Hence the topology π is exactly the topology of uniform convergence on the equicontinuous sets in $B(E, F)$.

We denote by $E\bar{\otimes}_\pi F$ (resp. $E\hat{\otimes}_\pi F$, resp. $E\hat{\otimes}_\pi F$) the sequential completion (resp. the bounding completion, resp. the completion) of $E\otimes_\pi F$.

Let $B_\beta(E, F)$ be the space $B(E, F)$ equipped with the topology of bibounded convergence, i.e. the topology of uniform convergence on the products $A\times B$ of bounded sets A in E and B in F. Then the canonical mappings $(E\hat{\otimes}_\pi F)'_\beta\to(E\otimes_\pi F)'_\beta\to B_\beta(E, F)$ are always continuous bijections.

If E and F are both (DF)-spaces, then these mappings are topological isomorphisms and $E\otimes_\pi F$ and $E\hat{\otimes}_\pi F$ are also (DF)-spaces (Grothendieck [6], Chap. I, p. 43).

If E is an (FG)-space (=Grothendieck Fréchet space=nuclear Fréchet space) and F is a Fréchet space, then we have again the topological isomorphisms $(E\hat{\otimes}_\pi F)'_\beta=(E\otimes_\pi F)'_\beta\cong B_\beta(E, F)$ and $E\hat{\otimes}_\pi F$ is a Fréchet space (Grothendieck [6], Chap. II, p. 75).

If $S: E\to H$ and $T: F\to K$ are continuous linear mappings, then the tensor product $S\otimes T: E\otimes_\pi F\to H\otimes_\pi K$ is also a continuous linear mapping and therefore induces continuous linear mappings $S\bar{\otimes}_\pi T: E\bar{\otimes}_\pi F\to H\bar{\otimes}_\pi K$, $S\hat{\otimes}_\pi T: E\hat{\otimes}_\pi F\to H\hat{\otimes}_\pi K$ and $S\hat{\otimes}_\pi T: E\hat{\otimes}_\pi F\to H\hat{\otimes}_\pi K$.

The topology π is called the projective tensor product topology because if E (resp. F) has the projective limit topology associated with a family of linear mappings $S_\mu: E\to E_\mu$ into locally convex spaces E_μ (resp. $T_\nu: F\to F_\nu$ into locally convex spaces F_ν), then $E\otimes_\pi F$ has the projective limit topology associated with the family $S_\mu\otimes T_\nu: E\otimes F\to E_\mu\otimes_\pi F_\nu$.

If $S: E\to H$ and $T: F\to K$ are surjective homomorphisms of locally convex spaces, then $S\otimes_\pi T: E\otimes_\pi F\to H\otimes_\pi K$ is also a surjective homomorphism and therefore $S\bar{\otimes}_\pi T$ (resp. $S\hat{\bar{\otimes}}_\pi T$, resp. $S\hat{\otimes}_\pi T$) is a topological homomorphism with sequentially dense (resp. boundedly dense, resp. dense) image (Grothendieck [6], Chap. I, p. 38). Hence if every quotient of $E\bar{\otimes}_\pi F$ etc. is sequentially complete etc., then it follows that $S\bar{\otimes}_\pi T$ etc. is surjective. This is the case if E and F are metrizable or if $E\bar{\otimes}_\pi F$ etc. is a reflexive (DF)-space (see [7], § 34).

The identity mapping $E\otimes_\pi F\to E\otimes_\varepsilon F$ is clearly continuous. It is a topological isomorphism if E or F is a Grothendieck space (=a nuclear space). In that case we write $E\bar{\otimes}F$, $E\hat{\bar{\otimes}}F$ and $E\hat{\otimes}F$ for $E\bar{\otimes}_\pi F=E\bar{\otimes}_\varepsilon F$ etc.

Conversely a locally convex space E is a Grothendieck space if $E\otimes_\pi F=E\otimes_\varepsilon F$ for $F=l^1$ or c_0 (Grothendieck [6], Chap. II, p. 42).

The other topology on $E\otimes F$ is obtained if we replace the continuity of the canonical bilinear mapping $E\times F\to E\otimes F$ by the separate continuity.

The inductive tensor product topology ι on the tensor product $E\otimes F$ of locally convex spaces E and F is by definition the strongest locally convex topology on $E\otimes F$ such that the canonical bilinear mapping $E\times F\to E\otimes F$ is separately continuous.

The topology ι is also a unique locally convex topology on $E\otimes F$ such that for any locally convex space H the canonical bilinear mapping $E\times F\to E\otimes F$ induces an isomorphism from the space $L(E\otimes F, H)$ of continuous linear mappings onto the space $B^s(E, F; H)$ of separately continuous bilinear mappings.

The dual $(E\otimes_\iota F)'$ is identified with the space $B^s(E, F)$ of all separately continuous bilinear functionals and the equicontinuous sets in $(E\otimes_\iota F)'$ with the separately equicontinuous sets in $B^s(E, F)$. The topology ι is therefore the topology of uniform convergence on the separately equicontinuous sets in $B^s(E, F)$ (Grothendieck [6], Chap. I, p. 73).

The sequential completion $E\bar{\otimes}_\iota F$ etc. are defined similarly. Continuous linear mappings $S: E\to H$ and $T: F\to K$ induce continuous linear mappings $S\otimes T: E\otimes_\iota F\to H\otimes_\iota K$, $S\bar{\otimes}_\iota T: E\bar{\otimes}_\iota F\to H\bar{\otimes}_\iota K$ etc.

The topology ι is called the inductive tensor product topology because if E (resp. F) has the inductive limit topology associated with a family of linear mappings $S_\mu: E_\mu\to E$ from locally convex spaces E_μ (resp. $T_\nu: F_\nu\to F$ from locally convex spaces F_ν), then $E\otimes_\iota F$ has the inductive limit locally convex topology associated with the family of linear mappings $S_\mu\otimes T_\nu: E_\mu\otimes_\iota F_\nu\to E\otimes F$ (Grothendieck [6], Chap. I, p. 76).

The topology ι is always stronger than the topology π. They are identical under the following circumstances:

PROPOSITION 2.1. *Suppose that locally convex spaces E and F satisfy one of the following conditions*:

(a) *E is metrizable and barrelled and F is metrizable;*

(b) *E is a barrelled (DF)-space such that each absolutely convex closed bounded set contains a countable dense subset and F is a (DF)-space;*

(c) *E and F are barrelled (DF)-spaces.*

Then $E\otimes_\pi F=E\otimes_\iota F$ and we have the following canonical isomorphisms of locally convex spaces:

$$B_\beta^s(E, F)=B_\beta(E, F)\cong L_\beta(E, F')\cong L_\beta(F, E'). \tag{2.2}$$

PROOF. The proofs in cases (a) and (c) are well-known (see [7], § 40.2).

In case (b) we have only to prove that every separately equicontinuous family $\mathfrak{M}$ in $B^s(E, F)$ is equi-hypocontinuous ([7], § 40.2 (10), p. 160). Since E is barrelled, $\mathfrak{M}._B=\{h(\cdot, b);\ h\in\mathfrak{M},\ b\in B\}$ is an equicontinuous set in E' for any bounded set B in F. Let A be an absolutely convex bounded set in E. Then the semi-norm $q(f)=\sup\{|h(a, f)|;\ h\in\mathfrak{M},\ a\in A\}$ on F is bounded on each bounded set B in F. If D is a countable dense set in A, then $q(f)$ is also the supremum of the countable family of continuous semi-norms $\left\{\sup_{h\in\mathfrak{M}}|h(d, f)|;\ d\in D\right\}$. Hence $q(f)$ is continuous, i.e. $\mathfrak{M}_{A}.=\{h(a, \cdot);\ h\in\mathfrak{M},\ a\in A\}$ is equicontinuous.

We note that every Montel (DF)-space E satisfies condition (b) because every closed bounded set A in E is a metrizable compact set ([9], Chap. I § 12, p. 95).

If E is an (FG)-space or a (DFG)-space (=complete Grothendieck (DF)-space), then the strong dual E' is also a Grothendieck space (Grothendieck [7], Chap. II, p. 40) and hence we have the canonical isomorphism

$$E'\hat{\otimes}F=E'\varepsilon F\cong L_\beta(E, F) \tag{2.3}$$

for any complete locally convex space F.

Thus we obtain the following theorem due to Grothendieck ([7], Chap. II, p. 76).

THEOREM 2.2. *Assume one of the following:*

(a) *E is an (FG)-space and F is a Fréchet space;*

(b) *E is a (DFG)-space and F is a complete (DF)-space.*

Then we have the following canonical isomorphisms of locally convex spaces:

$$E\hat{\otimes}_\varepsilon F=E\hat{\otimes}_\pi F=E\hat{\otimes}_\iota F; \tag{2.4}$$

$$(E\hat{\otimes}F)'_\beta\cong B_\beta(E, F)=B_\beta^s(E, F)\cong L_\beta(E, F')\cong L_\beta(F, E')\cong E'\hat{\otimes}F'. \tag{2.5}$$

Suppose that $E=\varinjlim E_\nu$ is an (LFG)-space represented as the strict inductive limit of a sequence of (FG)-spaces E_ν and $F=\varinjlim F_\nu$ is an (LF)-space represented as the strict inductive limit of a sequence of Fréchet spaces F_ν.

Then the canonical mapping $E_\nu\hat{\otimes}F_\nu\to E_{\nu+1}\hat{\otimes}F_{\nu+1}$ is a topological imbedding as the canonical mapping $E_\nu\varepsilon F_\nu\to E_{\nu+1}\varepsilon F_{\nu+1}$. Hence $\varinjlim E_\nu\hat{\otimes}F_\nu$ is an (LF)-space. In particular it is complete.

Since $E_\nu\otimes_\iota F_\nu=E_\nu\otimes_\varepsilon F_\nu$ by Theorem 2.2, we have a canonical continuous injection from

$$E\otimes_\iota F=\varinjlim_{\substack{\mu\to\infty\\ \nu\to\infty}} E_\mu\otimes_\iota F_\nu=\varinjlim_{\nu\to\infty} E_\nu\otimes_\iota F_\nu$$

into $\varinjlim E_\nu\hat{\otimes}F_\nu$. It is actually a topological imbedding with dense image, so that the completed inductive tensor product $E\hat{\otimes}_\iota F$ is identified with the (LF)-space $\varinjlim E_\nu\hat{\otimes}F_\nu$. In fact, let r be a continuous semi-norm on $E\otimes_\iota F$. Then its restriction r_ν to $E_\nu\otimes F_\nu$ is uniquely extended to a continuous semi-norm $\hat{r}_\nu$ on $E_\nu\hat{\otimes}F_\nu$. Since the imbedding $E_\nu\hat{\otimes}F_\nu\to E_{\nu+1}\hat{\otimes}F_{\nu+1}$ is topological, $\hat{r}_\nu$ is equal to the restriction of $\hat{r}_{\nu+1}$ to $E_\nu\hat{\otimes}F_\nu$. Let $\hat{r}$ be the semi-norm on $\varinjlim E_\nu\hat{\otimes}F_\nu$ defined to be $\hat{r}_\nu$ on $E_\nu\hat{\otimes}F_\nu$. Clearly $\hat{r}$ is continuous and extends r.

Thus we have proved the first part of the following.

THEOREM 2.3. *Suppose that the strict inductive limits of Fréchet spaces*

$$E=\varinjlim E_\nu\,, \tag{2.6}$$

$$F=\varinjlim F_\nu \tag{2.7}$$

are an (LFG)-space and an (LF)-space respectively. Then we have the following canonical isomorphisms of locally convex spaces:

$$E\hat{\otimes}_\iota F\cong\varinjlim E_\nu\hat{\otimes}F_\nu\,; \tag{2.8}$$

$$(E\hat{\otimes}_\iota F)'_\beta\cong B^s_\beta(E,\,F)\cong L_\beta(E,\,F')\cong L_\beta(F,\,E')\cong E'\hat{\otimes}F'\,. \tag{2.9}$$

PROOF. Let $A\subset E$ and $B\subset F$ be bounded sets. Then $A\otimes B$ is a bounded set in an $E_\nu\hat{\otimes}F_\nu\subset E\hat{\otimes}_\iota F$. Hence the canonical bijection

$$i:(E\hat{\otimes}_\iota F)'_\beta\to B^s_\beta(E,\,F)$$

is continuous.

Let C be a bounded set in $E\hat{\otimes}_\iota F$. Then it is a bounded set in an $E_\nu\hat{\otimes}F_\nu$. Hence it follows from Theorem 2.2 that there are bounded sets $A\subset E_\nu$ and $B\subset F_\nu$ such that C is included in the absolutely convex closed hull of $A\otimes B$. Consequently i is an open mapping.

Since E and F are barrelled, every $h\in B^s(E,\,F)$ is hypocontinuous. Hence we have canonical isomorphisms of linear spaces

$$B^s(E,\,F)\cong L(E,\,F'_\beta)\cong L(F,\,E'_\beta)\,.$$

The topological isomorphisms

$$B^s_\beta(E, F) \cong L_\beta(E, F'_\beta) \cong L_\beta(F, E'_\beta)$$

follow immediately from the definitions of topologies.

Lastly since E is a Montel space such that $E'_\beta = \varprojlim (E_\nu)'_\beta$ is a complete Grothendieck space and F'_β is complete, we have by Proposition 1.6 the canonical isomorphism

$$E'_\beta \hat{\otimes} F'_\beta = E'_\beta \varepsilon F'_\beta \cong L_\beta(E, F'_\beta).$$

Theorems 2.2 and 2.3 are abstract formulations of Schwartz' kernel theorems.

3. Vector valued ultradifferentiable functions.

We always assume that the sequence M_p satisfies conditions (M. 0), (M. 1), (M. 2)′ and (M. 3)′ and that Ω is an open set in $\boldsymbol{R}^n$.

The space $\mathcal{E}^{(M_p)}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega)$) of all ultradifferentiable functions of class (M_p) (resp. of class $\{M_p\}$) on Ω is represented as

$$\mathcal{E}^{(M_p)}(\Omega) = \varprojlim_{K \Subset \Omega} \varinjlim_{h \to 0} \mathcal{E}^{\{M_p\}, h}(K) \tag{3.1}$$

(resp.

$$\mathcal{E}^{\{M_p\}}(\Omega) = \varprojlim_{K \Subset \Omega} \varinjlim_{h \to \infty} \mathcal{E}^{\{M_p\}, h}(K)), \tag{3.2}$$

where $\mathcal{E}^{\{M_p\}, h}(K)$ is the Banach space of all infinitely differentiable functions φ on the regular compact set K in $\boldsymbol{R}^n$ such that

$$\|\varphi\|_{\mathcal{E}^{\{M_p\}, h}(K)} = \sup_{\substack{x \in K \\ \alpha}} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty, \tag{3.3}$$

and a natural locally convex topology in $\mathcal{E}^{(M_p)}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega)$) is defined by this representation.

Similarly the spaces $\mathcal{D}^{(M_p)}(\Omega)$ and $\mathcal{D}^{\{M_p\}}(\Omega)$ of ultradifferentiable functions with compact support in Ω are represented as

$$\mathcal{D}^{(M_p)}(\Omega) = \varinjlim_{K \Subset \Omega} \varprojlim_{h \to 0} \mathcal{D}^{\{M_p\}, h}_K \tag{3.4}$$

and

$$\mathcal{D}^{\{M_p\}}(\Omega) = \varinjlim_{K \Subset \Omega} \varinjlim_{h \to \infty} \mathcal{D}^{\{M_p\}, h}_K \tag{3.5}$$

as locally convex spaces with the Banach space $\mathcal{D}^{\{M_p\}, h}_K$ of all infinitely differentiable functions φ with compact support in K on $\boldsymbol{R}^n$ such that

$$\|\varphi\|_{\mathcal{D}^{\{M_p\}, h}_K} = \sup_{\substack{x \in \boldsymbol{R}^n \\ \alpha}} \frac{|D^\alpha \varphi(x)|}{h^{|\alpha|} M_{|\alpha|}} < \infty. \tag{3.6}$$

We denote by $*$ either (M_p) or $\{M_p\}$. The strong dual $\mathcal{D}^{*\prime}(\Omega)$ of $\mathcal{D}^*(\Omega)$ is by definition the locally convex space of all ultradistributions of class $*$ on

Ω. Since partitions of unity of class $*$ exist, the support of an ultradistribution is defined similarly to the case of distribution [I]. The strong dual $\mathcal{E}^{*\prime}(\Omega)$ of $\mathcal{E}^*(\Omega)$ is then identified with the subspace of $\mathcal{D}^{*\prime}(\Omega)$ composed of all elements with compact support, though the topology is different.

We have shown in [I] that $\mathcal{E}^{(M_p)}(\Omega)$ is an (FG)-space, $\mathcal{E}^{(M_p)\prime}(\Omega)$ a (DFG)-space (=complete Grothendieck (DF)-space), $\mathcal{D}^{(M_p)}(\Omega)$ an (LFG)-space (=the strict inductive limit of a sequence of (FG)-spaces), $\mathcal{D}^{(M_p)\prime}(\Omega)$ a (DLFG)-space (=the strong dual of an (LFG)-space), $\mathcal{D}^{\{M_p\}}(\Omega)$ a (DFG)-space and $\mathcal{D}^{\{M_p\}\prime}(\Omega)$ an (FG)-space ([I], Theorems 2.6, 5.4 and 5.12). In particular these spaces are complete bornologic Grothendieck spaces. As for the spaces $\mathcal{E}^{\{M_p\}}(\Omega)$ and $\mathcal{E}^{\{M_p\}\prime}(\Omega)$ we proved in [I] that they are complete bornologic Grothendieck spaces and in [II] that they are a (DLFG)-space and an (LFG)-space respectively if M_p satisfies (M.2) and (M.3).

Actually the latter is true without conditions (M.2) and (M.3). Since $\mathcal{E}^{\{M_p\}}(\Omega)$ is reflexive, we have only to prove that the strong dual $\mathcal{E}^{\{M_p\}\prime}(\Omega)$ is the strict inductive limit of a sequence of (FG)-spaces.

Let S be a subset of Ω. We denote by $\mathcal{D}^{*\prime}_S(\Omega)$ (resp. $\mathcal{E}^{*\prime}_S(\Omega)$) the subspace of $\mathcal{D}^{*\prime}(\Omega)$ (resp. $\mathcal{E}^{*\prime}(\Omega)$) composed of all elements with support in S. We know that if S is a closed set, then $\mathcal{D}^{*\prime}_S(\Omega)$ (resp. $\mathcal{E}^{*\prime}_S(\Omega)$) is a closed linear subspace of $\mathcal{D}^{*\prime}(\Omega)$ (resp. of $\mathcal{E}^{*\prime}(\Omega)$) ([I], Theorem 5.8).

PROPOSITION 3.1. *If K is a compact set in Ω, then*

$$\mathcal{E}^{*\prime}_K(\Omega)=\mathcal{D}^{*\prime}_K(\Omega) \tag{3.7}$$

as locally convex spaces.

PROOF. Clearly both sides are equal as vector spaces. The identity mapping $i: \mathcal{E}^{*\prime}_K(\Omega)\to\mathcal{D}^{*\prime}_K(\Omega)$ is continuous as a restriction of the continuous imbedding $\mathcal{E}^{*\prime}(\Omega)\to\mathcal{D}^{*\prime}(\Omega)$. Let χ be a function in $\mathcal{D}^*(\Omega)$ which is equal to one on a neighborhood of K. Since the multiplication by χ is a continuous linear mapping from $\mathcal{D}^{*\prime}(\Omega)$ into $\mathcal{E}^{*\prime}(\Omega)$, its restriction $i^{-1}: \mathcal{D}^{*\prime}_K(\Omega)\to\mathcal{E}^{*\prime}_K(\Omega)$ is continuous.

It follows from the proposition that the locally convex space (3.7) does not depend on the open set Ω. In fact, if Ω_1 is an open neighborhood of K in Ω, then the identity mapping $\mathcal{E}^{*\prime}_K(\Omega_1)\to\mathcal{E}^{*\prime}_K(\Omega)$ is continuous as a restriction of the continuous imbedding $\mathcal{E}^{*\prime}(\Omega_1)\to\mathcal{E}^{*\prime}(\Omega)$ and the identity mapping $\mathcal{D}^{*\prime}_K(\Omega)\to\mathcal{D}^{*\prime}_K(\Omega_1)$ is continuous as a restriction of the continuous restriction $\mathcal{D}^{*\prime}(\Omega)\to\mathcal{D}^{*\prime}(\Omega_1)$.

Therefore we write the space simply $\mathcal{E}^{*\prime}_K$. If $K\subset L$ are two compact sets, then the inclusion $\mathcal{E}^{*\prime}_K\to\mathcal{E}^{*\prime}_L$ is a topological imbedding onto a closed linear subspace because these spaces are considered as closed linear subspaces of a suitable space $\mathcal{E}^{*\prime}(\Omega)$.

PROPOSITION 3.2. *If S is a closed set in Ω, then*

(3.8) $$\mathcal{E}^{*\prime}_{S}(\Omega)=\varinjlim_{K\Subset\Omega} \mathcal{E}^{*\prime}_{K\cap S}$$

as the strict inductive limit of locally convex spaces.

PROOF. Continuous imbeddings $\mathcal{E}^{*\prime}_{K\cap S}\to\mathcal{E}^{*\prime}_{S}(\Omega)$ induce the continuous identity mapping

$$\varinjlim_{K\Subset\Omega} \mathcal{E}^{*\prime}_{K\cap S}\to\mathcal{E}^{*\prime}_{S}(\Omega).$$

To prove that this is an open mapping, let V be an absolutely convex neighborhood of zero in $\varinjlim \mathcal{E}^{*\prime}_{K\cap S}$. Choose a partition of unity χ_i, $i=1, 2, \cdots$, of class $*$ on Ω. Since $K_i=\operatorname{supp}\chi_i$ is compact, there is an absolutely convex bounded set B_i in $\mathcal{E}^*(\Omega)$ such that

$$V\cap\mathcal{E}^{*\prime}_{K_i\cap S}\supset B_i^\circ\cap\mathcal{E}^{*\prime}_{K_i\cap S},$$

where B_i° is the polar of B_i in $\mathcal{E}^{*\prime}(\Omega)$. Then

$$B=\sum_{i=1}^{\infty} 2^i\chi_i B_i$$

is a bounded set in $\mathcal{E}^*(\Omega)$ as a locally finite sum of bounded sets. Thus $B^\circ\cap\mathcal{E}^{*\prime}_{S}(\Omega)$ is a neighborhood of zero in $\mathcal{E}^{*\prime}_{S}(\Omega)$. To prove that this is included in V, let f be an arbitrary element of $B^\circ\cap\mathcal{E}^{*\prime}_{S}(\Omega)$. Since the support of f is compact,

$$f=\sum_{i=1}^{\infty}\chi_i f$$

is actually a finite sum. Noting that $f\in B^\circ\subset(2^i\chi_i B_i)^\circ$, we have $\chi_i f\in 2^{-i}B_i^\circ\cap \mathcal{E}^{*\prime}_{K_i\cap S}\subset 2^{-i}V$. Hence f belongs to V.

$\mathcal{E}^{\{M_p\}\prime}_K$ is an (FG)-space as a closed linear subspace of the (FG)-space $\mathcal{D}^{\{M_p\}\prime}(\Omega)$. Propositions 3.1 and 3.2 hold actually when M_p satisfies only (M. 1) and (M. 3)′. In that case $\mathcal{D}^{\{M_p\}\prime}(\Omega)$ is an (FS)-space (=Schwartz Fréchet space) and $\mathcal{E}^{\{M_p\}}(\Omega)$ is yet a reflexive space ([I], Theorem 5.12). Therefore we have the following.

THEOREM 3.3. *When M_p satisfies* (M. 1) *and* (M. 3)′, $\mathcal{E}^{\{M_p\}}(\Omega)$ *is a (DLFS)-space and* $\mathcal{E}^{\{M_p\}\prime}(\Omega)$ *is an (LFS)-space. If M_p satisfies* (M. 1), (M. 2)′ *and* (M. 3)′, *then* $\mathcal{E}^{\{M_p\}}(\Omega)$ *is a (DLFG)-space and* $\mathcal{E}^{\{M_p\}\prime}(\Omega)$ *is an (LFG)-space.*

It is clear from the definition of $\mathcal{E}^{(M_p)}(\Omega)$ that a function $\varphi\in C^\infty(\Omega)$ belongs to $\mathcal{E}^{(M_p)}(\Omega)$ if and only if

(3.9) $$p_{h^pM_p,K}(\varphi)=\sup_{\substack{x\in K\\ \alpha}}\frac{|D^\alpha\varphi(x)|}{h^pM_p}$$

is finite for any compact set K in Ω and $h>0$ and that the topology of $\mathcal{E}^{(M_p)}(\Omega)$ is determined by those semi-norms.

To obtain a similar family of semi-norms for the space $\mathcal{E}^{\{M_p\}}(\Omega)$, we start with the following.

LEMMA 3.4. *Let* $a_p \geqq 0$, $p=0, 1, 2, \cdots$, *be a sequence of numbers.*

(i) *There are constants* h *and* C *such that*

$$a_p \leqq Ch^p, \qquad p=0, 1, 2, \cdots, \tag{3.10}$$

if and only if

$$\sup_p \frac{a_p}{h_1 h_2 \cdots h_p} < \infty \tag{3.11}$$

for any sequence $h_p > 0$ *monotonously increasing to infinity.*

(ii) *There are a constant* C *and a sequence* $h_p > 0$ *monotonously increasing to infinity such that*

$$a_p \leqq \frac{C}{h_1 \cdots h_p}, \qquad p=0, 1, 2, \cdots, \tag{3.12}$$

if and only if

$$\sup_p h^p a_p < \infty \tag{3.13}$$

for any $h > 0$.

PROOF. We have only to prove the if parts.

(i) Suppose that $\sup_p (a_p/h^p) = \infty$ for any $h > 0$. Then there is a p_1 such that $a_{p_1} > 1$. Let $h_1 = h_2 = \cdots = h_{p_1} = 1$. Suppose that $h_1, \cdots, h_{p_n} = n$ are chosen so that $a_{p_n}/(h_1 \cdots h_{p_n}) > n$. Then we can find a $p_{n+1} > p_n$ such that

$$a_{p_{n+1}}/(h_1 \cdots h_{p_n}(n+1)^{p_{n+1}-p_n}) > n+1 .$$

Let $h_{p_n+1} = \cdots = h_{p_{n+1}} = n+1$. (3.11) does not hold for this sequence.

(ii) Suppose that

$$C_h = \sup_p h^p a_p < \infty$$

for any $h > 0$. Let $C = C_1$ and

$$H_p = C \sup_{h \geqq 1} (h^p / C_h), \qquad p=0, 1, 2, \cdots .$$

If $a_q > 0$, we have

$$a_q (h^p / C_h) \leqq a_q h^q / C_h \leqq 1$$

for any $0 \leqq p \leqq q$. Hence it follows that H_p is finite for all p except in the trivial case in which $a_p = 0$ except for a finite number of p.

We have $H_0 = 1$ and H_p is a logarithmically convex sequence such that H_p / h^p tends to infinity for any $h \geqq 1$. Hence if we write $H_p = h_1 h_2 \cdots h_p$, h_p tends monotonously to infinity, and we have (3.12).

PROPOSITION 3.5. *A function* $\varphi \in C^{\infty}(\Omega)$ *belongs to* $\mathcal{E}^{\{M_p\}}(\Omega)$ *if and only if*

$$p_{H_pM_p,K}(\varphi)=\sup_{\substack{x\in K\\ \alpha}}\frac{|D^{\alpha}\varphi(x)|}{H_{|\alpha|}M_{|\alpha|}} \tag{3.14}$$

is finite for any compact set K *in* Ω *and any sequence*

$$H_p=h_1h_2\cdots h_p,\ h_p\nearrow\infty . \tag{3.15}$$

The locally convex topology of $\mathcal{E}^{\{M_p\}}(\Omega)$ *is determined by the family of semi-norms* $p_{M_pH_p,K}$ *as* K *ranges over the compact sets in* Ω *and* H_p *over the sequences* (3.15).

PROOF. The first part follows from Lemma 3.4 (i).

Clearly $p_{H_pM_p,K}$ is a continuous semi-norm on $\mathcal{E}^{\{M_p\},h}(K)$ for any $h>0$. Hence it is a continuous semi-norm on $\mathcal{E}^{\{M_p\}}(\Omega)$.

On the other hand, since $\mathcal{E}^{\{M_p\}}(\Omega)$ is a reflexive space, every continuous semi-norm p is bounded by

$$p^B(\varphi)=\sup_{f\in B}|\langle\varphi, f\rangle|$$

for a bounded set B in the dual $\mathcal{E}^{\{M_p\}\prime}(\Omega)$.

By [I], Proposition 5.11 there is a compact set K' in Ω such that every $f\in B$ has support in K' and B is bounded in $\mathcal{D}^{\{M_p\}\prime}(\Omega)$. Hence it follows from the first structure theorem of ultradistributions ([I], Theorem 8.7) that every $f\in B$ is written

$$f=\sum_{\alpha} D^{\alpha}f_{\alpha}$$

with measures f_{α} on Ω such that for every compact set K and $L>0$

$$\|f_{\alpha}\|_{C(K)'}\leqq C_L L^{|\alpha|}/M_{|\alpha|}$$

with a constant C_L independent of $f\in B$.

Let χ be a function in $\mathcal{D}^{\{M_p\}}(\Omega)$ which is equal to one on a neighborhood of K' and let K be the support of χ. Then we have

$$p^B(\varphi)=\sup_{f\in B}|\langle\chi\varphi, f\rangle|\leqq\sup_{f\in B}\left\{\sum_{\alpha}\|D^{\alpha}(\chi\varphi)\|_{C(K)}\|f_{\alpha}\|_{C(K)'}\right\}.$$

Applying Lemma 3.4 (ii) to

$$a_p=\sup\{4^pM_p\|f_{\alpha}\|_{C(K)'}\,;\ f\in B,\ |\alpha|=p\},$$

we find a constant C and a sequence $H_p=h_1\cdots h_p$ with $h_p\nearrow\infty$ such that

$$H_{|\alpha|}M_{|\alpha|}\|f_{\alpha}\|_{C(K')}\leqq 4^{-|\alpha|}C,\quad f\in B.$$

Hence we have by [I], Proposition 2.7

$$p^B(\varphi)\leqq\sum_{\alpha}2^{-|\alpha|}C\sup_{\substack{x\in K\\ \alpha}}\frac{|D^{\alpha}(\chi\varphi)|}{2^{|\alpha|}H_{|\alpha|}M_{|\alpha|}}\leqq 2^nCp_{H_pM_p,K}(\chi)p_{H_pM_p,K}(\varphi).$$

If we take

$$H_p = h^p, \quad h > 0, \tag{3.16}$$

instead of (3.15), the semi-norms $p_{H_pM_p,K}$ become the semi-norms $p_{h^pM_p,K}$ defining the locally convex topology of $\mathcal{E}^{(M_p)}(\Omega)$.

We will later prove that the following family of semi-norms defines the locally convex topology of $\mathcal{D}^*(\Omega)$:

Let $\emptyset = K_0 \Subset K_1 \Subset K_2 \Subset \cdots \Subset K_\nu \Subset \cdots$ be a sequence of compact sets in Ω such that $\bigcup K_\nu = \Omega$. Then we take arbitrary sequences

$$H_p^{(0)}, H_p^{(1)}, \cdots, H_p^{(\nu)}, \cdots,$$

$$C_0, \quad C_1, \quad \cdots, C_\nu, \quad \cdots,$$

of sequences

$$H_p^{(\nu)} = (h^{(\nu)})^p, \quad h^{(\nu)} > 0 \quad (\text{resp. } h_1^{(\nu)} \cdots h_p^{(\nu)}, \; h_p^{(\nu)} \nearrow \infty),$$

and numbers $C_\nu > 0$, and define the semi-norm $p_{(H_p^{(\nu)}M_p, C_\nu)}$ by

$$p_{(H_p^{(\nu)}M_p, C_\nu)}(\varphi) = \sup_\nu \sup_{\substack{x \in K_\nu \\ \alpha}} C_\nu \left| \frac{D^\alpha \varphi(x)}{H_{|\alpha|}^{(\nu)} M_{|\alpha|}} \right|. \tag{3.17}$$

PROPOSITION 3.6. *$\mathcal{E}^*(\Omega)$, $\mathcal{D}^*(\Omega)$ and their strong duals $\mathcal{E}^{*\prime}(\Omega)$ and $\mathcal{D}^{*\prime}(\Omega)$ have the weak sequential approximation property.*

PROOF. Let E be one of the spaces $\mathcal{E}^{(M_p)}(\Omega)$, $\mathcal{E}^{\{M_p\}}(\Omega)$, $\mathcal{D}^{(M_p)}(\Omega)$ and $\mathcal{D}^{\{M_p\}}(\Omega)$. We prove that there exists a threefold sequence $T_{\lambda\mu\nu} \in E' \otimes E$ such that

$$\lim_{\lambda\to\infty} \lim_{\mu\to\infty} \lim_{\nu\to\infty} T_{\lambda\mu\nu}\varphi = \varphi \tag{3.18}$$

for any $\varphi \in E$. Since E is a barrelled space, every simply convergent sequence in $L_\sigma(E, E)$ is equicontinuous and the convergence is uniform on each precompact set in E by the Banach-Steinhaus theorem. Hence we will have the proposition for E.

Let K_λ be a sequence of compact sets in Ω as above, let χ_λ be a function in $\mathcal{D}^*(\operatorname{int} K_{\lambda+1})$ which is equal to 1 on K_λ and let $T_\lambda : E \to E$ be the multiplication by χ_λ. Then we have clearly $T_\lambda \varphi \to \varphi$ for any $\varphi \in E$.

Next we take a function $\iota(x) \in \mathcal{D}^{(M_p)}(\boldsymbol{R}^n)$ with $\int \iota(x)dx = 1$ and define $T_{\lambda\mu}$: $E \to E$ by the convolution

$$T_{\lambda\mu}\varphi(x) = \iota_\mu * (T_\lambda \varphi) = \mu^n \int \iota(\mu y)(T_\lambda \varphi)(x-y)dy.$$

As shown in the proof of [I], Theorem 6.10 $(T_\lambda\varphi)(x-y)$ is a continuous function in y with values in $\mathcal{D}^*(\Omega)$ in the variable x. Hence $T_{\lambda\mu}\varphi$ converges to $T_\lambda\varphi$ as $\mu \to \infty$ in the topology of $\mathcal{D}^*(\Omega)$ and hence in E.

We note that if μ is sufficiently large, then the convolution with $\iota_\mu(x)=\mu^n\iota(\mu x)$ is a continuous linear mapping from $C_0(K_{\lambda+2})$ into $\mathcal{D}^{(M_p)}(\Omega)$ and hence into E, where $C_0(K_{\lambda+2})$ is the Banach space of all functions on $K_{\lambda+2}$ which vanish on the boundary.

By the proof of Proposition 1.3 we can find a sequence $S_\nu\in C_0(K_{\lambda+1})'\otimes C_0(K_{\lambda+2})$ such that $S_\nu\varphi\to\varphi$ in $C_0(K_{\lambda+2})$ for any $\varphi\in C_0(K_{\lambda+1})$. Hence if we define $T_{\lambda\mu\nu}\in E'\otimes E$ by

$$T_{\lambda\mu\nu}\varphi=\iota_\mu*(S_\nu T_\lambda\varphi),$$

we have (3.18), completing the proof of the weak sequential approximation property for E.

The dual $T'_{\lambda\mu\nu}: E'\to E'$ belongs to $E\otimes E'$ and we have the strong convergence

$$\lim_{\lambda\to\infty}\lim_{\mu\to\infty}\lim_{\nu\to\infty}T'_{\lambda\mu\nu}f=f$$

for any $f\in E'$ because E is a Montel space. Since E' is also a barrelled space, this proves the weak sequential approximation property for E'.

The following proposition shows that $E=\mathcal{E}^*(\Omega)$ and $\mathcal{D}^*(\Omega)$ satisfies the assumption (a) of Theorem 1.12.

PROPOSITION 3.7. *For each ultradistribution f in $\mathcal{E}^{(M_p)\prime}(\Omega)$ (resp. $\mathcal{E}^{\{M_p\}\prime}(\Omega)$, resp. $\mathcal{D}^{(M_p)\prime}(\Omega)$, resp. $\mathcal{D}^{\{M_p\}\prime}(\Omega)$) there is a double sequence $f_{\lambda\mu}\in\mathcal{D}^{(M_p)}(\Omega)$ such that*

$$\lim_{\lambda\to\infty}\lim_{\mu\to\infty}f_{\lambda\mu}=f$$

in the strong topology.

Proof is similar to the above. We have only to take

$$f_{\lambda\mu}=\iota_\mu*\iota_\mu*(\chi_\lambda f)$$

(cf. [I], Theorem 6.10).

According to [II] we say that a subset K of $\boldsymbol{R}^n$ has the cone property if for each $x\in K$ there are a neighborhood $U\cap K$ of x, a unit vector e in $\boldsymbol{R}^n$ and a constant $\varepsilon_0>0$ such that $(U\cap K)+\varepsilon e$ is in the interior of K for any $0<\varepsilon<\varepsilon_0$.

PROPOSITION 3.8. *If K is a compact set in $\boldsymbol{R}^n$ with the cone property, then $\mathcal{D}^*_K$ has the weak sequential approximation property.*

PROOF. Let $\sum\chi_j(x)=1$ be a partition of unity of class $*$ subordinate to the open covering $\{U\}$ of K in $\boldsymbol{R}^n$ by open sets U which appear in the cone property. Each $\varphi\in\mathcal{D}^*_K$ is written as the finite sum $\sum\chi_j\varphi$. Let U_j, e_j and ε_j be as in the definition of the cone property and define $T^j_\lambda\varphi$ to be the translation $(\chi_j\varphi)(x-\lambda^{-1}\varepsilon_j e_j)$. $T^j_\lambda\varphi$ belongs to $\mathcal{D}^*(\operatorname{int}K)$ and converges to $\chi_j\varphi$ in $\mathcal{D}^*_K$. If we define $T^j_{\lambda\mu\nu}\in(\mathcal{D}^*_K)'\otimes\mathcal{D}^*_K$ as in the proof of Proposition 3.6, then we have similarly

$$\sum_j \lim_{\lambda\to\infty} \lim_{\mu\to\infty} \lim_{\nu\to\infty} T^j{}_{\lambda\mu\nu}\varphi=\varphi .$$

DEFINITION 3.9. Let F be a locally convex space. A function φ on Ω with values in F is said to be an *ultradifferentiable function of class* (M_p) (resp. *of class* $\{M_p\}$) if it is infinitely differentiable as an F-valued function and if for each continuous semi-norm q on F, compact set K in Ω and constant $h>0$ there is a constant C (resp. there are constants h and C) such that

$$\sup_{x\in K} q(D^\alpha\varphi(x))\leqq Ch^{|\alpha|}M_{|\alpha|}, \qquad |\alpha|=0, 1, 2, \cdots . \tag{3.19}$$

In view of Lemma 3.4 this holds if and only if

$$q_{H_pM_p,K}(\varphi)=\sup_{\substack{x\in K\\ \alpha}} q\Big(\frac{D^\alpha\varphi(x)}{H_{|\alpha|}M_{|\alpha|}}\Big) \tag{3.20}$$

is finite for any continuous semi-norm q on F, compact set K in Ω and

$$H_p=h^p, \quad h>0, \quad (\text{resp. } H_p=h_1\cdots h_p,\ h_p\nearrow\infty). \tag{3.21}$$

We denote by $\mathcal{E}^{(M_p)}(\Omega\,;F)$ (resp. $\mathcal{E}^{\{M_p\}}(\Omega\,;F)$) the space of all F-valued ultradifferentiable functions on Ω of class (M_p) (resp. of class $\{M_p\}$) and endow it with the locally convex topology defined by the semi-norms $q_{H_pM_p,K}$.

THEOREM 3.10. *If F is a sequentially complete (resp. boundedly complete, resp. complete) locally convex space, then a function $\varphi: \Omega\to F$ belongs to $\mathcal{E}^*(\Omega\,;F)$ if and only if $\langle\varphi(x), f'\rangle$ belongs to $\mathcal{E}^*(\Omega)$ for any $f'\in F'$ and we have the canonical isomorphisms of locally convex spaces*

$$\mathcal{E}^*(\Omega\,;F)\cong\mathcal{E}^*(\Omega)\varepsilon F\cong\mathcal{E}^*(\Omega)\bar{\otimes}F \tag{3.22}$$
$$(\textit{resp. } \cong\mathcal{E}^*(\Omega)\hat{\otimes}F,\ \textit{resp. } \cong\mathcal{E}^*(\Omega)\hat{\bar{\otimes}}F).$$

PROOF. If φ belongs to $\mathcal{E}^*(\Omega\,;F)$, then $\langle\varphi(x), f'\rangle$ clearly belongs to $\mathcal{E}^*(\Omega)$ for any $f'\in F'$.

Conversely suppose that $\varphi: \Omega\to F$ is a function such that $\langle\varphi(x), f'\rangle$ belongs to $\mathcal{E}^*(\Omega)$ for any $f'\in F'$. Since φ is scalarly infinitely differentiable, it follows from Grothendieck's lemme (see Schwartz [16], Appendice Lemme II and the remark after the lemma) that φ is infinitely differentiable. We have also

$$\langle D^\alpha\varphi, f'\rangle=D^\alpha\langle\varphi, f'\rangle \tag{3.23}$$

for any f'. Hence for each compact set K in Ω and sequence H_p of the form (3.21) the set

$$\{D^\alpha\varphi(x)/(H_{|\alpha|}M_{|\alpha|})\,;\ x\in K,\ \alpha\in \boldsymbol{Z}_+^n\} \tag{3.24}$$

in F is weakly bounded and hence bounded. Consequently we have $q_{H_pM_p,K}(\varphi)<\infty$ for any continuous semi-norm q on F, proving that $\varphi\in\mathcal{E}^*(\Omega\,;F)$.

The boundedness of (3.24) implies also that $\{\langle\varphi(x), f'\rangle ; f'\in A\}$ is bounded in $\mathcal{E}^*(\Omega)$ for any equicontinuous set A in F'. Since $\mathcal{E}^*(\Omega)$ is a Montel space, it is relatively compact.

In view of Proposition 3.7 it follows from Theorem 1.12 that $\mathcal{E}^*(\Omega ; F)=\mathcal{E}^*(\Omega)\varepsilon F$ as a vector space. To prove that this is a topological isomorphism, let $A\subset\mathcal{E}^{*\prime}(\Omega)$ and $B\subset F'$ be the equicontinuous sets defined by

$$A=\{e'\in\mathcal{E}^{*\prime}(\Omega) ; |\langle\varphi, e'\rangle|\leqq 1 \text{ for all } \varphi \text{ with } p_{H_pM_p,K}(\varphi)\leqq 1\},$$

$$B=\{f'\in F' ; |\langle f, f'\rangle|\leqq 1 \text{ for all } f \text{ with } q(f)\leqq 1\},$$

where K is a compact set in Ω, H_p a sequence of the form (3.21) and q a continuous semi-norm on F. Then we have by (3.23)

$$\begin{aligned}&\sup\{|\langle\langle\varphi(x), f'\rangle, e'\rangle| ; e'\in A,\ f'\in B\}\\ &=\sup\{p_{H_pM_p,K}(\langle\varphi(x), f'\rangle) ; f'\in B\}\\ &=\sup\left\{\left|\left\langle\frac{D^\alpha\varphi(x)}{H_{|\alpha|}M_{|\alpha|}}, f'\right\rangle\right| ; x\in K,\ \alpha\in\boldsymbol{Z}_+^n,\ f'\in B\right\}\\ &=q_{H_pM_p,K}(\varphi).\end{aligned}$$

The rest of the theorem follows from Propositions 3.6 and 1.4.

The support of a $\varphi\in\mathcal{E}^*(\Omega ; F)$ is defined in the same way as the scalar case. If K is a compact set in Ω,

$$\mathcal{D}_K^*(\Omega ; F)=\{\varphi\in\mathcal{E}^*(\Omega ; F) ; \operatorname{supp}\varphi\subset K\}$$

is a closed linear subspace of $\mathcal{E}^*(\Omega ; F)$.

It is easy to see that if $K\Subset\Omega_1\subset\Omega$, the restriction mapping $\mathcal{D}_K^*(\Omega ; F)\to\mathcal{D}_K^*(\Omega_1 ; F)$ is a topological isomorphism. Thus $\mathcal{D}_K^*(\Omega ; F)$ does not depend on the open neighborhood Ω of K, so that we write it also $\mathcal{D}_K^*(F)$.

THEOREM 3.11. *If F is a sequentially complete (resp. boundedly complete, resp. complete) locally convex space, and K is a compact set with the cone property, then we have the canonical isomorphisms of locally convex spaces*

(3.25) $$\mathcal{D}_K^*(F)\cong\mathcal{D}_K^*\varepsilon F\cong\mathcal{D}_K^*\tilde{\otimes}F \quad (\textit{resp. } \cong\mathcal{D}_K^*\hat{\otimes}F, \textit{ resp. } \cong\mathcal{D}_K^*\hat{\otimes}F).$$

PROOF. Since $\mathcal{D}_K^*$ is a closed linear subspace of $\mathcal{E}^*(\Omega)$, the injection $\mathcal{D}_K^*\to\mathcal{E}^*(\Omega)$ induces a topological imbedding $\mathcal{D}_K^*\varepsilon F\to\mathcal{E}^*(\Omega ; F)\cong\mathcal{E}^*(\Omega)\varepsilon F$ onto a closed linear subspace. The image is clearly included in $\mathcal{D}_K^*(F)$. Since $\mathcal{D}_K^*\varepsilon F$ is canonically isomorphic to $\mathcal{D}_K^*\tilde{\otimes}F$ (resp. $\mathcal{D}_K^*\hat{\otimes}F$, $\mathcal{D}_K^*\hat{\otimes}F$) by Propositions 1.4 and 3.8, we have only to prove that the sequential closure of $\mathcal{D}_K^*\otimes F$ in $\mathcal{D}_K^*(F)$ is equal to $\mathcal{D}_K^*(F)$. But it is proved in the same way as in Propositions 3.6 and 3.8 that for each $\varphi\in\mathcal{D}_K^*(F)$ $\chi_j\varphi$ is approximated by functions of the form $\iota_\mu*(T_\lambda^j\varphi)$, where $T_\lambda^j\varphi$ has a compact support L in $\operatorname{int}K$. Then $T_\lambda^j\varphi$ is approximated by a sequence of elements in $C_0(L)\otimes F$ in the topology of $C_0(L ; F)$ as Theorem 1.10.

DEFINITION 3.12. We define the locally convex space $\mathring{\mathcal{D}}^*(\Omega; F)$ of ultradifferentiable functions of class $*$ with compact support in Ω and with values in F as the strict inductive limit

$$\mathring{\mathcal{D}}^*(\Omega; F)=\varinjlim_{K\Subset\Omega} \mathcal{D}_K^*(F). \tag{3.26}$$

PROPOSITION 3.13. *Let $\emptyset=K_0\Subset K_1\Subset K_2\Subset\cdots\Subset K_\nu\Subset\cdots\subset\Omega$ be a sequence of compact sets such that $\bigcup K_\nu=\Omega$. Then the locally convex topology of $\mathring{\mathcal{D}}^{(M_p)}(\Omega; F)$ (resp. $\mathring{\mathcal{D}}^{\{M_p\}}(\Omega; F)$) is defined by the following family of continuous semi-norms $q_{(H_p^{(\nu)}M_p, q_\nu)}$:*

Let

$$H_p^{(0)}, H_p^{(1)}, \cdots, H_p^{(\nu)}, \cdots,$$

$$q_0, \quad q_1, \quad \cdots, q_\nu, \quad \cdots,$$

be an arbitrary sequence of sequences

$$H_p^{(\nu)}=(h^{(\nu)})^p, \quad h^{(\nu)}>0 \quad (resp.\ H_p^{(\nu)}=h_1^{(\nu)}\cdots h_p^{(\nu)},\ h_p^{(\nu)}\nearrow\infty), \tag{3.27}$$

and an arbitrary sequence of continuous semi-norms q_ν on F. Then the semi-norm $q_{(H_p^{(\nu)}M_p, q_\nu)}$ is defined by

$$q_{(H_p^{(\nu)}M_p, q_\nu)}(\varphi)=\sup_\nu \sup_{\substack{x\notin K_\nu\\ \alpha}} q_\nu\Big(\frac{D^\alpha\varphi(x)}{H_{|\alpha|}^{(\nu)}M_{|\alpha|}}\Big). \tag{3.28}$$

Moreover, an infinitely differentiable function $\varphi: \Omega\to F$ belongs to $\mathring{\mathcal{D}}^{(M_p)}(\Omega; F)$ (resp. $\mathring{\mathcal{D}}^{\{M_p\}}(\Omega; F)$) if and only if $q_{(H_p^{(\nu)}M_p, q_\nu)}(\varphi)$ is finite for any $H_p^{(\nu)}$ and q_ν.

PROOF. Since each compact set K is included in a K_ν, $q_{(H_p^{(\nu)}M_p, q_\nu)}$ is a continuous semi-norm on $\mathcal{D}_K^*(F)$ and hence on $\mathring{\mathcal{D}}^*(\Omega; F)$.

Conversely let r be a continuous semi-norm on $\mathring{\mathcal{D}}^*(\Omega; F)$. We take a partition of unity $\sum_{\nu=1}^\infty \chi_\nu(x)=1$ so that $\operatorname{supp}\chi_\nu\subset K_{\nu+1}\setminus K_{\nu-1}$. For each $\varphi\in\mathring{\mathcal{D}}^*(\Omega; F)$ we have

$$r(\varphi)\leqq \sum_{\nu=1}^\infty 2^{-\nu}r(2^\nu\chi_\nu\varphi)\leqq\sup_\nu r(2^\nu\chi_\nu\varphi).$$

Since $2^\nu\chi_\nu\varphi\in\mathcal{D}_{K_{\nu+1}}^*(F)$, there are a sequence $H_p^{(\nu-1)}$ defined by (3.27) and a continuous semi-norm $q_{\nu-1}$ on F such that

$$r(2^\nu\chi_\nu\varphi)\leqq q_{\nu-1, 2^pH_p^{(\nu-1)}M_p, K_{\nu+1}}(2^\nu\chi_\nu\varphi).$$

Hence we have

$$r(2^\nu\chi_\nu\varphi)\leqq \sup_{\substack{x\notin K_{\nu-1}\\ \alpha}} r_{\nu-1}\Big(\frac{D^\alpha\varphi}{H_{|\alpha|}^{(\nu-1)}M_{|\alpha|}}\Big)$$

for the continuous semi-norm

$$r_{\nu-1}(f)=2^\nu p_{H_p^{(\nu-1)}M_p, K_{\nu+1}}(\chi_\nu)q_{\nu-1}(f).$$

Thus we have

$$r(\varphi)\leqq q_{(H_p^{(\nu)}M_p, r_\nu)}(\varphi), \qquad \varphi\in\mathring{\mathcal{D}}^*(\Omega\,;F).$$

Suppose that $q_{(H_p^{(\nu)}M_p, q_\nu)}(\varphi)$ is finite for any $H_p^{(\nu)}$ and q_ν. Then $q_{H_pM_p, K}(\varphi)$ is finite for any K and q. Hence φ belongs to $\mathcal{E}^*(\Omega\,;F)$. If $\operatorname{supp}\varphi$ is not compact, we can easily find a sequence q_ν of continuous semi-norms on F such that $q_{(H_pM_p, q_\nu)}(\varphi)=\infty$.

Taking $F=\boldsymbol{C}$ (or $\boldsymbol{R}$), we find that the semi-norms $p_{(H_p^{(\nu)}M_p, C_\nu)}$ of (3.17) form a basis of continuous semi-norms on $\mathcal{D}^*(\Omega)$.

Now suppose that an F-valued function φ on Ω is scalarly in $\mathcal{D}^*(\Omega)$, i.e. $\langle\varphi(x), f'\rangle\in\mathcal{D}^*(\Omega)$ for any $f'\in F'$. Then by Theorem 3.10 φ is infinitely differentiable as a function with values in the sequential completion $\bar{F}$ of F. Moreover, let K_ν be a sequence of compact sets in Ω as in Proposition 3.13. Then it follows, similarly to Theorem 3.10, that

$$\bigcup_\nu \bigcup_{\substack{x\in K \\ \alpha}} C_\nu\Big(\frac{D^\alpha\varphi(x)}{H_{|\alpha|}^{(\nu)}M_{|\alpha|}}\Big)$$

is bounded in $\bar{F}$ for any sequence $H_p^{(\nu)}$ of sequences (3.27) and sequence C_ν of positive constants. Hence

$$q_{(H_p^{(\nu)}M_p, C_\nu)}(\varphi)=\sup_\nu \sup_{\substack{x\in K_\nu \\ \alpha}} C_\nu q\Big(\frac{D^\alpha\varphi(x)}{H_{|\alpha|}^{(\nu)}M_{|\alpha|}}\Big) \tag{3.29}$$

is finite for any continuous norm q on $\bar{F}$.

Conversely if an F-valued function φ is infinitely differentiable in $\bar{F}$ and if $q_{(H_p^{(\nu)}M_p, C_\nu)}(\varphi)$ is finite for any $H_p^{(\nu)}$, C_p and q, then clearly φ is scalarly in $\mathcal{D}^*(\Omega)$.

DEFINITION 3.14. We define $\mathcal{D}^{(M_p)}(\Omega\,;F)$ (resp. $\mathcal{D}^{\{M_p\}}(\Omega\,;F)$) to be the space of all infinitely differentiable functions φ on Ω with values in the locally convex space F such that $q_{(H_p^{(\nu)}M_p, C_\nu)}(\varphi)$ is finite for any sequence $H_p^{(\nu)}$ of sequences (3.27), sequence C_ν of positive constants and continuous semi-norm q on F, and we endow it with the locally convex topology defined by the semi-norms $q_{(H_p^{(\nu)}M_p, C_\nu)}$.

THEOREM 3.15. *If F is sequentially complete (resp. boundedly complete, resp. complete), then a function $\varphi:\Omega\to F$ belongs to $\mathcal{D}^*(\Omega: F)$ if and only if $\langle\varphi(x), f'\rangle$ belongs to $\mathcal{D}^*(\Omega)$ for any $f'\in F'$ and we have the canonical isomorphisms of locally convex spaces*

$$\mathcal{D}^*(\Omega\,;F)\cong\mathcal{D}^*(\Omega)\varepsilon F\cong\mathcal{D}^*(\Omega)\tilde{\otimes}F \tag{3.30}$$

$$(\textit{resp.}\ \cong\mathcal{D}^*(\Omega)\hat{\otimes}F,\ \textit{resp.}\ \cong\mathcal{D}^*(\Omega)\hat{\otimes}F).$$

PROOF. We have only to replace the semi-norm $p_{H_pM_p, K}$ by $p_{(H_p^{(\nu)}M_p, C_\nu)}$ in the proof of Theorem 3.10.

If $\varphi\in\mathcal{D}^*(\Omega;F)$, then it follows from the finiteness of $q_{(H_p^{(\nu)}M_p,C_\nu)}(\varphi)$ that for each equicontinuous set B in F' there is a compact set K in Ω such that the support of $\langle\varphi(x),f'\rangle$ is included in K for any $f'\in B$ but φ need not have a compact support in general.

Evidently we have the following.

PROPOSITION 3.16. (i) *Suppose that the locally convex space F has the property that if $f_\nu\in F$ is a sequence such that for each continuous semi-norm q on F $q(f_\nu)=0$ for sufficiently large ν, then $f_\nu=0$ for sufficiently large ν. Then $\dot{\mathcal{D}}^*(\Omega;F)$ and $\mathcal{D}^*(\Omega;F)$ are identical as linear spaces.*

(ii) *Suppose that for each sequence q_ν of continuous semi-norms on F there are a continuous semi-norm q and constants C_ν such that $q_\nu(f)\leqq C_\nu q(f)$. Then $\dot{\mathcal{D}}^*(\Omega;F)$ and $\mathcal{D}^*(\Omega;F)$ are identical as locally convex spaces.*

The condition of (i) is satisfied if F is the union of a family F_μ of linear subspaces with continuous norm such that every sequence converging to zero in F is included in one of F_μ. This is the case if F is the strict inductive limit of a sequence of locally convex spaces F_ν with continuous norm.

(DF)-spaces satisfy the condition of (ii) (Grothendieck [5], Lemme 2, [9], Proposition II. 3.7). Hence we obtain the following.

PROPOSITION 3.17. *If F is a complete (DF)-space, we have the topological isomorphisms*

$$\dot{\mathcal{D}}^{\{M_p\}}(\Omega;F)=\mathcal{D}^{\{M_p\}}(\Omega;F)\cong\mathcal{D}^{\{M_p\}}(\Omega)\hat{\otimes}F\cong\mathcal{D}^{\{M_p\}}(\Omega)\hat{\otimes}_\iota F. \tag{3.31}$$

On the other hand, if F is a Fréchet space, then we have by Theorems 3.11, 2.2 and 2.3 topological isomorphisms

$$\dot{\mathcal{D}}^{(M_p)}(\Omega;F)\cong\varinjlim_{K\Subset\Omega}\mathcal{D}^{(M_p)}{}_K\hat{\otimes}F=\varinjlim_{K\Subset\Omega}\mathcal{D}^{(M_p)}{}_K\hat{\otimes}_\iota F\cong\mathcal{D}^{(M_p)}(\Omega)\hat{\otimes}_\iota F. \tag{3.32}$$

This may be different from $\mathcal{D}^{(M_p)}(\Omega;F)$ and even if it coincides with $\mathcal{D}^{(M_p)}(\Omega;F)$ its topology may be different from that of $\mathcal{D}^{(M_p)}(\Omega;F)$.

More generally let $F=\varinjlim F_\nu$ be an (LF)-space and define

$$\dot{\mathcal{D}}^{(M_p)}{}_K(F)=\varinjlim_{\nu}\mathcal{D}^{(M_p)}{}_K(F_\nu), \tag{3.33}$$

$$\dot{\mathcal{D}}^{(M_p)}(\Omega;F)=\varinjlim_{K\Subset\Omega}\dot{\mathcal{D}}^{(M_p)}{}_K(F). \tag{3.34}$$

Then we have topological isomorphisms

$$\dot{\mathcal{D}}^{(M_p)}{}_K(F)\cong\mathcal{D}^{(M_p)}{}_K\hat{\otimes}_\iota F, \tag{3.35}$$

$$\dot{\mathcal{D}}^{(M_p)}(\Omega;F)\cong\mathcal{D}^{(M_p)}(\Omega)\hat{\otimes}_\iota F. \tag{3.36}$$

4. Vector valued ultradistributions.

We assume as before that M_p is a sequence satisfying conditions (M. 0), (M. 1), (M. 2)′ and (M. 3)′, Ω is an open set in $\boldsymbol{R}^n$ and F is a locally convex space. The asterisk $*$ denotes either (M_p) or $\{M_p\}$.

DEFINITION 4.1. Similarly to Schwartz [17] we call an element of

$$\mathcal{D}^{*\prime}(\Omega; F)=L_\beta(\mathcal{D}^*(\Omega), F) \tag{4.1}$$

an *ultradistribution of class* $*$ *on* Ω *with values in* F.

Since $\mathcal{D}^*(\Omega)$ is a Montel space, we have by (1.10) the topological isomorphism

$$\mathcal{D}^{*\prime}(\Omega; F)\cong\mathcal{D}^{*\prime}(\Omega)\varepsilon F. \tag{4.2}$$

The complete Grothendieck space $\mathcal{D}^{*\prime}(\Omega)$ has the weak sequential approximation property by Proposition 3.6. Hence, if F is sequentially complete (resp. boundedly complete, resp. complete), then we have the canonical topological isomorphism

$$\mathcal{D}^{*\prime}(\Omega; F)\cong\mathcal{D}^{*\prime}(\Omega)\bar{\otimes} F \quad (\text{resp. } \cong\mathcal{D}^{*\prime}(\Omega)\tilde{\otimes} F, \text{ resp. } \cong\mathcal{D}^{*\prime}(\Omega)\hat{\otimes} F). \tag{4.3}$$

We write the value of an $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega; F)$ at a $\varphi\in\mathcal{D}^*(\Omega)$ as $\int\varphi(x)\boldsymbol{f}(x)dx$. If $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega; F)$ and $f'\in F'$, we denote by $\langle\boldsymbol{f}, f'\rangle$ the ultradistribution

$$\varphi\mapsto\left\langle\int\varphi(x)\boldsymbol{f}(x)dx,\ f'\right\rangle.$$

Under the isomorphism

$$\mathcal{D}^{*\prime}(\Omega)\varepsilon F\cong L_\varepsilon(F'_c, \mathcal{D}^{*\prime}(\Omega))$$

an $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega; F)$ corresponds to the element

$$\langle\boldsymbol{f}, \cdot\rangle\in L_\varepsilon(F'_c, \mathcal{D}^{*\prime}(\Omega)).$$

DEFINITION 4.2. More generally let $\mathcal{Q}$ be a space of ultradistributions, i.e. a linear subspace of a $\mathcal{D}^{*\prime}(\Omega)$ equipped with a locally convex topology stronger than the induced topology from $\mathcal{D}^{*\prime}(\Omega)$. Then we define the space $\mathcal{Q}(F)$ of *ultradistributions of type* $\mathcal{Q}$ *with values in* F by

$$\mathcal{Q}(F)=L_\varepsilon(\mathcal{Q}'_c, F)\cong\mathcal{Q}\varepsilon F. \tag{4.4}$$

In view of Theorems 1.10, 3.10, 3.11 and 3.15 this is compatible with our older notations $C(\Omega; F)$, $\mathcal{E}^*(\Omega; F)$, $\mathcal{D}^*_K(F)$ and $\mathcal{D}^*(\Omega; F)$ at least when F is sequentially complete.

If $\mathcal{Q}$ is imbedded in $\mathcal{D}^{*\prime}(\Omega)$ with the continuous injection $S: \mathcal{Q}\to\mathcal{D}^{*\prime}(\Omega)$, then $\mathcal{Q}(F)$ is imbedded in $\mathcal{D}^{*\prime}(\Omega; F)$ under $S\varepsilon 1_F$. Thus we identify $\mathcal{Q}(F)$ with a linear subspace of $\mathcal{D}^{*\prime}(\Omega; F)$. If an $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega; F)$ belongs to $\mathcal{Q}(F)$, then it is scalarly in $\mathcal{Q}$ i.e. $\langle\boldsymbol{f}, f'\rangle$ belongs to $\mathcal{Q}$ for any $f'\in F$. If moreover $\langle\boldsymbol{f}, \cdot\rangle: F'_c\to\mathcal{Q}$

is continuous, then $\boldsymbol{f}$ belongs to $\mathcal{G}(F)$. In some cases the continuity holds automatically.

PROPOSITION 4.3. *Assume one of the following conditions*:

(a) *$\mathcal{G}$ is a webbed space and F'_c is ultrabornologic*;

(b) *$\mathcal{G}=\mathcal{H}'_c$, where $\mathcal{H}$ is a metrizable locally convex space containing $\mathcal{D}^*(\Omega)$ continuously as a dense linear subspace and F is sequentially complete*;

(c) *$\mathcal{G}$ is a semi-Montel space with continuous imbeddings $\mathcal{D}^*(\Omega)\subset\mathcal{G}\subset\mathcal{D}^{*\prime}(\Omega)$ such that the sequential closure (resp. the bounding closure, resp. the closure) of $\mathcal{D}^*(\Omega)$ in $\mathcal{G}'_c$ is equal to $\mathcal{G}'_c$, $\mathcal{G}$ has a fundamental system of absolutely convex neighborhoods of 0 which are closed under the induced topology in $\mathcal{G}$ from the topology of $\mathcal{D}^{*\prime}(\Omega)$ and F is sequentially complete (resp. boundedly complete, resp. complete).*

Then an $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega;F)$ belongs to $\mathcal{G}(F)$ whenever $\langle\boldsymbol{f},f'\rangle\in\mathcal{G}$ for all $f'\in F'$.

PROOF. (a) The linear mapping $\langle\boldsymbol{f},\cdot\rangle: F'_c\to\mathcal{D}^{*\prime}(\Omega)$ is continuous and has the image in $\mathcal{G}$. Hence it has a closed graph as a mapping from F'_c into $\mathcal{G}$, so that it is continuous by De Wilde's closed graph theorem [3], [7].

Proofs in cases (b) and (c) are the same as those of Propositions 14 and 15 of Schwartz [17], Chap. I, pp. 53-54 respectively.

Let $\mathcal{G}$ and $\mathcal{H}$ be spaces of ultradistributions. Every continuous linear mapping $S:\mathcal{G}\to\mathcal{H}$ induces continuous linear mapping $S\varepsilon 1_F:\mathcal{G}(F)\to\mathcal{H}(F)$. Hence elementary operations defined for ultradistributions (see [I], § 6) are extended to vector valued ultradistributions.

The multiplication by a function $a\in\mathcal{E}^*(\Omega)$ is defined by

$$\int\varphi(x)(a\boldsymbol{f})(x)dx=\int(a(x)\varphi(x))\boldsymbol{f}(x)dx, \tag{4.5}$$

and is a continuous linear mapping $\mathcal{D}^{*\prime}(\Omega;F)\to\mathcal{D}^{*\prime}(\Omega;F)$, $\mathcal{E}^{*\prime}(\Omega;F)\to\mathcal{E}^{*\prime}(\Omega;F)$ etc.

The differential operator $P(D)$ of constant coefficients, and if M_p satisfies (M.2), the ultradifferential operator of class $*$

$$P(D)=\sum_{|\alpha|=0}^{\infty}a_\alpha D^\alpha \tag{4.6}$$

are defined by

$$\int\varphi(x)(P(D)\boldsymbol{f})(x)dx=\int(P(-D)\varphi(x))\boldsymbol{f}(x)dx \tag{4.7}$$

and are continuous linear mappings $\mathcal{D}^{*\prime}(\Omega;F)\to\mathcal{D}^{*\prime}(\Omega;F)$, $\mathcal{E}^{*\prime}(\Omega;F)\to\mathcal{E}^{*\prime}(\Omega;F)$ etc. Similarly to [I], Theorem 6.8 we can also prove that if $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega;F)$ (resp. $\mathcal{E}^{*\prime}(\Omega;F)$) then

$$P(D)\boldsymbol{f}=\sum_{|\alpha|=0}^{\infty} a_{\alpha} D^{\alpha} \boldsymbol{f}$$

converges absolutely in $\mathscr{D}^{*\prime}(\Omega; F)$ (resp. $\mathscr{E}^{*\prime}(\Omega; F)$).

The convolution with a scalar function is defined similarly. Suppose that Ω, Ω_1 and Ω_2 are open sets in $\boldsymbol{R}^n$ satisfying

$$\Omega=\Omega_1-\Omega_2=\{x_1-x_2;\ x_i\in\Omega_i\}.$$

If $\psi\in\mathscr{D}^*(\Omega_2)$ and $\boldsymbol{f}\in\mathscr{D}^{*\prime}(\Omega; F)$, then $\psi*\boldsymbol{f}\in\mathscr{E}(\Omega_1; F)$ is defined by

$$(\psi*\boldsymbol{f})(x)=\int\psi(x-y)\boldsymbol{f}(y)dy$$

and the linear mapping $\psi*: \mathscr{D}^{*\prime}(\Omega; F)\to\mathscr{E}(\Omega_1; F)$ is continuous. If M_p satisfies (M.2), then $\mathscr{E}(\Omega_1; F)$ may be replaced by $\mathscr{E}^*(\Omega_1; F)$.

We have

$$\int\varphi(x)(\psi*\boldsymbol{f})(x)dx=\int(\check{\psi}*\varphi)(x)\boldsymbol{f}(x)dx\,, \tag{4.8}$$

for any $\varphi\in\mathscr{D}^*(\Omega_1)$, where $\check{\psi}(x)=\psi(-x)$. Thus if we take for ψ the function ι_μ in the proof of Proposition 3.6, we find that $\iota_\mu*\iota_\mu*\boldsymbol{f}\in\mathscr{E}^*(\Omega_1; F)$ converges to $\boldsymbol{f}$ in $\mathscr{D}^{*\prime}(\Omega_1)$ on every relatively compact open subset Ω_1 of Ω.

If M_p satisfies (M.2) and if $g\in\mathscr{E}^{*\prime}(\Omega_2)$ and $\boldsymbol{f}\in\mathscr{D}^{*\prime}(\Omega; F)$, then $g*\boldsymbol{f}\in\mathscr{D}^{*\prime}(\Omega_1; F)$ is defined by

$$\int_{\Omega_1}\varphi(x)(g*\boldsymbol{f})(x)dx=\int_{\Omega}(\varphi*\check{g})(x)\boldsymbol{f}(x)dx\,, \qquad \varphi\in\mathscr{D}^{*\prime}(\Omega_1)\,, \tag{4.9}$$

and $g*: \mathscr{D}^{*\prime}(\Omega; F)\to\mathscr{D}^{*\prime}(\Omega_1; F)$ is continuous.

There are many other cases in which convolutions with scalar ultradistributions are defined.

Since the multiplication by partitions of unity of class $*$ is possible, the notion of support is defined for vector valued ultradistributions of class $*$ in the same way as the scalar case.

Let K be a compact set in Ω. Then $\mathscr{E}_K^{*\prime}$ is a closed linear subspace of $\mathscr{E}^{*\prime}(\Omega)$ and also of $\mathscr{D}^{*\prime}(\Omega)$ and hence $\mathscr{E}_K^{*\prime}(F)$ is looked upon as a closed linear subspace of $\mathscr{E}^{*\prime}(\Omega; F)$ and of $\mathscr{D}^{*\prime}(\Omega; F)$. It is easy to see that $\mathscr{E}_K^{*\prime}(F)$ is then exactly the subspace of all elements with support in K. Hence the space of all F-valued ultradistributions with compact support is written

$$\hat{\mathscr{E}}^{*\prime}(\Omega; F)=\varinjlim_{K\Subset\Omega}\mathscr{E}_K^{*\prime}(F)\,. \tag{4.10}$$

This is continuously imbedded in $\mathscr{E}^{*\prime}(\Omega; F)$ but it may be strictly smaller, and even when it coincides as a set its inductive limit topology may be strictly stronger. Similarly to Proposition 3.16 we have the following.

PROPOSITION 4.4. (i) *Suppose that the locally convex space F has the property that if $f_\nu \in F$ is a sequence such that for each continuous semi-norm q on F $q(f_\nu)=0$ for sufficiently large ν, then $f_\nu=0$ for sufficiently large ν. Then $\mathcal{E}^{*\prime}(\Omega; F)$ and $\hat{\mathcal{E}}^{*\prime}(\Omega; F)$ are identical as linear spaces.*

(ii) *Suppose that for each sequence q_ν of continuous semi-norms on F there are a continuous semi-norm q on F and constants C_ν such that $q_\nu(f) \leqq C_\nu q(f)$. Then $\mathcal{E}^{*\prime}(\Omega; F)$ and $\hat{\mathcal{E}}^{*\prime}(\Omega; F)$ are identical as locally convex spaces.*

PROOF. (i) Let $K_1 \Subset K_2 \Subset \cdots$ be a sequence of compact sets in Ω such that $\bigcup K_\nu = \Omega$. If the support of $\boldsymbol{f} \in \mathcal{D}^{*\prime}(\Omega; F)$ is not compact, we can find a sequence $\varphi_\nu \in \mathcal{D}^*(\Omega)$ with $\operatorname{supp} \varphi_\nu \cap K_\nu = \emptyset$ such that $f_\nu = \int \varphi_\nu(x) \boldsymbol{f}(x) dx \neq 0$. Suppose to the contrary that $\boldsymbol{f}$ belongs to $\mathcal{E}^{*\prime}(\Omega; F)$. Then $a_\nu f_\nu$ converges to zero in F for any sequence of numbers a_ν because $a_\nu \varphi_\nu$ converges to zero in $\mathcal{E}^*(\Omega)$. Let q be an arbitrary continuous semi-norm on F. We have $q(a_\nu f_\nu) = |a_\nu| q(f_\nu) \to 0$ for any a_ν only if $q(f_\nu)=0$ except for a finite number of ν. Hence we have $f_\nu=0$ for sufficiently large ν by the assumption.

(ii) F satisfies the condition of (i). In fact, if $f_\nu \in F$ are all different from zero, then we can find continuous semi-norms q_ν on F such that $q_\nu(f_\nu) \neq 0$. Hence if q is a continuous semi-norm such that $q_\nu(f) \leqq C_\nu q(f)$, then we have $q(f_\nu) \neq 0$ for all ν.

Evidently the identity mapping $\hat{\mathcal{E}}^{*\prime}(\Omega; F) \to \mathcal{E}^{*\prime}(\Omega; F)$ is continuous. To prove that the inverse is continuous let r be a continuous semi-norm on $\hat{\mathcal{E}}^{*\prime}(\Omega; F)$.

If we take a sequence of compact sets K_ν in Ω and a partition of unity $\chi_\nu(x)$ as in the proof of Proposition 3.13, then we have similarly

$$r(\boldsymbol{f}) \leqq \sum_{\nu=1}^{\infty} 2^{-\nu} r(2^\nu \chi_\nu \boldsymbol{f}) \leqq \sup_\nu r(2^\nu \chi_\nu \boldsymbol{f}).$$

Since $2^\nu \chi_\nu \boldsymbol{f}$ belongs to $\mathcal{E}^{*\prime}_{K_{\nu+1}}(F)$, which may be regarded as a closed linear subspace of $\mathcal{E}^{*\prime}(\Omega; F) = \mathcal{E}^{*\prime}(\Omega) \varepsilon F$, there are continuous semi-norms p_ν on $\mathcal{E}^{*\prime}(\Omega)$ and q_ν on F such that

$$r(2^\nu \chi_\nu \boldsymbol{f}) \leqq (p_\nu \varepsilon q_\nu)(2^\nu \chi_\nu \boldsymbol{f}).$$

Hence it follows from the assumption that

$$r(\boldsymbol{f}) \leqq \sup_\nu (\bar{p}_\nu \varepsilon q)(\chi_\nu \boldsymbol{f}),$$

where $\bar{p}_\nu$ is a constant times p_ν and q is a continuous semi-norm on F. In view of Proposition 3.2

$$p(e) = \sup_\nu \bar{p}_\nu(\chi_\nu e)$$

is a continuous semi-norm on $\mathcal{E}^{*\prime}(\Omega)$ and we have

$$r(\boldsymbol{f}) \leqq (p\varepsilon q)(\boldsymbol{f}),$$

proving that r is a continuous semi-norm on $\mathcal{E}^{*\prime}(\Omega; F)$.

Suppose moreover that $*=(M_p)$ and F is a complete (DF)-space. Then $\mathcal{E}_K^{*\prime}$ and $\mathcal{E}^{*\prime}(\Omega)$ are (DFG)-spaces so that we have

$$\mathcal{E}_K^{(M_p)\prime}(F) \cong \mathcal{E}_K^{(M_p)\prime} \hat{\otimes} F = \mathcal{E}_K^{(M_p)\prime} \hat{\otimes}_\varepsilon F, \tag{4.11}$$

$$\mathring{\mathcal{E}}^{(M_p)\prime}(\Omega; F) = \mathcal{E}^{(M_p)\prime}(\Omega; F) \cong \mathcal{E}^{(M_p)\prime}(\Omega) \hat{\otimes}_\varepsilon F. \tag{4.12}$$

On the other hand, if $*=\{M_p\}$ and F is a Fréchet space, then $\mathcal{E}_K^{*\prime}(F)$ is an (FG)-space and hence we have by Theorems 2.2 and 2.3

$$\mathcal{E}_K^{\{M_p\}\prime}(F) \cong \mathcal{E}_K^{\{M_p\}\prime} \hat{\otimes} F = \mathcal{E}_K^{\{M_p\}\prime} \hat{\otimes}_\varepsilon F, \tag{4.13}$$

$$\mathring{\mathcal{E}}^{\{M_p\}\prime}(\Omega; F) = \varinjlim_{K \Subset \Omega} \mathcal{E}_K^{\{M_p\}\prime}(F) \cong \mathcal{E}^{\{M_p\}\prime}(\Omega) \hat{\otimes}_\varepsilon F. \tag{4.14}$$

DEFINITION 4.5. A vector valued ultradistribution $\boldsymbol{f} \in \mathcal{D}^{*\prime}(\Omega; F)$ is said to be *bounded* if it map a neighborhood of zero in $\mathcal{D}^*(\Omega)$ into a bounded set in F. An $\boldsymbol{f} \in \mathcal{D}^{*\prime}(\Omega; F)$ is said to be *locally bounded* if its restriction $\boldsymbol{f}|_{\Omega_1}$ to every relatively compact open subset Ω_1 of Ω is bounded.

If F is a metrizable locally convex space, then every $\boldsymbol{f} \in \mathcal{D}^{\{M_p\}\prime}(\Omega; F)$ is bounded ([7], § 40.2, [9], Chap. II § 3). If F is a (DF)-space, then every $\boldsymbol{f} \in \mathcal{D}^{(M_p)\prime}(\Omega; F)$ is locally bounded ([9], Chap. II § 3). In particular, scalar valued ultradistributions are always locally bounded. However, $\delta(x-y)$ corresponding to the identity mapping $\mathcal{D}^*(\Omega) \to \mathcal{D}^*(\Omega)$ is not locally bounded as an element in $\mathcal{D}^{*\prime}(\Omega; \mathcal{D}^*(\Omega))$.

The dual $(E \otimes_\pi F)' = (E \bar{\otimes}_\pi F)' = (E \hat{\otimes}_\pi F)' = B(E, F)$ may be identified with the set of all linear mappings $L: E \to F'$ which maps a neighborhood of zero in E into an equicontinuous set in F'. Hence if F is sequentially complete, every element in the dual $(\mathcal{D}^*(\Omega; F))' = (\mathcal{D}^*(\Omega) \bar{\otimes} F)'$ is identified with a bounded ultradistribution in $\mathcal{D}^{*\prime}(\Omega; F')$. If F is barrelled, then conversely every bounded ultradistribution in $\mathcal{D}^{*\prime}(\Omega; F')$ is identified with an element in $(\mathcal{D}^*(\Omega; F))'$.

An $\boldsymbol{f} \in \mathcal{D}^{*\prime}(\Omega; F) = L(\mathcal{D}^*(\Omega), F)$ is locally bounded if and only if its restriction to $\mathcal{D}_K^*$ is bounded for every compact set K in Ω. Hence it follows from Theorem 3.11 that if F is sequentially complete, then every element in the dual $(\mathring{\mathcal{D}}^*(\Omega; F))' = \left(\varinjlim_{K \Subset \Omega} \mathcal{D}_K^*(F)\right)'$ is identified with a locally bounded ultradistribution in $\mathcal{D}^{*\prime}(\Omega; F')$ and if, moreover, F is barrelled, then conversely every locally bounded ultradistribution in $\mathcal{D}^{*\prime}(\Omega; F')$ is identified with an element in $(\mathring{\mathcal{D}}^*(\Omega; F))'$.

To represent the whole ultradistributions $\mathcal{D}^{*\prime}(\Omega; F)$ as the dual we can use the canonical isomorphisms

$$(E \otimes_\varepsilon F)' = (E \hat{\otimes}_\varepsilon F)' = B^s(E, F) \cong L(E_\tau, F'_\tau). \tag{4.15}$$

If F is a reflexive locally convex space, we have canonical isomorphisms

(4.16) $$\mathcal{D}^{*\prime}(\Omega ; F)=L(\mathcal{D}^*(\Omega), F)\cong B^s(\mathcal{D}^*(\Omega), F')\cong(\mathcal{D}^*(\Omega)\hat{\otimes}_\varepsilon F')'.$$

Thus if we define

(4.17) $$\dot{\mathcal{D}}^*(\Omega ; F')=\mathcal{D}^*(\Omega)\hat{\otimes}_\varepsilon F',$$

generalizing (3.36), then we have

(4.18) $$\mathcal{D}^{*\prime}(\Omega ; F)\cong(\dot{\mathcal{D}}^*(\Omega ; F'))'.$$

When F is a reflexive locally convex space, J.-L. Lions and E. Magenes [12] define the ultradistributions of class M_p with values in F to be the continuous linear functionals on the locally convex space

$$\mathcal{D}_{M_p}(\Omega ; F')=\varinjlim_{K\Subset\Omega}\varinjlim_{h\to\infty}\mathcal{D}^{\{M_p\},h}{}_K\varepsilon F'.$$

Since the imbeddings $\mathcal{D}^{\{M_p\},h}{}_K\to\mathcal{D}^{\{M_p\},k}{}_K$ are nuclear for k/h sufficiently large, the space is the same as

$$\varinjlim_{K\Subset\Omega}\varinjlim_{h\to\infty}\mathcal{D}^{\{M_p\},h}{}_K\hat{\otimes}_\pi F'.$$

Moreover, the π topology coincides with the ε topology on $E\otimes F$ if E is a normed space and F is a barrelled space. Hence we have the topological imbeddings

$$\mathcal{D}^{\{M_p\}}(\Omega)\otimes_\varepsilon F'\subset\mathcal{D}_{M_p}(\Omega ; F')\subset\mathcal{D}^{\{M_p\}}(\Omega)\hat{\otimes}_\varepsilon F'.$$

Therefore their ultradistributions $(\mathcal{D}_{M_p}(\Omega ; F'))'$ are identical with our ultradistributions $\mathcal{D}^{\{M_p\}\prime}(\Omega ; F)=L(\mathcal{D}^{\{M_p\}}(\Omega), F)$ in spite of their remark on p. 363 of [12].

Schwartz [17] Chap. I, pp. 84-86 obtained the structure theorem of bounded distributions. Similarly we have the following two structure theorems of locally bounded ultradistributions corresponding to our two structure theorems ([I], §§ 8 and 10) for scalar valued ultradistributions.

Lions and Magenes [12] give a structure theorem similar to our first one for all ultradistributions in $(\mathcal{D}_{M_p}(\Omega ; F'))'$. However, there seems to be a gap in their proof of the same nature as we pointed out in [I] for C. Roumieu's proof of the structure theorem of scalar valued ultradistributions.

THEOREM 4.6. *Suppose that F is a sequentially complete locally convex space. If an $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega ; F)$ is locally bounded, then its restriction $\boldsymbol{f}|_\omega$ to each relatively compact open set ω in Ω is represented as*

(4.19) $$\boldsymbol{f}|_\omega=\sum_{|\alpha|=0}^\infty D^\alpha \boldsymbol{f}_\alpha,$$

where $\boldsymbol{f}_\alpha$ are continuous functions on $\bar{\omega}$ with values in F such that the set

(4.20) $$R=\{H_{|\alpha|}M_{|\alpha|}\boldsymbol{f}_{\alpha}(x)\,;\ x\in\bar{\omega},\ |\alpha|=0,1,2,\cdots\}$$

is bounded in F *for a sequence*

(4.21) $$H_p=h^p,\ h>0,\quad or\quad h_1h_2\cdots h_p,\ 0<h_p\nearrow\infty,$$

according as $*=(M_p)$ *or* $*=\{M_p\}$.

Conversely if $\boldsymbol{f}_\alpha\in C(\bar{\omega}\,;F)$ *and* R *is bounded for a sequence* (4.21), *then the series* (4.19) *converges absolutely in the sense of Mackey in* $\mathcal{D}^{*\prime}(\omega\,;F)$ *and represents a bounded ultradistribution on* ω.

PROOF. First we consider the case of scalar valued ultradistributions. Let $f\in\mathcal{D}^{*\prime}(\Omega)$. By Proposition 3.5 we can find a sequence H_p of (4.21) and a constant C such that

(4.22) $$|\langle\varphi,f\rangle|\leqq C\sup\left|\frac{D^\alpha\varphi(x)}{H_{|\alpha|}M_{|\alpha|}}\right|,\qquad \varphi\in\mathcal{D}^*_{\bar{\omega}}.$$

Under the mapping $\varphi\mapsto(D^\alpha\varphi/H_{|\alpha|}M_{|\alpha|}\,;\ |\alpha|=0,1,\cdots)$ $\mathcal{D}^*_{\bar{\omega}}$ is imbedded in the Banach space $C_0(\bigcup_\alpha\bar{\omega})$ of continuous functions vanishing at infinity on the disjoint union of $\bar{\omega}$. By the Hahn-Banach theorem the linear functional f can be continued to a continuous linear functional on $C_0(\bigcup\bar{\omega})$, so that there are complex measures $\mu_\alpha\in C(\bar{\omega})'$ satisfying

$$\sum_\alpha H_{|\alpha|}M_{|\alpha|}\|\mu_\alpha\|_{C(\bar{\omega})'}\leqq C$$

and

$$\langle\varphi,f\rangle=\sum_\alpha\langle D^\alpha\varphi,\mu_\alpha\rangle,\qquad \varphi\in\mathcal{D}^*_{\bar{\omega}}.$$

Let $N(x)\in C(\boldsymbol{R}^n)$ be a fundamental solution of $\Delta^nN(x)=\delta(x)$ and define

$$g_\alpha(x)=(-1)^{|\alpha|}\int_{\bar{\omega}}N(x-y)\mu_\alpha(dy).$$

Then there is a constant B_0 depending only on ω such that

$$\sum_\alpha H_{|\alpha|}M_{|\alpha|}\|g_\alpha\|_{C(\bar{\omega})}\leqq B_0C$$

and we have

$$\begin{aligned}\langle\varphi,f\rangle&=\sum_\alpha\langle D^\alpha\varphi,(-1)^{|\alpha|}\Delta^ng_\alpha\rangle\\&=\sum_\alpha\langle\varphi,\Delta^nD^\alpha g_\alpha\rangle.\end{aligned}$$

Let

$$\Delta^n=\sum_\beta c_\beta D^\beta$$

and

$$f_\alpha=\sum_\beta c_\beta g_{\alpha-\beta}.$$

Then we have

(4.23) $$f=\sum_{\alpha} D^{\alpha}f_{\alpha}\,.$$

In view of condition (M.2)′ there are constants H_1 and B_1 such that

(4.24) $$\sum_{\alpha} H_1^{|\alpha|}H_{|\alpha|}M_{|\alpha|}\|f_{\alpha}\|_{C(\bar{\omega})}\leqq B_0B_1C\,.$$

Next let $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega\,;F)$ be a locally bounded ultradistribution with values in F. For each relatively compact open set ω in Ω there are a continuous semi-norm p on $\mathcal{D}^{*}_{\bar{\omega}}$ and an absolutely convex closed bounded set B such that

$$\int\varphi(x)\boldsymbol{f}(x)dx\in p(\varphi)B\,,\qquad \varphi\in\mathcal{D}^{*}_{\bar{\omega}}\,.$$

Since $\mathcal{D}^{*}_{\bar{\omega}}$ is a Grothendieck space and since the normed space F_B generated by B is complete, we can find an equicontinuous sequence f_i in $(\mathcal{D}^{*}_{\bar{\omega}})'$, a sequence b_i in B and a sequence $\lambda_i\geqq 0$ in l^1 such that

(4.25) $$\int\varphi(x)\boldsymbol{f}(x)dx=\sum_{i=1}^{\infty}\lambda_i\langle\varphi,\,f_i\rangle b_i\,.$$

f_i satisfy (4.22) for a sequence H_p of (4.21) and a constant C independent of i. Hence we have

$$f_i=\sum_{\alpha} D^{\alpha}f_{i\alpha}$$

with $f_{i\alpha}\in C(\bar{\omega})$ satisfying (4.24).

Let

$$\boldsymbol{f}_{\alpha}(x)=\sum_{i=1}^{\infty}\lambda_i f_{i\alpha}(x)b_i\,.$$

Then $\boldsymbol{f}_{\alpha}$ are clearly continuous functions on $\bar{\omega}$ with values in F_B such that

$$\{H_1^{|\alpha|}H_{|\alpha|}M_{|\alpha|}\boldsymbol{f}_{\alpha}(x)\,;\ x\in\bar{\omega},\ |\alpha|=0,1,\cdots\}$$

is bounded in F_B and (4.19) holds.

Lastly suppose that $\boldsymbol{f}_{\alpha}\in C(\bar{\omega}\,;F)$ and R defined by (4.20) is bounded for a sequence (4.21).

Let B be the absolute convex closed hull of R in F. The semi-norm p on $\mathcal{D}^{*}_{\bar{\omega}}$ defined by

$$p(\varphi)=\sup_{x,\alpha}\left|\frac{D^{\alpha}\varphi(x)}{2^{-|\alpha|}H_{|\alpha|}M_{|\alpha|}}\right|$$

is continuous and we have

$$\int\varphi(x)D^{\alpha}\boldsymbol{f}_{\alpha}(x)dx=\int(-D)^{\alpha}\varphi(x)\boldsymbol{f}_{\alpha}(x)dx\in 2^{-|\alpha|}|\omega|\,p(\varphi)B\,,\qquad \varphi\in\mathcal{D}^{*}_{\bar{\omega}}.$$

Hence (4.19) converges absolutely in the operator norm of continuous linear mappings from the normed space $(\mathcal{D}^{*}_{\bar{\omega}})_p$ with norm p into F_B. Since F_B is a Banach space the series converges and represents a bounded ultradistribution with values in F_B.

THEOREM 4.7. *Suppose that* M_p *satisfies also* (M.2) *and* (M.3) *and that* F *is a sequentially complete locally convex space. If* $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega;F)$ *is a locally bounded ultradistribution, then for each relatively compact open set* ω *in* Ω *there are a function* $\boldsymbol{g}\in C(\bar{\omega};F)$ *and an ultradifferential operator* $P(D)$ *of class* $*$ *such that*

$$\boldsymbol{f}|_{\omega}=P(D)\boldsymbol{g}. \tag{4.26}$$

Conversely if $\boldsymbol{g}\in C(\bar{\omega};F)$ *and* $P(D)$ *is an ultradifferential operator of class* $*$, *then* $P(D)\boldsymbol{g}$ *is a bounded ultradistribution in* $\mathcal{D}^{*\prime}(\omega;F)$.

PROOF is similar to the previous theorem. If $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega;F)$ is locally bounded and ω is a relatively compact open set in Ω, then we have (4.25) for an equicontinuous sequence f_i in $(\mathcal{D}^*_{\bar{\omega}})'$, a bounded sequence b_i in F and a sequence $\lambda_i\geqq 0$ in l^1. By the second structure theorem of [I] we can find an ultradifferential operator $P(D)$ of class $*$ and a bounded sequence of continuous functions g_i on $\bar{\omega}$ such that

$$f_i=P(D)g_i.$$

Then $\boldsymbol{g}(x)=\sum_{i=1}^{\infty}\lambda_i g_i(x)b_i$ is a continuous function on $\bar{\omega}$ with values in F and we have (4.26).

The converse is evident.

Lastly we prove the structure theorem of vector valued ultradistributions with support at the origin.

Let Ω be an open neighborhood of 0 in $\boldsymbol{R}^n$. Then $\mathcal{E}^{*\prime}_{\{0\}}(F)=\mathcal{E}^{*\prime}_{\{0\}}\varepsilon F$ is identified with the closed linear subspace $\{\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega;F);\ \operatorname{supp}\boldsymbol{f}\subset\{0\}\}$ of $\mathcal{D}^{*\prime}(\Omega;F)$ and also with the closed linear subspace $\{\boldsymbol{f}\in\mathcal{E}^{*\prime}(\Omega;F);\ \operatorname{supp}\boldsymbol{f}\subset\{0\}\}$ of $\mathcal{E}^{*\prime}(\Omega;F)$ as a locally convex space. In fact, since $\mathcal{E}^{*\prime}_{\{0\}}$ is a closed linear subspace of $\mathcal{D}^{*\prime}(\Omega)$ and of $\mathcal{E}^{*\prime}(\Omega)$, $\mathcal{E}^{*\prime}_{\{0\}}(F)$ is identified with a closed linear subspace of $\mathcal{D}^{*\prime}(\Omega;F)=\mathcal{D}^{*\prime}(\Omega)\varepsilon F$ and of $\mathcal{E}^{*\prime}(\Omega;F)=\mathcal{E}^{*\prime}(\Omega)\varepsilon F$. If $\boldsymbol{f}$ is in $\mathcal{E}^{*\prime}_{\{0\}}(F)$, then $\langle\boldsymbol{f},f'\rangle$ belongs to $\mathcal{E}^{*\prime}_{\{0\}}$ for any $f'\in F'$, so that we have

$$\left\langle\int\varphi(x)\boldsymbol{f}(x)dx,\ f'\right\rangle=\int\varphi(x)\langle\boldsymbol{f}(x),\ f'\rangle dx=0$$

for any $\varphi\in\mathcal{D}^*(\Omega\setminus\{0\})$. This implies $\operatorname{supp}\boldsymbol{f}\subset\{0\}$. Conversely if $\boldsymbol{f}\in\mathcal{D}^{*\prime}(\Omega;F)$ or $\mathcal{E}^{*\prime}(\Omega;F)$ has the support at the origin, then we have

$$\int\varphi(x)\langle\boldsymbol{f}(x),\ f'\rangle dx=\left\langle\int\varphi(x)\boldsymbol{f}(x)dx,\ f'\right\rangle=0$$

for any $f'\in F'$ and $\varphi\in\mathcal{D}^*(\Omega\setminus\{0\})$ and hence $\langle\boldsymbol{f}(x),\cdot\rangle$ belongs to $L(F'_c,\mathcal{E}^{*\prime}_{\{0\}})$.

THEOREM 4.8. *Suppose that* M_p *satisfies* (M.2) *and* (M.3) *and that* F *is a sequentially complete locally convex space. Then every* $\boldsymbol{f}\in\mathcal{E}^{*\prime}_{\{0\}}(F)$ *is uniquely* *decomposed as*

$$f(x)=\sum_{\alpha} D^\alpha\delta(x)\otimes f_\alpha \tag{4.27}$$

with a family of elements $f_\alpha\in F$ *satisfying the following condition:*

For any continuous semi-norm q *on* F *there exist a sequence* H_p *of* (4.21) *and a constant* C *such that*

$$q(f_\alpha)\leqq\frac{C}{H_{|\alpha|}M_{|\alpha|}}. \tag{4.28}$$

Conversely if $f_\alpha\in F$ *satisfy the above estimates, then* (4.27) *converges absolutely in* $\mathcal{E}^{*\prime}_{\{0\}}(F)$ *and represents an* $f\in\mathcal{E}^{*\prime}_{\{0\}}(F)$.

PROOF. First we prove the converse part. Since $\mathcal{E}^{*\prime}_{\{0\}}(F)$ is a sequentially complete linear subspace of $\mathcal{E}^{*\prime}(\boldsymbol{R}^n;F)$, we have only to prove that the series (4.27) converges absolutely in $\mathcal{E}^{*\prime}(\boldsymbol{R}^n;F)=\mathcal{E}^{*\prime}(\boldsymbol{R}^n)\varepsilon F$. Its topology is defined by the family of semi-norms $p^B\varepsilon q$, where p^B is the supremum of the absolute values on a bounded set B in $\mathcal{E}^*(\Omega)$ and q is a continuous semi-norm on F. Suppose that (4.28) holds. Then we have

$$\begin{aligned}(p^B\varepsilon q)(D^\alpha\delta(x)\otimes f_\alpha)&=q(f_\alpha)\sup_{\varphi\in B}\left|\int\varphi(x)D^\alpha\delta(x)dx\right|\\&\leqq\frac{C}{H_{|\alpha|}M_{|\alpha|}}\sup_{\varphi\in B}|D^\alpha\varphi(0)|\\&\leqq 2^{-|\alpha|}C\sup_{\varphi\in B}p_{2^{-p}H_pM_p,\{0\}}(\varphi),\end{aligned}$$

where $p_{2^{-p}H_pM_p,\{0\}}$ denotes the continuous semi-norm (3.9) or (3.14) on $\mathcal{E}^*(\boldsymbol{R}^n)$. Hence (4.27) converges absolutely in $\mathcal{E}^{*\prime}_{\{0\}}(F)$.

The proof of the direct part is similar to that of the structure theorem of ultradistributions with support in a submanifold in [II]. We consider the Fourier-Laplace transform

$$\mathcal{F}f(\zeta)=\int e^{-ix\zeta}f(x)dx \tag{4.29}$$

of $f\in\mathcal{E}^{*\prime}_{\{0\}}(F)$. It is scalarly holomorphic on $\boldsymbol{C}^n$. Hence it belongs to the space $\mathcal{O}(\boldsymbol{C}^n;F)$ of all entire functions with values in F. In fact, since every continuous linear functional on $\mathcal{O}(\boldsymbol{C}^n)$ is represented by a measure with compact support in $\boldsymbol{C}^n$, condition (a) of Theorem 1.12 is satisfied. $\mathcal{O}(\boldsymbol{C}^n)$ is also a Montel Fréchet space.

We will prove that for any continuous semi-norm q of F there exist a sequence H_p of (4.21) and a constant C such that

$$q(\mathcal{F}f(\zeta))\leqq C\exp(HM)(\zeta), \tag{4.30}$$

where

$$(HM)(\zeta)=\sup_p \log(|\zeta|^p/(H_pM_p)).$$

Then it follows that the coefficients

$$f_\alpha=\frac{1}{(2\pi i)^n}\oint\frac{\mathcal{F}\boldsymbol{f}(\zeta)}{\zeta_1^{\alpha_1+1}\cdots\zeta_n^{\alpha_n+1}}d\zeta_1\cdots d\zeta_n$$

of the Taylor expansion

$$\mathcal{F}\boldsymbol{f}(\zeta)=\sum_\alpha f_\alpha\zeta^\alpha$$

satisfy the estimates

$$q(f_\alpha)\leqq\inf_{r_1\cdots r_n>0}\frac{C\exp(HM)(r)}{r^\alpha}\leqq\frac{Cn^{|\alpha|/2}}{H_{|\alpha|}M_{|\alpha|}}.$$

$\mathcal{F}\boldsymbol{f}(\zeta)$ coincides with the Fourier-Laplace transform of $\sum D^\alpha\delta(x)\otimes f_\alpha$. Hence we have (4.27).

Applying $\boldsymbol{f}$ to polynomials, we have the uniqueness of $f_\alpha\in F$.

The proof of (4.30) is similar to that of the Paley-Wiener theorem for ultradistributions in [II].

Since $\boldsymbol{f}$ belongs to $\mathcal{E}^{*\prime}(\boldsymbol{R}^n; F)=L(\mathcal{E}^*(\boldsymbol{R}^n), F)$, there exist a sequence H_p of (4.21), a constant C and a compact set K in $\boldsymbol{R}^n$ such that

(4.31) $$q\Big(\int\varphi(x)\boldsymbol{f}(x)dx\Big)\leqq Cp_{H_pM_p,K}(\varphi),\qquad \varphi\in\mathcal{E}^*(\boldsymbol{R}^n).$$

Applying this to $\varphi(x)=e^{-ix\xi}$, $\xi\in\boldsymbol{R}^n$, we obtain

(4.32) $$q(\mathcal{F}\boldsymbol{f}(\xi))\leqq C\exp(HM)(\xi),\qquad \xi\in\boldsymbol{R}^n.$$

On the other hand, since $\operatorname{supp}\boldsymbol{f}\subset\{0\}$, we can find for any $\varepsilon>0$ a constant C_ε such that

(4.33) $$q(\mathcal{F}\boldsymbol{f}(\zeta))\leqq C_\varepsilon\exp\varepsilon|\zeta|,\qquad \zeta\in\boldsymbol{C}^n.$$

In fact, let χ be a function in $\mathcal{D}^{(M_p)}(\boldsymbol{R}^n)$ which takes value 1 on a neighborhood of 0 and with support in the ε ball B. Then it follows from (4.31) that

$$\begin{aligned}q(\mathcal{F}\boldsymbol{f}(\zeta))&=q\Big(\int e^{-ix\zeta}\chi(x)\boldsymbol{f}(x)dx\Big)\leqq Cp_{H_pM_p,K}(e^{-ix\zeta}\chi(x))\\&\leqq Cp_{2^{-p}H_pM_p,B}(e^{-ix\zeta})p_{2^{-p}H_pM_p,K}(\chi(x))\\&\leqq C_\chi\exp\{(HM)(2\zeta)+\varepsilon|\zeta|\}.\end{aligned}$$

Since $(HM)(\zeta)=o(\zeta)$ as $|\zeta|\to\infty$ by (4.7) of [I], we have (4.33) with ε replaced by 2ε.

Lastly we derive (4.30) with a smaller H_p from (4.32) and (4.33) by the Phragmén-Lindelöf theorem.

Let A be the equicontinuous set of all $f'\in F'$ such that $|\langle f, f'\rangle|\leqq q(f)$ for any $f\in F$. Then for each $f'\in A$, $\xi\in\boldsymbol{R}^n$ and unit vector ξ_0 in $\boldsymbol{R}^n$ the holomorphic function

$$F(z)=\langle \mathcal{I}\boldsymbol{f}(\xi+\xi_0 z),\ f'\rangle$$

on the upper half plane $\operatorname{Im} z\geqq 0$ satisfies

$$|F(x)|\leqq C\exp(HM)(\xi+\xi_0 x)\,, \qquad x\in \boldsymbol{R}.$$

and for any $\varepsilon>0$ there is a constant C_ε such that

$$|F(z)|=C_\varepsilon \exp\{\varepsilon|\xi|+\varepsilon|z|\}, \quad \operatorname{Im} z\geqq 0\,.$$

Let $\xi=x_0\xi_0+\xi'$ be the orthogonal decomposition and set

$$P(z)=\prod_{p=1}^{\infty}\left(1-\frac{i(z+x_0+i|\xi'|)}{h_p m_p}\right),$$

where $m_p=M_p/M_{p-1}$ and $h_p=h$ in the case $*=(M_p)$. Then we have by (10.5) of [I]

$$|P(x)^{-1}F(x)|\leqq C\,, \qquad x\in \boldsymbol{R}\,,$$

and

$$|P(z)^{-1}F(z)|\leqq C_\varepsilon \exp\{\varepsilon|\xi|+\varepsilon|z|\}, \quad \operatorname{Im} z\geqq 0\,.$$

Hence we have by the Phragmén-Lindelöf theorem

$$|F(z)|\leqq C|P(z)|\,, \qquad \operatorname{Im} z\geqq 0\,.$$

In case $*=(M_p)$, it follows from Proposition 4.6 of [I] that there exist constants $h_1\leqq h$ and C_1 such that

$$\begin{aligned}|P(z)|&\leqq C_1 \exp M((z+x_0+i|\xi'|)/h_1)\\ &\leqq C_1 \exp\,(H_1 M)(\xi+\xi_0 z), \quad \operatorname{Im} z\geqq 0\,,\end{aligned}$$

where $H_{1,p}=(h_1/\sqrt{2})^p$.

In case $*=\{M_p\}$, the same proposition shows that

$$P(z)\leqq C_1 \exp M(\varepsilon(|z+x_0+i|\xi'||))$$

for a constant C_1 and an increasing continuous function $\varepsilon(\rho)\geqq 0$ on $[0,\infty)$ such that $\varepsilon(\rho)/\rho\to 0$ as $\rho\to\infty$ (see the proof of Theorem 9.4 in [I]). Hence the proof is reduced to the following improvement of Lemma 3.12 in [I].

LEMMA 4.9. *Let M_p be a sequence of positive numbers satisfying* (M.0), (M.1) *and $m_p=M_p/M_{p-1}\nearrow\infty$ as $p\to\infty$. If $\varepsilon(\rho)\geqq 0$ is a continuous increasing function on $[0,\infty)$ such that $\varepsilon(\rho)/\rho\to 0$ as $\rho\to\infty$, then there is a monotonously increasing sequence of positive numbers h_p tending to ∞ such that*

$$M(\varepsilon(\rho))\leqq C(HM)(\rho)\,, \qquad 0\leqq\rho<\infty\,, \tag{4.34}$$

for a constant C.

PROOF. By increasing $\varepsilon(\rho)$ if necessary we may assume without loss of generality that

$$\varepsilon(\rho)\nearrow\infty \quad \text{and} \quad \varepsilon(\rho)/\rho\searrow 0 \qquad \text{as } \rho\to\infty. \tag{4.35}$$

Moreover, we may assume $\varepsilon(0)=0$ and $C=1$. Then define h_p as the unique solution of

$$\varepsilon(h_p m_p)=m_p, \qquad p=1, 2, \cdots. \tag{4.36}$$

By (4.35) we have $h_p m_p\nearrow\infty$ and $h_p=h_p m_p/\varepsilon(h_p m_p)\nearrow\infty$.

Moreover, if $\rho\geqq h_p m_p$ and $q\leqq p$, then we have $\varepsilon(\rho)\leqq\rho/h_q$. The supremum

$$M(\varepsilon(\rho))=\sup_p \log\prod_{q=1}^{p}\frac{\varepsilon(\rho)}{m_q}$$

is attained at p if $m_p\leqq\varepsilon(\rho)\leqq m_{p+1}$ or if $h_p m_p\leqq\rho\leqq h_{p+1}m_{p+1}$, so that we have

$$M(\varepsilon(\rho))=\log\prod_{q=1}^{p}\frac{\varepsilon(\rho)}{m_q}\leqq\log\prod_{q=1}^{p}\frac{\rho}{h_q m_q}\leqq(HM)(\rho).$$

This completes the proof of Theorem 4.8.

5. The kernel theorems.

In this section we assume that M_p is a sequence satisfying conditions (M.0), (M.1), (M.2) and (M.3)'. The condition (M.2) plays an important role.

Let Ω' and Ω'' be open sets in $\boldsymbol{R}^{n'}$ and $\boldsymbol{R}^{n''}$ respectively. We denote by x (resp. y) a point in Ω' (resp. Ω''). K' and K'' stand for compact sets with the cone property in Ω' and Ω'' respectively. We determine various topological tensor products of spaces of ultradistributions and obtain an ultradistribution version of Schwartz' theory of kernels [14], [15], [16] and [17].

If $\mathcal{F}$ and $\mathcal{G}$ are function spaces on Ω' and Ω'' respectively, then the tensor product $\mathcal{F}\otimes\mathcal{G}$ is identified with the space of all linear combinations of products $\varphi(x)\psi(y)$ of $\varphi\in\mathcal{F}$ and $\psi\in\mathcal{G}$. Under this identification we have proved in [II] the following canonical isomorphisms of locally convex spaces:

$$\mathcal{E}^*(\Omega')\hat{\otimes}\mathcal{E}^*(\Omega'')=\mathcal{E}^*(\Omega'\times\Omega''); \tag{5.1}$$

$$\mathcal{D}^*_{K'}\hat{\otimes}\mathcal{D}^*_{K''}=\mathcal{D}^*_{K'\times K''}; \tag{5.2}$$

$$\mathcal{D}^{\{M_p\}}(\Omega')\hat{\otimes}\mathcal{D}^{\{M_p\}}(\Omega'')=\mathcal{D}^{\{M_p\}}(\Omega'\times\Omega''). \tag{5.3}$$

Since $\mathcal{D}^{(M_p)}(\Omega')=\varinjlim_{K'\Subset\Omega'}\mathcal{D}^{(M_p)}_{K'}$ and $\mathcal{D}^{(M_p)}(\Omega'')=\varinjlim_{K''\Subset\Omega''}\mathcal{D}^{(M_p)}_{K''}$ are (DFG)-spaces, we have by Theorem 2.3

$$\mathcal{D}^{(M_p)}(\Omega')\hat{\otimes}_\iota\mathcal{D}^{(M_p)}(\Omega'')=\mathcal{D}^{(M_p)}(\Omega'\times\Omega''). \tag{5.4}$$

We have also proved in [II] the *kernel theorem* for ultradistributions:

$$\mathcal{D}^{*\prime}(\Omega'\times\Omega'')\cong B^{s}_{\beta}(\mathcal{D}^{*}(\Omega'),\ \mathcal{D}^{*}(\Omega''))$$
$$\cong L_{\beta}(\mathcal{D}^{*}(\Omega'),\ \mathcal{D}^{*\prime}(\Omega''))\cong L_{\beta}(\mathcal{D}^{*}(\Omega''),\ \mathcal{D}^{*\prime}(\Omega'))$$
$$\cong\mathcal{D}^{*\prime}(\Omega')\hat{\otimes}\mathcal{D}^{*\prime}(\Omega''). \tag{5.5}$$

This follows from (5.4) and (5.3) by Theorems 2.3 and 2.2 respectively. If $*=\{M_p\}$, the spaces in (5.5) are also isomorphic to $B_{\beta}(\mathcal{D}^{*}(\Omega'),\ \mathcal{D}^{*}(\Omega''))$.

Let $f\in\mathcal{D}^{*\prime}(\Omega')$ and $g\in\mathcal{D}^{*\prime}(\Omega'')$. Then the tensor product $f\otimes g$ in $\mathcal{D}^{*\prime}(\Omega')\hat{\otimes}\mathcal{D}^{*\prime}(\Omega'')$ is identified under the isomorphism (5.5) with the ultradistribution $f(x)g(y)\in\mathcal{D}^{*\prime}(\Omega'\times\Omega'')$ defined by

$$\iint\chi(x,\ y)f(x)g(y)dxdy$$
$$=\int\left(\int\chi(x,\ y)f(x)dx\right)g(y)dy$$
$$=\int\left(\int\chi(x,\ y)g(y)dy\right)f(x)dx,\qquad \chi\in\mathcal{D}^{*}(\Omega'\times\Omega''). \tag{5.6}$$

In fact, if $\chi\in\mathcal{D}^{*}(\Omega')\otimes\mathcal{D}^{*}(\Omega'')\subset\mathcal{D}^{*}(\Omega'\times\Omega'')$, this is nothing but the definition of isomorphism (5.5). Each $\chi\in\mathcal{D}^{*}(\Omega'\times\Omega'')$ is an element in $\mathcal{D}^{*}_{K'\times K''}$, so that there is a net $\chi_{\nu}\in\mathcal{D}^{*}_{K'}\otimes\mathcal{D}^{*}_{K''}$ converging to χ in the topology of $\mathcal{D}^{*}_{K'\times K''}$. Then $\chi_{\nu}(\cdot,\ y)$ converges to $\chi(\cdot,\ y)$ in $\mathcal{D}^{*}_{K'}$ for every fixed y. Hence $\int\chi_{\nu}(x,\ y)f(x)dx$ converges pointwise to $\int\chi(x,\ y)f(x)dx$.

On the other hand, we have by (5.2) $\mathcal{D}^{*}_{K'\times K''}\cong L_{\varepsilon}((\mathcal{D}^{*}_{K'})'_{c},\ \mathcal{D}^{*}_{K''})$, so that $\int\chi_{\nu}(x,\ y)f(x)dx$ converges to $\int\chi(x,\ y)f(x)dx$ in $\mathcal{D}^{*}_{K''}$. Similarly $\int\chi_{\nu}(x,\ y)g(y)dy$ converges to $\int\chi(x,\ y)g(y)dy$ in $\mathcal{D}^{*}_{K'}$. Hence we have (5.6) as the limit of (5.6) for χ_{ν}.

We always regard $f\otimes g$ as an ultradistribution in $\mathcal{D}^{*\prime}(\Omega'\times\Omega'')$ under this identification.

If $\Omega'_1\subset\Omega'$ and $\Omega''_1\subset\Omega''$ are open subsets, then we have

$$f\otimes g|_{\Omega'_1\times\Omega''_1}=f|_{\Omega'_1}\otimes g|_{\Omega''_1} \tag{5.7}$$

immediately from the definition. This implies

$$\operatorname{supp} f\otimes g=\operatorname{supp} f\times\operatorname{supp} g \tag{5.8}$$

because the open sets $\Omega'_1\times\Omega''_1$ form a basis of open sets in $\Omega'\times\Omega''$.

Let $h\in\mathcal{D}^{*\prime}(\Omega'\times\Omega'')$. The bilinear form

$$\iint \varphi(x)\psi(y)h(x, y)dxdy$$

on $\mathcal{D}^*(\Omega')\times\mathcal{D}^*(\Omega'')$, the linear mapping $T: \mathcal{D}^*(\Omega')\to\mathcal{D}^{*\prime}(\Omega'')$ defined by

$$\int \psi(y)(T\varphi)(y)dy = \iint \varphi(x)\psi(y)h(x, y)dxdy$$

and its dual $T': \mathcal{D}^*(\Omega'')\to\mathcal{D}^{*\prime}(\Omega')$ are the images of h under the first three isomorphisms of (5.5). We write

$$(T\varphi)(y)=\int \varphi(x)h(x, y)dy \tag{5.9}$$

and call the ultradistribution $h(x, y)$ the *kernel* of the continuous linear mapping T.

From now on we consider smaller classes of linear mappings T and determine corresponding classes of kernels h.

When the class $*$ is fixed the smallest class in regularity is following one.

THEOREM 5.1. *Under the correspondence* (5.9) *we have the isomorphism of locally convex spaces*

$$L_\beta(\mathcal{E}^{*\prime}(\Omega'), \mathcal{E}^*(\Omega''))\cong\mathcal{E}^*(\Omega'\times\Omega'') . \tag{5.10}$$

PROOF. Since $\mathcal{E}^{*\prime}(\Omega')$ is a complete bornologic Montel space with the strong dual $\mathcal{E}^*(\Omega')$ and since $\mathcal{E}^*(\Omega'')$ is a complete Grothendieck space, we have by Proposition 1.7 the isomorphism

$$L_\beta(\mathcal{E}^{*\prime}(\Omega'), \mathcal{E}^*(\Omega''))\cong\mathcal{E}^*(\Omega')\hat{\otimes}\mathcal{E}^*(\Omega'') . \tag{5.11}$$

The right hand side is naturally isomorphic to $\mathcal{E}^*(\Omega'\times\Omega'')$ by (5.1).

In view of Theorem 3.10 the space of kernels $\mathcal{E}^*(\Omega')\hat{\otimes}\mathcal{E}^*(\Omega'')$ may also be identified with the space $\mathcal{E}^*(\Omega'; \mathcal{E}^*(\Omega''))$ of vector valued ultradifferentiable functions and with $\mathcal{E}^*(\Omega''; \mathcal{E}^*(\Omega'))$.

An $h(x, y)$ in $\mathcal{E}^*(\Omega'\times\Omega'')$ is called a *regularizing kernel of class* $*$.

Similarly Proposition 1.7, Theorem 3.10 and Definition 4.1 imply the following.

THEOREM 5.2. *We have the following isomorphisms of locally convex spaces*:

$$\begin{aligned} L_\beta(\mathcal{D}^*(\Omega'), \mathcal{E}^*(\Omega''))&\cong\mathcal{D}^{*\prime}(\Omega')\hat{\otimes}\mathcal{E}^*(\Omega'') \\ &\cong\mathcal{D}^{*\prime}(\Omega'; \mathcal{E}^*(\Omega''))\cong\mathcal{E}^*(\Omega''; \mathcal{D}^{*\prime}(\Omega')); \end{aligned} \tag{5.12}$$

$$\begin{aligned} L_\beta(\mathcal{E}^{*\prime}(\Omega'), \mathcal{D}^{*\prime}(\Omega''))&\cong\mathcal{E}^*(\Omega')\hat{\otimes}\mathcal{D}^{*\prime}(\Omega'') \\ &\cong\mathcal{E}^*(\Omega'; \mathcal{D}^{*\prime}(\Omega''))\cong\mathcal{D}^{*\prime}(\Omega''; \mathcal{E}^*(\Omega')) . \end{aligned} \tag{5.13}$$

An ultradistribution $h(x, y)$ in the spaces of (5.12) (resp. (5.13)) is called a *semi-regular kernel in* $y\in\Omega''$ (resp. *in* $x\in\Omega'$) *of class* $*$. It is called *a regular kernel of class* $*$ if it is semi-regular both in x and in y.

Next we determine the smallest class in support.

THEOREM 5.3. (i) *Under the correspondence* (5.9) *we have the isomorphisms of locally convex spaces*

$$L_\beta(\mathcal{E}^{(M_p)}(\Omega'), \mathcal{E}^{(M_p)\prime}(\Omega''))\cong\mathcal{E}^{(M_p)\prime}(\Omega')\hat{\otimes}\mathcal{E}^{(M_p)\prime}(\Omega'')\cong\mathcal{E}^{(M_p)\prime}(\Omega'\times\Omega''). \tag{5.14}$$

(ii) *Under* (5.9) *we have the isomorphism of locally convex spaces*

$$L_\beta(\mathcal{E}^{\{M_p\}}(\Omega'), \mathcal{E}^{\{M_p\}\prime}(\Omega''))\cong\mathcal{E}^{\{M_p\}\prime}(\Omega')\hat{\otimes}\mathcal{E}^{\{M_p\}\prime}(\Omega''). \tag{5.15}$$

The right hand side is the same as

$$\mathcal{E}^{\{M_p\}\prime}(\Omega'\times\Omega'')\cong\mathcal{E}^{\{M_p\}\prime}(\Omega')\hat{\otimes}_\iota\mathcal{E}^{\{M_p\}\prime}(\Omega'') \tag{5.16}$$

as a linear space but has a strictly weaker locally convex topology.

PROOF. We have again by Proposition 1.7 the topological isomorphism

$$L_\beta(\mathcal{E}^*(\Omega'), \mathcal{E}^{*\prime}(\Omega''))\cong\mathcal{E}^{*\prime}(\Omega')\hat{\otimes}\mathcal{E}^{*\prime}(\Omega'').$$

If $*=(M_p)$, then $\mathcal{E}^*(\Omega')$ and $\mathcal{E}^*(\Omega'')$ are (FG)-space, so that we have by Theorem 2.2

$$\mathcal{E}^{(M_p)\prime}(\Omega')\hat{\otimes}\mathcal{E}^{(M_p)\prime}(\Omega'')\cong(\mathcal{E}^*(\Omega')\hat{\otimes}\mathcal{E}^*(\Omega''))'\cong\mathcal{E}^{*\prime}(\Omega'\times\Omega'').$$

In any case $\mathcal{E}^{*\prime}(\Omega')\hat{\otimes}\mathcal{E}^{*\prime}(\Omega'')=\mathcal{E}^{*\prime}(\Omega')\varepsilon\mathcal{E}^{*\prime}(\Omega'')$ is continuously imbedded in $\mathcal{D}^{*\prime}(\Omega'\times\Omega'')\cong\mathcal{D}^{*\prime}(\Omega')\varepsilon\mathcal{D}^{*\prime}(\Omega'')$. We prove that the image coincides with the space $\mathcal{E}^{*\prime}(\Omega'\times\Omega'')$ of ultradistributions with compact support also in the case where $*=\{M_p\}$.

We note that $\mathcal{E}^{*\prime}(\Omega'')$ has a continuous norm q. An example is

$$q(g)=\sup_{\substack{\zeta\in C^n\\|\zeta|\le 1}}|\langle e^{-ix\zeta}, g(x)\rangle|.$$

If $T\in L(\mathcal{E}^*(\Omega'), \mathcal{E}^{*\prime}(\Omega''))$, then $q(T\varphi)$ is a continuous semi-norm on $\mathcal{E}^*(\Omega')$ so that there exist a compact set K' in Ω', a sequence H_p of (4.21) and a constant C such that

$$q(T\varphi)\le Cp_{H_pM_p,K'}(\varphi), \qquad \varphi\in\mathcal{E}^*(\Omega').$$

Since q is a norm, we have

$$(T\varphi)(y)=\int\varphi(x)h(x, y)dx=0$$

for all $\varphi\in\mathcal{D}^*(\Omega'\setminus K)$. This implies supp $h\subset K'\times\Omega''$. Similarly we have supp h

$\subset \Omega' \times K''$ for a compact set K'' in Ω''. Hence supp $h \subset K' \times K''$ is compact.

Conversely if $h(x, y) \in \mathcal{E}^{*\prime}(\Omega' \times \Omega'')$, then the corresponding linear mapping T clearly belongs to $L(\mathcal{E}^{*}(\Omega'), \mathcal{E}^{*\prime}(\Omega''))$.

Next we show that (5.16) is a topological isomorphism. Since $\mathcal{E}^{\{M_p\}\prime}(\Omega')$ and $\mathcal{E}^{\{M_p\}\prime}(\Omega'')$ are (LFG)-space by Theorem 3.3, it follows from Theorem 2.3 that

$$\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes}_\iota \mathcal{E}^{\{M_p\}\prime}(\Omega'') \cong \lim_{\substack{\longrightarrow \\ K' \Subset \Omega' \\ K'' \Subset \Omega''}} \mathcal{E}_{K'}^{\{M_p\}\prime} \hat{\otimes} \mathcal{E}_{K''}^{\{M_p\}\prime} \tag{5.17}$$

and

$$(\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes}_\iota \mathcal{E}^{\{M_p\}\prime}(\Omega''))'_\beta \cong \mathcal{E}^{\{M_p\}}(\Omega') \hat{\otimes} \mathcal{E}^{\{M_p\}}(\Omega'') \cong \mathcal{E}^{\{M_p\}}(\Omega' \times \Omega'') . \tag{5.18}$$

$\mathcal{E}_{K'}^{\{M_p\}\prime} \hat{\otimes} \mathcal{E}_{K''}^{\{M_p\}\prime} = \mathcal{E}_{K'}^{\{M_p\}\prime} \varepsilon \mathcal{E}_{K''}^{\{M_p\}\prime}$ is an (FG)-space as a closed linear subspace of $\mathcal{D}^{\{M_p\}\prime}(\Omega' \times \Omega'') = \mathcal{D}^{\{M_p\}\prime}(\Omega') \varepsilon \mathcal{E}^{\{M_p\}\prime}(\Omega'')$. (If K' and K'' are compact sets with the cone property, then we can actually prove

$$\mathcal{E}_{K'}^{*\prime} \hat{\otimes} \mathcal{E}_{K''}^{*\prime} = \mathcal{E}_{K' \times K''}^{*\prime} \tag{5.19}$$

in the same way as the proof of (5.2) in [II].) Hence $\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes}_\iota \mathcal{E}^{\{M_p\}\prime}(\Omega'')$ is reflexive as an (LFG)-space. Taking the strong dual of (5.18), we have (5.16).

Since

$$\mathcal{E}_{K'}^{\{M_p\}\prime} \hat{\otimes} \mathcal{E}_{K''}^{\{M_p\}\prime} \subset \mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes} \mathcal{E}^{\{M_p\}\prime}(\Omega'')$$

is a topological imbedding onto a closed linear subspace, we have a continuous linear bijection

$$\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes}_\iota \mathcal{E}^{\{M_p\}\prime}(\Omega'') \longrightarrow \mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes} \mathcal{E}^{\{M_p\}\prime}(\Omega'')$$

as its inductive limit.

To show that the topologies are different we prove that the dual $(\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes} \mathcal{E}^{\{M_p\}\prime}(\Omega''))' \cong B(\mathcal{E}^{\{M_p\}\prime}(\Omega'), \mathcal{E}^{\{M_p\}\prime}(\Omega''))$ is strictly smaller than the dual

$$(\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes}_\iota \mathcal{E}^{\{M_p\}\prime}(\Omega''))' = \mathcal{E}^{\{M_p\}}(\Omega' \times \Omega'') .$$

Since $\mathcal{E}^{\{M_p\}\prime}(\Omega')$ and $\mathcal{E}^{\{M_p\}\prime}(\Omega'')$ are reflexive Grothendieck spaces, every $\chi(x, y) \in (\mathcal{E}^{\{M_p\}\prime}(\Omega') \hat{\otimes} \mathcal{E}^{\{M_p\}\prime}(\Omega''))'$ is represented as

$$\chi(x, y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x) \psi_i(y)$$

with a sequence λ_i in l^1, a bounded sequence φ_i in $\mathcal{E}^{\{M_p\}}(\Omega')$ and a bounded sequence ψ_i in $\mathcal{E}^{\{M_p\}}(\Omega'')$ (Grothendieck [6], Chap. II, p. 39). Hence it follows that for each compact set K' in Ω' there is an $h>0$ independent of $y \in \Omega''$ such that $\chi(\cdot, y)$ restricted to K' belong to $\mathcal{E}^{\{M_p\},h}(K')$ for any $y \in \Omega''$. But there are $\chi \in \mathcal{E}^{\{M_p\}}(\Omega' \times \Omega'')$ which do not satisfy this condition.

A kernel $h(x, y)$ in $\mathcal{E}^{*\prime}(\Omega' \times \Omega'')$ is said to be *compatifying*.

THEOREM 5.4. (i) *The following isomorphism of locally convex spaces holds under the correspondence* (5.9):

$$L_\beta(\mathcal{D}^*(\Omega'), \mathcal{E}^{*\prime}(\Omega'')) \cong \mathcal{D}^{*\prime}(\Omega') \hat{\otimes} \mathcal{E}^{*\prime}(\Omega''). \tag{5.20}$$

An $h(x, y) \in \mathcal{D}^{*\prime}(\Omega' \times \Omega'')$ *belongs to the space of kernels on the right hand side if and only if* supp h *is proper over* Ω' *i.e. if for any compact set* K' *in* Ω' supp $h \cap K' \times \Omega''$ *is compact.*

(ii) *The following isomorphism of locally convex spaces holds under the correspondence* (5.9):

$$L_\beta(\mathcal{E}^*(\Omega'), \mathcal{D}^{*\prime}(\Omega'')) \cong \mathcal{E}^{*\prime}(\Omega') \hat{\otimes} \mathcal{D}^{*\prime}(\Omega''). \tag{5.21}$$

An $h(x, y) \in \mathcal{D}^{*\prime}(\Omega' \times \Omega'')$ *belongs to the space of kernels on the right hand side if and only if* supp h *is proper over* Ω''.

PROOF. The topological isomorphisms (5.20) and (5.21) are proved in the same way as before.

Suppose that $h(x, y)$ is a kernel in $\mathcal{D}^{*\prime}(\Omega') \hat{\otimes} \mathcal{E}^{*\prime}(\Omega'')$ and K' is a compact set in Ω'. If $\chi(x) \in \mathcal{D}^*(\Omega')$ is a function which is equal to 1 on a neighborhood of K', then the product $\chi(x)h(x, y)$ is a kernel of a continuous linear mapping from $\mathcal{E}^*(\Omega')$ into $\mathcal{E}^{*\prime}(\Omega'')$, so that it belongs to $\mathcal{E}^{*\prime}(\Omega' \times \Omega'')$ by Theorem 5.3. Hence there is a compact set K'' in Ω'' such that $\mathrm{supp}(\chi(x)h(x, y)) \subset \Omega' \times K''$. Since $h(x, y) = \chi(x)h(x, y)$ on a neighborhood of $K' \times \Omega''$, we have supp $h \cap K' \times \Omega'' \subset K' \times K''$.

In case $* = (M_p)$, this may also be proved from the fact that $T : \mathcal{D}^*_{K'} \to \mathcal{E}^{*\prime}(\Omega'')$ is bounded as a continuous linear mapping from a Fréchet space into a (DF)-space.

Conversely suppose that supp $h \cap K' \times \Omega''$ is compact for any compact set K' in Ω'. Then the corresponding linear mapping T maps $\mathcal{D}^*_{K'}$ continuously into $\mathcal{E}^{*\prime}(\Omega'')$ for any compact set K' in Ω' and therefore T belongs to $L(\mathcal{D}^*(\Omega'), \mathcal{E}^{*\prime}(\Omega''))$.

Similar is the proof of (ii).

The kernels $h(x, y)$ of Theorem 5.4 (i) are called *semi-compact kernels in* $y \in \Omega''$ of class $*$ and the kernels of Theorem 5.4 (ii) *semi-compact kernels in* $x \in \Omega'$ of class $*$. A kernel $h(x, y) \in \mathcal{D}^{*\prime}(\Omega' \times \Omega'')$ is called a *compact kernel* if it is semi-compact both in x and in y.

In Theorems 5.1–5.4 we assumed the continuity of linear mappings T. However, if T is a linear mapping defined by (5.9) with a kernel $h(x, y) \in \mathcal{D}^{*\prime}(\Omega' \times \Omega'')$, then the continuity holds automatically whenever T maps the space on the left into the space on the right.

For example, suppose that a linear mapping $T : \mathcal{D}^*(\Omega') \to \mathcal{D}^{*\prime}(\Omega'')$ with the kernel $h(x, y) \in \mathscr{D}^{*\prime}(\Omega' \times \Omega'')$ maps every $\varphi \in \mathscr{D}^*(\Omega')$ into $\mathscr{E}^*(\Omega'')$ and that for every $g \in \mathscr{E}^{*\prime}(\Omega'')$ the ultradistribution $\varphi \mapsto \langle T\varphi, g\rangle$ belongs to $\mathscr{E}^*(\Omega')$. Then the kernel $h(x, y)$ belongs to $\mathscr{E}^*(\Omega') \varepsilon \mathscr{E}^*(\Omega'')$.

In fact, the first assumption implies by De Wilde's closed graph theorem that $T \in L(\mathscr{D}^*(\Omega'), \mathscr{E}^*(\Omega''))$. Then we can apply Proposition 4.3 with $\mathscr{G} = \mathscr{E}^*(\Omega')$ and $F = \mathscr{E}^*(\Omega'')$.

THEOREM 5.5. *An ultradistribution $h(x, y) \in \mathscr{D}^{*\prime}(\Omega' \times \Omega'')$ is a regular compact kernel of class $*$ if and only if it belongs to $\mathscr{D}^{*\prime}(\Omega') \hat{\otimes} \mathscr{D}^*(\Omega'') \cap \mathscr{D}^*(\Omega') \hat{\otimes} \mathscr{D}^{*\prime}(\Omega'') \cap \mathscr{E}^{*\prime}(\Omega') \hat{\otimes} \mathscr{E}^*(\Omega'') \cap \mathscr{E}^*(\Omega') \hat{\otimes} \mathscr{E}^{*\prime}(\Omega'')$.*

PROOF. Suppose that $h(x, y)$ is a regular compact kernel. The continuous linear mapping $T: \mathscr{D}^*(\Omega') \to \mathscr{D}^{*\prime}(\Omega'')$ with the kernel $h(x, y)$ maps $\mathscr{D}^*(\Omega')$ into $\mathscr{E}^*(\Omega'') \cap \mathscr{E}^{*\prime}(\Omega'') = \mathscr{D}^*(\Omega'')$. Hence it follows from Proposition 4.3 (or the closed graph theorem) that $h(x, y)$ belongs to $\mathscr{D}^{*\prime}(\Omega') \varepsilon \mathscr{D}^*(\Omega'')$. Similarly the dual $T': \mathscr{D}^*(\Omega'') \to \mathscr{D}^{*\prime}(\Omega')$ maps $\mathscr{D}^*(\Omega'')$ into $\mathscr{D}^*(\Omega')$, so that $h(x, y)$ belongs to $\mathscr{D}^*(\Omega') \varepsilon \mathscr{D}^{*\prime}(\Omega'')$.

Since $\operatorname{supp} h$ is proper over Ω', $T \in L(\mathscr{E}^{*\prime}(\Omega'), \mathscr{D}^{*\prime}(\Omega''))$ maps $\mathscr{E}^{*\prime}(\Omega')$ into $\mathscr{E}^{*\prime}(\Omega'')$. Hence we have $T \in L(\mathscr{E}^{*\prime}(\Omega'), \mathscr{E}^{*\prime}(\Omega'')) \cong \mathscr{E}^*(\Omega') \varepsilon \mathscr{E}^{*\prime}(\Omega'')$ by the closed graph theorem. Similarly we have $h \in \mathscr{E}^{*\prime}(\Omega') \varepsilon \mathscr{E}^*(\Omega'')$.

The converse is trivial.

6. ε tensor products of locally convex sheaves.

The kernels appearing in Theorems 5.1 and 5.2 have a local property. To formulate it we recall the definition of sheaves.

A sheaf $\mathscr{F}$ of linear spaces over a topological space X is a collection of linear spaces $\mathscr{F}(U)$ defined for all open sets U in X together with linear mappings $\rho_V^U: \mathscr{F}(U) \to \mathscr{F}(V)$ defined for all pairs of open sets U, V with $U \supset V$ which satisfies the following five conditions:

(i) $\mathscr{F}(\phi) = \{0\}$;

(ii) $\rho_U^U = 1$ (the identity mapping);

(iii) If $U \supset V \supset W$ are three open sets, then $\rho_W^U = \rho_W^V \circ \rho_V^U$;

(iv) If $\{V_i\}$ is an open covering of an open set U and if $f \in \mathscr{F}(U)$ satisfies $\rho_{V_i}^U(f) = 0$ for all i, then $f = 0$;

(v) If $\{V_i\}$ is an open covering of an open set U and if $f_i \in \mathscr{F}(V_i)$, defined for all i, satisfy

$$\rho_{V_i \cap V_j}^{V_i}(f_i) = \rho_{V_i \cap V_j}^{V_j}(f_j)$$

for all i and j, then there is an $f \in \mathscr{F}(U)$ such that $f_i = \rho_{V_i}^U(f)$.

The mappings ρ_V^U are called restriction mappings and $\rho_V^U(f)$ is often written $f|_V$.

Definition 6.1. We call a sheaf $\mathcal{F}$ of linear spaces over X a *locally convex sheaf* if each $\mathcal{F}(U)$ is endowed with a locally convex topology and the following two conditions are satisfied:

(vi) The restriction mappings $\rho_V^U: \mathcal{F}(U)\to\mathcal{F}(V)$ are continuous;

(vii) If $\{V_i\}$ is an open covering of an open set U, then the imbedding $\mathcal{F}(U)\subset\prod\mathcal{F}(V_i)$ defined by $f\mapsto(f|_{V_i})$ is topological.

A locally convex sheaf $\mathcal{F}$ is said to be complete (or boundedly complete or sequentially complete) if every $\mathcal{F}(U)$ is so.

If U is locally compact, then we can take an open covering $\{V_i\}$ of relatively compact open sets and hence every continuous semi-norm $p(f)$ on $\mathcal{F}(U)$ depends only on the restriction $f|_V$ to a relatively compact open set V determined by p.

Proposition 6.2. *The sheaf $\mathcal{E}^*$ of ultradifferentiable functions and the sheaf $\mathcal{D}^{*\prime}$ of ultradistributions over $\boldsymbol{R}^n$ are complete locally convex sheaves.*

Proof. We have only to verify condition (vii). Let $\sum\chi_j(x)=1$ be a partition of unity of class $*$ subordinate to the open covering $\{V_i\}$ of the open set U and let p be a continuous semi-norm on $\mathcal{F}(U)$ with $\mathcal{F}=\mathcal{E}^*$ or $\mathcal{D}^{*\prime}$. Since $p(f)$ depends only on the behavior of f on a relatively compact set V in U, $p(\chi_j f)$ vanishes identically except for a finite number of j, say $j=1, 2, \cdots, m$. Suppose that $\operatorname{supp}\chi_j\Subset V_{i(j)}$. Then $q_j(g)=p(\chi_j g)$ is a continuous semi-norm on $\mathcal{F}(V_{i(j)})$ and we have

$$p(f)\leqq\sum_{j=1}^{m}p(\chi_j f)=\sum_{j=1}^{m}p(\chi_j(f|_{V_{i(j)}}))=\sum_{j=1}^{m}q_j(f|_{V_{i(j)}}).$$

In the definition of locally convex sheaves we assumed that $\mathcal{F}(U)$ was defined for all open set U. We have, however, the following.

Proposition 6.3. *Let $\mathcal{U}$ be a base of open sets in X such that $U\cap V\in\mathcal{U}$ whenever U and $V\in\mathcal{U}$. If a system of locally convex spaces $\mathcal{F}(U)$, defined for all $U\in\mathcal{U}$, and linear mappings $\rho_V^U: \mathcal{F}(U)\to\mathcal{F}(V)$, defined for all $U, V\in\mathcal{U}$ with $U\supset V$, satisfies conditions* (i)-(vii) *whenever $U, V, \cdots$ are open sets in $\mathcal{U}$, then there is a unique locally convex sheaf $\mathcal{F}$ which extends $\{\mathcal{F}(U), \rho_V^U; U, V\in\mathcal{U}\}$.*

If $\mathcal{F}(U)$ are complete (or boundedly complete or sequentially complete) for all $U\in\mathcal{U}$, then so is $\mathcal{F}$.

Proof. Let W be an arbitrary open set in X. We take an open covering $\{U_i\}$ of W consisting of elements in $\mathcal{U}$ and define $\mathcal{F}(W)$ to be the space of all elements (f_i) in $\prod\mathcal{F}(U_i)$ such that $f_i|_{U_i\cap U_j}=f_j|_{U_i\cap U_j}$ for all i, j and endow it with the relative topology from the direct product $\prod\mathcal{F}(U_i)$.

If $\{V_j\}$ is another open covering of W by elements in $\mathcal{U}$, then it follows from properties (i)-(v) that for each $(f_i)\in\mathcal{F}(W)$ there is a unique $(g_j)\in\prod\mathcal{F}(V_j)$ such that $f_i|_{U_i\cap V_j}=g_j|_{U_i\cap V_j}$ for all i, j. Moreover g_j satisfy $g_j|_{V_j\cap V_k}=g_k|_{V_j\cap V_k}$ for all j, k. Hence $\mathcal{F}(W)$ does not depend on the choice of open covering $\{U_i\}$.

Since $\prod\mathcal{F}(U_i)\subset\prod\mathcal{F}(U_i\cap V_j)$ and $\prod\mathcal{F}(V_j)\subset\prod\mathcal{F}(U_i\cap V_j)$ are both topological imbeddings by property (vii), the locally convex topology of $\mathcal{F}(W)$ does not depend on the open covering either.

The restriction mappings $\rho_W^V: \mathcal{F}(V)\to\mathcal{F}(W)$ are naturally defined by taking open coverings $\{U_i\}$ of V and $\{U_j'\}$ of W by elements of $\mathcal{U}$ so that $\{U_i\}\supset\{U_j'\}$.

Since the above are the only possible choice of $\mathcal{F}(W)$ and ρ_W^V, the uniqueness is clear.

Suppose that $\mathcal{F}(U)$ are complete (resp. boundedly complete, resp. sequentially complete) for all $U\in\mathcal{U}$ and that W is an open set with the covering $\{U_i\}\subset\mathcal{U}$. If $f^\nu=(f_i^\nu)$ is a Cauchy net (resp. a bounded Cauchy net, resp. a Cauchy sequence) in $\mathcal{F}(W)$, then every component f_i^ν is a Cauchy net etc. in $\mathcal{F}(U_i)$ and hence converges to an f_i. By condition (vi) $f=(f_i)$ belongs to $\mathcal{F}(W)$, and f^ν converges to f.

Subsheaves and direct products of locally convex sheaves are locally convex sheaves.

A subsheaf $\mathcal{F}$ of a locally convex sheaf $\mathcal{G}$ is said to be closed (or boundedly closed, or sequentially closed) if the linear subspace $\mathcal{F}(U)$ of $\mathcal{G}(U)$ is so for every open set U or equivalently for every U in a base $\mathcal{U}$ of open sets.

If $\mathcal{G}$ is a complete (resp. boundedly complete, resp. sequentially complete) locally convex sheaf, then every closed (resp. boundedly closed, resp. sequentially closed) subsheaf $\mathcal{F}$ of $\mathcal{G}$ is complete (resp. boundedly complete, resp. sequentially complete).

The direct product of complete (resp. boundedly complete, resp. sequentially complete) locally convex sheaves is complete (resp. boundedly complete, resp. sequentially complete).

THEOREM 6.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally convex sheaves over locally compact spaces X and Y respectively. Then there is a unique locally convex sheaf $\mathcal{F}\varepsilon\mathcal{G}$ over $X\times Y$ such that*

$$(\mathcal{F}\varepsilon\mathcal{G})(U\times V)=\mathcal{F}(U)\varepsilon\mathcal{G}(V) \tag{6.1}$$

for all open sets U in X and V in Y.

If $\mathcal{F}$ and $\mathcal{G}$ are complete (or boundedly complete or sequentially complete), then so is $\mathcal{F}\varepsilon\mathcal{G}$.

PROOF. In view of Propositions 6.3 and 1.1 it suffices to prove that the locally convex spaces $\mathcal{F}(U)\varepsilon\mathcal{G}(V)$ and the linear mappings $\rho_{U'}^U\varepsilon\rho_{V'}^V: \mathcal{F}(U)\varepsilon\mathcal{G}(V)\to\mathcal{F}(U')\varepsilon\mathcal{G}(V')$ satisfy conditions (i)-(vii) as $U\times V$ and $U'\times V'\subset U\times V$ range over the base $\mathcal{V}=\{U\times V;\ U$ open in X, V open in $Y\}$ of open sets in $X\times Y$.

We have $U\times V\supset U'\times V'$ if and only if $U\supset U'$ and $V\supset V'$. Hence conditions (i)-(iii) follow from the same conditions for $\mathcal{F}$ and $\mathcal{G}$. Condition (vi) is trivial.

In order to prove conditions (iv), (v) and (vii), we first consider the case where $U\times V\in\mathcal{V}$ is covered by $U\times V_j\in\mathcal{V}$. Conditions (iv) and (v) for $\mathcal{G}$ say that

$$0\longrightarrow\mathcal{G}(V)\xrightarrow{\ \varepsilon\ }\prod_j\mathcal{G}(V_j)\xrightarrow{\ \delta\ }\prod_{j,k}\mathcal{G}(V_j\cap V_k)$$

is exact, where $\varepsilon(f)=(g|_{V_j})$ and $\delta((g_j))=(g_k|_{V_j\cap V_k}-g_j|_{V_j\cap V_k})$.

We have to prove that

$$0\longrightarrow L_\varepsilon((\mathcal{F}(U))'_c,\ \mathcal{G}(V))\xrightarrow{\ \varepsilon\ }\prod_j L_\varepsilon((\mathcal{F}(U)'_c,\ \mathcal{G}(V_j))$$
$$\xrightarrow{\ \delta\ }\prod_{j,k}L_\varepsilon((\mathcal{F}(U))'_c,\ \mathcal{G}(V_j\cap V_k))$$

is exact. But this follows from the above exactness. In fact, if $\varepsilon(L)=0$ for an $L\in L_\varepsilon((\mathcal{F}(U))'_c,\ \mathcal{G}(V))$, then $\varepsilon(L)f'=\varepsilon(Lf')=0$ for all $f'\in(\mathcal{F}(U))'$ and hence we have $Lf'=0$ for all $f'\in(\mathcal{F}(U))'$. Similarly if $\delta((L_j))=0$ for an $(L_j)\in\prod L_\varepsilon((\mathcal{F}(U))'_c,\ \mathcal{G}(V_j))$, then there is a unique linear mapping $L:(\mathcal{F}(U))'\to\mathcal{G}(V)$ such that $L_jf'=(Lf')|_{V_j}$ for all $f'\in(\mathcal{F}(U))'$ and j. Since $\mathcal{G}(V)$ is topologically imbedded in $\prod\mathcal{G}(V_j)$ and since L_j are continuous, $L:(\mathcal{F}(U))'_c\to\mathcal{G}(V)$ is continuous.

Lastly condition (vii) follows from Proposition 1.5 and the remark preceding the proposition.

The same proof works when $U\times V$ is covered by $U_i\times V\in\mathcal{V}$. Consequently conditions (iv), (v) and (vii) hold also for all coverings of $U\times V$ of the form $U_i\times V_j$ in which indices i and j move independently.

To consider the general case, we remark that if, in general, an open set U is covered by open sets U_k and if conditions (iv), (v) and (vii) hold for a refinement $\{V_i\}$ of $\{U_k\}$, then the conditions hold for the original covering $\{U_k\}$. In fact, for each i choose a $k=k(i)$ such that $V_i\subset U_k$. Then the topological imbedding $\mathcal{F}(U)\to\prod\mathcal{F}(V_i)$ is factorized as

$$\mathcal{F}(U)\xrightarrow{\ \varepsilon\ }\prod\mathcal{F}(U_k)\xrightarrow{\ \rho\ }\prod\mathcal{F}(V_i)$$

with the continuous linear mapping ρ defined by $\rho((f_k))=(f_{k(i)}|_{V_i})$. Hence ε is a topological imbedding. Condition (v) is proved similarly.

Now suppose that $U\times V$ is covered by $U_k\times V_k\in\mathcal{V}$. The covering may not have a refinement of the product type $U_i\times V_j$ but if we restrict ourselves to a relatively compact open subset $U'\times V'$ of $U\times V$, then clearly it has an open covering of the product type which is a refinement of $\{U_k\times V_k\}$. So take relatively compact open coverings $\{U'_i\}$ of U and $\{V'_j\}$ of V and let $\{U''_l\times V''_l\}$ be the union of the open coverings of $U'_i \times V'_j$ as above. Then conditions (iv), (v) and (vii) hold for the covering $\{U''_l \times V''_l\}$ of $U \times V$. Since it is a refinement of the original covering, the conditions hold for the original covering.

Applying the theorem to $\mathcal{E}^*$ and $\mathcal{D}^{*\prime}$, we find that the spaces of kernels in the kernel theorem and Theorems 5.1 and 5.2 are spaces of sections over $\Omega' \times \Omega''$ of locally convex sheaves $\mathcal{D}_x^{*\prime}\varepsilon\mathcal{D}_y^{*\prime}$, $\mathcal{E}_x^*\varepsilon\mathcal{E}_y^*$, $\mathcal{D}_x^{*\prime}\varepsilon\mathcal{E}_y^*$ and $\mathcal{E}_x^*\varepsilon\mathcal{D}_y^{*\prime}$, where the subscripts x and y stand for independent variables of the topological spaces on which locally convex sheaves $\mathcal{E}^*$ and $\mathcal{D}^{*\prime}$ are defined. The kernel theorem says

(6.2) $$\mathcal{D}_x^{*\prime}\varepsilon\mathcal{D}_y^{*\prime} = \mathcal{D}_{x,y}^{*\prime}$$

and Theorem 5.1

(6.3) $$\mathcal{E}_x^*\varepsilon\mathcal{E}_y^* = \mathcal{E}_{x,y}^* .$$

We write

(6.4) $$\mathcal{E}_x^*\varepsilon\mathcal{D}_y^{*\prime} = \mathcal{E}_x^*\mathcal{D}_y^{*\prime} ,$$

(6.5) $$\mathcal{D}_x^{*\prime}\varepsilon\mathcal{E}_y^* = \mathcal{D}_x^{*\prime}\mathcal{E}_y^* .$$

These are subsheaves of $\mathcal{D}_{x,y}^{*\prime}$ but are not its topological subsheaves.

Let Ω be an open set in $\boldsymbol{R}^n = \boldsymbol{R}^{n'} \times \boldsymbol{R}^{n''}$, where x and y stand for variables in $\boldsymbol{R}^{n'}$ and $\boldsymbol{R}^{n''}$ respectively. An ultradistribution $h \in \mathcal{D}_{x,y}^{*\prime}(\Omega)$ belongs to $\mathcal{D}_x^{*\prime}\mathcal{E}_y^*(\Omega)$ if and only if for each point $(\mathring{x}, \mathring{y}) \in \Omega$ there are open neighborhoods Ω' of $\mathring{x}$ and Ω'' of $\mathring{y}$ with $\Omega' \times \Omega'' \subset \Omega$ such that $h|_{\Omega' \times \Omega''} \in \mathcal{D}^{*\prime}(\Omega') \hat{\otimes} \mathcal{E}^*(\Omega'')$. In view of Theorem 5.2 this is the case if

(6.6) $$\int \varphi(x) h(x, y) dx \in \mathcal{E}^*(\Omega'')$$

for any $\varphi \in \mathcal{D}^*(\Omega')$.

Conversely if $h \in \mathcal{D}_x^{*\prime}\mathcal{E}_y^*(\Omega)$, then we have (6.6) for any $\varphi \in \mathcal{D}^*(\Omega')$ as far as $\Omega' \times \Omega'' \subset \Omega$.

Let $\mathcal{F}$ and $\mathcal{G}$ be locally convex sheaves over a topological space X. A sheaf homomorphism $T: \mathcal{F} \to \mathcal{G}$ is a collection of linear mappings $T(U): \mathcal{F}(U) \to \mathcal{G}(U)$ defined for all open sets U in X such that $\rho_V^U \circ T(U) = T(V) \circ \rho_V^U$ whenever $U \supset V$. It is called continuous if all $T(U)$ are continuous.

In order that a sheaf homomorphism $T: \mathcal{F} \to \mathcal{G}$ be continuous, it is sufficient that $T(U): \mathcal{F}(U) \to \mathcal{G}(U)$ are continuous for all U in a base of open sets in X. Moreover, we have the following proposition which is proved in a similar way to Proposition 6.3.

PROPOSITION 6.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be locally convex sheaves over X and let $\mathcal{U}$ be a base of open sets in X which is closed under intersection. If $T(U): \mathcal{F}(U) \to \mathcal{G}(U)$ are continuous linear mappings defined for all $U \in \mathcal{U}$ and such that*

$\rho_V^U \circ T(U) = T(V) \circ \rho_V^U$ whenever U and $V \in \mathcal{V}$ satisfy $U \supset V$, then there is a unique continuous sheaf homomorphism $T: \mathcal{F} \to \mathcal{G}$ which extends $\{T(U); U \in \mathcal{V}\}$. If $\{T_\alpha(U)\}$ is equicontinuous for every $U \in \mathcal{V}$, then $\{T_\alpha(U)\}$ is equicontinuous for every open set U.

In particular, we have the following.

PROPOSITION 6.6. *Let $\mathcal{F}$ and $\mathcal{H}$ (resp. $\mathcal{G}$ and $\mathcal{K}$) be locally convex sheaves over a locally compact space X (resp. Y). If $S: \mathcal{F} \to \mathcal{H}$ and $T: \mathcal{G} \to \mathcal{K}$ are continuous sheaf homomorphisms, then there is a unique continuous sheaf homomorphism $S \varepsilon T: \mathcal{F} \varepsilon \mathcal{G} \to \mathcal{H} \varepsilon \mathcal{K}$ such that $(S \varepsilon T)(U \times V)$ coincides with*

$$S(U) \varepsilon T(V): \mathcal{F}(U) \varepsilon \mathcal{G}(V) \longrightarrow \mathcal{H}(U) \varepsilon \mathcal{K}(V)$$

for all open sets U in X and V in Y.

Let $a(x) \in \mathcal{E}^*(\Omega')$ and $b(y) \in \mathcal{E}^*(\Omega'')$. Then the multiplications $a(x) \cdot : \mathcal{D}^{*\prime} \to \mathcal{D}^{*\prime}$ by $a(x)$ and $b(y) \cdot : \mathcal{E}^* \to \mathcal{E}^*$ by $b(y)$ are continuous sheaf homomorphisms over Ω' and Ω'' respectively. The ε tensor product $a(x) \cdot \varepsilon b(y) \cdot : \mathcal{D}_x^{*\prime} \mathcal{E}_y^* \to \mathcal{D}_x^{*\prime} \mathcal{E}_y^*$ is the multiplication by the product $a(x) b(y) \in \mathcal{E}^*(\Omega' \times \Omega'')$.

More generally we have the following.

THEOREM 6.7. *Let Ω be an open set in $\boldsymbol{R}^n = \boldsymbol{R}^{n'} \times \boldsymbol{R}^{n''}$, where $x \in \boldsymbol{R}^{n'}$ and $y \in \boldsymbol{R}^{n''}$. If $a(x, y) \in \mathcal{E}^*(\Omega)$, then the multiplication by a is a continuous sheaf homomorphism*

$$a(x, y) \cdot : \mathcal{D}_x^{*\prime} \mathcal{E}_y^* \longrightarrow \mathcal{D}_x^{*\prime} \mathcal{E}_y^*$$

over Ω.

Moreover, the bilinear mapping $(a(x, y), h(x, y)) \mapsto a(x, y) h(x, y)$ is hypocontinuous on $\mathcal{E}^*(\Omega) \times \mathcal{D}_x^{*\prime} \mathcal{E}_y^*(\Omega)$ into $\mathcal{D}_x^{*\prime} \mathcal{E}_y^*(\Omega)$.

PROOF. Let E be a bounded set in $\mathcal{E}^*(\Omega)$. First we prove that the multiplications by $a(x, y) \in E$ are continuous sheaf homomorphisms $\mathcal{D}_x^{*\prime} \mathcal{E}_y^* \to \mathcal{D}_x^{*\prime} \mathcal{E}_y^*$ over Ω and that they are equicontinuous linear mappings from $\mathcal{D}_x^{*\prime} \mathcal{E}_y^*(\Omega)$ into itself.

Since the multiplications are sheaf homomorphisms $\mathcal{D}_{x,y}^{*\prime} \to \mathcal{D}_{x,y}^{*\prime}$, we see from Proposition 6.5 that it is sufficient to prove that the multiplications by $a \in E$ are equicontinuous linear mappings

$$a(x, y): \mathcal{D}^{*\prime}(\Omega') \varepsilon \mathcal{E}^*(\Omega'') \longrightarrow \mathcal{D}^{*\prime}(\Omega') \varepsilon \mathcal{E}^*(\Omega'') \tag{6.7}$$

for any open sets Ω' in $\boldsymbol{R}^{n'}$ and Ω'' in $\boldsymbol{R}^{n''}$ such that $\Omega' \times \Omega'' \subset \Omega$.

Since $\mathcal{D}^{*\prime}(\Omega') \varepsilon \mathcal{E}^*(\Omega'') = \mathcal{D}^{*\prime}(\Omega') \hat{\otimes}_\pi \mathcal{E}^*(\Omega'')$, this is the same as to prove that the bilinear mappings $(f(x), \phi(y)) \mapsto a(x, y) f(x) \phi(y)$ are equicontinuous on $\mathcal{D}^{*\prime}(\Omega') \times \mathcal{E}^*(\Omega'')$ into $\mathcal{D}^{*\prime}(\Omega') \varepsilon \mathcal{E}^*(\Omega'')$.

We have for any $\varphi(x)\in\mathcal{D}^*(\Omega')$

$$\int\varphi(x)a(x, y)f(x)\phi(y)dx$$

$$=\int[a(x, y)\phi(y)][\varphi(x)f(x)]dx\in\mathcal{E}^*(\Omega'').$$

Hence the ultradistributions $a(x, y)f(x)\phi(y)$ belong to $\mathcal{D}^{*\prime}(\Omega')\varepsilon\mathcal{E}^*(\Omega'')$.

In order to prove the equicontinuity we show that for any continuous semi-norms p on $\mathcal{D}^{*\prime}(\Omega')$ and q on $\mathcal{E}^*(\Omega'')$ there are continuous semi-norms $\bar{p}$ on $\mathcal{D}^{*\prime}(\Omega')$ and $\bar{q}$ on $\mathcal{E}^*(\Omega'')$ independent of $a\in E$ such that

$$(p\varepsilon q)(a(x, y)f(x)\phi(y))\leqq\bar{p}(f)\bar{q}(\phi),\quad f\in\mathcal{D}^{*\prime}(\Omega'),\ \phi\in\mathcal{E}^*(\Omega'').$$

Let A (resp. B) be the set of all $\varphi(x)\in\mathcal{E}^*(\Omega')$ such that $\left|\int\varphi(x)f(x)dx\right|\leqq p(f)$, $f\in\mathcal{D}^{*\prime}(\Omega')$ (resp. of all $g(y)\in\mathcal{E}^{*\prime}(\Omega'')$ such that $\left|\int\phi(y)g(y)dy\right|\leqq q(\phi)$, $\phi\in\mathcal{E}^*(\Omega'')$). Then

$$(p\varepsilon q)(a(x, y)f(x)\phi(y))$$

$$=\sup\left\{\left|\iint a(x, y)\varphi(x)f(x)\phi(y)g(y)dxdy\right|;\ \varphi\in A,\ \phi\in B\right\}.$$

A is a bounded set in $\mathcal{D}^*_{K'}$ for a compact set K' in Ω' and B is a bounded set in $\mathcal{E}^{*\prime}_{K''}$ for a compact set K'' in Ω''.

The bilinear functionals $\iint a(x, y)\bar{f}(x)\bar{g}(y)dxdy$ are equicontinuous on $\mathcal{E}^{*\prime}_{K'}\times\mathcal{E}^{*\prime}_{K''}$ because they are separately equicontinuous by Theorem 5.1 and $\mathcal{E}^{*\prime}_{K'}$ and $\mathcal{E}^{*\prime}_{K''}$ are both (DFG)-spaces or (FG)-spaces according as $*=(M_p)$ or $\{M_p\}$. Hence there are continuous semi-norms r on $\mathcal{E}^{*\prime}_{K'}$ and s on $\mathcal{E}^{*\prime}_{K''}$ independent of $a\in E$ such that

$$\left|\iint a(x, y)\bar{f}(x)\bar{g}(y)dxdy\right|\leqq r(\bar{f})s(\bar{g}),\ \bar{f}\in\mathcal{E}^{*\prime}_{K'},\ \bar{g}\in\mathcal{E}^{*\prime}_{K''}.$$

On the other hand, the multiplication is a hypocontinuous bilinear mapping $\mathcal{D}^*_{K'}\times\mathcal{D}^{*\prime}(\Omega')\to\mathcal{D}^{*\prime}_{K'}$ and $\mathcal{E}^*(\Omega'')\times\mathcal{E}^{*\prime}_{K''}\to\mathcal{E}^{*\prime}_{K''}$. Consequently there are continuous semi-norms $\bar{p}$ on $\mathcal{D}^{*\prime}(\Omega')$ and $\bar{q}$ on $\mathcal{E}^*(\Omega'')$ such that

$$(p\varepsilon q)(a(x, y)f(x)\phi(y))\leqq\sup\{r(\varphi f)s(\phi g);\ \varphi\in A,\ g\in B\}\leqq\bar{p}(f)\bar{q}(\phi).$$

Next we prove that for every $h(x, y)\in\mathcal{D}^{*\prime}_x\mathcal{E}^*_y(\Omega)$ the multiplication $a(x, y)\mapsto a(x, y)h(x, y)$ is continuous from $\mathcal{E}^*_{x,y}(\Omega)$ into $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y(\Omega)$. Since it is a sheaf homomorphism $\mathcal{E}^*_{x,y}\to\mathcal{D}^{*\prime}_x\mathcal{E}^*_y$, we may again restrict ourselves to the case $\Omega=\Omega'\times\Omega''$. Then we have to prove that for any continuous semi-norms p on $\mathcal{D}^{*\prime}(\Omega')$ and q on $\mathcal{E}^*(\Omega'')$ the semi-norm $(p\varepsilon q)(a(x, y)h(x, y))$ in $a\in\mathcal{E}^*(\Omega'\times\Omega'')$

is continuous.

Let $A\subset\mathcal{D}^*(\Omega')$ and $B\subset\mathcal{E}^{*\prime}(\Omega'')$ be the bounded sets associated to p and q as above. Then we have

$$(p\varepsilon q)(a(x, y)h(x, y))$$
$$=\sup\left\{\left|\iint(a(x, y)h(x, y))\varphi(x)g(y)dxdy\right|;\ \varphi\in A,\ \psi\in B\right\}.$$

In view of the first part of our proof the integrand is equal to the product of $a(x, y)\in\mathcal{E}^*(\Omega'\times\Omega'')$ and $h(x, y)\varphi(x)g(y)\in\mathcal{E}^{*\prime}_{K'\times K''}$. The latter is the image of $h(x, y)$ under the ε product of continuous linear mappings $\varphi(x)\cdot: \mathcal{D}^{*\prime}(\Omega')\to\mathcal{D}^{*\prime}(\Omega')$ and $g(y)\cdot: \mathcal{E}^*(\Omega'')\to\mathcal{D}^{*\prime}(\Omega'')$. Since $A\cdot$ and $B\cdot$ are equicontinuous in $L(\mathcal{D}^{*\prime}(\Omega'),\ \mathcal{D}^{*\prime}(\Omega'))$ and in $L(\mathcal{E}^*(\Omega''),\ \mathcal{D}^{*\prime}(\Omega''))$ respectively, it follows from Proposition 1 of Schwartz [17], Chap. I, p. 20 that $\{h(x, y)\varphi(x)g(y);\ \varphi\in A,\ g\in B\}$ is bounded in $\mathcal{D}^{*\prime}(\Omega'\times\Omega'')=\mathcal{D}^{*\prime}(\Omega')\varepsilon\mathcal{D}^{*\prime}(\Omega'')$ and therefore in $\mathcal{E}^{*\prime}_{K'\times K''}$. Hence $(p\varepsilon q)(a(x, y)\times h(x, y))$ is a continuous semi-norm on $\mathcal{E}^*(\Omega'\times\Omega'')$.

Since $\mathcal{E}^*_{x,y}(\Omega)$ is a barrelled space, the mappings $a(x, y)\mapsto a(x, y)h(x, y)$ are equicontinuous in $L(\mathcal{E}^*_{x,y}(\Omega),\ \mathcal{D}^{*\prime}_x\mathcal{E}^*_y(\Omega))$ whenever h are in a bounded set in $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y(\Omega)$. This completes the proof of Theorem 6.7.

If $\varphi(x)\in\mathcal{E}^*(\Omega')$ and $g(y)\in\mathcal{D}^{*\prime}(\Omega'')$, then it follows from Theorem 6.4 that the multiplication by $\varphi(x)g(y)$ is a continuous sheaf homomorphism $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\to\mathcal{D}^{*\prime}_{x,y}$ over $\Omega'\times\Omega''$. But an arbitrary $k(x, y)\in\mathcal{E}^*_x\mathcal{D}^{*\prime}_y(\Omega'\times\Omega'')$ is not necessarily a continuous multiplier from $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y$ into $\mathcal{D}^{*\prime}_{x,y}$. For example, $\delta(x-y)$ belongs to $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y(\boldsymbol{R}^n\times\boldsymbol{R}^n)\cap\mathcal{E}^*_x\mathcal{D}^{*\prime}_y(\boldsymbol{R}^n\times\boldsymbol{R}^n)$, but its square does not exist in $\mathcal{D}^{*\prime}_{x,y}(\boldsymbol{R}^n\times\boldsymbol{R}^n)$.

We note, however, that if $h(x, y)\in\mathcal{D}^{*\prime}_x\mathcal{E}^*_y(\Omega'\times\Omega'')$, then the bilinear mapping $(\varphi(x), g(y))\mapsto\varphi(x)g(y)h(x, y)$ is separately continuous on $\mathcal{E}^*_x(\Omega')\times\mathcal{D}^{*\prime}_y(\Omega'')$ into $\mathcal{D}^{*\prime}_{x,y}(\Omega'\times\Omega'')$, so that the product $k(x, y)h(x, y)$ has a meaning for $k(x, y)\in\mathcal{E}^*_x(\Omega')\hat{\otimes}_\iota\mathcal{D}^{*\prime}_y(\Omega'')$.

We have so far considered ε tensor products of two locally convex sheaves but it is easy to extend the theory to ε tensor products of more than two factors (cf. Schwartz [17], Chap. I, §1). We have the associativity and commutativity similarly to the case of algebraic tensor products. Hence if each factor is either $\mathcal{E}^*$ or $\mathcal{D}^{*\prime}$, the ε tensor product is isomorphic to $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y$, where x and y are independent variables in some $\boldsymbol{R}^{n'}$ and $\boldsymbol{R}^{n''}$ respectively.

Let $\mathcal{F}$ (resp. $\mathcal{G}$) be a locally convex sheaf over a locally compact space X with independent variable ξ (resp. Y with independent variable η) and let U (resp. V) be an open set in $X\times\boldsymbol{R}^n$ (resp. $\boldsymbol{R}^n\times Y$), where $\boldsymbol{R}^n=\boldsymbol{R}^{n'}\times\boldsymbol{R}^{n''}$ with independent variables x in $\boldsymbol{R}^{n'}$ and y in $\boldsymbol{R}^{n''}$. Then for any $f(\xi, x, y)\in\mathcal{F}\varepsilon\mathcal{E}^*_{x,y}(U)$ and $g(x, y, \eta)\in\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\varepsilon\mathcal{G}(V)$ we can define the product $f(\xi, x, y)g(x, y, \eta)\in\mathcal{F}\varepsilon\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\varepsilon\mathcal{G}((U\times Y)\cap(X\times V))$ by Theorems 6.7 and 6.4. The bilinear mapping $(f, g)\mapsto fg$ is hypocontinuous.

Let $\mathcal{F}$ be a locally convex sheaf over a locally compact space X with independent variable ξ, let U be an open set in $X\times\boldsymbol{R}^n$ and let V be the projection of U into X. Suppose that a $g(\xi, x)\in\mathcal{F}\varepsilon\mathcal{D}_x^{*\prime}(U)$ has a proper support over V or that for each compact set K in V $\operatorname{supp} g\cap(X\times K)$ is compact in U. Then we can define the integral

$$\int g(\xi, x)dx\in\mathcal{F}(U)$$

as follows: Let V_1 be a relatively compact open set in V and let $U_1=U\cap(V_1\times\boldsymbol{R}^n)$. By extension by zero $g_1=g|_{U_1}$ may be regarded as an element in $\mathcal{F}(V_1)\varepsilon\mathcal{E}^{*\prime}(\boldsymbol{R}^n)$. In fact, there is a compact set K in $\boldsymbol{R}^n$ such that $\operatorname{supp} g_1\subset V_1\times K$. Let L be a compact neighborhood of K and choose a function $\chi(x)\in\mathcal{D}_L^*$ which is identically 1 on a neighborhood of K. Then we have

$$\begin{aligned} g_1&=(1\varepsilon\chi(x)\cdot)g_1+(1\varepsilon(1-\chi(x)\cdot))g_1\\ &=(1\varepsilon\chi(x)\cdot)g_1\end{aligned}$$

because $1\varepsilon(1-\chi(x)\cdot)$ is a sheaf homomorphism. Since $\chi(x)\cdot:\mathcal{D}^{*\prime}(\boldsymbol{R}^n)\to\mathcal{E}_L^{*\prime}$ is continuous, g_1 belongs to $\mathcal{F}(V_1)\varepsilon\mathcal{E}_L^{*\prime}\subset\mathcal{F}(V_1)\varepsilon\mathcal{E}^{*\prime}(\boldsymbol{R}^n)$.

Thus $f_1(\xi)=\int g_1(\xi, x)dx\in\mathcal{F}(V_1)$ is determined as the image of g_1 under $1\varepsilon I$: $\mathcal{F}(V_1)\varepsilon\mathcal{E}^{*\prime}(\boldsymbol{R}^n)\to\mathcal{F}(V_1)\varepsilon\boldsymbol{C}=\mathcal{F}(V_1)$, where I is the inner product with the function identically equal to one.

If V_2 is an open subset of V_1, then the restriction mapping $\mathcal{F}\varepsilon\mathcal{D}^{*\prime}(V_1\times\boldsymbol{R}^n)\to\mathcal{F}\varepsilon\mathcal{D}^{*\prime}(V_2\times\boldsymbol{R}^n)$ is equal to $\rho\varepsilon 1$, where ρ is the restriction mapping $\mathcal{F}(V_1)\to\mathcal{F}(V_2)$, so that we have

$$f_1|_{V_2}=\int g_2(\xi, x)dx\,,$$

with $g_2=g|_{U\cap(V_2\times\boldsymbol{R}^n)}$. Hence if V_j is a relatively compact open covering of V and $f_j(\xi)\in\mathcal{F}(V_j)$ are defined as above, then $f_j(\xi)$ are compatible and determine a unique

$$f(\xi)=\int g(\xi, x)dx\in\mathcal{F}(U)\,.$$

When $\mathcal{F}$ is a locally convex sheaf over X and F is a closed set in an open subset U of X, we denote by $\mathcal{F}_F(U)$ the subspace of all $f\in\mathcal{F}(U)$ with $\operatorname{supp} f\subset F$. It is a closed linear subspace of $\mathcal{F}(U)$.

When X is locally compact and U is an open set in $X\times\boldsymbol{R}^n$, we define $\mathcal{F}\mathcal{E}^{*\prime}(U)$ to be the set of all $q(\xi, x)\in\mathcal{F}\varepsilon\mathcal{D}^{*\prime}(U)$ such that $\operatorname{supp} g$ is proper over the projection V of U into X and endow it with the inductive limit locally convex topology of the representation

$$\mathcal{F}\mathcal{E}^{*\prime}(U)=\varinjlim_{F}(\mathcal{F}\varepsilon\mathcal{D}^{*\prime})_F(U)$$

as F ranges over the closed sets in U which are proper over V.

From the definition of integrals we have immediately the following.

PROPOSITION 6.8. *Let U be an open set in $X\times\boldsymbol{R}^n$, let V be its projection in X and let F be a closed set in U which is proper over V. Then the integral $g(\xi, x)\mapsto\int g(\xi, x)dx$ is a continuous linear mapping from $(\mathcal{F}\varepsilon\mathcal{D}^{*\prime})_F(U)$ into $\mathcal{F}(V)$ and from $\mathcal{F}\mathcal{E}^{*\prime}(U)$ into $\mathcal{F}(V)$.*

Suppose that $\mathcal{F}$ and $\mathcal{F}_1$ (resp. $\mathcal{G}$ and $\mathcal{G}_1$) are locally convex sheaves over a locally compact space X (resp. Y) such that $\mathcal{F}_1$ (resp. $\mathcal{G}_1$) is a subsheaf of $\mathcal{F}$ (resp. $\mathcal{G}$) with the continuous imbedding $S: \mathcal{F}_1\to\mathcal{F}$ (resp. $T: \mathcal{G}_1\to\mathcal{G}$). Then the ε tensor product $\mathcal{F}_1\varepsilon\mathcal{G}_1$ is naturally looked upon as a subsheaf of $\mathcal{F}\varepsilon\mathcal{G}$ under the continuous sheaf injection $S\varepsilon T$.

With the same assumption the multiplications $\mathcal{F}\varepsilon\mathcal{E}^*_{x,y}(U)\times\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\varepsilon\mathcal{G}(V)\to\mathcal{F}\varepsilon\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\varepsilon\mathcal{G}((U\times Y)\cap(X\times V))$ and $\mathcal{F}_1\varepsilon\mathcal{E}^*_{x,y}(U)\times\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\varepsilon\mathcal{G}_1(V)\to\mathcal{F}_1\varepsilon\mathcal{D}^{*\prime\prime}_x\mathcal{E}^*_y\varepsilon\mathcal{G}_1((U\times Y)\cap(X\times V))$ defined above are compatible in the sense that $S\varepsilon 1\varepsilon T(f(\xi, x, y)\times g(x, y, \eta))=(S\varepsilon 1)f(\xi, x, y)\cdot(1\varepsilon T)g(x, y, \eta)$. Similarly the integrals $\mathcal{F}\mathcal{E}^{*\prime}(U)\to\mathcal{F}(V)$ and $\mathcal{F}_1\mathcal{E}^{*\prime}(U)\to\mathcal{F}_1(V)$ are compatible.

Let $h(x, y)$ be the kernel of a continuous linear mapping $T: \mathcal{D}^*_x(\Omega')\to\mathcal{D}^{*\prime}_y(\Omega'')$. As we remarked earlier, it is regularizing (resp. semi-regular in y, resp. semi-regular in x, resp. regular) of class $*$ if and only if it is a section of $\mathcal{E}^*_{x,y}$ (resp. $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y$, resp. $\mathcal{E}^*_x\mathcal{D}^{*\prime}_y$, resp. $\mathcal{D}^{*\prime}_x\mathcal{E}^*_y\cap\mathcal{E}^{*\prime}_x\mathcal{D}^{*\prime}_y$) over $\Omega'\times\Omega''$. It is compactifying (resp. compact in y, resp. compact in x, resp. compact) if and only if $\operatorname{supp} h$ is compact (resp. proper over Ω', resp. proper over Ω'', resp. proper both over Ω' and over Ω'').

Then the notation

$$(T\varphi)(y)=\int\varphi(x)h(x, y)dx$$

has actually the meaning of the integral in x of the product of $\varphi\in\mathcal{D}^*_x(\Omega')$ etc. and $h\in\mathcal{D}^{*\prime}_x\mathcal{D}^{*\prime}_y(\Omega'\times\Omega'')$ etc. as defined above.

Furthermore, let $k(y, z)$ be the kernel of a continuous linear mapping $S: \mathcal{D}^*_y(\Omega'')\to\mathcal{D}^{*\prime}_z(\Omega''')$. We say that the mappings T and S (or the kernels $h(x, y)$ and $k(y, z)$) are composable if $h(x, y)$ or $k(y, z)$ is semi-regular in y and if $h(x, y)$ or $k(y, z)$ is semi-compact in y. Then the product ST is defined as a continuous linear mapping from $\mathcal{D}^*_x(\Omega')$ into $\mathcal{D}^{*\prime}_z(\Omega''')$, the product $h(x, y)k(y, z)$ is defined as an element of $\mathcal{D}^{*\prime}_x\varepsilon\mathcal{D}^{*\prime}_y\varepsilon\mathcal{D}^{*\prime}_z(\Omega'\times\Omega''\times\Omega'')$ such that the support is proper over $\Omega'\times\Omega'''$, and the kernel of ST is calculated as the integral

$$\int h(x, y)k(y, z)dy\,.$$

Hence we obtain similar results to Schwartz [14], [17] Chap. I § 4 on compositions of kernels.

7. Ultradifferential operators.

In this section we assume that the sequence M_p satisfies conditions (M.0), (M.1), (M.2) and (M.3). Our problem is to determine continuous sheaf homomorphisms $T: \mathcal{F}\to\mathcal{G}$, where $\mathcal{F}$ and $\mathcal{G}$ are the sheaf $\mathcal{E}^*$ of ultradifferentiable functions or the sheaf $\mathcal{D}^{*\prime}$ of ultradistributions of class $*$.

PROPOSITION 7.1. *If $T: \mathcal{F}\to\mathcal{G}$ is a continuous sheaf homomorphism over an open set Ω in $\boldsymbol{R}^n$, then there is a unique continuous linear mapping $T(\Omega): \mathcal{F}_*(\Omega)\to\mathcal{G}(\Omega)$ whose kernel $h(x, y)$ has support in the diagonal $\Delta=\{(x, x)\in\Omega^2\}$ and such that*

$$(T(U)\varphi)(y)=\int_U\varphi(x)h(x, y)dx \tag{7.1}$$

for every open set U in Ω and $\varphi(x)\in\mathcal{F}(U)$, where $\mathcal{E}^*_*(\Omega)=\mathcal{D}^*(\Omega)$ and $\mathcal{D}^{*\prime}_*(\Omega)=\mathcal{E}^{*\prime}(\Omega)$.

Conversely if $h(x, y)$ is the kernel of a continuous linear mapping $T(\Omega)$: $\mathcal{F}_*(\Omega)\to\mathcal{G}(\Omega)$ such that $\operatorname{supp} h\subset\Delta$, then (7.1) defines a unique continuous sheaf homomorphism $T: \mathcal{F}\to\mathcal{G}$ over Ω.

PROOF. First we prove the converse part. So suppose that $h(x, y)$ is the kernel of a continuous linear mapping $T(\Omega): \mathcal{F}_*(\Omega)\to\mathcal{G}(\Omega)$ such that $\operatorname{supp} h\subset\Delta$. Then it follows from theorems in § 5 that $h(x, y)$ belongs to $\mathcal{F}'_*\varepsilon\mathcal{G}(\Omega\times\Omega)$, where $\mathcal{E}^{*\prime}_*=\mathcal{D}^{*\prime}$ and $(\mathcal{D}^{*\prime}_*)'=\mathcal{E}^*$. Therefore if $\varphi\in\mathcal{F}(U)$, the product $\varphi(x)h(x, y)$ belongs to $\mathcal{D}^{*\prime}\varepsilon\mathcal{G}(U\times\Omega)$ and the mapping $\varphi\mapsto\varphi h$ is continuous. Since its support is in $\Delta\cap(U\times\Omega)$, it follows from Proposition 6.8 that integral (7.1) belongs to $\mathcal{G}(U)$ and $T(U): \mathcal{F}(U)\to\mathcal{G}(U)$ is continuous. If V is an open subset of U, then we have by the definition of integral

$$(T(U)\varphi)|_V=T(V)(\varphi|V)\,.$$

The uniqueness is trivial.

Next suppose that $T: \mathcal{F}\to\mathcal{G}$ is a continuous sheaf homomorphism. Then $T(\Omega): \mathcal{F}_*(\Omega)\to\mathcal{G}(\Omega)$ is clearly a continuous linear mapping. Let $h(x, y)$ be its kernel:

$$(T(\Omega)\varphi)(y)=\int_\Omega\varphi(x)h(x, y)dx\,, \qquad \varphi\in\mathcal{F}_*(\Omega)\,.$$

Since T is a sheaf homomorphism, we have $\operatorname{supp}(T(\Omega)\varphi)\subset\operatorname{supp}\varphi$ for any $\varphi\in\mathcal{F}(\Omega)$. Hence if U and V are disjoint open sets in Ω, then we have for any $\varphi\in\mathcal{D}^*(U)$ and $\psi\in\mathcal{D}^*(V)$

$$\iint \varphi(x)\psi(y)h(x, y)dxdy=\int\psi(y)(T(\Omega)\varphi)(y)dy=0.$$

This implies $h|_{U\times V}=0$. Hence supp h is included in the diagonal.

By the converse part the kernel $h(x, y)$ induces a continuous sheaf homomorphism S. The proof will be completed if we show that $T=S$. Let $\varphi\in\mathcal{F}(U)$ and let x be a point in U. Multiplying φ by a function in $\mathcal{D}^*(U)$, we can find a $\psi\in\mathcal{F}_*(\Omega)$ such that $\varphi|_V=\psi|_V$ for an open neighborhood V of x in U. Then

$$\begin{aligned}(T(U)\varphi)|_V&=T(V)(\varphi|_V)=T(V)(\psi|_V)=(T(\Omega)\psi)|_V\\&=\int\psi(x)h(x, y)dx|_V=(S(\Omega)\psi)|_V=S(V)(\psi|_V)=(S(U)\varphi)|_V.\end{aligned}$$

Since φ and x are arbitrary, we have $T(U)=S(U)$.

Thus our problem is reduced to the characterization of all elements in $\mathcal{D}^{*\prime}_{\Delta}(\Omega\times\Omega)$, $(\mathcal{D}^{*\prime}\mathcal{E}^*)_{\Delta}(\Omega\times\Omega)$, $(\mathcal{E}^*\mathcal{D}^{*\prime})_{\Delta}(\Omega\times\Omega)$ and $\mathcal{E}^*_{\Delta}(\Omega\times\Omega)$. The last one consists only of zero, so that we have:

THEOREM 7.2. *There are no other continuous sheaf homomorphisms $\mathcal{D}^{*\prime}\to\mathcal{E}^*$ than zero.*

To apply Theorem 4.8 we change the independent variables (x, y) in $\Omega\times\Omega$ into (ξ, η) in $\tilde{\Omega}=\{(\xi, \eta);\ \eta\in\Omega,\ \xi+\eta\in\Omega\}$ by

$$\begin{cases}\xi=x-y,\\ \eta=y.\end{cases}\tag{7.2}$$

We will discuss coordinate transformations of ultradistributions in the next part of our series. In this case, however, the theory is very simple because the Jacobian of the transformation is identically equal to 1. Let Φ be the transformation (7.2). If $\psi(\xi, \eta)\in\mathcal{D}^*(\tilde{\Omega})$, we define $\Phi^*\psi\in\mathcal{D}^*(\Omega\times\Omega)$ by

$$(\Phi^*\psi)(x, y)=\psi(x-y, y).$$

It is easily proved that $\Phi^*:\ \mathcal{D}^*(\tilde{\Omega})\to\mathcal{D}^*(\Omega\times\Omega)$ is an isomorphism of locally convex spaces. Therefore it induces the isomorphism

$$\Phi:\ \mathcal{D}^{*\prime}(\Omega\times\Omega)\cong\mathcal{D}^{*\prime}(\tilde{\Omega})\tag{7.3}$$

defined by $\langle\psi, \Phi f\rangle=\langle\Phi^*\psi, f\rangle$ or by

$$\int\psi(\xi, \eta)(\Phi f)(\xi, \eta)d\xi d\eta=\int\psi(x-y, y)f(x, y)dxdy.\tag{7.4}$$

Hence we obtain the following characterization of continuous sheaf homomorphisms $\mathcal{E}^*\to\mathcal{D}^{*\prime}$ by the structure theorem of vector valued ultradistributions with support at the origin.

THEOREM 7.3. *Every continuous sheaf homomorphism* $T:\mathcal{E}^*\to\mathcal{D}^{*\prime}$ *over an open set* Ω *in* $\boldsymbol{R}^n$ *is uniquely represented as*

$$(7.5)\qquad (T\varphi)(x)=\sum_{\alpha} a_\alpha(x)D^\alpha\varphi(x)\,,\qquad \varphi\in\mathcal{E}^*(\Omega_1)\,,$$

for any open subset Ω_1 *of* Ω *with ultradistributions* $a_\alpha(x)\in\mathcal{D}^{*\prime}(\Omega)$ *such that for any continuous semi-norm* q *on* $\mathcal{D}^{*\prime}(\Omega)$ *there exist a sequence* H_p *of* (4.21) *and a constant* C *satisfying*

$$(7.6)\qquad q(a_\alpha)\leqq\frac{C}{H_{|\alpha|}M_{|\alpha|}}\,.$$

Conversely if a family of ultradistributions $a_\alpha(x)\in\mathcal{D}^{*\prime}(\Omega)$ *satisfy the above condition, then* (7.5) *converges absolutely in* $\mathcal{D}^{*\prime}(\Omega_1)$ *uniformly in* φ *in every bounded set in* $\mathcal{E}^*(\Omega_1)$ *and represents a unique continuous sheaf homomorphism* $T:\mathcal{E}^*\to\mathcal{D}^{*\prime}$ *over* Ω.

PROOF. By the coordinate transformation (7.2) the kernel $h(x, y)$ of a continuous sheaf homomorphism T is transformed to an ultradistribution $k(\xi, \eta)\in\mathcal{D}^{*\prime}(\tilde{\Omega})$ with support in the closed set $\xi=0$. By extension by zero $k(\xi, \eta)$ may be regarded as an element of $\mathcal{E}^{*\prime}_{\{0\}}(\mathcal{D}^{*\prime}(\Omega))$. Hence it follows from Theorem 4.8 that

$$(7.7)\qquad h(x, y)=\sum_{\alpha}(-1)^{|\alpha|}D^\alpha\delta(x-y)\otimes a_\alpha(y)$$

with $a_\alpha(x)\in\mathcal{D}^{*\prime}(\Omega)$ as in the statement of the theorem. Since

$$\int(-1)^{|\alpha|}D^{|\alpha|}\delta(x-y)\varphi(x)dx=D^\alpha\varphi(x)\,,$$

we have (7.5).

Conversely suppose that $a_\alpha(x)\in\mathcal{D}^{*\prime}(\Omega)$ satisfy the condition of the theorem. Then (7.7) converges absolutely in $\mathcal{D}^{*\prime}_{\Delta}(\Omega\times\Omega)$ and represents the kernel of a continuous sheaf homomorphism $\mathcal{E}^*\to\mathcal{D}^{*\prime}$ over Ω. Let B be a bounded set in $\mathcal{E}^*(\Omega_1)$. We prove that (7.5) converges absolutely in $\mathcal{D}^{*\prime}(\Omega_1)$ uniformly in $\varphi\in B$. For every continuous semi-norm q on $\mathcal{D}^{*\prime}(\Omega_1)$ there is a compact set K in Ω_1 such that $q(\varphi)=0$ whenever $\varphi|_K=0$. Hence, if we take a $\chi(x)\in\mathcal{D}^*(\Omega_1)$ which is equal to 1 on a neighborhood of K, then we have $q(a_\alpha D^\alpha\varphi)=q(a_\alpha D^\alpha(\chi\varphi))$. Since χB is a bounded set in $\mathcal{D}^*(\Omega_1)$, the absolute convergence of (7.7) in $\mathcal{D}^{*\prime}(\Omega_1\times\Omega_1)$ $\cong L_\beta(\mathcal{D}^*(\Omega_1), \mathcal{D}^{*\prime}(\Omega_1))$ implies the absolute convergence of (7.5) in $\mathcal{D}^{*\prime}(\Omega_1)$ which is uniform in φ in B.

To obtain a similar characterization of continuous sheaf homomorphisms $\mathcal{E}^* \to \mathcal{E}^*$ we prepare the following.

THEOREM 7.4. *The transformation Φ defined by* (7.2) *induces the isomorphism of locally convex spaces*

$$\Phi: \mathcal{D}_x^{*\prime}\mathcal{E}_y^*(\Omega\times\Omega) \cong \mathcal{D}_\xi^{*\prime}\mathcal{E}_\eta^*(\tilde{\Omega}). \tag{7.8}$$

PROOF. We prove the continuity. Since $\mathcal{D}_x^{*\prime}\mathcal{E}_y^*(\Omega\times\Omega) = \mathcal{D}^{*\prime}(\Omega)\hat{\otimes}_\pi\mathcal{E}^*(\Omega)$ and since $\mathcal{D}_\xi^{*\prime}\mathcal{E}_\eta^*$ is a complete locally convex sheaf, it suffices to prove that for any open sets U and V in $\boldsymbol{R}^n$ with $U\times V \Subset \tilde{\Omega}$ the bilinear mapping $(f(x), \varphi(y)) \mapsto \Phi(f\varphi)|_{U\times V}$ is continuous on $\mathcal{D}^{*\prime}(\Omega)\times\mathcal{E}^*(\Omega)$ into $\mathcal{D}_\xi^{*\prime}\mathcal{E}_\eta^*(U\times V)$.

$\Phi(f\varphi)$ belongs to $\mathcal{D}_\xi^{*\prime}\mathcal{E}_\eta^*(U\times V)$ because we have for any $\psi(\xi)\in\mathcal{D}^*(U)$

$$\int\psi(\xi)\Phi(f\varphi)(\xi, \eta)d\xi = \int\psi(x-\eta)f(x)\varphi(\eta)dx \in \mathcal{E}^*(V).$$

To prove the continuity let $A\subset\mathcal{D}^*(U)$ and $B\subset\mathcal{E}^{*\prime}(V)$ be arbitrary bounded sets. We have to show that there are continuous semi-norms p on $\mathcal{D}^{*\prime}(\Omega)$ and q on $\mathcal{E}^*(\Omega)$ such that

$$\begin{aligned}\left|\iint\psi(\xi)\Phi(f\varphi)(\xi, \eta)g(\eta)d\xi d\eta\right| &= \left|\iint\psi(x-y)f(x)\varphi(y)g(y)dxdy\right| \\ &\leqq p(f)q(\varphi) \quad \text{for any } \psi\in A \text{ and } g\in B\end{aligned} \tag{7.9}$$

(cf. Proof of Theorem 6.7).

Since A is bounded, it follows from Lemma 3.4 that there are constants C, h and $1\leqq h_p\nearrow\infty$ (resp. C and h) such that

$$|D^\alpha\psi(x)| \leqq Ch^{|\alpha|}M_{|\alpha|}/(h_1\cdots h_{|\alpha|}) \qquad (\text{resp. } Ch^{|\alpha|}M_{|\alpha|})$$

for any $\psi\in A$ in case $*=(M_p)$ (resp. $*=\{M_p\}$). In view of condition (M.2) we have therefore

$$\begin{aligned}|D_y^\beta D_x^\alpha\psi(x-y)| &= |D^{\alpha+\beta}\psi(x-y)| \\ &\leqq AC\frac{(Hh)^{|\alpha|}M_{|\alpha|}}{h_1\cdots h_{|\alpha|}}\frac{(Hh)^{|\beta|}M_{|\beta|}}{h_1\cdots h_{|\beta|}} \\ (\text{resp.} \quad &\leqq AC(Hh)^{|\alpha|}M_{|\alpha|}(Hh)^{|\beta|}M_{|\beta|}).\end{aligned}$$

Consequently we have

$$\left|D_y^\beta\int\psi(x-y)f(x)dx\right| \leqq p(f)(Hh)^{|\beta|}M_{|\beta|}/(h_1\cdots h_{|\beta|})$$

$$(\text{resp. } p(f)(Hh)^{|\beta|}M_{|\beta|})$$

for any $y \in V$, where p is the continuous semi-norm on $\mathcal{D}^{*\prime}(\Omega)$ defined to be the supremum of $|\langle \chi, f\rangle|$ for all $\chi \in \mathcal{D}^*(U+V)$ such that

$$|D^\alpha \chi(x)| \leqq AC(Hh)^{|\alpha|}M_{|\alpha|}/(h_1 \cdots h_{|\alpha|})$$
$$(\text{resp. } AC(Hh)^{|\alpha|}M_{|\alpha|}).$$

Let $\bar{A}$ be the bounded set in $\mathcal{E}^*(V)$ of all $\bar{\psi}(y)$ such that

$$\sup_{y \in V} |D^\beta \bar{\psi}(y)| \leqq (Hh)^{|\beta|}M_{|\beta|}/(h_1 \cdots h_{|\beta|})$$
$$(\text{resp. } (Hh)^{|\beta|}M_{|\beta|}).$$

Then $\bar{A}B = \{\bar{\psi}(y)g(y);\ \bar{\psi} \in \bar{A},\ g \in B\}$ is also bounded in $\mathcal{E}^{*\prime}(V)$. Hence we have (7.9) for the continuous semi-norm

$$q(\varphi) = \sup\{|\langle \varphi, \bar{g}\rangle|;\ \bar{g} \in \bar{A}B\}$$

on $\mathcal{E}^*(V)$.

Similarly we can prove the continuity of the inverse of Φ in (7.8).

THEOREM 7.5. *Every continuous sheaf homomorphism* $T: \mathcal{E}^* \to \mathcal{E}^*$ *over an open set* Ω *in* $\boldsymbol{R}^n$ *is uniquely represented as*

$$(7.10) \qquad (T\varphi)(x) = \sum_\alpha a_\alpha(x) D^\alpha \varphi(x), \qquad \varphi \in \mathcal{E}^*(\Omega_1),$$

for any open subset Ω_1 *of* Ω *with ultradifferentiable functions* $a_\alpha(x) \in \mathcal{E}^*(\Omega)$ *satisfying the following condition:*

In case $* = (M_p)$, *for any compact set* K *in* Ω *and* $h > 0$ *there are constants* L *and* C *(In case* $* = \{M_p\}$, *for any compact set* K *in* Ω *and* $L > 0$ *there are constants* h *and* C*) such that*

$$(7.11) \qquad \sup_{x \in K} |D^\beta a_\alpha(x)| \leqq \frac{CL^{|\alpha|}h^{|\beta|}M_{|\beta|}}{M_{|\alpha|}}.$$

Conversely if a family of ultradifferentiable functions $a_\alpha(x) \in \mathcal{E}^*(\Omega)$ *satisfy the above condition, then* (7.10) *converges absolutely in* $\mathcal{E}^*(\Omega_1)$ *uniformly in* φ *in every bounded set in* $\mathcal{E}^*(\Omega_1)$ *and represents a continuous sheaf homomorphism* $T: \mathcal{E}^* \to \mathcal{E}^*$ *over* Ω.

PROOF is exactly the same as that of Theorem 7.3. We have only to show that a family of ultradifferentiable functions $a_\alpha(x) \in \mathcal{E}^*(\Omega)$ satisfy the condition of the theorem if and only if for any continuous semi-norm q on $\mathcal{E}^*(\Omega)$ there are a sequence H_p of (4.21) and a constant C such that (7.6) holds. Since (3.9) (resp. (3.14)) gives a base of continuous semi-norms on $\mathcal{E}^*(\Omega)$, this means that for any compact set K in Ω and sequence H'_p of (4.21) there are a sequence H_p of (4.21) and a constant C such that

(7.12) $$\sup_{x\in K}|D^\beta a_\alpha(x)|\leqq\frac{CH'_{|\beta|}M_{|\beta|}}{H_{|\alpha|}M_{|\alpha|}}.$$

In case $*=(M_p)$ we have $H'_{|\beta|}=h^{|\beta|}$ and $H_{|\alpha|}=L^{-|\alpha|}$, and hence the condition of the theorem. In case $*=\{M_p\}$, it follows from Lemma 3.4 that (7.12) holds for an H_p if and only if for any $L>0$ there is a constant C such that

(7.13) $$\sup_{x\in K}|D^\beta a_\alpha(x)|\leqq\frac{CL^{|\alpha|}H'_{|\beta|}M_{|\beta|}}{M_{|\alpha|}}.$$

Employing Lemma 3.4 again, we find that (7.13) holds for any H'_p if and only if there are constants h and C such that (7.11) holds.

DEFINITION 7.6. An *ultradifferential operator of class* $*$ on an open set Ω in $\boldsymbol{R}^n$ is by definition a continuous sheaf homomorphism $\mathcal{E}^*\to\mathcal{E}^*$ over Ω as characterized in Theorem 7.5.

Let

$$h(x, y)=\sum_\alpha(-1)^{|\alpha|}D^\alpha\delta(x-y)\otimes a_\alpha(y)$$

be the kernel of an ultradifferential operator (7.10) of class $*$ on Ω. Then the dual

$$h'(x, y)=h(y, x)=\sum_\alpha a_\alpha(x)\otimes D^\alpha\delta(x-y)$$

belongs to $(\mathcal{E}^*_x\mathcal{D}^{*\prime}_y)_\Delta(\Omega\times\Omega)$. Every element in $(\mathcal{E}^*_x\mathcal{D}^{*\prime}_y)_\Delta(\Omega\times\Omega)$ is obtained in this way because the dual gives an isomorphism $(\mathcal{D}^{*\prime}_x\mathcal{E}^*_y)_\Delta(\Omega\times\Omega)\cong(\mathcal{E}^*_x\mathcal{D}^{*\prime}_y)_\Delta(\Omega\times\Omega)$. Hence we get the following.

THEOREM 7.7. *Every continuous sheaf homomorphism* $T:\mathcal{D}^{*\prime}\to\mathcal{D}^{*\prime}$ *over an open set* Ω *in* $\boldsymbol{R}^n$ *is uniquely represented as*

(7.14) $$(Tf)(x)=\sum_\alpha D^\alpha(a_\alpha(x)f(x)),\qquad f\in\mathcal{D}^{*\prime}(\Omega_1)$$

for any open subset Ω_1 *of* Ω *with ultradifferentiable functions* $a_\alpha(x)\in\mathcal{E}^*(\Omega)$ *satisfying the condition of Theorem* 7.5.

Conversely, if a family of ultradifferentiable functions $a_\alpha(x)\in\mathcal{E}^*(\Omega)$ *satisfy the condition, then* (7.14) *converges absolutely in* $\mathcal{D}^{*\prime}(\Omega_1)$ *uniformly in* f *in every bounded set in* $\mathcal{D}^{*\prime}(\Omega_1)$ *and represents a continuous sheaf homomorphism* T: $\mathcal{D}^{*\prime}\to\mathcal{D}^{*\prime}$ *over* Ω.

Added in Proof.

H. O. Fattorini has proved Theorems 4.6 and 4.7 by different methods in the paper structure theorems for vector valued ultradistributions, J. Funct. Anal. **39** (1980), 381-407.

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