On degeneration of rational surfaces

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In this paper we shall prove the following:

THEOREM 1. *Let K be a complete discrete valuation field whose residue field k is algebraically closed and of characteristic zero. Let X be a 2-dimensional smooth projective variety over K such that $X\times_K\bar{K}$ is birationally equivalent to the projective space $\boldsymbol{P}^2_{\bar{K}}$ over $\bar{K}$, where $\bar{K}$ denotes the algebraic closure of K. Then X has at least one K-rational point.*

The surface X as in Theorem 1 is called a *rational surface* over K. We note that a rational surface over K is not necessarily birationally equivalent to $\boldsymbol{P}^2_K$ over K (Example 1). In the geometric language our theorem means the following: Let $D=\{z\in \boldsymbol{C};\ |z|<1\}$ be a disc, let X_D be a projective variety over D, and let $\pi\colon X_D\to D$ be the projection. We assume the following conditions:

(1) X_D is non-singular,

(2) The fibers $\pi^{-1}(z)$ for $z\in D^*=D\backslash\{0\}$ are non-singular rational surfaces, and

(3) Any irreducible component of $\pi^{-1}(0)$ is non-singular and cross normally each other.

Then $\pi^{-1}(0)$ has a component of multiplicity one. We note that the family X_D is not necessarily birationally equivalent to a smooth family of rational surfaces over D.

Theorem 1 was raised as a problem by T. Mabuchi, who encountered this problem during his study of 3-folds with negative Kodaira dimension. The author would like to thank him for stimulating discussions.

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1. Preliminaries

A field K is called a C_1-*field*, if every homogeneous polynomial of degree d in m variables over K with $d<m$ has a non-trivial zero in K.

THEOREM 2 (Theorem 10 of [2]). *Let K be a complete discrete valuation field whose residue field k is algebraically closed. Then K is a C_1-field.*

PROPOSITION 3 ([5] Chap. X, Prop. 10). *If K is a C_1-field, then the Brauer group of K is zero, i.e., the following holds: Let X be a smooth projective variety over K such that $\bar{X}=X\times_K\bar{K}$ is isomorphic to $\boldsymbol{P}_{\bar{K}}^n$ over $\bar{K}$, where $\bar{K}$ denotes the algebraic closure of K. Then X is isomorphic to $\boldsymbol{P}_K^n$.*

PROPOSITION 4 ([5] Chap. X, Application after Prop. 11). *If K is a C_1-field, then every principal homogeneous space over a torus over K is trivial.*

From now on we assume that K is a field as in Theorem 1. For an algebraic variety X over K, we denote by $X(K)$ the set of K-rational points on X.

COROLLARY 5. *Let X and X' be smooth projective surfaces over K. Assume that X and X' are birationally equivalent over K. Then $X(K)\neq\emptyset$ if and only if $X'(K)\neq\emptyset$.*

PROOF. We can reduce it to the following case: there is a monoidal transformation $f\colon X'\to X$ with center p on X. First, if $q\in X'(K)$, then $f(q)\in X(K)$. On the other hand, if $r\in X(K)$ and $r\neq p$, then $f^{-1}(r)\in X'(K)$. If $p\in X(K)$, then the exceptional curve $E=f^{-1}(p)$ is isomorphic to $\boldsymbol{P}_K^1$, and hence $X'(K)\supset E(K)\neq\emptyset$.
Q.E.D.

Example 1. By Theorem A of [3], cubic surfaces $x_0^3+x_1^3+x_2^3+a_ix_3^3$, where $a_i\in K^\times\backslash(K^\times)^3$ for $i=1,2$ are birationally equivalent, if and only if $a_1a_2^{-1}\in(K^\times)^3$. Thus there are cubic surfaces over K which are not birationally equivalent to $\boldsymbol{P}_K^2$ over K.

2. Proof of Theorem 1

A *rational surface* X over a field K is defined to be a smooth 2-dimensional projective variety over K such that $X\times_K\bar{K}$ is birationally equivalent to $\boldsymbol{P}_{\bar{K}}^2$, where $\bar{K}$ is the algebraic closure of K.

THEOREM 6 ([1] Theorem 1). *Any minimal rational surface X is one of the following:*

(1) $\boldsymbol{P}_K^2$.

(2) *a quadric Q in $\boldsymbol{P}_K^3$ such that* $\operatorname{Pic} Q\cong\boldsymbol{Z}$.

(3) $\operatorname{Pic} X\cong\boldsymbol{Z}\times\boldsymbol{Z}$, *and there is a morphism $f\colon X\to C$ such that C and the generic fiber X_η are smooth curves of genus 0.*

(4) $\operatorname{Pic} X\cong\boldsymbol{Z}$, *and it is generated by the ample anticanonical sheaf Ω_X^{-1}.*

Let K be a complete discrete valuation field and let X be a rational surface as in Theorem 1. By Corollary 5 we may assume that X is minimal. We shall check that $X(K)\neq\emptyset$ in each case of Theorem 6.

case (1): trivial.

case (2): This follows from Theorem 2.

case (3): By Proposition 3, C is isomorphic to $\boldsymbol{P}_K^1$. Using Bertini's theorem, we choose a point $p\in C(K)$ such that $f^{-1}(p)$ is smooth. Then $f^{-1}(p)$ is again isomorphic to $\boldsymbol{P}_K^1$. Thus $X(K)\supset f^{-1}(p)(K)\neq\emptyset$.

case (4): In this case $\bar{X}=X\times_K\bar{K}$ is a del Pezzo surface (cf. Section 24 of [4]). Let $d=(\Omega_X.\Omega_X)$ be the degree of X. We know that $1\leq d\leq 9$.

If $d=9$, then $\bar{X}$ is isomorphic to $\boldsymbol{P}_{\bar{K}}^2$. Hence $X(K)\neq\emptyset$ by Proposition 3.

If $d=8$, then $\bar{X}$ is isomorphic to either $\boldsymbol{P}_{\bar{K}}^1\times\boldsymbol{P}_{\bar{K}}^1$ or $\boldsymbol{F}_{\bar{K}}^1$, where the latter is obtained by blowing up one point from $\boldsymbol{P}_{\bar{K}}^2$.

If $d\leq 7$, then $\bar{X}$ is obtained from $\boldsymbol{P}_{\bar{K}}^2$ by blowing up $(9-d)$-points $x_1,\dots,x_{9-d}$ of "general position" on $\boldsymbol{P}_{\bar{K}}^2$. Let h: $\bar{X}\to\boldsymbol{P}_{\bar{K}}^2$ be the projection. An exceptional curve of the first kind E on $\bar{X}$ is defined to be a non-singular rational curve with $(E.E)=-1$. Let $\boldsymbol{E}$ be the set of all exceptional curves of the first kind on $\bar{X}$. The Galois group $G=\mathrm{Gal}(\bar{K}/K)$ acts on $\bar{X}$ and each $g\in G$ sends an exceptional curve of the first kind to another. Since $\mathrm{Pic}\,X\cong\boldsymbol{Z}$, all the exceptional curves on $\bar{X}$ are conjugate under the action of G. Thus, there is an element $g\in G$ which induces a cyclic simply transitive action on $\boldsymbol{E}$.

THEOREM 7 (Theorem 26.2 of [3]).

(1) *The image $h(E)$ of an arbitrary exceptional curve E of the first kind on $\bar{X}$ is one of the following type*:

(i) *one of the points x_i,*

(ii) *a line passing through two of the points x_i,*

(iii) *a conic passing through five of the points x_i,*

(iv) *a cubic passing through seven of the points x_i such that one of them is a double point,*

(v) *a quartic passing through eight of the points x_i such that three of them are double points,*

(vi) *a quintic passing through eight of the points x_i such that six of them are double points,*

(vii) *a sextic passing through eight of the points x_i such that seven of them are double points and one is a triple point.*

(2) *The number $n=n(d)$ of exceptional curves of the first kind on $\bar{X}$ is given by the following table:*

d	1	2	3	4	5	6	7	8
n	240	56	27	16	10	6	3	1

First, we treat the following cases:

(i) $d=3$,

(ii) $d=6$,

(iii) $d=8$ and $\bar{X}=\boldsymbol{P}_{\bar{K}}^{1}\times\boldsymbol{P}_{\bar{K}}^{1}$.

subcase (i): In this case X is a cubic surface in $\boldsymbol{P}_{K}^{3}$. Then $X(K)\neq\varnothing$ by Theorem 2.

subcase (ii): Let D be the union of all the exceptional curves on $\bar{X}$. Then $X\backslash D$ becomes a principal homogeneous space over some 2-dimensional K-torus by Theorem 30.3.1 of [4]. By Proposition 4, it is a K-torus. Thus $X\backslash D$ has a K-rational point (for example, the origin).

subcase (iii): $\mathrm{Pic}\,\bar{X}$ has two generators L_1 and L_2 corresponding to the generators of lines on $\bar{X}$. Since $\mathrm{Pic}\,X\cong\boldsymbol{Z}$, the Galois group $G=\mathrm{Gal}\,(\bar{K}/K)$ interchanges them. Put $K_1=K(t^{1/2})$, where t is a uniformizing element of K with respect to the given valuation. Then the unique subgroup $G_1=\mathrm{Gal}\,(\bar{K}/K_1)$ of index 2 in G acts on $\mathrm{Pic}\,\bar{X}$ trivially. Putting $X_1=X\times_K K_1$, we obtain $\mathrm{Pic}\,X_1\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$. Thus X_1 falls into the class (3) of Theorem 6. Hence $X_1(K_1)\neq\varnothing$. Let $P\in X_1(K_1)$. If $P\in X(K)$, then we are done. If not, let Q be the conjugate point of P over K. There are two cases: (a) P and Q are on a line L of self-intersection zero on $\bar{X}$. Then L is K-rational and has a K-rational point by Proposition 3. (b) P and Q are not on a line of self-intersection zero. Then, blowing up P and Q, we obtain a del Pezzo surface X' of degree 6 over K. By the subcase (ii), $X'(K)\neq\varnothing$. Therefore, $X(K)\neq\varnothing$. Q.E.D.

We shall complete the proof of Theorem 1 by the following:

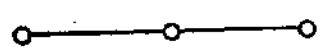
LEMMA 8. *In the remaining cases, X is not minimal.*

PROOF. If $d=8$, then the only exceptional curve E on $\bar{X}$ is invariant under the action of G. Thus E can be blown down.

The elements of $\boldsymbol{E}$ and their intersections give a configuration. We shall show that there can be no simply transitive cyclic action on $\boldsymbol{E}$ which preserves the above configuration, if $d=7$, 5, 4, 2 or 1. In the following drawings, a terminal denotes an element of $\boldsymbol{E}$ and a cord between them denotes an intersection.

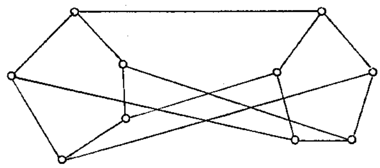
A double cord and a triple cord denote a double and a triple intersections, respectively. Let g denote the simply transitive cyclic action on $\boldsymbol{E}$ induced by the Galois action of G. Thus $\boldsymbol{E}=\{g^kE\}_{0\le k<n(d)}$ for an $E\in\boldsymbol{E}$, and $g^{n(d)}=\mathrm{id}$.

If $d=7$, then the configuration of $\boldsymbol{E}$ is the following:



It is trivial that there is no possibility of g.

If $d=5$, then the configuration is as follows:

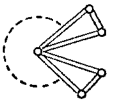


Let $S=\{k\in \boldsymbol{Z}/10\boldsymbol{Z};\ (g^kE.E)=1\}$. Then $\sharp S=3$. Since $(g^{-k}E.E)=(E.g^kE)$, if $k\in S$, then $-k\in S$. Thus $5\in S$. Let k be another element of S. Then the set $\{E, g^kE, g^{5+k}E, g^5E\}$ makes a square in the configuration, which does not exist, a contradiction.

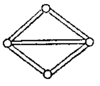
Let $d=4$ and let $S=\{k\in \boldsymbol{Z}/16\boldsymbol{Z};\ (g^kE.E)=1\}$ as above. Then $\sharp S=5$. If $k\in S$, then $-k\in S$. Thus $8\in S$. We also know that if $i,j\in S$, $0\neq i\neq j\neq 0$, then $(g^iE.g^jE)=0$. Hence $i-j\notin S$. On the other hand, any $E'\in\boldsymbol{E}$ can be joined with E by a succession of at most two cords. Thus $S+S$ coincides with the whole $\boldsymbol{Z}/16\boldsymbol{Z}$. Then the possible S are the following: (a) $\{1,3,8,13,15\}$, (b) $\{1,5,8,11,15\}$, (c) $\{3,7,8,9,13\}$, and (d) $\{5,7,8,9,11\}$. The replacements $g_1=g^5$, $g_2=g^3$ and $g_3=g^7$ reduce (b), (c) and (d) to (a), respectively. Thus we consider only the case (a). Let $\boldsymbol{E}_1=\{g^kE;\ k\in S\}$ and $\boldsymbol{E}_2=\boldsymbol{E}\backslash(\boldsymbol{E}_1\cup\{E\})$. Then we observe that for each $E'\in\boldsymbol{E}_2$ there are exactly two cords joining E' with $\boldsymbol{E}_1$. But since $14\equiv 15+15\equiv 13+1 \pmod{16}$, there are three cords joining $g^{14}E\in\boldsymbol{E}_2$ with $\boldsymbol{E}_1$, a contradiction.

Let $d=2$. In this case there appear double cords. There is only one $E'\in\boldsymbol{E}$ such that $(E.E')=2$. Hence $E'=g^{28}E$ by symmetry. Let $S=\{k\in \boldsymbol{Z}/56\boldsymbol{Z};\ (g^kE.E)=1\}$. Then $\sharp S=27$. If $k\in S$, then $-k\in S$. Hence the order of S must be even, a contradiction.

Finally, let $d=1$. In this case there appear triple cords. We consider only the double cords. Let $S=\{k\in \boldsymbol{Z}/240\boldsymbol{Z};\ (g^kE.E)=2\}$. Then $\sharp S=56$. The configuration of $\{E\} \cup \{g^k E;\ k \in S\}$ with respect to double cords is as follows:



There are 28 pairs $\{a_i, b_i\}$ $(i=1,\dots,28)$ in S such that $(g^{a_i}E.\,E)=(g^{b_i}E.\,E)=(g^{a_i}E.\ g^{b_i}E)=2$ and $(g^{a_i}E.\ g^{a_j}E)=(g^{a_i}E.\ g^{b_j}E)=(g^{b_i}E.\ g^{b_j}E)=0$ for $i\neq j$. Pick a pair $\{a_i, b_i\}$. Then $a_i-b_i \in S$. If $a_i-b_i \not\equiv b_i$ (mod 240), then $\{a_i-b_i, a_i\}$ gives another pair and we have a configuration as follows:



But this is a contradiction. Hence $a_i \equiv 2b_i$ (mod 240). Similarly, $b_i \equiv 2a_i$ (mod 240). Thus $3a_i \equiv 3b_i \equiv 0$ (mod 240). This contradicts the fact that $\sharp S=56$. Q.E.D.

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