A proof of the holomorphic Lefschetz formula for higher dimensional fixed point sets by the heat equation method

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Introduction

Let X be a compact complex manifold of dimension m and $f: X \to X$ a holomorphic mapping. The holomorphic Lefschetz number of f, denoted by $L(f)$, is defined by

$$L(f)=\sum_{p=0}^{m}(-1)^p \operatorname{trace} H^p f \tag{1}$$

where $H^p f$ is the linear endomorphism on the $(0, p)$th Dolbeault cohomology group of X induced from f. When X is endowed with a hermitian metric g, we have an analytic expression of $L(f)$ as follows. Let $\square_p=\bar{\partial}\bar{\partial}^*+\bar{\partial}^*\bar{\partial}: \wedge^{0,p}X \to \wedge^{0,p}X$ denote the complex Laplacian acting on the space of smooth $(0, p)$-forms on X and $e_p(t; x, y)$, $t>0$, the fundamental solution of the heat operator $\square_p+\frac{d}{dt}$. Then $L(f)$ is represented as

$$L(f)=\int_X \sum_{p=0}^{m}(-1)^p \operatorname{trace}(f\times 1)^* e_p(t; x, x)\,\mathrm{dvol}. \tag{2}$$

(See, for example, K.)

Assume that each component Y of the fixed point set of f is nondegenerate, i.e. Y is a regular complex submanifold and the kernel of $I-df: T_yX \to T_yX$ is exactly equal to T_yY for any $y \in Y$. Fix a smooth fibration $\pi: \mathrm{Tub}(Y) \to Y$ of a tubular neighbourhood of Y and integrate the (m, m)-form

$$\phi(t)=\sum_{p=0}^{m}(-1)^p \operatorname{trace}(f\times 1)^* e_p(t; x, x)\,\mathrm{dvol}$$

along the fiber. Then by [G1] and [L] we obtain an asymptotic expansion formula

$$\pi_*\phi(t)=\sum_{k=0}^{n_Y}\varphi_k(f, g; \pi)t^{k-n_Y}+O(t) \tag{3}$$

where $n_Y=\dim_C Y$ and $\varphi_k(f, g; \pi)$ is the smooth (n_Y, n_Y)-form on Y which is uniquely determined by the behavior of the mapping f and the metric g near Y and a choice of the fibration $\pi:\mathrm{Tub}(Y)\to Y$. By formula (2) and (3) and an estimate $\left|\int_{X-\cup_Y \mathrm{Tub}(Y)} \phi(t)\right| = O(e^{-c/t})$, we have

$$L(f)=\sum_Y \int_Y \varphi_{n_Y}(f, g; \pi) \tag{4}$$

and

$$\int_Y \varphi_k(f, g; \pi)=0 \qquad \text{for} \quad k<n_Y. \tag{5}$$

On the other hand, under the assumption that each component of the fixed point set is nondegenerate, Toledo and Tong proved

$$L(f)=\sum_Y \left[\mathrm{Todd}(Y)\prod_i \left\{\sum_{p\geq 0}(-1)^p \lambda_i^p \,\mathrm{ch}(\wedge^p N_i^*(Y))\right\}^{-1}\right][Y]. \tag{6}$$

Here $\mathrm{Todd}(Y)$ denotes the Todd class of Y and ch is the Chern character. $\{\lambda_i\}$ are the eigenvalues of df different from 1 and $N_i(Y)$ is the uniquely determined subbundle of $TX|_Y$ by the conditions;

i) $N_i(Y)$ is df-invariant,

ii) the restriction of $\lambda_i I-df$ to $N_i(Y)$ is nilpotent,

iii) the rank of $N_i(Y)$ is equal to the multiplicity of λ_i. ([T-T])

Let $\omega(f, g)$ be the Chern form associated to the cohomology class in formula (6) via the Weil homomorphism. It is natural to ask whether the following equality

$$\pi_*\phi(t)=\omega(f, g)+O(t) \tag{7}$$

holds or not. When X is a Kähler manifold and f is a holomorphic isometry, especially the identity mapping, this equality holds, ([G1], [G2] and [P]). In the case f is the identity mapping and X is not Kählerian Gilkey showed that the difference of $\omega(\mathrm{id}, g)$ and $\phi_m(\mathrm{id}, g)$ is written by a non-zero transgression form [G3]. Hence $\phi_m(\mathrm{id}, g)$ does not always coincide with $\omega(\mathrm{id}, g)$. Therefore equality (7) does not always hold. The aim of this paper is to give an answer for the following problem:

(P) For what kind of mappings and their nondegenerate components of the fixed point set can we construct a hermitian metric g such that equality (7) holds for some fibrations?

At first we try to obtain a sufficient condition for a hermitian metric g to give equality (7) for some fibrations. In connection with this we shall give an answer of somewhat local nature as follows.

THEOREM 1. *Suppose a metric g and a mapping f satisfy the following conditions*:

(C-1) $g_Y=g_0\oplus(\bigoplus_i h_i)$ *where g_0 is a Kähler metric on Y and h_i is a hermitian metric on $N_i(Y)$.*

(C-2) *Let $\Omega=\sum g_{a\bar{b}}dz^a\wedge d\bar{z}^b$ be the fundamental two form with respect to g. Then for each $y\in Y$, $d\Omega(y)=\nabla d\Omega(y)=0$ where ∇ denotes the riemannian connection on X.*

(C-3) *Let ξ, ζ and η be holomorphic vectors at $y\in Y$. If one of ξ, ζ and η is tangential to Y then $g(\nabla_{f_*\xi}f_*\zeta-f_*\nabla_\xi\zeta, \bar{\eta})=0$.*

Then there is an open covering $\{U_\lambda\}$ of Tub(Y) *and smooth fibrations $\pi_\lambda: U_\lambda\to U_\lambda\cap Y$ such that for any partition of unity $\{\eta_\lambda\}$ subordinate to $\{U_\lambda\}$ we have*

$$\pi_{\lambda*}[\eta_\lambda\phi(t)]=\eta_\lambda\int_Y\omega(f, g)+O(t).$$

Next we shall give a condition for a mapping f such that we can construct a hermitian metric which satisfies the conditions in Theorem 1.

THEOREM 2. *Let Y be a nondegenerate component of $f: X\to X$. Assume*

i) *Y admits a Kähler metric,*

ii) *$N_i(Y)$ admits a hermitian metric such that the restriction of df to $N_i(Y)$ is a parallel endomorphism,*

iii) *there is a torsion free connection ∇ around Y and a subbundle V of $TX|_Y$ such that $V\oplus TY=TX_Y$ and $\nabla df(V\otimes V)\subset V$. Then there is a hermitian metric g around Y such that g and f satisfy the conditions in Theorem 1.*

For some cases we shall verify above conditions.

THEOREM 3. 1) *If df induce a semisimple endomorphism of $N(Y)=\oplus N_i(Y)$ then the second condition in Theorem 2 is valid.*

2) *If $I-df\otimes df: N(Y)\otimes N(Y)\to N(Y)\otimes N(Y)$ is nonsingular then the third condition in Theorem 2 is valid.*

At first we review results in [G1] and [L] about asymptotic expansion formula (3) (§1). In §2 we construct a nice fibration stated in Theorem 1. Then we can conclude that a smooth form $\varphi_k(f, g; \pi)$ in (3) is determined by a Kähler metric g_0 on Y, a hermitian metric h_i on $N_i(Y)$ and a bundle endomorphism $df|_{N(Y)}: N(Y)\to N(Y)$ (§3). Finally according to the idea of Gilkey [G2], we can prove $\varphi_k(f, g; \pi)$

$=0$ for $k<n_Y$ and φ_{n_Y} represents some Chern classes by the Weil homomorphism (§4). In §§5-6 we shall prove Theorem 2 and Theorem 3.

The method in this paper works for other elliptic complexes and sometimes they are much simpler than ours. For example Donnely and Patodi proved the G-signature theorem [D-P] and Gilkey proved the Lefschetz formula for the de Rham complex [G2]. Baisically our proof is somewhat similar with their proves, however the fact that φ_{n_Y} is a Chern form is much difficult to show and our proof for that is much different from theirs.

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§1 Asymptotic expansion formula

In this section we let (X, g) be a compact Riemannian manifold of dimension $2m$ and $E\rightarrow X$ a hermitian vector bundle over X. Let $\square:\Gamma(E)\rightarrow\Gamma(E)$ be a second order selfadjoint nonnegative elliptic operator and $K(\square:t)\in\Gamma(E\boxtimes(E^*\otimes\Omega))$, $t>0$, denotes the fundamental solution of the Cauchy problem for the heat operator $\square+\frac{t}{dt}$. Here $\Gamma(\ )$ denotes the space of smooth sections and Ω the volume bundle of X. $E\boxtimes(E^*\otimes\Omega)\rightarrow X\times X$ is the exterior tensor product of E and $E^*\otimes\Omega$. Let $f:X\rightarrow X$ be a smooth selfmapping and $\Phi\in\Gamma(E\otimes f^{-1}E^*)$. Restrict $K(\square;t)$ to the set $\{(f(x), x); x\in X\}\subset X\times X$ and regard it as a smooth section of $f^{-1}E\otimes E^*\otimes\Omega\rightarrow X$. Then by the contraction of E and E^*, $\psi(\Phi;\square;t)=\operatorname{trace}\Phi K(\square;t)$ is a volume element of X. Set

$$L(\Phi, \square; t)=\int_X \psi(\Phi, \square; t). \tag{1.1}$$

Assume the fixed point set of f consist of only one nondegenerate component of Y of dimension $2n$, i.e. Y is a regular $2n$ dimensional submanifold and $I-df$ induces nonsingular endomorphism on the normal bundle of Y. Then we have

LEMMA 1.1 ([L][G2]). *For any smooth fibration $\pi:\mathrm{Tub}(Y)\rightarrow Y$ of the tublar neighbourhood of Y,*

$$\pi_*\psi(\Phi, \square; t)\sim\sum_{k\geq 0}\varphi_k(\Phi, \square, \pi)t^{k-n},\ t\downarrow 0.$$

Now we are concerned in properties of ψ and φ_k. We choose a local coordinate system $Z=(z^1, \cdots, z^{2m})$ on $U\subset X$ such that

i) $Y\cap U=\{z^{2n+1}=\cdots=z^{2m}=0\}$

ii) $\pi^{-1}(y)=\{z^i=z^i(y),\ 1\leq i\leq 2n\}$ for any $y\in Y\cap U$.

Moreover we fix a local frame field of E on U. Then we have

LEMMA 1.2. *$\varphi_k(\Phi,\Box,\pi)$ is written by a universal polynomial of following variables;*

derivatives of a symbol of $\Box$ and components of metric g,

derivatives of components of Φ and f by coordinate functions, $\{z^{2n+1},\dots,z^{2m}\}$,

$$\sqrt{\det(g_{ij})_{1\leq i,j\leq 2m}}^{-1},\ \sqrt{\det(g_{ij})_{1\leq i,j\leq 2n}}^{-1},\ \det(I-df|_{N(Y)})^{-1}.$$

This lemma is direct consequence from the proof of Lemma 1.1.

Next we apply a product data to ϕ and φ_k. Set

$$\begin{aligned}&X=X_1\times X_2,\ E=E_1\boxtimes E_2\to X_1\times X_2,\\ &\Box=\Box_1\otimes 1+1\otimes\Box_2:\Gamma(E_1)\otimes\Gamma(E_2)\to\Gamma(E_1)\otimes\Gamma(E_2),\\ &f=f_1\times f_2:X_1\times X_2\to X_1\times X_2 \quad\text{and}\quad \Phi=\Phi_1\times\Phi_2.\end{aligned}$$

Then because of the uniqueness of the fundamental solution, $K(\Box;t)=K(\Box_1;t)\otimes K(\Box_2;t)$. Furthermore we have

LEMMA 1.3 (Product formula).

$$\begin{aligned}&\phi(\Phi,\Box;t)=\phi(\Phi_1,\Box_1;t)\phi(\Phi_2,\Box_2;t)\\ &\varphi_k(\Phi,\Box,\pi)=\sum_{p+q=k}\varphi_p(\Phi_1,\Box_1,\pi_1)\varphi_q(\Phi_2,\Box_2,\pi_2).\end{aligned}$$

Let $\Box_c=c^{-2}\Box$. Both $K(\Box,c^{-2}t)$ and $K(\Box_c,t)$ denote the Schwartz kernel of the operator $e^{-c^{-2}t\Box}$. Therefore we get

LEMMA 1.4 (Weight formula).

$$\begin{aligned}&\phi(\Phi,\Box_c;t)=\phi(\Phi,\Box;c^{-2}t),\\ &\varphi_k(\Phi,\Box_c;\pi)=c^{-2k+2n}\varphi_k(\Phi,\Box,\pi).\end{aligned}$$

In this paper we apply these formulas to the Dolbeault complex;

$$E=\bigoplus_{p\geq 0}\wedge^{0,p}X,\ \Box=\bar\partial^*\bar\partial+\bar\partial\bar\partial^* \quad\text{and}\quad \Phi=\bigoplus_{p\geq 0}(-1)^p f^*.$$

§2 Local invariants with a data of mappings

In what follows we always use the following convention on indices; indices $a,b,\dots$ run from 1 to m, indices $i,j,\dots$ run from 1 to n and indices $u,v,\dots$ run from $n+1$ to m. Also we assume $\alpha,\beta,\dots$ denote multiindices from 1 to n and $\mu,\nu,\dots$ from $n+1$ to m.

Let X, Y, g and f be as in Theorem 1. We choose a holomorphic local coordinate system $W=(w^1,\dots,w^m)$ defined on $V\subset X$ such that $U=V\cap Y=\{w^{n+1}=\cdots=w^m=0\}$ and $f^i_u=0$ along U. Here and the following part of this paper we use the notation;

$$g_{a\bar b/\alpha\bar\beta\mu\bar\nu}=\left(\frac{\partial}{\partial w}\right)^{\alpha}\left(\frac{\partial}{\partial \bar w}\right)^{\beta}\left(\frac{\partial}{\partial w}\right)^{\mu}\left(\frac{\partial}{\partial \bar w}\right)^{\nu} g\left(\frac{\partial}{\partial w^a},\frac{\partial}{\partial \bar w^b}\right)$$

$$f^a_{b/\mu}=\left(\frac{\partial}{\partial w}\right)^{\mu}\frac{\partial f^a}{\partial w^b}.$$

For any $y\in U$, define

$$D_y:=\left\{x=(w^1,\dots,w^m)\in V\;\middle|\; w^i+\frac{1}{2}g^{ik}g_{uk/v}(y)w^uw^v + \frac{1}{2}\left(\frac{\partial}{\partial w^j}\right)(g^{ik}g_{uk/v})(y)w^uw^vw^j=w^i(y)\quad \text{for any}\quad i,\ 1\leqq i\leqq n\right\}.$$

LEMMA 2.1. *The union of complex disks, $\bigcup_{y\in U} D_y$, gives rise to a smooth fibration of V.*

PROOF. Define $\varphi: U\times V\to \boldsymbol{C}^n$ by

$$\varphi^i(y,(w^1,\dots,w^m))=w^i+\frac{1}{2}g^{ik}g_{uk/v}(y)w^uw^v+\frac{1}{2}\left(\frac{\partial}{\partial w^j}\right)(g^{ik}g_{uk/v})(y)w^uw^vw^j-w^i(y).$$

Then $\varphi^{-1}(0)=\bigcup_{y\in U}(y, D_y)$. Identify U with the diagonal set in $U\times U$. Since $\varphi(U)=0$ and $J\varphi=$the Jacobian matrix of $\varphi=(-I_n, I_n, 0)$ on U, $\varphi^{-1}(0)$ is a smooth submanifold of $U\times V$ around U. Let $\pi_V: U\times V\to V$ be the projection to the second factor. Because $\operatorname{Ker} J\pi_V\cap \operatorname{Ker} J\varphi=\{0\}$ along U, we take a sufficiently small neighbourhood V_0 of U in $\varphi^{-1}(0)$, if necessary, such that $\pi_V: V_0\to V$ is an into diffeomorphism. Furthermore the fibration of $V_0\subset\varphi^{-1}(0)$ inherits to one of V by disks D_y. Q.E.D.

Let $\pi: V\to U$ be a projection with $\pi(D_y)=y$ and fix any $y\in U$.

LEMMA 2.2. *There is a holomorphic local coordinate system $Z=(z^1,\dots,z^m)$ around y with $Z(y)=0$ such that*

1) $\pi^{-1}(y)=\{z^1=\cdots=z^n=0\}$ *and* $U\cap Y=\{z^{n+1}=\cdots=z^m=0\}$
2) $f^u_i=0$ *along* U.
3) $g_{a\bar b}(0)=\delta_{ab}$
4) $f^i_{uv}(0)=g_{u\bar i/v}(0)=f^i_{juv}(0)=g_{i\bar j/uv}(0)=0$
5) $f^u_{iv}(0)=g_{v\bar u/i}(0)=g_{u\bar v/w}(0)=0$

6) $g_{i\bar{j}/\alpha}(0)=0$ *for any* α *with* $|\alpha|\leqq 2n$.

PROOF. At first we define a holomorphic local coordinate system Z_1 by

$$z_1^i=w^i+\frac{1}{2}g^{ik}g_{uk/v}(y)w^u w^v+\frac{1}{2}\left(\frac{\partial}{\partial w^j}\right)(g^{ik}g_{uk/v})(y)w^u w^v w^j-w^i(y)$$

$$z_1^u=w^u.$$

Then with respect to Z_1 we can verify 1), 2) and 4). By a linear coordinate transformation we obtain a local coordinate Z_2 with 1), 2), 3) and 4). Define Z_3 by

$$z_3^i=z_2^i+\frac{1}{2}g_{2\,j\bar{i}/k}(0)z_2^j z_2^k$$

$$z_3^u=z_2^u+\frac{1}{2}g_{2\,v\bar{u}/w}(0)z_2^v z_2^w+g_{2\,v\bar{u}/i}(0)z_2^i z_2^v.$$

Then we obtain the formula 1), 2), 3), 4) and 5). To complete the proof we define local coordinate systems Z_k, 4 k $2n+2$, inductively by

$$z_k^i=z_{k-1}^i+\sum_{|\alpha|=k-1}\frac{g_{k-1}(\alpha,\bar{i})}{\alpha!}(z_{k-1})^\alpha \qquad z_k^u=z_{k-1}^u,$$

where $g(\alpha,\bar{i})$ denotes $g_{j\bar{i}/\beta}$, $(j\beta)=\alpha$. Now the lemma follows by the direct computation. Q.E.D.

Fix a holomorphic local coordinate system as in Lemma 2.2. Let

$$\tilde{\pi}:(z^1,\dots,z^m)\in V\to(z^1,\dots,z^n,0,\dots,0)\in U$$

and define a smooth function $\eta:D_y\to\boldsymbol{R}$ with $\eta(y)=0$ by $dv=\eta dv'$. Here dv (resp. dv') denotes a measure of the fiber D_y induced from the fibration π (resp. $\tilde{\pi}$). Then for any $\eta_0\in C^\infty(V)$, we have

$$\text{(2.1)}\qquad \pi_*[\eta_0\phi(t)](y)=\left[\int_{D_y}\eta_0\frac{\phi(t)}{\mathrm{dvol}_X}dv\right]\mathrm{dvol}_Y$$

$$=\left[\int_{D_y}\tilde{\eta}\frac{\phi(t)}{\mathrm{dvol}_X}dv'\right]\mathrm{dvol}_Y,\ \tilde{\eta}=\eta_0\eta.$$

Construct a parametrix of the heat kernel under the coordinate Z and apply the asymptotic expansion formula in Lemma 1.1, we have

$$(2.1)=\sum_{k=0}^{n}\tilde{P}_k^{mn}(g,f,\tilde{\eta},Z)t^{k-n}\,\mathrm{dvol}_Y+O(t).$$

By Lemma 1.2 $\tilde{P}_k^{mn}$ is a polynomial of variables;

$$g_{ab/\alpha\bar\beta\mu\bar\nu},\ f^a_{b/\mu},\ \overline{f^a_{b/\mu}},\quad J=|\det(I-f^u_v)|^{-1}\quad\text{and}\quad \tilde\eta_{/\mu\bar\nu}.$$

Moreover in consequence of the lemmas in §1 we have the following

LEMMA 2.3. *$\tilde P^{mn}_k$ satisfies the following conditions:*
(invariance) Let W be a holomorphic local coordinate system obtained from Z by a $U(n)\times U(m-n)$ linear coordinate transformation. Then

$$\tilde P^{mn}_k(g,f,\tilde\eta,Z)=\tilde P^{mn}_k(g,f,\tilde\eta,W).$$

(homogenity) Define the order of variables by

$$\operatorname{ord}(g_{ab/\alpha\bar\beta\mu\bar\nu})=|\alpha|+|\beta|+|\mu|+|\nu|,\ \operatorname{ord}(f^a_{b/\mu})=|\mu|\ \textit{and}\ \operatorname{ord}(\tilde\eta_{/\mu\bar\nu})=|\mu|+|\nu|.$$

Then $\tilde P^{mn}_k$ is a homogenious polynomial of order $2k$.
(regularity) Let g_0 denote the flat metric on $\boldsymbol{C}$. Assume g is locally isometric to the product metric $g_1\times g_0$ on $M_1\times\boldsymbol{C}$ and $f=f_1\times\mathrm{id}_{\boldsymbol{C}}:M_1\times\boldsymbol{C}\to M_1\times\boldsymbol{C}$. Then $\tilde P^{mn}_k(g,f,\tilde\eta,Z)=0$.

PROOF. The invariance is obvious by the definition of $\tilde P^{mn}_k$. For the proof of the homogenity we are sufficient to verify $\tilde P^{mn}_k(\lambda^2 g,f,\tilde\eta,\lambda Z)=\lambda^{-2k}P^{mn}_k(g,f,\tilde\eta,Z)$. Because $\square_{\lambda^2 g}=\lambda^{-2}\square_g$, this formula follows from Lemma 1.4. As for the regularity, by Lemma 1.3, we have

$$\phi(g_1\times g_0,f_1\times\mathrm{id}_C)(t)=\phi(g_1,f_1)(t)\wedge\phi(g_0,\mathrm{id}_C)(t).$$

Then it follows the lemma by $\phi(g_0,\mathrm{id}_C)(t)=0$. Q.E.D.

§3 Reduction to the normal bundle

In this section we treat the regular invariant polynomials more abstructly.

Let $\mathcal{G}$ be a set of germs of hermitian metrics g at $0\in\boldsymbol{C}^{n+d}$ and holomorphic mappings $f:(\boldsymbol{C}^{m+d},0)\to(\boldsymbol{C}^{m+d},0)$, which satisfy the following conditions:

i) $d\Omega=\nabla d\Omega=0$ along $\boldsymbol{C}^n=\{(z^1,\dots,z^n,0,\dots,0)\}\subset\boldsymbol{C}^{n+d}$.

ii) $f(z^i,0)=(z^i,0)$ and $f^u_i=0$ along $\boldsymbol{C}^n$.

iii) $g_{i\bar u}=g_{u\bar i}=0$ along $\boldsymbol{C}^n$.

iv) the lower derivatives of the components of g and f at 0 satisfy the equations in Lemma 2.2.

We regard $g_{a\bar b/\alpha\bar\beta\mu\bar\nu}$ and $f^a_{b/\mu}$ as variables which take complex values over $\mathcal{G}$. We say $P:\mathcal{G}\to\boldsymbol{C}$ to be a regular invariant polynomial homogenious of order $2k$ iff P is defined by a polynomial of derivatives of metrics and mappings satisfy the conditions in Lemma 2.3.

PROPOSITION 3.1. *Let P be a regular invariant polynomial homogenious of order $2k \leqq 2n$. Then if $k<n$, $P=0$, and if $k=n$ P is a polynomial of variables;*

$$g_{i\bar{j}/k\bar{l}},\ g_{u\bar{v}/i\bar{j}},\ f^u_v \text{ and } \overline{f^u_v}.$$

To prove this proposition, we prepare some notions of variables. At first we divide variables into three types as follows;
(type 1) the variables which are invariant under the symmetrization of indices $(1, \ldots, n)$;

$g_{i\bar{j}/\alpha\bar{\beta}\mu\bar{\nu}}$ with $|\alpha|=0$ or $|\mu|+|\nu|\leqq 2$
$g_{i\bar{u}/\alpha\bar{\beta}\mu\bar{\nu}}$ with $|\alpha|=0$ or $|\mu|+|\nu|=1$ or both $|\mu|+|\nu|=2$ and $|\mu|\geqq 1$
$g_{u\bar{u}/\alpha\bar{\beta}\mu\bar{\nu}}$, $f^i_{j/\mu}$, $\overline{f^i_{j/\mu}}$, $f^u_{j/\mu}$ and $\overline{f^i_{u/\mu}}$

(type 2) the variables which are not invariant under the symmetrization of indices $(1, \ldots, n)$;

$$g_{i\bar{u}/\alpha\bar{\beta}\mu\bar{\nu}} \text{ which is not of type 1}$$

(type 3) the variables which do not contain indices i, $1 \leqq i \leqq n$.

Let A be any monomial. Define

$\deg_i(A)$ = the number of times an index i appears in A
$\deg_{\bar{i}}(A)$ = the number of times an index $\bar{i}$ appears in A
$i(A)=\sum_{i=1}^{n} \deg_i(A) \qquad \bar{i}(A)=\sum_{i=1}^{n} \deg_{\bar{i}}(A)$
$L_1(A)$ = the number of variables of type 1 contained in A
$L_2(A)$ = the number of variables of type 2 contained in A.

Then we have

LEMMA 3.2. *Let P be a regular invariant polynomial. Then for any monomials A in P, $\deg_i(A) \geqq 1$ for any i, $1 \leqq i \leqq n$.*

PROOF. Divide P into two polynomials, $P=P_1+P_2$, where P_1 consists of monomials A with $\deg_1(A)\geqq 1$ and P_2 consists of monomials A with $\deg_1(A)=0$. Apply a product data $g_0 \times g_1$ and $\mathrm{id}_C \times f_1$ in Lemma 2.3 and choose a local coordinate Z such that z^1 is a canonical coordinate of $\boldsymbol{C}$. Then

$$P(g_0 \times g_1, \mathrm{id}_C \times f_1, Z)=P_2(g_1, f_1, Z').$$

Since P is regular, $P_2=0$. Q.E.D.

The next lemma will be proved in an appendix.

LEMMA 3.3. *Let P be a regular invariant polynomial. Then for any mono-*

mials A in P, we have

$$\deg_i(A)=\deg_{\bar{i}}(A) \quad \textit{for any} \quad i,\ 1\leqq i\leqq n,\ L_1(A)+2L_2(A)\geqq n.$$

Now we return to the proof of Proposition 3.1. Let A be any monomials in P. Put

$$A=\prod X_p \prod Y_q \prod Z_r,$$

where X_p, Y_q and Z_r are variables of type 1, 2 and 3 respectively. Then by Lemma 3.3, we have

$$\begin{aligned} 0&=i(A)-\bar{i}(A)\\ &=\sum(i(X_p)-\bar{i}(X_p))+\sum(i(Y_p)-\bar{i}(Y_p))+\sum(-\bar{i}(Z_r)). \end{aligned}$$

LEMMA 3.4. $i(X_p)-\bar{i}(X_p)\geqq 2-\mathrm{ord}(X_p)$
$i(Y_q)-\bar{i}(Y_q)\geqq 5-\mathrm{ord}(Y_q)$
$-\bar{i}(Z_r)\geqq -\mathrm{ord}(Z_r)$.

PROOF. Let $X=q_{i\bar{j}/\alpha\bar{\beta}\mu\bar{\nu}}$ a variable of type 1. Assume $i(X)-\bar{i}(X)=|\alpha|-|\beta|=2|\alpha|+|\mu|+|\nu|-\mathrm{ord}(X)\leqq 1-\mathrm{ord}(X)$. Then $|\alpha|=0$ and $|\mu|+|\nu|\leqq 1$. Because $g_{i\bar{j}/u\bar{\beta}}=g_{u\bar{j}/i\bar{\beta}}=0$ and $g_{i\bar{j}/\bar{\beta}}=0$, $X=0$. As for other variables of type 1 we obtain the inequallity by the same argument. Let $Y=g_{i\bar{b}/\alpha\bar{\beta}\mu\bar{\nu}}$ and assume $i(Y)-\bar{i}(Y)\leqq 4-\mathrm{ord}(Y)$. Then Y satisfies one of the following;

i) $|\alpha|=2$, $|\mu|=|\nu|=0$ and $b=j$
ii) $|\alpha|=1$, $|\mu|+|\nu|\leqq 2$ and $b=j$
iii) $|\alpha|=1$, $|\mu|+|\nu|\leqq 1$ and $b=u$.

For any cases Y is a variable of type 1. Q.E.D.

Then we have

$$\begin{aligned} 0&\geqq 2L_1(A)+5L_2(A)-\mathrm{ord}(A)\\ &\geqq 2n+L_2(A)-\mathrm{ord}(A)\geqq L_2(A)\geqq 0. \end{aligned}$$

Therefore $L_2(A)=0$, $\mathrm{ord}(A)=2n$, $\mathrm{ord}(X_p)=2$ and $\mathrm{ord}(Z_r)=0$. Thus proof of Proposition 3.1 is completed.

Our polynomial $\tilde{P}_k^{mn}$ include other variables $\tilde{\eta}_{/\mu\bar{\nu}}$ and J. But variables $\tilde{\eta}_{/\mu\bar{\nu}}$ appear in each monomials of $\tilde{P}_k^{mn}$ exactly one time. Hence we can conclude

$$\begin{aligned} &\tilde{P}_k^{mn}(g, f, \tilde{\eta}, Z)=0 \quad \text{for} \quad k<n, \text{ and}\\ &\tilde{P}_n^{mn}(g, f, \tilde{\eta}, Z)\mathrm{dvol}_Y=\eta_0 P^{mn}(g, f). \end{aligned}$$

Here $P^{mn}(g, f)$ is a polynomial of components of curvature tensors with coefficients f_v^u and $\bar{f}_v^u$ which independent on choices of local coordinate on Y and local frames of $N(Y)$. Then we have

$$\pi_*[\eta_0\phi(t)]=\eta_0 P^{mn}(g,f)+O(t)$$

and $L(f)=\int_Y P^{mn}(g,f)$.

§4 Representation by the characteristic forms

Let Y be a closed complex manifold of dimension n and V_i, $1\leqq i\leqq k$, be a family of holomorphic vector bundles of rank d_i over Y. Let P be any map from Kähler metrics on Y hermitian metrics on V_i and bundle endomorphisms of V_i to smooth (n,n)-forms on Y. We say P to be a regular invariant form iff P is defined by a regular invariant polynomial in terms of components of the curvature tensors and bundle endomorphisms. Let $\mathcal{P}_{n;d_1,\dots,d_k}$ denote the set of such regular invariant forms. For any positive integers $(n_0;n_1,\dots,n_k)$ with $\sum_{i=0}^{k} n_i=n$, we define a map $\mathcal{P}_{n_0;n_1,\dots,n_k}\to\mathcal{P}_{n_0}\times\mathcal{Q}_{n_1;d_1}\times\cdots\times\mathcal{Q}_{n_k;d_k}$ as follows. Here $\mathcal{Q}_{n;d}$ denote the set of all invariant forms in terms of components of the curvature tensor for V and bundle maps of V. $\mathcal{P}_n$ denote the set of all regular invariant forms in terms of components of the curvature tensor for TY.

Let (Y_0,g_0) be a Kähler manifold of dimension n_0 and (Y_i,g_i) a flat Kähler manifold of dimension n_i. Let (V_i,h_i,T_i) be a family of a hermitian vector bundle over Y_i and a bundle map of V_i covers the identity mapping of Y_i. For any $P\in\mathcal{P}_{n_0;n_1,\dots,n_k}$ we apply a Kähler metric $g=g_0\times g_1\times\cdots\times g_k$ and $\prod_{i=1}^{k}(h_i,T_i)$. Then by the invariance of P, we have

$$P(g_0\times g_1\times\cdots\times g_k;(h_1,T_1),\dots,(h_k,T_k))=P_0(g_0)\wedge Q_1(h_1,T_1)\wedge\cdots\wedge Q_k(h_k,T_k)$$

where $P_0\in\mathcal{P}_{n_0}$ and $Q_i\in\mathcal{Q}_{n_i;d_i}$. Define

$$\Pi_{n_0;n_1,\dots,n_k}(P)=(P_0;Q_1,\dots,Q_k)\in\mathcal{P}_{n_0}\times\mathcal{Q}_{n_1;d_1}\times\cdots\times\mathcal{Q}_{n_k;d_k}.$$

LEMMA 4.1.
$$\Pi=\bigoplus_{\Sigma n_i=n}\Pi_{n_0;n_1,\dots,n_k}:\mathcal{P}_{n_0;n_1,\dots,n_k}\longrightarrow\bigoplus_{\Sigma n_i=n}\mathcal{P}_{n_0}\times\mathcal{Q}_{n_1;d_1}\times\cdots\times\mathcal{Q}_{n_k;d_k}$$
is an injective endomorphism.

PROOF. Let P be any regular invariant forms. Decompose P

$$P=\sum_{\Sigma n_i=n}P_{n_0;n_1,\dots,n_k}$$

where $P_{n_0;n_1,\dots,n_k}$ consists of monomials A such that the components of the curvature tensor for h_i appears in A exactly n_i times. Obviously $P_{n_0;n_1,\dots,n_k}$ is an invariant form. Assume $P_0=P_{n_0;n_1,\dots,n_k}$ is a non zero form. By the argument of Gilkey in [G2], we can find a monomial A in P_0 such that $A=A_0A_1\cdots A_k$ with $\deg_i(A)=\deg_i(A_j)$ for i, $n_0+\cdots+n_{j-1}+1\leqq i\leqq n_0+\cdots+n_j$. Then A_0 can be regarded as a monomial of $\mathcal{P}_{n_0}$ component of $\Pi(P_0)$ and A_i of $\mathcal{Q}_{n_i;d_i}$ component of $\Pi(P_0)$. Therefore $\Pi(P_0)\neq 0$ and Π is injective. Q.E.D.

Let $Q\in\mathcal{Q}_{n;d}$ and (h, T) be a metric and mapping on V. Then there is a polynomial map $\bar{Q}:M_d\times\cdots\times M_d\to \boldsymbol{C}$,

$$\bar{Q}(F_1, F_2; W_1, \dots, W_n) \text{ for } F_i,\ W_j\in M_d=\{d\times d \text{ matrices}\}\cong \boldsymbol{C}^{d^2},$$

which is symmetric multilinear for the latter n matrixes such that $Q(h, T)=\bar{Q}(T, T^*; \Omega, \dots, \Omega)$. Here T^* denote the adjoint of T, $T^{*u}_{\ v}=h^{ua}\bar{T}^{\ b}_{a}h_{vb}$, and Ω denote the curvature tensor for h. Moreover $\bar{Q}$ is invariant under the adjoint action of $GL(d;\boldsymbol{C})$,

$$\bar{Q}(uF_1u^{-1}, uF_2u^{-1}; uW_1u^{-1}, \dots, uW_nu^{-1})=\bar{Q}(F_1, F_2; W_1, \dots, W_n).$$

LEMMA 4.2. *Let $Q\in\mathcal{Q}_{n;d}$. Assume for any hermitian vector bundles $\pi:(V, h)\to Y$ and any semisimple endomorphism $T:V\to V$ with $\nabla T=0$, $Q(h, T)[Y]=0$. Then $\bar{Q}=0$.*

PROOF. Fix $y_0\in Y$ and choose a local frame such that the connection form ω vanishes at y_0. Then by the formula

$$\nabla T=\omega T+dT-T\omega=0 \quad\text{and}\quad d\omega T-\omega dT-dT\omega-Td\omega=0$$

we have $\Omega(y_0)T(y_0)=T(y_0)\Omega(y_0)$. Hence we can diagonalize both Ω and T simultaneously. Let $\{\lambda_i\}$ and $\{\mu_j\}$ be the eigenvalues of Ω and T respectively. Then

$$Q(h, T)=\sum_j P_j(\mu_1, \lambda', \mu')\lambda_1^j \quad\text{with}\quad \lambda'=(\lambda_2, \dots, \lambda_d) \quad \mu'=(\mu_2, \dots, \mu_d).$$

Assume $Q\neq 0$. Then for some j, $P_j(\mu_1, \lambda', \mu')\neq 0$. By induction, we choose (Y_0, V_0, h_0, T_0) such that $P_j(\mu_1)(h_0, T_0)[Y_0]\neq 0$. Let $Y_1=\boldsymbol{C}P_j$ be the complex projective space and V_1 the hyperplane bundle over Y_1. Set $T_1:V_1\to V_1$ a scalar multiplication by μ_1. Therefore

$$Q(h_0\oplus h_1, T_0\oplus T_1)[Y_0\times Y_1]=P_j(\mu_1)(h_0, T_0)[Y_0](C_1(V_1))^j[Y_1]\neq 0.$$

Thus Q vanishes for any semisimple endomorphisms T. Because the set of all semisimple matrixes is dence in M_d, there are no algebraic relation between the components of T. Hence $\bar{Q}=0$ as a polynomial. Q.E.D.

Now we return to the problem to determine the formula P^{mn}. By the definition of $\Pi_{n_0;n_1,\dots,n_k}(P^{mn})=P_{n_0}Q_{n_1;d_1}\cdots Q_{n_k;d_k}$, we have

$$P_n(g)=P^{nn}(g)$$
$$Q_{n;d}(h,T)=P^{n+d,n}(\text{flat}\oplus h,T),$$

where flat$\oplus h$ denote a hermitian metric on the total space of V over the flat Kähler manifold. It should be noted that for any Kähler metric g we can find the invariant form $Q_{n;d}(h,T)$ in $P^{n+d,n}(g\oplus h,T)$.

Assume $T:V\to V$ is any semisimple holomorphic endomorphism of the holomorphic vector bundle over Y. We set

$$T=\bigoplus_i \lambda_i I: V_i\to V_i \text{ and } h_i \text{ a hermitian metric on } V_i.$$

Then $Q_{n;d}(\oplus h_i,T)$ is an invariant polynomial in terms of the components of curvature tensors for h_i with constant coefficients. By the Gilkey's theorem in [G2], $Q_{n;d}(\oplus h_i,T)$ is a linear combination of the Chern forms for V_i. Because the local index, $\int_Y P^{mn}(g,f)$, does not depend on the choice of metrics, $Q_{n;d}(h,T)[Y]$ is independent on h with $\nabla T\equiv 0$. Therefore by Lemma 4.2, $Q_{n;d}(h,T)$ is a linear combination of the Chern forms for h with coefficients of the eigenvalues of T. On the other hand P_n is the Todd form for the Kähler metric on TY, [P1][G2]. Hence we can regard $P_{n_0}Q_{n_1;d_1}\cdots Q_{n_k;d_k}$ as an element of $\mathcal{P}_{n;d_1,\dots,d_k}$. By the injectivity of Π, Lemma 4.1, we obtain

$$P^{nm}(g,f)=\sum_{\Sigma n_i=n} P_{n_0}(g_0)\wedge Q_{n_1;d_1}(h_1,df_1)\wedge\cdots\wedge Q_{n_k;d_k}(h_k,df_k).$$

We set $\sum_{n\geq 0}P_n=\text{Todd}$ and $\sum_{n\geq 0}Q_{n;d}=Q_d$. Then we have

$$P^{nm}=[\text{Todd}\,Q_{d_1}\cdots Q_{d_k}]_{(n,n)}$$

and the proof of Theorem 1 is completed.

§5 Proof of Theorem 2

In this section we use the convention on indices in §2.

Let Y be a nondegenerate component of $f:X\to X$. We say a holomorphic local coordinate system $Z=(z^1,\dots,z^m)$ defined on V around $y\in Y$ to be admissible iff

$$U=V\cap Y=\{z^{n+1}=\cdots=z^m=0\}$$
$$f^u_i=0 \quad \text{along} \quad U.$$

Let g_0 be a Kähler metric on Y and h_i a hermitian metric of $N_i(Y)$ with $df_i=0$. Let ∇ be a torsion free connection such that $df(N(Y)\otimes N(Y))\subset N(Y)$. Fix any admissible coordinate $Z=\{z^a\}$, then we have

$$\nabla df\left(\frac{\partial}{\partial z^u}\,\frac{\partial}{\partial z^v}\right)=\sum\,(f^s_u f^t_v\omega^i_{st}+\omega^i_{uv}-f^i_{uv})\frac{\partial}{\partial z^i}$$
$$+\left(\text{terms of } \frac{\partial}{\partial \bar{z}^i},\ \frac{\partial}{\partial z^w} \text{ and } \frac{\partial}{\partial \bar{z}^w}\right),$$

where ω denotes the components of the connection ∇. Therefore we obtain

$$f^s_u f^t_v\omega^i_{st}+\omega^i_{uv}-f^i_{uv}=0. \tag{5.1}$$

Now we will construct a hermitian metric on the coordinate neighbourhood of Z, V, which satisfies the conditions (P-1), (P-2) and (P-3) in Theorem 1.

Step 1. We define $g_{a\bar{b}}$ on $U=V\cap Y$ by

i) $g_{i\bar{j}}(z^i, 0)=g_{0i\bar{j}}(z^i)$

ii) $g_{i\bar{u}}(z^i, 0)=0$

iii) $g_{u\bar{v}}(z^i, 0)=(\bigoplus h_i)_{u\bar{v}}(z^i)$.

Then g satisfies (P-1) in Theorem 1.

Step 2. Set

(*) $g_{b\bar{a}/c}=g_{c\bar{a}/b}$

(**) $f^a_{bc}+f^d_b f^e_c\Gamma^a_{de}-f^a_d\Gamma^d_{bc}=0$ with $\Gamma^a_{bc}=g^{a\bar{d}}g_{b\bar{d}/c}$.

We consider these two equations dividing into some cases according to the indices a, b and c are whether tangential or vertical.

1) The case $(a, b, c)=(i, j, k)$:

(*) $g_{j\bar{i}/k}=g_{k\bar{i}/j}$

(**) $\Gamma^i_{jk}-\Gamma^i_{jk}=0$.

(*) follows from the assumption that g_0 is a Kähler metric.

2) The case $(a, b, c)=(u, i, j)$:

(*) $g_{j\bar{u}/k}=g_{k\bar{u}/j}$

(**) $f^u_{jk}+\Gamma^u_{jk}-f^u_v\Gamma^v_{jk}=0$.

Because $g_{i\bar{u}}=0$ along U, $g_{i\bar{u}/k}=\Gamma^u_{ik}=0$. Furthermore since Z is admissible $f^u_{jk}=0$. Thus we obtain both equations.

3) The case $(a, b, c)=(i, j, u)$:

(∗) $g_{j\bar{i}/u}=g_{u\bar{i}/j}$

(∗∗) $f_u^v\Gamma_{jv}^i-\Gamma_{ju}^i=0.$

Since $g_{u\bar{i}/j}=0$ on U, we need to define

$$g_{i\bar{j}/u}=g_{i\bar{j}/\bar{u}}=0. \tag{5.2}$$

Then we obtain (∗∗) because $\Gamma_{ju}^i=g^{i\bar{k}}g_{j\bar{k}/u}=0.$

4) The case $(a,b,c)=(i,u,v)$:

(∗) $g_{u\bar{i}/v}=g_{v\bar{i}/u}$

(∗∗) $f_{uv}^i+f_u^sf_v^t\Gamma_{st}^i-\Gamma_{uv}^i=0.$

By (5.1), we are sufficient to define

$$g_{u\bar{i}/v}=g_{k\bar{i}}\omega_{uv}^k \quad \text{and} \quad g_{i\bar{u}/\bar{v}}=g_{i\bar{k}}\overline{\omega_{uv}^k}. \tag{5.3}$$

Since ∇ is torsionfree, we obtain both equations.

5) The case $(a,b,c)=(u,v,i)$:

(∗) $g_{v\bar{u}/i}=g_{i\bar{u}/v}$

(∗∗) $f_{vi}^u+f_v^s\Gamma_{is}^u-f_t^u\Gamma_{is}^t=0.$

(∗∗) follows from the assumption $\nabla df_i=0$. For (∗), we define

$$g_{i\bar{u}/v}=g_{v\bar{u}/i} \quad \text{and} \quad g_{u\bar{i}/\bar{v}}=g_{u\bar{v}/\bar{i}}. \tag{5.4}$$

For the case $(a,b,c)=(u,v,w)$, we construct a hermitian metric $\tilde{g}$ on the neighbourhood of Y such that with respect to any admissible coordinates $\tilde{g}_{u\bar{v}/w}=\tilde{g}_{w\bar{v}/u}$ along Y. Set

$$\tilde{g}_{u\bar{v}/w}|_Y=D_{u\bar{v}w} \quad \text{and} \quad \tilde{g}_{u\bar{v}/\bar{w}}|_Y=D_{u\bar{v}\bar{w}}.$$

Define $g_{a\bar{b}}$ by

$$\begin{aligned}
g_{i\bar{j}}(z^i,z^u)&=g_{0i\bar{j}}(z^i,0)+O(|z^u|^2)\\
g_{i\bar{u}}(z^i,z^u)&=g_{v\bar{u}/i}(z^i,0)z^v+g_{i\bar{k}}\overline{\omega_{uv}^k}(z^i)\bar{z}^v+O(|z^u|^2)\\
g_{u\bar{v}}(z^i,z^u)&=g_{u\bar{v}}(z^i,0)+D_{u\bar{v}w}(z^i)z^w+D_{u\bar{v}\bar{w}}(z^i)\bar{z}^w+O(|z^u|^2).
\end{aligned}$$

Then g and f satisfy the conditions (P-1), (P-3) and $d\Omega=0$.

Step 3. Because $d\Omega=0$, the equation $\nabla d\Omega=0$ is equivalent to

(∗∗∗) $g_{a\bar{b}/c\bar{d}}=g_{c\bar{b}/a\bar{d}}$ and $g_{a\bar{b}/\bar{c}\bar{d}}=g_{a\bar{c}/\bar{b}\bar{d}}$.

Let $C_{iuv}=g_{j\bar{i}}\omega_{uv}^j|_Y$ and $h=\oplus h_i$. Define

$$\begin{aligned}
g_{i\bar{j}}&=g_{0i\bar{j}}+h_{u\bar{v}/i\bar{j}}z^u\bar{z}^v+\frac{1}{2}C_{\bar{j}uv/i}z^uz^v+\frac{1}{2}\overline{C_{\bar{i}uv/j}}\bar{z}^u\bar{z}^v\\
g_{i\bar{v}}&=h_{u\bar{v}/i}z^u+\overline{C_{\bar{i}uv}}\bar{z}^u+\frac{1}{2}D_{u\bar{v}w/i}z^uz^w+D_{u\bar{v}\bar{w}/i}z^u\bar{z}^w\\
g_{u\bar{v}}&=h_{u\bar{v}}+D_{u\bar{v}w}z^w+D_{u\bar{v}\bar{w}}\bar{z}^w.
\end{aligned}$$

Then the equation $\nabla d\Omega=0$ follows by direct computation.

Let W be an another admissible coordinate and g' a hermitian metric constructed for W. Since the second order derivatives of the metric determined by the globally defined data g_0, h, and $\tilde{g}$, g' coincide with g up to third order terms in z^u. Therefore using a partition of unity, we obtain a hermitian metric around Y which satisfies each conditions in Theorem 1. Thus the proof of Theorem 2 is completed.

§6 Proof of Theorem 3

Assume $N(Y)=\oplus N_i(Y)$ and $df|_{N_i(Y)}=\lambda_i Id$. Let h_i be any hermitian metric on $N_i(Y)$ and $h=\oplus h_i$. Then $\nabla df=0$ and we obtain the first assertion of Theorem 3.

To prove the second assertion, we fix a holomorphic local coordinate system Z as in §5. If $I-df\otimes df$ is nonsingular we can find a family of smooth functions $\{C^i_{uv}\}$ such that

$$C^i_{uv}-f^s_u f^t_v C^i_{st}=f^i_{uv}.$$

For another coordinate W, we also obtain a family $\{D^i_{uv}\}$. Then these two families satisfy the following equation;

$$C^i_{uv}=\frac{\partial z^i}{\partial w^j}\frac{\partial^2 w^j}{\partial z^u \partial z^v}+\frac{\partial z^i}{\partial w^j}\frac{\partial w^s}{\partial z^u}\frac{\partial w^t}{\partial z^v}D^i_{st}.$$

Hence we can construct a connection such that its components Γ^i_{uv} is equal to C^i_{uv}. By definition of C^i_{uv} we have $\nabla df(V\otimes V)\subset V$ and the proof of Theorem 3 is completed.

§7 Some remarks

1. This results can be extended to the following situations. Let X and f be as in Theorem 1. Assume we are given ξ, a holomorphic vector bundle over X, and $\varphi: f^*\xi\to\xi$, a holomorphic bundle endomorphism on ξ. For the Dolbeault complex over X with coefficient in ξ,

$$\longrightarrow\Gamma(\wedge^{0,p}X\otimes\xi)\xrightarrow{\bar\partial}\Gamma(\wedge^{0,p+1}X\otimes\xi)\longrightarrow,$$

we can define the Lefschetz number $L(f,\varphi)$. Then we have the same results for $L(f,\varphi)$ as in Theorem 1. There are no difficulty in our proof for this generalization.

2. In [P2], Patodi announced the same results about the holomorphic Lefschetz formula. Instead of our condition (C-3), he assumed that $N_i(Y)$ is decomposed to holomorphic subbundles,

$$N_i(Y) = \sum E_{ij}$$

such that $df_i - \lambda_i I$ maps E_{ij} into E_{ij+1}, I can not understand the meaning of this assumption and of course I do not have a chance to know his results in detail. But under such assumption we can construct a hermitian metric of $N(Y)$ such that $\nabla df = 0$.

Appendix

Let V be a hermitian vector space of dimension n. Let $\mathcal{B}V$ denote the set of all unitary basis of V and $\mathcal{T}V$ the set of covariant tensors

$$\mathcal{T}V = \bigoplus_{p,q} (\overset{p}{\otimes} V^*) \otimes (\overset{q}{\otimes} \bar{V}^*).$$

We say a map $P: \mathcal{B}V \times \mathcal{T}V \to \boldsymbol{C}$ to be a polynomial map iff P is defined by an element of a polynomial algebra

$$\tilde{\mathcal{P}} = \boldsymbol{C}\left[\begin{matrix} X(\alpha, \bar{\beta}), & Y(i\alpha, \bar{j}\bar{\beta}); & \alpha \text{ and } \beta \text{ denote multiindices from} \\ & & 1 \text{ to } n \text{ and } 1 \leqq i, j \leqq n. \end{matrix}\right]$$

where $X(\alpha, \bar{\beta})$ and $Y(i\alpha, \bar{j}\bar{\beta})$ are regarded as variables which take complex values for $(\boldsymbol{e}, T) \in \mathcal{B}V \times \mathcal{T}V$ by

$$X(\alpha, \bar{\beta})(\boldsymbol{e}, T) = T(e^\alpha \otimes \bar{e}^\beta)$$
$$Y(i\alpha, \bar{j}\bar{\beta})(\boldsymbol{e}, T) = T(e_i \otimes e^\alpha \otimes \bar{e}_j \otimes \bar{e}^\beta).$$

Let $u \in U(V) \cong U(n)$. We define $uP \in \tilde{\mathcal{P}}$ for any $P \in \tilde{\mathcal{P}}$ by

$$uP(\boldsymbol{e}, T) = P(u\boldsymbol{e}, T).$$

If $uP = P$ for any $u \in U(n)$, P is said to be an invariant polynomial. Let A be a monomial in $\tilde{\mathcal{P}}$,

$$A = X(\alpha_1, \bar{\alpha}_1') \cdots X(\alpha_s, \bar{\alpha}_s') Y(i_1\beta_1, \bar{j}_1\bar{\beta}_1') \cdots Y(i_t\beta_t, \bar{j}_t\bar{\beta}_t').$$

Define

$L_1(A) = s =$ the number of variables $X(\alpha, \bar{\beta})$ contained in A

$L_2(A) = t =$ the number of variables $Y(i\alpha, \bar{j}\bar{\beta})$ contained in A

$$\deg_i(A) = \sum_{h=1}^{s} \alpha_h(i) + \sum_{k=1}^{t} (\beta_k(i) + \delta_{i, i_k})$$

$$\deg_{\bar{i}}(A)=\sum_{h=1}^{s}\alpha'_h(i)+\sum_{k=1}^{t}(\beta'_k(i)+\delta_{i,j_k}).$$

LEMMA A.1. *Let P be an invariant polynomial and A a monomial in P. Then for any index i, $1\leqq i\leqq n$,*

$$\deg_i(A)=\deg_{\bar{i}}(A).$$

PROOF. Let $u=\begin{pmatrix} e^{\sqrt{-1}\theta} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}\in U(n)$.

Then $uA=e^{\sqrt{-1}\theta(\deg_1(A)-\deg_{\bar{1}}(A))}A$. Since θ is arbitrary and P is an invariant polynomial, $\deg_1(A)-\deg_{\bar{1}}(A)=0$. Q.E.D.

LEMMA A.2. *Let P be an invariant polynomial. Assume for each monomials A in P and i, $1\leqq i\leqq n$, we have* $\deg_i(A)>0$. *Then P consists of monomials A with $L_1(A)+2L_2(A)\geqq n$.*

To prove this lemma we prepair the following

SUBLEMMA. *Let A be a monomial in P and A_1 a monomial which is obtained from A by changing a single index i to j. Then there is a monomial $B\neq A$ in P such that B is obtained from A_1 by changing a single index j to i or $\bar{i}$ to $\bar{j}$.*

PROOF. Set

$$u=\begin{pmatrix} s & -\bar{t} & & & 0 \\ t & \bar{s} & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \text{ with } s\bar{s}+t\bar{t}=1.$$

Then uA is a linear combination of some monomials with coefficient in polynomials of s, $\bar{s}$, t and $\bar{t}$. Exchange $s\bar{s}$ by $1-t\bar{t}$. Then

$$uA=A+t\bar{s}\sum c_jA_j+(\text{higher order terms in } t,\ \bar{t})$$

Here A_j is a monomial obtained from A by changing a single index 1 to 2 or $\bar{2}$ to $\bar{1}$. By Lemma A.1 A_j is not a monomial in P. Because $uP=P$, it proves the sublemma.

PROOF OF LEMMA A.2. Assume

$$A=X(1,\dots,1,*)\cdots X(k,\dots,k,*)X(k+1,\dots,k+1,j,\dots,*)A'$$

is a monomial in P. By the sublemma there exist a monomial B in P such that

$$B=X(1,\dots,1,\bar{*})\cdots X(k,\dots,k,\bar{*})X(k+1,\dots,k+1,k+1,\dots,\bar{*})B'.$$

We use this argument successively to obtain a monomial A_0 in P,

$$A_0=X(1,\dots,1,\bar{*})\cdots X(s,\dots,s,\bar{*})Y(i_1j_1,\dots,j_1,\bar{*})\cdots Y(i_tj_t,\dots,j_t,\bar{*})A_0',$$

where A_0' is a monomial such that each indices i which appears in A_0' is contained in $(1,\dots,s,i_1,\dots,i_t,j_1,\dots,j_t)$. By the assumption $\deg_i(A_0)>0$ we have $L_1(A_0)+2L_2(A_0)\geqq n$. Since L_1 and L_2 are invariant under the action of $U(n)$, $L_1(A)+2L_2(A)\geqq n$ for every A in P. Thus the proof is completed. Q.E.D.

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