On the propagation of micro-analyticity along the boundary

By Akira KANEKO*)

In this paper we give results on propagation of micro-analyticity of the solution of a class of linear partial differential equations with constant coefficients $p(D)u=0$ up to its boundary values. Let us shortly recall the former results. Assume that p is of order m and with respect to which $x_1=0$ is non-characteristic. Let p_m be the principal part of p, put $\nu=(1, 0, \cdots, 0)$ and let $V^+_{\nu,A}(p)\subset S^{n-2}$ be the closure of the set of points $\xi'\in S^{n-2}$ for which a root of the equation $p_m(\zeta_1, \xi')=0$ for ζ_1 has positive imaginary part. This is called the set of boundary characteristic directions from the positive side. The set of points added by the closure operation is the so called glancing region. Then the result in the most generic case says that the singular spectrum (S. S. for short) of the boundary values of real analytic solutions of $p(D)u=0$ on $x_1>0$ is contained in $V^+_{\nu,A}(p)$ (see Kaneko [6]). There are also results corresponding to the equations with real analytic coefficients where $V^+_{\nu,A}(p)$ should be replaced by the set $V^+_{S,A}$ with $S=\{x_1=0\}$ defined in a more complicated way. (See Kaneko [7]. See also Schapira [17], Kataoka [11].)

It is known that some points in the glancing region are superfluous in the above estimate of S. S. of boundary values. For such delicate results see Schapira [18], Kaneko [8], Kataoka [11].

Here we give a new phenomenon which takes place fairly generally in the glancing region. Let us explain it in a fixed system of coordinates: Assume that the frozen operator $p(D_1, 0, D_n)$ is semihyperbolic to $x_1<0$, that is, assume that all the roots of $p_m(\zeta_1, 0, 1)=0$ have non-positive imaginary part. Let u be a local real analytic solution of $p(D)u=0$ on $x_1>0$, and let $u_j(x')$, $j=0, \cdots, m-1$ be its boundary values. Then

$$\bigcup_{j=0}^{m-1} \text{S. S. } u_j(x')\cap(\{x_n=\text{const.}\}\times\{\sqrt{-1}\,dx_n\infty\}) \tag{0.1}$$

cannot be compact. That is, the micro-analyticity propagates along the level surface $x_n=\text{const.}$ in the boundary.

---

*) Partially supported by the Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture of Japan.

This assertion is best possible in the sense that if there exists a root of $p_m(\zeta_1, 0, 1)=0$ with positive imaginary part, then we can construct a real analytic solution on $x_1>0$ for which (0.1) is compact. We can micro-localize this assertion to the obvious form (Theorem 2.5), which improves definitively our former work Theorem 4.1 of [8]. Also we can somewhat specify the way of propagation under a stronger assumption on the operator (Theorems 2.6–2.7). For the wave equation, however, Theorem 4.2 in [8] is naturally ultimate, because it asserts the propagation in the bicharacteristic level. (See also note 2 of [8] added in proof.)

There are vast studies on the propagation of micro-differentiability up to the boundary which have its origin in the work of Lax-Nirenberg [15]. There are detailed studies on the glancing region mainly for operators of the second order. See e.g. Ivriĭ [1], Melrose-Sjöstrand [14], Wakabayashi [20] etc. Recently these studies are extended to the case of micro-analyticity. See e.g. Sjöstrand [24], [25]. There are also studies concerning the propagation of micro-differentiability along the boundary. See Andersson-Melrose [21], Eskin [22] where the infinitely flat case is not contained. Sjöstrand [24] treats the micro-analytic version seemingly in more general situation. It should be noted that the characteristic feature of our study is that a priori we do not pose any boundary conditions on the solution throughout. This is because the starting point of our study has been the problem of continuation of regular solutions, and not the boundary value problem itself.

The method of this paper is based on the Fourier analysis which extends those developed in [4] or [6]. Especially [6] will be considered as the direct predecessor of the present article, although §4 of [8] is the one as for the treated problem.

The deepness of our present result can be known by its application to the continuation of real analytic solutions given in §3. That is, all the results hitherto obtained by the method of Fourier analysis can be explained by our result from the standpoint of local theory, but except for the only one process: the irreducible decomposition of the operator. A survey lecture about this subject will be found in [23].

§1. Review of representation of boundary values via Fundamental Principle.

First we rapidly recall a part of our preceding paper stating the relation between the hyperfunction boundary value theory and the Fundamental Principle of Ehrenpreis-Palamodov. See §1, §2 in [6]. From now on, however, we abandon the notation of Palamodov which we have employed in our previous papers.

Consequently there will be some changes of signs in the formulas related with the Fourier transformation.

Let $p(D)$ be a linear partial differential operator of order m with constant coefficients, where $D=(D_1, \cdots, D_n)$ with $D_i=-\sqrt{-1}\,\partial/\partial x_i$. Assume that $x_1=0$ is non-characteristic with respect to $p(D)$. Let U be a relatively compact, convex open neighborhood of the origin of $\boldsymbol{R}^n$. Put

$$U^{\pm}=U\cap\{\pm x_1>0\}, \qquad K=U\cap\{x_1=0\}, \qquad L=\bar{K}.$$

We denote by U' the set K considered as an $(n-1)$-dimensional open set in $\boldsymbol{R}^{n-1}$. Now let u be a hyperfunction solution of $p(D)u=0$ on U^+. Then there exist a unique extension $[u]\in\mathcal{B}(U)$, called the canonical extension of u, and unique coefficients $u_j(x')\in{}'\mathcal{B}(U')$ satisfying

$$\begin{aligned}&\operatorname{supp}[u]\subset\{x_1\geqslant 0\},\\ &p(D)[u]=\sum_{j=0}^{m-1}u_j(x')D_1^{m-1-j}\delta(x_1).\end{aligned} \tag{1.1}$$

The coefficients $u_j(x')$ are called the boundary values of u (with respect to a certain boundary system which we will not specify). Here as usual we have distinguished the hyperfunction of $n-1$ variables $x'=(x_2, \cdots, x_n)$ by the symbol ${}'\mathcal{B}$.

Let now $[u]\in\mathcal{B}(U)$ be an arbitrary extension of u satisfying $\operatorname{supp}[u]\subset\{x_1\geqslant 0\}$. Then we have $\operatorname{supp}p(D)[u]\subset K$. Let $v=[[p(D)[u]]]\in\mathcal{B}(\boldsymbol{R}^n)$ denote one of its extensions which is in the space $\mathcal{B}[L]$ of the hyperfunctions with support in L. This gives rise to the following well defined mapping

$$\begin{aligned}\mathcal{B}_p(U^+)\ni u\longmapsto\ &[[p(D)[u]]]\bmod p(D)\,\mathcal{B}[L]+\mathcal{B}[L\setminus K]\\ &\in\mathcal{B}[L]/(p(D)\,\mathcal{B}[L]+\mathcal{B}[L\setminus K]).\end{aligned}$$

This is "locally" injective in the sense that, if the image of u is equal to zero in the quotient space, then $u\equiv 0$ on a neighborhood of K by virtue of Holmgren's uniqueness theorem. (And, what we need in fact is the information of u in the very neighborhood of K.) The Fourier transform and the restriction to the variety:

$$N(p)=\{\zeta\in\boldsymbol{C}^n;\ p(\zeta)=0\},$$

give another description of this mapping. Namely, we obtain an entire function $F(\zeta)=\hat{v}(\zeta)$ and then a holomorphic function $F(\zeta)|_{N(p)}$ on the variety $N(p)$, satisfying the following inequality: Given $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$(1.2) \qquad |F(\zeta)| \leqslant C_\varepsilon \exp(\varepsilon|\zeta| + H_L(\operatorname{Im}\zeta)),$$

where

$$H_L(\operatorname{Im}\zeta) = \sup_{x \in L}(-\operatorname{Re}\sqrt{-1}\,x\cdot\zeta) = \sup_{x \in L} x \cdot \operatorname{Im}\zeta.$$

(Note that we are now employing the standard definition for the Fourier transform:

$$F(\zeta) = \hat{v}(\zeta) = \int_{R^n} e^{-\sqrt{-1}x\cdot\zeta} v(x)dx.)$$

The function $F(\zeta)|_{N(p)}$ is determined from u modulo the image of $\mathcal{B}[L \setminus K]$ as the ambiguity which, however, does not allow the characterization by the growth order. To each representative $F(\zeta)|_{N(p)}$ corresponds a unique entire function of the form

$$(1.3) \qquad \hat{f}(\zeta) = \zeta_1^{m-1}\hat{f}_0(\zeta') + \zeta_1^{m-2}\hat{f}_1(\zeta') + \cdots + \hat{f}_{m-1}(\zeta')$$

satisfying $\hat{f}(\zeta)|_{N(p)} = F(\zeta)|_{N(p)}$ and still the inequality (1.2). This is a very special case of the Fundamental Principle. In fact, in this case the coefficients $\hat{f}_j(\zeta')$ are determined from $F(\zeta)|_{N(p)}$ via the roots $\zeta_1 = \tau_j(\zeta')$, $j=1, \cdots, m$ of the equation $p(\zeta_1, \zeta')=0$ by means of the usual interpolation formula. (For the sake of simplicity we are describing as if $p(D)$ were irreducible. For the modification to the general case, see Remark 1.4 in [6].) Hence the coefficients $\hat{f}_j(\zeta')$ satisfy the same inequality (1.2). Note that $H_L(\operatorname{Im}\zeta)$ is in fact a function of $\operatorname{Im}\zeta'$, because L is contained in $x_1=0$. Lemma 1.1 in [6] asserts that $\hat{f}_j(\zeta')$ agrees with the Fourier transform of an extension of the boundary value $u_j(x')$ to $'\mathcal{B}[\overline{U'}]$.

The above description of the boundary values permits us to obtain $\hat{f}(\zeta)$ by various ways. In fact even the ambiguity of the form $p(D)\mathcal{B}_*(\boldsymbol{R}^n)$ vanishes when restricted to $N(p)$ after the Fourier transform, where $\mathcal{B}_*(\boldsymbol{R}^n)$ denotes the space of all the hyperfunctions with compact support. Thus let $[u]$ be any modification of u on a neighborhood of K; more precisely, let $[u] \in \mathcal{B}(U)$ be such that

$$\operatorname{supp}[u] \subset \{x_1 \geqslant 0\},$$
$$[u] = u \text{ on a neighborhood of } \partial U^+ \setminus K.$$

Then we obtain as yet an extension $v = [[p(D)[u]]]$ with compact support, whose ambiguity is only in $p(D)\mathcal{B}_*(\boldsymbol{R}^n) + \mathcal{B}[L \setminus K]$. Therefore $F(\zeta)|_{N(p)} = \hat{v}(\zeta)|_{N(p)}$ agrees with $\hat{f}(\zeta)|_{N(p)}$ in (1.3) modulo $\widehat{\mathcal{B}[L \setminus K]}|_{N(p)}$. Choosing $[u]$ in a suitable way we can thus reflect the information of the regularity of the solution u on $x_1 > 0$ upon $\hat{f}(\zeta)$, that is, upon the boundary values. In the preceding paper [6], for a real analytic solution u, we have employed the cutting $(1-\chi)u$ by a Gevrey class function χ as an example of such an extension.

In this paper, we introduce a more delicate construction for such an extension. We essentially consider two typical situations. The first is the following:

$$U' = U'' \times \{-r < x_n < r\}, \tag{1.4}$$

where U'' is a relatively compact convex open neighborhood of $0 \in \mathbf{R}^{n-2}_{x''}$, $x'' = (x_2, \cdots, x_{n-1})$, while the real analytic solution $u \in \mathcal{A}_p(U^+)$ can be continued real analytically on a neighborhood of the part of the boundary:

$$\{0\} \times \partial U'' \times \{-r \leqslant x_n \leqslant r\}. \tag{1.5}$$

In this case, to utilize this regularity assumption we specify the way of defining $F(\zeta)$ as follows: On a neighborhood of (1.5), the canonical extension $[u]$ is of the form $Y(x_1)u$, where we are writing u also for the extended real analytic function. Hence, for the characteristic function $\chi_{U''}(x'')$ of U'', the product $\chi_{U''}(x'')p(D)[u]$ is meaningful there. Let $[[\chi_{U''}(x'')p(D)[u]]] \in \mathcal{B}[L]$ be an extension with minimal support. We then obtain

$$F(\zeta) = \widehat{[[\chi_{U''}(x'')p(D)[u]]]}, \tag{1.6}$$

as a special representative. From the definition formula (1.1) we see that

$$\chi_{U''}(x'')p(D)[u] = \sum_{j=0}^{m-1} \chi_{U''}(x'')u_j(x')D_1^{m-1-j}\delta(x_1),$$

hence that

$$F(\zeta)\,|_{N(p)} = \hat{f}(\zeta)\,|_{N(p)},$$

with $\hat{f}(\zeta)$ of the form (1.3), where the coefficient $\hat{f}_j(\zeta')$ is the Fourier image of some extension $f_j(x') \in {}'\mathcal{B}[\overline{U'}]$ of $\chi_{U''}(x'')u_j(x')$.

Now let $\psi_1(x_1, x_n)$ be a Gevrey class function defined in $\{-r < x_n < r\}$ such that $\operatorname{supp}\psi_1$ is contained in a neighborhood of $\{0\} \times \{-r < x_n < r\}$ and that $\psi_1 \equiv 1$ on a smaller neighborhood of this set. Choose also a Gevrey class function $\psi_2(x'')$ on $\mathbf{R}^{n-2}_{x''}$ such that

$$\operatorname{supp}\psi_2(x'') \subset \overline{U''},$$
$$\psi_2 \equiv 1 \text{ in } U''_\delta = \{x'' \in U'';\ \operatorname{dis}(x'', \partial U'') > \delta\},$$

where δ is chosen in such a way that u admits the analytic continuation up to the part of the boundary $\{0\} \times (\overline{U''} \setminus U''_\delta) \times \{-r \leqslant x_n \leqslant r\}$. Clearly we can choose them in such a way that, setting $\chi=\phi_1\phi_2$,

$$\overline{\operatorname{supp} p(D)((1-\chi)u)}\cap\partial U^+\subset L\setminus K_\delta,$$

where $K_\delta=\{0\}\times U''_\delta\times\{-r<x_n<r\}$. Choose an extension $[[p(D)((1-\chi)u)]]$ of $p(D)\times((1-\chi)u)\in\mathcal{B}(U^+)$ with minimal support such that

$$\begin{aligned}&\operatorname{supp}[[p(D)((1-\chi)u)]]\subset\{x_1\geqslant 0\},\\&[[p(D)((1-\chi)u)]]=Y(x_1)p(D)((1-\chi)u)\\&\qquad\text{on a neighborhood of } \{0\}\times(\overline{U''}\setminus U''_\delta)\times\{-r<x_n<r\}.\end{aligned}$$

Then the argument in §2 of [6] shows that

$$\hat{f}(\zeta)=F(\zeta)\equiv\widehat{[[p(D)((1-\chi)u)]]}\quad\operatorname{mod}\ \widehat{\mathcal{B}[L\setminus K_\delta]}$$

on $N(p)$. The reason why the ambiguity is not in $\widehat{\mathcal{B}[L\setminus K]}$ as in §2 of [6] is that our function χ is not equal to 1 on a whole neighborhood of K. We will show below that in view of the present construction related with the mentioned regularity of u, we can assert a little more concerning this ambiguity. With a slight abuse of notation, introduce the following sets for later use:

$$\begin{aligned}&\partial_\pm K=\{0\}\times\overline{U''}\times\{\pm r\},\\&\partial_l K=\{0\}\times\partial U''\times\{-r\leqslant x\leqslant r\}.\end{aligned}$$

Then the original difference

$$[[\chi_{U''}(x'')p(D)[u]]]-[[p(D)((1-\chi)u)]]$$

is a hyperfunction in $p(D)\mathcal{B}_*(\boldsymbol{R}^n)+\mathcal{B}[L\setminus K_\delta]$ such that along $(L\setminus K_\delta)\cap\{-r<x_n<r\}$ the direction of the singular spectrum of the second component is contained in $\xi_n=0$, where ξ denotes the fibre coordinates of $\sqrt{-1}\,S^*_\infty\boldsymbol{R}^n$. We will show this in a little more general situation.

Let therefore $J(D')$ be a local operator with constant coefficients in $D'=(D_2,\cdots,D_n)$. (For a short account of this notion see §1 of [2].) Then $J(D')u$ is again an element of $\mathcal{A}_p(U)$. Hence we can apply the above construction to this solution.

LEMMA 1.1. *We have*

$$J(\zeta')F(\zeta)=\widehat{[[p(D)((1-\chi)J(D')u)]]}+\hat{w}\tag{1.7}$$

on $N(p)$, *where* w *is a hyperfunction such that* $\operatorname{supp} w\subset L\setminus K_\delta$ *and that along* $(L\setminus K_\delta)\cap\{-r<x_n<r\}$ *it has the form*

$$w=\sum_{j=0}^{m-1}w_j(x')D_1^{m-1-j}\delta(x_1)\tag{1.8}$$

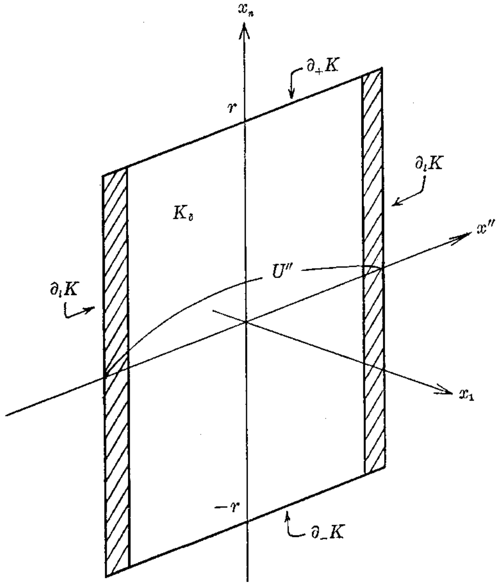

Fig. 1. (of K)

with $\operatorname{supp} w_j(x') \subset (\overline{U''} \setminus U''_\delta) \times \{-r<x_n<r\}$, $\text{S.S.}w_j(x') \subset \{\xi_n=0\}$.

PROOF. Since we are specifying the ambiguity only along the lateral part, it suffices to examine in $-r<x_n<r$. Thus we can replace $[[\chi_{U''}(x'')p(D)[u]]]$ by $\chi_{U''}(x'')p(D)[u]$ and $[[p(D)((1-\chi)J(D')u)]]$ by $Y(x_1)p(D)((1-\chi)J(D')u)$. Thus we have

$$\begin{aligned}
&J(D')[[\chi_{U''}(x'')p(D)[u]]]-[[p(D)((1-\chi)J(D')u)]]\\
&=J(D')(\chi_{U''}(x'')p(D)[u])-\chi_{U''}(x'')(Y(x_1)p(D)((1-\chi)J(D')u))\\
&=\chi_{U''}(x'')p(D)J(D')[u]+(J(D')(\chi_{U''}(x'')p(D)[u])-\chi_{U''}(x'')J(D')p(D)[u])\\
&\quad-(\chi_{U''}(x'')p(D)(Y(x_1)(1-\chi)J(D')u)\\
&\quad-(Y(x_1)p(D)((1-\chi)J(D')u)-p(D)(Y(x_1)(1-\chi)J(D')u)).
\end{aligned}$$

Here the second term in the last side has support in $\{0\}\times\partial U''\times \boldsymbol{R}$, hence $p(D)[u]$ can be replaced by $p(D)(Y(x_1)u)$ there, because of the analyticity of u. Consequently this term is singular only by the factors $\chi_{U''}(x'')$, $Y(x_1)$ or their derivatives, and only up to the order m in D_1. Hence it has the form (1.8) with the coefficients $w_j(x')$ singular only in the micro-directions satisfying $\xi_n=0$, i.e. lacking the term with dx_n. Similarly, the fourth term has support in $L\setminus K_\delta$ and satisfies the same regularity.

Finally the first and the third terms combine themselves to

$$\begin{aligned}(1.9)\qquad &\chi_{U''}(x'')p(D)(J(D')[u])-\chi_{U''}(x'')p(D)(Y(x_1)(1-\chi)J(D')u)\\ &=p(D)(\chi_{U''}(x'')(J(D')[u]-Y(x_1)(1-\chi)J(D')u))\\ &\quad+(\chi_{U''}(x'')p(D)(J(D')[u])-p(D)(\chi_{U''}(x'')J(D')[u]))\\ &\quad-(\chi_{U''}(x'')p(D)(Y(x_1)(1-\chi)J(D')u)-p(D)(\chi_{U''}(x'')Y(x_1)(1-\chi)J(D')u)).\end{aligned}$$

Here the hyperfunction under the operator $p(D)$ in the first term has support in $\operatorname{supp}\chi\cap\{x_1\geqslant 0\}$. Hence this term belongs to $p(D)\mathcal{B}_*(\boldsymbol{R}^n)$ (modulo $\mathcal{B}[\partial_+K\cup\partial_-K]$). The second and the third terms have supports in $\{x_1\geqslant 0\}\times\partial U''\times\boldsymbol{R}$, because the factor $\chi_{U''}(x'')$ is at least once differentiated. Since the support of the whole hyperfunction (1.9) has been contained in $\operatorname{supp}\chi\cap\{x_1\geqslant 0\}$, we conclude that the sum of these terms has support in fact contained in $\{0\}\times\partial U''\times\boldsymbol{R}$. By the same reason as above, this sum is singular only in the micro-directions satisfying $\xi_n=0$, and of the form (1.8). Thus applying the Fourier transform and restricting to $N(p)$, we obtain the lemma.

Now let φ be a Gevrey class function such that $\operatorname{supp}\varphi$ is contained in the ε-neighborhood of $\partial_+K\cup\partial_-K$ and that $\varphi\equiv 1$ on the $\varepsilon/2$-neighborhood. Put

$$\begin{aligned}&v=(1-\varphi)Y(x_1)p(D)((1-\chi)J(D')u),\\ &w^++w^-=[[p(D)((1-\chi)J(D')u)]]-v,\end{aligned}$$

where $w^\pm$ indicates the component whose support is contained in the ε-neighborhood of $\partial_\pm K$ respectively. Then v is a Gevrey class function except on $x_1=0$ only by the factor $Y(x_1)$, with

$$\operatorname{supp} v\subset\{0\leqslant x_1\leqslant\varepsilon\}\times\overline{U''}\times\{-r\leqslant x_n\leqslant r\}.$$

(We are assuming in the same time that $\overline{\operatorname{supp}\psi_1}$ is contained in the ε-neighborhood of $\{0\}\times\{-r\leqslant x_n\leqslant r\}$.) We can choose the Gevrey regularity of ψ_1, ψ_2, φ so that $\hat{v}(\zeta)$ satisfies, for given $A>0$, $0<q<1$,

$$(1.10)\qquad |\hat{v}(\zeta)|\leqslant C\exp(-A|\operatorname{Re}\zeta'|^q+\varepsilon(\operatorname{Im}\zeta_1)_++H_L(\operatorname{Im}\zeta')).$$

Also, by virtue of Lemma 1.1 we can assume, by modifying $w^\pm$ if necessary, that

$$(1.11)\qquad J(\zeta')F(\zeta)=\hat{v}(\zeta)+\hat{w}^+(\zeta)+\hat{w}^-(\zeta)+\hat{w}^l(\zeta)$$

without ambiguity on $N(p)$, where $w^l(x)$ is a hyperfunction with support in $(L\setminus K_\delta)\cap\{-r+\varepsilon\leqslant x_n\leqslant r-\varepsilon\}$ such that it has the form (1.8) with the coefficients $w_j(x')$ modified on $x_n=\pm(r-\varepsilon)$, hence micro-analytic except only the directions in $\xi_n=0$ along $(L\setminus K_\delta)\cap\{-r+\varepsilon<x_n<r-\varepsilon\}$. Let $\hat{f}_j(\zeta')$, $\hat{g}_j(\zeta')$, $\hat{h}_j^\pm(\zeta')$, $\hat{h}_j^l(\zeta')$, $j=0,\cdots,m-1$ be the

coefficients of the interpolation polynomial in ζ_1 corresponding to $F(\zeta)$, $\hat{v}(\zeta)$, $\hat{w}^{\pm}(\zeta)$, $\hat{w}^l(\zeta)$ respectively. Since this correspondence is $\mathcal{O}_{\zeta'}$-linear, we have

$$J(\zeta')\hat{f}_j(\zeta')=\hat{g}_j(\zeta')+\hat{h}_j^+(\zeta')+\hat{h}_j^-(\zeta')+\hat{h}_j^l(\zeta'), \qquad j=0, \cdots, m-1.$$

As mentioned before, $\hat{f}_j(\zeta')$ is in fact the Fourier transform of a hyperfunction $f_j(x')\in{}'\mathcal{B}[\overline{U'}]$ extending $u_j(x')\in{}'\mathcal{B}(U')$. This is the same thing as for $\hat{h}_j^l(\zeta')$. In fact it is the Fourier image of $h_j^l(x')$ which is the indicated modification of $w_j(x')$.

On the contrary, $\hat{g}_j(\zeta')$, $\hat{h}_j^{\pm}(\zeta')$ have no primitive images as hyperfunctions with compact support despite the notation. This is because the support of $v(x)$, $w^{\pm}(x)$ are not in $x_1=0$. Employing the interpolation formula we can estimate them as follows:

LEMMA 1.2. *Under the above construction, $\hat{g}_j(\zeta')$, $\hat{h}_j^{\pm}(\zeta')$ satisfy the following estimates*

$$(1.12)\qquad |\hat{g}_j(\zeta')|\leqslant C(1+|\zeta'|)^M \exp(-A|\mathrm{Re}\,\zeta'|^q+\varepsilon \sup_k (\mathrm{Im}\,\tau_k(\zeta'))_+ +H_L(\mathrm{Im}\,\zeta')),$$

$$(1.13)\qquad |h_j^{\pm}(\zeta')|\leqslant C_\gamma \exp (\gamma|\zeta'|+\varepsilon \sup_k (\mathrm{Im}\,\tau_k(\zeta'))_+ +H_{\partial_{\pm}K} (\mathrm{Im}\,\zeta')+\varepsilon|\mathrm{Im}\,\zeta'|),$$

where $\tau_k(\zeta')$ are the roots of $p(\zeta_1, \zeta')=0$ for ζ_1, M is a constant depending only on $p(D)$ and C_γ is a constant depending on $\gamma>0$ which may be arbitrarily small.

The proof will be obvious. These estimates are our fundamental information to the next section. Remark that they imply that these entire functions are rather the Fourier image of analytic functionals with support compact but not contained in the real axis.

Now we consider the second typical situation. This time we assume the following form:

$$U'=\{-r<x_2<r\}\times U^*,$$

where U^* is a relatively compact, convex open neighborhood of $0\in \boldsymbol{R}^{n-2}_{x^*}$, with $x^*=(x_3, \cdots, x_n)$. We also assume that the real analytic solution $u\in\mathcal{A}_p(U^+)$ can be continued real analytically on a neighborhood of the set $\partial_l K=\partial_l^+K\cup\partial_l^-K$, where

$$(1.14)\qquad \partial_l^{\pm}K=\{0\}\times\{x_2=\pm r\}\times\overline{U^*}.$$

Let $\phi_1(x_1, x^*)$ be a Gevrey class function in $\boldsymbol{R}\times U^*$ such that $\mathrm{supp}\,\phi_1$ is contained in a neighborhood of $\{0\}\times U^*$ and that $\phi_1\equiv 1$ on a smaller neighborhood of this set. Let $\phi_2(x_2)$ be another Gevrey class function on $\boldsymbol{R}$ such that

$$\operatorname{supp}\psi_2(x_2)\subset\{|x_2|\leqslant r\},$$
$$\psi_2\equiv 1 \quad \text{in} \quad \{|x_2|\leqslant r-\delta\}.$$

Here $\delta>0$ is a constant chosen in such a way that u admits the analytic continuation up to the part of the boundary $\{0\}\times\{r-\delta\leqslant|x_2|\leqslant r\}\times\overline{U^*}$. Let $\chi=\psi_1\psi_2$. Then by a similar argument as above we obtain that

$$F(\zeta)=\widehat{[[Y(r^2-x_2^2)p(D)[u]]]}$$

and that

$$F(\zeta)\equiv\widehat{[[p(D)((1-\chi)u)]]} \quad \bmod \widehat{\mathcal{B}[L\setminus K_\delta]}$$

on $N(p)$, where now $K_\delta=\{0\}\times\{|x_2|<r-\delta\}\times U^*$. Introduce the notation

$$\partial_r K=\{0\}\times\{-r<x_2<r\}\times\partial U^*.$$

In place of Lemma 1.1 we have now

LEMMA 1.3. *For any local operator $J(D')$ with constant coefficients, we have*

$$J(\zeta')F(\zeta)=\widehat{[[p(D)((1-\chi)J(D')u)]]}+\hat{w} \tag{1.15}$$

on $N(p)$, where w is a hyperfunction such that $\operatorname{supp} w\subset L\setminus K_\delta$ *and that along* $\{0\}\times\{r-\delta\leqslant|x_2|\leqslant r\}\times U^*$ *it has the form* (1.8) *with the coefficients $w_j(x')$ satisfying*

$$\operatorname{supp} w_j(x')\subset\{r-\delta\leqslant|x_2|\leqslant r\}\times U^*,$$
$$\text{S.S. } w_j(x')\subset\{\xi_n=0\}.$$

The proof is similar. Now let φ be a Gevrey class function such that $\operatorname{supp}\varphi$ is contained in the ε-neighborhood of $\partial_r K$ and that $\varphi\equiv 1$ on the $\varepsilon/2$-neighborhood. Put

$$v=(1-\varphi)Y(x_1)p(D)((1-\chi)J(D')u),$$
$$w=[[p(D)((1-\chi)J(D')u)]]-v.$$

Then v is a Gevrey class function except on $x_1=0$, which satisfies

$$\operatorname{supp} v\subset\{0\leqslant x_1\leqslant\varepsilon\}\times\{-r\leqslant x_2\leqslant r\}\times\overline{U^*}$$

and (1.10), if we choose the size of $\operatorname{supp}\psi_1$ and the regularity of ψ_1, ψ_2, φ in a similar way. Put

$$U_\varepsilon^*=\{x^*\in U^*;\ \operatorname{dis}(x^*,\partial U^*)>\varepsilon\}.$$

By virtue of Lemma 1.3 we have now instead of (1.11)

$$J(\zeta')F(\zeta)=\hat{v}(\zeta)+\hat{w}_1^+(\zeta)+\hat{w}_1^-(\zeta)+\hat{w}_r(\zeta)$$

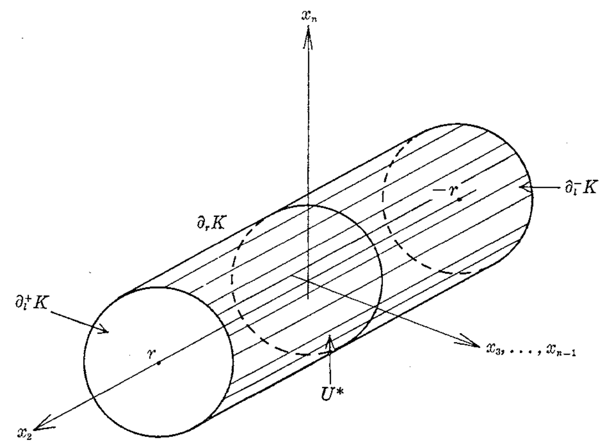

Fig. 2. (of K, where the x_1-axis is not written)

without ambiguity on $N(p)$, where $w_l^+(x)$, $w_l^-(x)$, $w_r(x)$ denote hyperfunctions with support in $\{0\}\times\{r-\delta\leqslant x_2\leqslant r\}\times\overline{U^*}$, $\{0\}\times\{-r\leqslant x_2\leqslant -r+\delta\}\times\overline{U^*}$, $\{0\leqslant x_1\leqslant \varepsilon\}\times\{-r\leqslant x_2 \leqslant r\}\times(\overline{U^*}\setminus U_\varepsilon^*)$ respectively, whose sum reproduces $w(x)$. Let $\hat{f}_j(\zeta')$, $\hat{g}_j(\zeta')$, $\hat{h}_j^r(\zeta')$, $\hat{h}_j^\pm(\zeta')$ denote the coefficients of the interpolation polynomial in ζ_1, corresponding to $F(\zeta)$, $\hat{v}(\zeta)$, $\hat{w}_r(\zeta)$, $\hat{w}_l^\pm(\zeta)$ respectively. Again $\hat{f}_j(\zeta')$, $\hat{h}_j^\pm(\zeta')$ are the Fourier image of some hyperfunctions $f_j(x')$, $h_j^\pm(x')$ of similar nature and especially $h_j^\pm(x')$ are micro-analytic outside the directions $\xi_n=0$ along $\{0\}\times\{r-\delta\leqslant \pm x_2\leqslant r\}\times U^*$ respectively. On the other hand, we have

LEMMA 1.4. *$\hat{g}_j(\zeta')$ satisfies the same estimate as* (1.12). *On the other hand, given any decomposition $\partial U^*=\bigcup_{i=1}^N L_i^*$, there exists a corresponding decomposition*

$$\hat{h}_j^r(\zeta')=\sum_{i=1}^N \hat{h}_{ji}^r(\zeta')$$

such that

$$(1.17)\qquad |\hat{h}_{ji}^r(\zeta')|\leqslant C_\gamma \exp\,(\gamma|\zeta'|+\varepsilon\sup_k(\operatorname{Im}\tau_k(\zeta'))_+ + H_{\mathrm{ch}(L_i^*)}(\operatorname{Im}\zeta^*)+r|\operatorname{Im}\zeta_2|+\varepsilon|\operatorname{Im}\zeta'|),$$

where C_γ is a constant depending on the small parameter $\gamma>0$, and ch(.) *denotes the convex hull.*

The last assertion follows from the corresponding decomposition of $w_r(x)$.

§2. Propagation of regularity up to the boundary.

Now we examine the fundamental estimates given in the preceding section.

The non-characteristic assumption of $p(D)$ gives the estimate

$$|\tau_k(\zeta')| \leqslant M|\zeta'|, \qquad k=1, \cdots, m, \tag{2.1}$$

for some constant $M>0$, but nothing better in general. Therefore the existence of the term $\varepsilon \sup_k(\operatorname{Im} \tau_k(\zeta'))_+$ does not allow us to consider $\hat{g}_j(\zeta')$, $\hat{h}_j^{\pm}(\zeta')$ as the Fourier image of hyperfunctions as the notation may imply. To overcome this difficulty we pose some condition on the roots $\tau_k(\zeta')$ and employ a damping factor related to it. To explain our way of argument in a simpler situation, we will first pose a stronger assumption:

PROPOSITION 2.1. *Let $U'=U''\times\{-r<x_n<r\}$. Assume that the roots $\tau_k(\zeta')$, $k=1, \cdots, m$ satisfy the following estimate: There exist positive constants b, c, C and a positive integer N such that*

$$\operatorname{Im} \tau_k(\zeta') \leqslant b|\operatorname{Im} \zeta'|+c|\operatorname{Re} \zeta_n|^{1-1/N}|\zeta''|^{1/N}+C, \qquad \text{for } \zeta' \in C^{n-1}. \tag{2.2}$$

Let u be a real analytic solution of $p(D)u=0$ in U^+. If all the boundary values of u are real analytic on a neighborhood of $\partial U''\times\{-r\leqslant x_n\leqslant r\}$, then they become micro-analytic to the direction $+\sqrt{-1}\, dx_n\infty$ everywhere in U'.

Here we have coherently written $\zeta''=(\zeta_2, \cdots, \zeta_{n-1})$.

PROOF. Let $u_j(x')$ be the boundary values of the given solution. Employing Proposition 1.6 of [8] as in the proof of Theorem 4.1 in [8], we can assume without loss of generality that $\text{S.S.}u_j(x')$, $j=0, \cdots, m-1$ contain only the direction $\sqrt{-1}dx_n\infty$, and show that they are in fact void. Apply the construction of the first type of §1, and let $\hat{f}_j(\zeta')$, $\hat{g}_j(\zeta')$, $\hat{h}_j^{\pm}(\zeta')$, $\hat{h}_j^{!}(\zeta')$ be the coefficients thus obtained. Then put

$$\hat{E}(\xi', \varepsilon)=\exp(-2c\varepsilon(\sqrt{1+|\xi''|^2})^{1/N}(\sqrt{1+|\xi'|^2})^{1-1/N}),$$

and consider

$$\begin{aligned} &J(\xi')\hat{f}_j(\xi')\hat{E}(\xi', \varepsilon) \\ &=\hat{g}_j(\xi')\hat{E}(\xi', \varepsilon)+\hat{h}_j^{+}(\xi')\hat{E}(\xi', \varepsilon)+\hat{h}_j^{-}(\xi')\hat{E}(\xi', \varepsilon)+\hat{h}_j^{!}(\xi')\hat{E}(\xi', \varepsilon), \end{aligned} \tag{2.3}$$

where $\varepsilon>0$ is a parameter common to the construction of §1. Put

$$E(x', \varepsilon)={}'\mathcal{F}^{-1}\hat{E}(\xi', \varepsilon),$$

where we have written ${}'\mathcal{F}$ the $(n-1)$-dimensional Fourier transformation. This is a function of x' in some Gevrey class and, in view of Lemma 2.3 in [4] (or by a direct consideration), real analytic except along the hypersurface $x_n=0$. It is even micro-analytic there outside the direction $\pm\sqrt{-1}\,dx_n\infty$. Note that as a hyperfunction of x', ε, $E(x', \varepsilon)$ contains ε as a complex holomorphic parameter for $\operatorname{Re}\varepsilon>0$. Then (2.3) is in fact the Fourier image of the following Fourier hyperfunction of $n-1$ variables x':

$$J(D')(f_j(x') * E(x', \varepsilon)), \tag{2.4}$$

(For a short account of the notion of Fourier hyperfunction see §1 of [2].) In order to see the analyticity of this hyperfunction, let us examine the right-hand side of (2.3). The last term of (2.3) is also the Fourier image of

$$w_j(x') * E(x', \varepsilon), \tag{2.5}$$

which, by the studied regularity of $w_j(x')$ (Lemma 1.1), becomes real analytic in $U''_\delta \times \{-r+\varepsilon<x_n<r-\varepsilon\}$ as the standard calculus of S.S. shows. On the other hand, in view of Lemma 1.2 and (2.2) the function $\hat{g}_j(\xi')$ satisfies the estimate

$$\begin{aligned}|\hat{g}_j(\xi')\hat{E}(\xi', \varepsilon)| \leqslant C'(1+|\xi'|)^M \exp(-A|\xi'|^q + c\varepsilon|\xi_n|^{1-1/N}|\xi''|^{1/N} \\ -2c\varepsilon(\sqrt{1+|\xi'|^2})^{1-1/N}(\sqrt{1+|\xi''|^2})^{1/N}).\end{aligned} \tag{2.6}$$

This shows that $\hat{g}_j(\xi')\hat{E}(\xi', \varepsilon)$ is the Fourier image of a Gevrey class function on $\boldsymbol{R}^{n-1}$.

Finally, concerning $\hat{h}_j^{\pm}(\zeta')$ we have the estimate

$$\begin{aligned}|\hat{h}_j^{\pm}(\zeta')\hat{E}(\zeta', \varepsilon)| \leqslant C'_\gamma \exp(\gamma|\zeta'|+H_{\partial_{\pm}K}(\operatorname{Im}\zeta')+\varepsilon(b+1)|\operatorname{Im}\zeta'| + c\varepsilon|\operatorname{Re}\zeta_n|^{1-1/N}|\zeta''|^{1/N} \\ -2c\varepsilon \operatorname{Re}((\sqrt{1+\zeta'^2})^{1-1/N}(\sqrt{1+\zeta''^2})^{1/N})),\end{aligned}$$

where we have written $\zeta''^2=\zeta_2^2+\cdots+\zeta_{n-1}^2$ etc. Let $\lambda>0$ be a small constant. On the complex neighborhood $\{|\operatorname{Im}\zeta''|\leqslant\lambda\sqrt{|\operatorname{Re}\zeta''|^2+1},\ |\operatorname{Im}\zeta_n|\leqslant\lambda\sqrt{|\operatorname{Re}\zeta_n|^2+1}\}$ of the real axis $\boldsymbol{R}^{n-1}\subset\boldsymbol{C}^{n-1}$, we have

$$\operatorname{Re}(1+\zeta''^2)=1+|\operatorname{Re}\zeta''|^2-|\operatorname{Im}\zeta''|^2 \geqslant (1-\lambda^2)(|\operatorname{Re}\zeta''|^2+1)$$

$$|\operatorname{Im}(1+\zeta''^2)| \leqslant 2|\operatorname{Re}\zeta''|\cdot|\operatorname{Im}\zeta''| \leqslant \frac{2\lambda}{1-\lambda^2}\operatorname{Re}(1+\zeta''^2),$$

and, since $|\operatorname{Im}\zeta'|\leqslant\lambda\sqrt{|\operatorname{Re}\zeta'|^2+2}$ there, we have similarly

$$|\operatorname{Im}(1+\zeta'^2)| \leqslant \frac{4\lambda}{1-2\lambda^2}\operatorname{Re}(1+\zeta'^2).$$

Then we have

$$\begin{aligned}&\arg(\sqrt{1+\zeta'^2})^{1-1/N}(\sqrt{1+\zeta''^2})^{1/N} \\ &= \frac{1}{2}\left\{\left(1-\frac{1}{N}\right)\tan^{-1}\frac{2\lambda}{1-2\lambda^2}+\frac{1}{N}\tan^{-1}\frac{2\lambda}{1-\lambda^2}\right\}=\theta(\lambda),\end{aligned}$$

hence,

$$\begin{aligned}\mathrm{Re}\,(\sqrt{1+\zeta'^2})^{1-1/N}(\sqrt{1+\zeta''^2})^{1/N} &\geqslant \cos\theta(\lambda)(\sqrt{|1+\zeta'^2|})^{1-1/N}(\sqrt{|1+\zeta''^2|})^{1/N}\\ &\geqslant (1-2\lambda^2)\cos\theta(\lambda)(\sqrt{1+|\mathrm{Re}\,\zeta'|^2})^{1-1/N}(\sqrt{1+|\zeta''|^2})^{1/N}.\end{aligned}$$

Since $\cos\theta(\lambda)$ approaches 1 as λ tends to 0, we thus conclude that, in such a neighborhood for small λ, we have

$$\begin{aligned}&c|\mathrm{Re}\,\zeta_n|^{1-1/N}|\zeta''|^{1/N}-2c\,\mathrm{Re}((\sqrt{1+\zeta'^2})^{1-1/N}(\sqrt{1+\zeta''^2})^{1/N})\\ &\leqslant -c'(\sqrt{1+|\mathrm{Re}\,\zeta'|^2})^{1-1/N}(\sqrt{1+|\zeta''|^2})^{1/N}\\ &\leqslant -c'|\mathrm{Re}\,\zeta''|,\end{aligned}$$

with some $c'>0$, hence,

$$|\hat{h}_j^{\pm}(\zeta')\hat{E}(\zeta',\varepsilon)|\leqslant C''_\gamma \exp(\gamma|\zeta'|+H_{\partial_\pm K}(\mathrm{Im}\,\zeta')+\varepsilon(b+1)|\mathrm{Im}\,\zeta'|-c'\varepsilon|\mathrm{Re}\,\zeta''|). \tag{2.7}$$

In view of Lemma 2.3 in [4], we see from this estimate that $\hat{h}_j^{\pm}(\xi')\hat{E}(\xi',\varepsilon)$ is the Fourier image of a Fourier hyperfunction which is real analytic outside the $\varepsilon(b+1)$-neighborhood of the hyperplane $x_n=\pm r$ respectively.

Summing up, the hyperfunction (2.4) of x' is in fact continuous in $U''_\delta\times\{-r+\varepsilon<x_n<r-\varepsilon\}$. Since $J(D')$ is arbitrary, this implies that

$$f_j(x')*E(x',\varepsilon) \tag{2.8}$$

is real analytic there by virtue of Theorem 3.3 in [2].

Now we will examine the uniformity of this analyticity with respect to ε. The elementary proof of the estimation of S.S. via the deformation of the contour of integral shows that, if ε runs in the set $\{\varepsilon_0\leqslant\varepsilon\leqslant\delta\}$ for some $\varepsilon_0>0$, then (2.5) for $J(D')=1$ is holomorphic in a fixed complex neighborhood of $U''_\delta\times\{-r+\delta<x_n<r-\delta\}$ with a modulus uniform in ε. On the other hand, the estimates (2.6), (2.7) hold uniformly with respect to ε in this region, if we replace ε by δ or ε_0. Thus the function (2.4) is bounded on every compact subset M of $U''_\delta\times\{-r+\delta<x_n<r-\delta\}$ locally uniformly with respect to ε. Therefore in view of Proposition 2.4 in [2] (see Lemma 2.2 below) the family of real analytic functions (2.8) constitutes a bounded set in the space of real analytic functions $\mathcal{A}(M)$, when ε runs in $\{\varepsilon_0\leqslant\varepsilon\leqslant\delta\}$. Thus by the structure theorem of the bounded sets of the (DFS) space $\mathcal{A}(M)$, (2.8) becomes holomorphic on a fixed complex neighborhood of M independent of ε in $\{\varepsilon_0\leqslant\varepsilon\leqslant\delta\}$. Since (2.8) is clearly holomorphic with respect to ε in $\mathrm{Re}\,\varepsilon>0$ for fixed real x', we can apply the Malgrange-Zerner theorem (see e.g. [13]) to conclude that (2.8) is real analytic in the joint variables (x',ε) in $U''_\delta\times\{-r+\delta<x_n$

$<r-\delta\}\times\{\varepsilon>0\}$. Note that this conclusion is valid also for complex values of the parameter ε satisfying $\mathrm{Re}\,\varepsilon>0$.

Recall that if we let ε tend to 0, then the Fourier hyperfunction (2.8) converges to $f_j(x')$. Employing the compactification $\boldsymbol{S}^{n-1}=\boldsymbol{R}^{n-1}\cup\{\infty\}$ as in [4], we see therefore that $f_j(x')$ is the first boundary value to $\varepsilon\to+0$ of the real analytic solution (2.8) of the following equation:

$$\left\{\frac{\partial^{2N}}{\partial\varepsilon^{2N}}+(2c)^{2N}(\Delta_{x''}-1)(1-\Delta_{x'})^{N-1}\right\}V(x',\varepsilon)=0. \tag{2.9}$$

(See Corollary 2.6 in [5]. For a more general comparison of the cohomological-topological boundary values see also § 2.3 of [10].) Thus in order to conclude that $f_j(x')$ is real analytic in

$$\Omega'_\delta=U''_\delta\times\{-r+\delta<x_n<r-\delta\},$$

it suffices to show that $V(x',\varepsilon)$ admits the analytic continuation up to this set in the boundary. This will be done in the sequel employing the local Bochner theorem. Recall first that $\mathrm{S.S.}f_j(x')$ contains only the direction $\sqrt{-1}\,dx_n\infty$ in Ω'_δ. This implies that $f_j(x')$ admits there the boundary value expression by unique term of the form $F_j(x'+\sqrt{-1}\,\Gamma'0)$, where Γ' denotes the half space $\{y_n>0\}$ in $\boldsymbol{R}^{n-1}_{y'}$ and $F_j(z')$ is a function holomorphic on an infinitesimal wedge with this breadth. To fix the idea let this wedge be of the form

$$\{z'=x'+\sqrt{-1}\,y';\ x'\in\Omega'_\delta,\ \lambda(|y''|)<y_n<B\}, \tag{2.10}$$

where $\lambda(t)$ is a non-negative convex continuous function of $t\geqslant0$ such that $\lambda(t)>0$ for $t>0$, $\lambda(0)=0$ and $\lambda(t)/t\to0$ as $t\to0$. Note also that by virtue of the micro-local regularity of $E(x',\varepsilon)$ remarked before $\mathrm{S.S.}f_j(x')*E(x',\varepsilon)$ contain only the direction $\sqrt{-1}\,dx_n\infty$ as a hyperfunction of x', ε. Further a closer study of $V(x',\varepsilon)=f_j(x')*E(x',\varepsilon)$ shows that it admits similar expression of the form $V((x',\varepsilon)+\sqrt{-1}\,\Gamma'\times\boldsymbol{R}0)$, where $\Gamma'\times\boldsymbol{R}$ denotes the half space $\{y_n>0\}$ in $\boldsymbol{R}^{n-1}_{y'}\times\boldsymbol{R}_\varepsilon$ and $V(z',\varepsilon)$ is holomorphic on an infinitesimal wedge of the product type $(2.10)\times\{\mathrm{Re}\,\varepsilon>0\}$. The studied real analyticity of $V(x',\varepsilon)$ then implies that this defining function $V(z',\varepsilon)$ admits the analytic continuation up to the real axis. This is why we have employed the same notation to the defining function.

On the other hand, $V(x',\varepsilon)$ is a solution of the differential equation (2.9). Thus the boundary value expression by unique term implies that the defining function $V(z', \varepsilon)$ itself is a solution of the corresponding holomorphic differential equation

$$(2.9)' \qquad \left\{\frac{\partial^{2N}}{\partial\varepsilon^{2N}}+(2c)^{2N}(\Delta_{z''}-1)(1-\Delta_{z'})^{N-1}\right\}V(z', \varepsilon)=0,$$

where ε is considered complex. Thus by virtue of Leray's precise version of the Cauchy-Kowalevsky theorem, $V(z', \varepsilon)$ can be continued up to a domain of the form

$$\{(z', \varepsilon)\in \boldsymbol{C}^n;\ x'\in\Omega'_{2\delta},\ \lambda(|y''|)+k(-\mathrm{Re}\,\varepsilon)_+<y_n<B'\},$$

where k is a constant depending on the operator. Together with the above information of analyticity, $V(z', \varepsilon)$ is therefore holomorphic on a domain of the form

$$(2.11) \quad \{(z', \varepsilon)\in \boldsymbol{C}^n;\ x'\in\Omega'_{2\delta},\ \lambda(|y''|)+k(-\mathrm{Re}\,\varepsilon)_+-\mu((\mathrm{Re}\,\varepsilon)_+)<y_n<B'',\ |\mathrm{Re}\,\varepsilon|<\delta\},$$

where $\mu(t)$ is a function of the same type as $\lambda(t)$. Note that this domain is independently defined of $\mathrm{Im}\,\varepsilon$.

We will now show that the domain (2.11) can be improved to

$$(2.13) \quad \{(z', \varepsilon)\in \boldsymbol{C}^n;\ x'\in\Omega'_{3\delta},\ \kappa(|y''|+(-\mathrm{Re}\,\varepsilon)_+)-\mu((\mathrm{Re}\,\varepsilon)_+)<y_n<B'',\ |\mathrm{Re}\,\varepsilon|<\delta\},$$

where $\kappa(t)$ is another function of the same type as $\lambda(t)$. For this purpose consider $V(x', \varepsilon)=f_j(x') * E(x', \varepsilon)$ more in detail. For the sake of simplicity we will first make a reduction to assume that $\mathrm{S.S.}f_j(x')$ contains only the direction $+\sqrt{-1}\,dx_n\infty$ everywhere on $\boldsymbol{R}^{n-1}$. In fact this is permitted by considering if necessary

$$(2.14) \qquad V(x', \varepsilon) \underset{x'}{*} J(D_{\omega'})W(x', \omega')|_{\omega'=\nu'}$$

instead of $V(x', \varepsilon)$, where

$$(2.15) \quad W(x', \omega')=\frac{(n-2)!}{(-2\pi\sqrt{-1})^{n-1}}\,\frac{\phi(x', \omega')e^{-x'^2}}{(x'\omega'+\sqrt{-1}\,(x'^2-(x'\omega')^2)/\sqrt{1+x'^2}+\sqrt{-1}0)^{n-1}}$$

is the component of a curved wave decomposition of $\delta(x')$ employed in Lemma 2.2 of [6] and $J(D_{\omega'})$ is a local operator with constant coefficients with respect to a system of local coordinates near $\omega'=\nu'=(0, \cdots, 0, 1)$. If this new solution (2.14) of the equation (2.9) can be continued up to $\varepsilon=0$ in the set $\Omega'_{3\delta}$, then its first boundary value $f_j(x') \underset{x'}{*} J(D_{\omega'})W(x', \omega')|_{\omega'=\nu'}$ will be real analytic. Since $J(D_{\omega'})$ is arbitrary, $f_j(x')$ will then be micro-analytic there to the direction $+\sqrt{-1}\,dx_n\infty$ by virtue of Lemma 1.1 in [8]. Because $f_j(x')$ has been micro-analytic to the other directions, it will actually become real analytic on $\Omega'_{3\delta}$. Remark that on $\mathrm{Re}\,\varepsilon\geqslant 0$ the hyperfunction (2.14) enjoys the same regularity property as $V(x', \varepsilon)$ indicated above. Note also that in view of the property of the component (2.15) studied in Lemma 2.3 of [6], we can even assume as a result of the above reduction the following estimate for the Fourier transform of $f_j(x')$: For every $\gamma>0$ there exists $C_\gamma>0$ such that

$$|\hat{f}_j(\xi')|\leqslant C_\gamma e^{-|\xi'|/C_\gamma}, \text{ on } |\xi''|\geqslant\gamma\xi_n,$$

whereas on the whole $\boldsymbol{R}_{\xi'}^{n-1}$ $\hat{f}_j(\xi')$ satisfies the infra-exponential estimate.

Now consider

$$\begin{aligned}V(x', \varepsilon)&={}'\mathcal{F}^{-1}(\hat{f}_j(\xi')\hat{E}(\xi', \varepsilon))\\&=\frac{1}{(2\pi)^{n-1}}\int_{\boldsymbol{R}^{n-1}}e^{\sqrt{-1}x'\xi'}\hat{f}_j(\xi')\hat{E}(\xi', \varepsilon)d\xi'\end{aligned}$$

for $\operatorname{Re}\varepsilon\leqslant 0$. Let x' run into a complex z' with $y_n=\operatorname{Im} z_n>0$. For every $\gamma>0$ we will have

$$\begin{aligned}&|e^{\sqrt{-1}z'\xi'}\hat{f}_j(\xi')\hat{E}(\xi', \varepsilon)|\\&\leqslant C'_\gamma \exp(-y'\xi'-|\xi'|/C_\gamma-2c\operatorname{Re}\varepsilon(\sqrt{1+|\xi''|^2})^{1/N}(\sqrt{1+|\xi'|^2})^{1-1/N}) \text{ on } |\xi''|\geqslant\gamma\xi_n.\end{aligned}$$

Hence the integral on the region $|\xi''|\geqslant\gamma\xi_n$ will become holomorphic on the neighborhood $2c|\operatorname{Re}\varepsilon|+|y'|<1/C_\gamma$ of the real axis (depending on γ but indifferent to the sign of y_n). On the other hand, for $\gamma>0$ sufficiently small we have

$$\begin{aligned}&|e^{\sqrt{-1}z'\xi'}\hat{f}_j(\xi')\hat{E}(\xi', \varepsilon)|\\&\leqslant C\exp(-y''\xi''-y_n\xi_n-2c\operatorname{Re}\varepsilon(\sqrt{1+|\xi''|^2})^{1/N}(\sqrt{1+|\xi'|^2})^{1-1/N})\\&\leqslant C'\exp(-(y_n-\gamma|y''|-3c\gamma|\operatorname{Re}\varepsilon|)|\xi_n|) \quad \text{on } |\xi''|\leqslant\gamma\xi_n.\end{aligned}$$

Thus the integral on the region $|\xi''|\leqslant\gamma\xi_n$ will become holomorphic on the wedge $y_n>\gamma|y''|+3c\,\gamma|\operatorname{Re}\varepsilon|$ which approaches the half space $y_n>0$ as $\gamma\to 0$. Summing up we conclude that there exists a convex continuous function $\kappa(t)$ of $t\geqslant 0$ satisfying $\kappa(t)>0$ for $t>0$, $\kappa(0)=0$ and $\kappa(t)/t\to 0$ as $t\to 0$ such that $V(z', \varepsilon)$ can be prolonged up to a domain of the form

$$\{(z', \varepsilon)\in\boldsymbol{C}^n;\ \kappa(|y''|+|\operatorname{Re}\varepsilon|)<y_n<B'''\}. \tag{2.16}$$

Now the union of (2.11) and (2.16) gives a domain of the form (2.13).

Finally we apply the local version of Bochner's tube theorem just as in the proof of Theorem 3.1 in [8], with the role of $\operatorname{Re}\varepsilon$ and $\operatorname{Im}\varepsilon$ interchanged. Then we conclude that $V(z', \varepsilon)$ can be prolonged from (2.13) to a neighborhood of $\Omega'_{4\delta}\times\{\operatorname{Re}\varepsilon=0\}$. Since δ is arbitrary this ends the fairly long proof of Proposition 2.1.

q. e. d.

The following lemma should rather be given in [2].

LEMMA 2.2. *Let $\mathcal{A}_{\mathcal{J}}(K)$ denote the topological linear space of real analytic functions on a neighborhood of the compact set K, endowed with the semi-norms*

$$\|f(x)\|_J=\sup_{x\in K}|J(D)f(x)|. \tag{2.17}$$

Then the bounded sets are common to $\mathcal{A}_{\mathcal{J}}(K)$, $\mathcal{A}(K)$, $\sigma(\mathcal{A}(K), \mathcal{B}[K])$. They come from bounded sets of holomorphic functions on some complex neighborhoods of K.

The proof is similar to Proposition 2.4 in [2]. In fact, a subset $B\subset\mathcal{A}_{\mathcal{J}}(K)$, on which the semi-norms (2.17) are bounded, produces a bounded set in $\sigma(\mathcal{A}(\{x_0\}), \mathcal{B}[\{x_0\}])$ for every fixed $x_0\in K$. Then by the (DFS) property of the space $\mathcal{A}(\{x_0\})$, the functions belonging to B can be continued to a fixed complex neighborhood of x_0, where they are uniformly bounded. Since x_0 is arbitrary, the assertion follows. (Recall that $\mathcal{A}_{\mathcal{J}}(K)\succ\sigma(\mathcal{A}(K), \mathcal{B}[K])$ is a delicate (maybe false) conjecture concerned with the representation of hyperfunctions by measures with accurate support. See pp. 332–333 in [2].)

Now we will improve the above result in various points.

REMARK 2.3.

1) It suffices to assume the inequality (2.1) only for $\mathrm{Re}\,\zeta_n>0$ (hence equivalently for $\mathrm{Re}\,\zeta_n\geqslant\lambda|\mathrm{Re}\,\zeta''|$ for some $\lambda>0$):

$$\mathrm{Im}\,\tau_k(\zeta')\leqslant b|\mathrm{Im}\,\zeta'|+c|\mathrm{Re}\,\zeta_n|^{1-1/N}|\zeta''|^{1/N}+C \quad \text{for} \quad \mathrm{Re}\,\zeta_n\geqslant\lambda|\mathrm{Re}\,\zeta''|. \tag{2.2$'$}$$

In fact, let $W(x', \omega')$ be the component (2.15) of the decomposition of $\delta(x')$. Put

$$\Delta'=\{\omega_n\geqslant 2\lambda|\omega''|\} \text{ and } W(x', \Delta')=\int_{\Delta'\cap S^{n-2}}W(x', \omega')d\omega'.$$

Then in order to prove the micro-analyticity of $f_j(x')$ to the direction $+\sqrt{-1}\,dx_n\infty$, it suffices to show the micro-analyticity of $f_j(x')*W(x', \Delta')$ to the same direction. We then consider $f_j(x')*W(x', \Delta')$ and hence $\hat{f}_j(\zeta')\hat{W}(\zeta', \Delta')$ etc. instead of $f_j(x')$ and $\hat{f}_j(\zeta')$ etc. Since $\hat{W}(\zeta', \Delta')$ decreases exponentially in the region $\mathrm{Re}\,\zeta_n\leqslant\lambda|\mathrm{Re}\,\zeta''|$ on a conical neighborhood of the real axis (cf. Lemma 2.3 in [6]), it annihilates there the influence of $\varepsilon\sup_k(\mathrm{Im}\,\tau_k(\zeta'))_+$, which may produce $\varepsilon M|\zeta'|$ in view of (2.1), for sufficiently small ε. Thus the estimate (2.2) assumed only for $\mathrm{Re}\,\zeta_n\geqslant\lambda|\mathrm{Re}\,\zeta''|$ will be sufficient to follow the above argument to deduce the real analyticity of (2.8).

2) Instead of the real analyticity of the solution u on U^+, and of the real

analyticity of the boundary values of u on a neighborhood of $\partial U''\times\{-r<x_n<r\}$, it suffices to assume their micro-analyticity to the directions $\rho^{-1}(+\sqrt{-1}\,dx_n\infty)$ resp. $+\sqrt{-1}\,dx_n\infty$, where ρ denotes the projection from $S^{n-1}\setminus\{\pm\sqrt{-1}\,dx_1\infty\}$ to the equator $\xi_1=0$. This follows directly from Theorem 1.10 in [8].

3) The set U'' need not be convex. In fact, first of all for general U' we can still assert, instead of Lemma 1.1 in [6], that $f_j(x')\in{}'\mathcal{B}[\mathrm{ch}(\overline{U'})]$ and that $f_j(x')|_{U'}$ agrees with the boundary value $u_j(x')$. Further, in the argument of §1, we have given an explicit construction (1.6) of $F(\zeta)$ along the non-convex part of the boundary $\{0\}\times\partial U''\times\{-r<x_n<r\}$. By this construction we still have $f_j(x')\in{}'\mathcal{B}[\overline{U'}]$, and we can follow the remaining part of §1 as well as the proof of Proposition 2.1, with trivial modifications such as to change $\partial_\pm K$ to $\mathrm{ch}(\partial_\pm K)$. Finally, Theorem 1.10 in [8], which we have cited just above, holds also for a non-convex open set U''. (See the note 1 added in proof of [8]. In the proof of the corresponding Proposition 1.8 in [8], we should then use a non-characteristic hypersurface of the form $x_1=t(x')$ which is no more convex, or choose a bounded domain X' such that $W'\Subset X'\Subset V'$, surrounded by a real analytic hypersurface not necessarily convex. The essential point is, however, that the hyperplanes $x_n=\text{const.}$ can be tangent to e.g. $\partial X'$ only near $x_n=\pm r$. This prevents the propagation of singularity inside W', and allows us to follow the same argument.)

In order to formulate our first main result, let us recall the definition of the following class of operators.

DEFINITION 2.4 (cf. [8], Definition 2.9). Let $\nu'\in R^{n-1}$ be a unit vector. We say that $p(D)$ is *partially $\sqrt{-1}\,\nu'dx'\infty$-semihyperbolic to $x_1<0$ (resp. to $x_1>0$)* if the roots $\tau_k^0(\nu')$, $j=1,\cdots,m$ of the homogeneous characteristic equation $p_m(\zeta_1,\nu')=0$ satisfy

$$\text{(2.18)}\qquad \mathrm{Im}\,\tau_k^0(\nu')\leqslant 0$$

(resp. $\mathrm{Im}\,\tau_k^0(\nu')\geqslant 0$).

A differential operator $p(D)$ which is partially $\sqrt{-1}\,\nu'dx'\infty$-semihyperbolic to $x_1<0$ is partially $-\sqrt{-1}\,\nu'dx'\infty$-semihyperbolic to $x_1>0$ as is easily seen by the substitution $\zeta\longmapsto-\zeta$. Recall also (Lemma 2.10 and its proof in [8]) that when $\nu'=(0,\cdots,0,1)$, the partial $\sqrt{-1}\,dx_n\infty$-semihyperbolicity to $x_1<0$ is equivalent to the inequality (2.2)′ for the roots $\tau_k(\zeta')$ of $p(\zeta_1,\zeta')=0$ or again to the inequality

$$\text{(2.19)}\qquad \mathrm{Im}\,\tau_k(\zeta')\leqslant\varepsilon|\mathrm{Re}\,\zeta_n|+b|\mathrm{Im}\,\zeta_n|+C_{\zeta'',\varepsilon},\quad\text{if}\quad \mathrm{Re}\,\zeta_n>0,$$

for them (maybe with a different constant b).

Summing up the above remarks we obtain

THEOREM 2.5. *Let $p(D)$ be an operator which is partially $\sqrt{-1}\,\nu'\,dx'\infty$-semihyperbolic to $x_1<0$. Let u be a local hyperfunction solution of $p(D)u=0$ on $x_1>0$ which is micro-analytic to the directions $\rho^{-1}(\sqrt{-1}\,\nu' dx'\infty)$. Let $u_j(x')$ be its boundary values to $x_1\to+0$. Then the set*

$$\bigcup_{j=0}^{m-1} \text{S.S.} u_j(x') \cap \{(x', \sqrt{-1}\,\nu' dx'\infty);\ \nu' x' = \text{const.}\} \tag{2.20}$$

cannot be compact. The same conclusion holds for a solution in $x_1<0$ of the operator partially $\sqrt{-1}\,\nu' dx'\infty$-semihyperbolic to $x_1>0$.

In fact, assume that (2.20) is compact. Then choose a system of coordinates of the hyperplane $x_1=0$ in such a way that $\nu'=(0, \cdots, 0, 1)$, and choose a neighborhood of this compact set of the form $U''\times\{-r<x_n<r\}$ such that the boundary values $u_j(x')$ are micro-analytic to the direction $\sqrt{-1}\,dx_n\infty$ along $\partial U''\times\{-r\leqslant x_n\leqslant r\}$. In view of the above remarks we can apply Proposition 2.1 to conclude that (2.20) is void as a matter of fact.

In a similar way, from the latter part of §1 we can obtain the following result a little precise concerning the mode of propagation. This is our second main result. For the sake of simplicity, we will give it in a fixed system of coordinates.

THEOREM 2.6. *Assume that the roots $\tau_k(\zeta')$ of $p(\zeta_1, \zeta')=0$ satisfy*

$$\operatorname{Im} \tau_k(\zeta') \leqslant \varepsilon|\operatorname{Re} \zeta^*| + b|\operatorname{Im} \zeta'| + C_{\zeta_2,\varepsilon}, \quad \text{for} \quad \operatorname{Re} \zeta_n \geqslant c|\operatorname{Re} \zeta''|. \tag{2.21}$$

(Here $\zeta^*=(\zeta_3, \cdots, \zeta_n)$.) Let u be a local hyperfunction solution of $p(D)u=0$ on $x_1>0$, which is micro-analytic to the directions $\rho^{-1}(+\sqrt{-1}\,dx_n\infty)$. Let $u_j(x')$ be its boundary values to $x_1\to+0$. Then the set

$$\bigcup_{j=0}^{m-1} \text{S.S.} u_j(x') \cap \{(x', +\sqrt{-1}\,dx_n\infty);\ x^* = \text{const.}\} \tag{2.22}$$

cannot be compact. The same conclusion holds for a solution in $x_1<0$ if the inequality (2.21) holds for $-\operatorname{Im} \tau_k(\zeta')$.

The proof is similar to that of Theorem 2.5. This time we employ the damping factor

$$'\mathcal{F}^{-1}(\exp(-2c\varepsilon(\sqrt{1+\xi_2^2})^{1/N}(\sqrt{1+\xi'^2})^{1-1/N})).$$

By a construction intermediate to these two situations, we can prove in general the following result.

THEOREM 2.7. *Assume that the roots $\tau_k(\zeta')$ of $p(\zeta_1, \zeta')=0$ satisfy*

$$\operatorname{Im}\tau_k(\zeta')\leqslant\varepsilon(|\operatorname{Re}\zeta_{n_0+1}|+\cdots+|\operatorname{Re}\zeta_n|)+b|\operatorname{Im}\zeta'|+C_{\zeta_2,\dots,\zeta_{n_0},\varepsilon}, \quad \text{for } \operatorname{Re}\zeta_n\geqslant c|\operatorname{Re}\zeta''|, \tag{2.23}$$

for some n_0 $(2\leqslant n_0\leqslant n-1)$. Let u be a local hyperfunction solution of $p(D)u=0$ on $x_1>0$ which is micro-analytic to the directions $\rho^{-1}(+\sqrt{-1}\,dx_n\infty)$. Let $u_j(x')$ be its boundary values to $x_1\to+0$. Then the set

$$\bigcup_{j=0}^{m-1} \text{S. S.}\, u_j(x')\cap\{(x', +\sqrt{-1}\,dx_n\infty);\ x_{n_0+1}=\cdots=x_n=\text{const.}\} \tag{2.24}$$

cannot be compact. The same conclusion holds for a solution in $x_1<0$ if (2.23) *holds for $-\operatorname{Im}\tau_k(\zeta')$.*

Example 2.8. Theorem 2.5 is applicable e. g. to the operator

$$(D_1+\sqrt{-1}\,D_n)(D_1^2-D_4^2+D_2D_n)+D_3^3 \qquad (n\geqslant 5), \tag{2.25}$$

of the general feature. Recall that if the roots $\tau_k^0(\xi')$ satisfy $\operatorname{Im}\tau_k^0(\xi')\leqslant 0$ for ξ' in a neighborhood of ν' instead of the only condition (2.18), then (2.20) always becomes void by virtue of the propagation of micro-analyticity up to the boundary (Theorem 3.7 in [8] or a corresponding work of Schapira). Therefore the interest of Theorem 2.5 lies in the region where some of $\operatorname{Im}\tau_k^0(\nu')$ vanish, that is, in the so called glancing region. Note that the case where $\operatorname{Im}\tau_k^0(\nu')$ all vanish is already covered by Theorem 4.1 of [8]. In view of Theorem 2.7, the conclusion for the example (2.25) can be improved up to the non-compactness of (2.24) for $n_0=3$. Since, however, we do not utilize here the properties of the operator other than the estimate of the roots, Theorem 2.7 is far from the sharp propagation along the boundary bicharacteristic strip when we apply it e. g. to the wave equation to which a precise result is known (see Theorem 4.2 in [8] and the note 2 added in proof thereafter).

The argument of this section relies much on the Fourier analysis. One may think that an argument of more local character would be more satisfactory in relation to the extension to the case of variable coefficients. On this point we will give the following remark.

REMARK 2.9. 1) We can prove the real analyticity of $f_j(x')*E(x', \varepsilon)$ in $\Omega'_\delta\times$

$\{\mathrm{Re}\,\varepsilon>0\}$ by virtue of Green's formula coupling the given solution $u(x)$ and the solution $F_j(x,\varepsilon)$ of the following boundary value problem:

$$(2.26)\quad \begin{cases} {}^t p(D)F_j(x,\varepsilon)=0, \\ D_1^k F_j(x,\varepsilon)|_{x_1\to+0}=\delta_{m-j-1,k}\{J(D_{\omega'})W_0(x',\omega')|_{\omega'=\nu'} * E(x',\varepsilon)\}(y'-x'), \\ \qquad\qquad k=0,\cdots,m-1. \end{cases}$$

Here $W_0(x',\omega')$ denotes the component of the plane wave decomposition of $\delta(x')$, $\nu'=(0,\cdots,0,1)$ and y' denotes the parameters. The defining function of the data is holomorphic on

$$\{(z',\varepsilon)\in \boldsymbol{C}^n;\ \mathrm{Im}\,z_n<0,\ \mathrm{Re}\,\varepsilon>0\},$$

that is, on the half space $y_n<0$ with the parameter ε aside. The solvability of (2.26) for such data under the assumption of partial $-\sqrt{-1}\,dx_n\infty$-semihyperbolicity to $x_1>0$ of ${}^tp$ is just guaranteed by Proposition 2.11 of [8]. On the application of Green's formula, we must fully utilize the micro-analyticity of $u(x)$ to the direction $\sqrt{-1}\,dx_n\infty$ even up to the part $\partial U''\times\{-r\leqslant x_n\leqslant r\}$ of the boundary and the fact that the convolution by $J(D_{\omega'})W_0(x',\omega')|_{\omega'=\nu'} * E(x',\varepsilon)$ causes the propagation of S.S. only along the conormal bundle of $x_n=$const.. The use of the *plane* wave decomposition tends to the constant coefficients. Note, however, that the defining function of the $-\sqrt{-1}\,dx_n\infty$-component of $E(y'-x',\varepsilon)$, which may be obtained e. g. by

$$'\mathcal{F}^{-1}(Y(\xi_n)\exp(-2c\varepsilon(\sqrt{1+|\xi''|^2})^{1/N}(\sqrt{1+|\xi'|^2})^{1-1/N}))(y'-x'),$$

is holomorphic only on the domain of the form

$$\left\{(z',\varepsilon)\in \boldsymbol{C}^n;\ -c_1\,\mathrm{Re}\,\varepsilon<y_n<-\frac{c_2}{(\mathrm{Re}\,\varepsilon)^N}|y''|^{N+1},\ \mathrm{Re}\,\varepsilon>0\right\}.$$

When we forget the parameter ε, this is an infinitesimal half space $y_n<0$ of fairly general type. The problem (2.26) could not be solved in general for such data. (See the example given in pp. 428–429 of [8].) The situation cannot be much improved by the convolution with the component of a *curved* wave decomposition, e. g. (2.15), of $\delta(x')$. Hence the above argument via the Fourier transformation does not seem to be paraphrased by a local argument on the real domain. This is the reason why we have unified the method of this paper by the Fourier analysis.

2) When $N=1$, the equation (2.9) becomes the partial Laplacian lacking D_n. Hence the real analyticity of $f_j(x') * E(x',\varepsilon)$ implies the micro-analyticity of the boundary value $f_j(x')$ to the direction $\sqrt{-1}\,dx_n\infty$, just by the local theory on prop-

agation of micro-analyticity up to the boundary (Schapira [18], Kaneko [8]). When N is general, the equation (2.9) does not seem to have such property. We might, however, utilize instead the "pseudo differential equation"

$$\left\{\frac{\partial}{\partial \varepsilon}+2c(\sqrt{1-\Delta_{x''}})^{1/N}(\sqrt{1-\Delta_{x'}})^{1-1/N}\right\}V(x', \varepsilon)=0.$$

The operation of this operator is global along the conormal bundle of $x_n=\text{const.}$. Therefore we will have to prepare anyway a kind of global theory to treat such an operator. The latter part of the proof of Proposition 2.1 just studies for this operator the propagation of micro-analyticity up to the boundary for a special solution.

§ 3. Applications to the continuation of real analytic solutions.

Now we apply the above results to the continuation of real analytic solutions. We have introduced in [6] a method for proving continuation of real analytic solutions based on the Holmgren type theorem and the propagation of micro-analyticity up to the boundary. First we will improve it in the following form for the full use of our results.

LEMMA 3.1. *Let $p(x, D)$ be a linear partial differential operator with real analytic coefficients. Assume that $x_1=0$ is non-characteristic with respect to p. Let $\varphi(x')=0$ be a non-singular hypersurface of class C^1 in $\boldsymbol{R}^{n-1}_{x'}=\{x_1=0\}$ passing through the origin. Assume that the operator enjoys the following propagation of micro-analyticity along the boundary concerning the local real analytic solution u of $p(x, D)u=0$ on $\pm x_1>0$: If the boundary values $u^{\pm}_j(x')$ of u are real analytic on $\varphi(x')<0$, then they become simultaneously micro-analytic at either of the points $(0, \pm\sqrt{-1}\,d\varphi(x')\infty)\in\sqrt{-1}\,S^*_{\infty}\boldsymbol{R}^{n-1}$. Then every real analytic solution u of $p(x, D)u=0$, defined on a neighborhood of 0 except on $\{x_1=0, \varphi(x')\geqslant 0\}$, can be continued as a hyperfunction solution up to a neighborhood of 0.*

The proof is just the same: Let $[u]_{\pm}$ be the canonical extension of $u|_{\{\pm x_1>0\}}$. Then we have

$$p(x,D)([u]_{+}+[u]_{-})=\sum_{j=0}^{m-1}u_j(x')D_1^{m-1-j}\delta(x_1),$$

where $u_j(x')=u^+_j(x')-u^-_j(x')$ have supports in $\varphi(x')\geqslant 0$. From the assumption of propagation of micro-analyticity, they become micro-analytic at either of $(0, \pm\sqrt{-1}\,d\varphi(x')\infty)$. Thus the Holmgren type theorem shows that $0\notin\operatorname{supp} u_j$, hence that $[u]_+ + [u]_-$ is the required extension of u as a hyperfunction solution.

As the first application we will reproduce Theorem 2.7, 2) in [3] which was obtained by a global process based on convex Fourier analysis.

DEFINITION 3.2. We say that $p(D)$ is *partially* $\sqrt{-1}\nu' dx'\infty$*-hyperbolic* if the roots $\tau_k^0(\nu')$, $k=1, \cdots, m$ of $p_m(\zeta_1, \nu')=0$ satisfy

$$\text{(3.1)} \qquad \operatorname{Im} \tau_k^0(\nu')=0.$$

The partial $\sqrt{-1}\nu' dx'\infty$-hyperbolicity is equivalent to the partial $\sqrt{-1}\nu' dx'\infty$-semihyperbolicity to both sides $\pm x_1>0$. Therefore when $\nu'=(0, \cdots, 0, 1)$ it is equivalent to the following inequality for the roots $\tau_k(\zeta')$ of $p(\zeta_1, \zeta')=0$:

$$\text{(3.2)} \qquad |\operatorname{Im} \tau_k(\zeta')| \leqslant \varepsilon |\operatorname{Re} \zeta_n| + b|\operatorname{Im} \zeta_n| + C_{\zeta'', \varepsilon} \text{ on } \operatorname{Re} \zeta_n>0.$$

THEOREM 3.3. *Let K be the intersection of a convex compact set in the hypersurface $x_1=0$ with its half space $\{\nu' x'<0\}$. Let U be a neighborhood of K in $\boldsymbol{R}^n$. Assume that $p(D)$ is partially $\sqrt{-1}\nu' dx'\infty$-hyperbolic. Then we have*

$$\mathcal{A}_p(U \setminus K)/\mathcal{A}_p(U)=0.$$

In fact, by virtue of Theorem 2.5 we can assert that the difference of the boundary values $u_j(x')=u_j^+(x')-u_j^-(x')$ is micro-analytic everywhere to the direction $\sqrt{-1}\nu' dx'\infty$. Thus we can apply Lemma 3.1 to every hypersurface of class C^1 in $x_1=0$ which is convex below and equal to $\nu' x'=$const. near K. By the method of sweeping out, we conclude that $u_j(x')\equiv 0$, hence that u can be extended to an element of $\mathcal{B}_p(U)$. It remains to recall that the convexity of K guarantees the propagation of interior regularity $\mathcal{B}_p(U) \cap \mathcal{A}(U \setminus K)=\mathcal{A}_p(U)$ (see e.g. Kawai [12], Theorem 5.1.1).

One may notice that the above theorem requires the inequality (3.2) only on $\operatorname{Re} \zeta_n>0$ whereas Theorem 2.7, 2) in [3] requires the same inequality only on $\operatorname{Im} \zeta_n>0$. These restrictions naturally come from the respective methods and might be significant for pseudo-differential operators. Since however we consider here only differential operators, both restrictions are superfluous as is easily seen by the substitution $\zeta \longmapsto -\zeta$.

More generally, our argument gives us the following result.

THEOREM 3.4. *Let L be a compact subset of $\boldsymbol{R}^n$ with $n \geqslant 3$ contained in the hyperplane $\{x_1=0\}$. Let $2 \leqslant n_0 \leqslant n-1$ and put*

$$K=L \cap \{x_{n_0+1}^2+\cdots+x_n^2<1\}.$$

Let U be an open neighborhood of K in $\boldsymbol{R}^n$. Assume that the roots $\tau_k(\zeta')$, $k=1, \cdots, m$ of the equation $p(\zeta_1, \zeta')=0$ satisfy

$$|\mathrm{Im}\,\tau_k(\zeta')| \leqslant \varepsilon(|\mathrm{Re}\,\zeta_{n_0}| + \cdots + |\mathrm{Re}\,\zeta_n|) + b|\mathrm{Im}\,\zeta'| + C_{\zeta_2,\ldots,\zeta_{n_0-1},\varepsilon} \quad \text{for } \zeta' \in \boldsymbol{C}^{n-1}. \tag{3.3}$$

Then the image of the natural mapping

$$\mathcal{A}_p(U\setminus K)/\mathcal{A}_p(U) \to \mathcal{B}_p(U\setminus K)/\mathcal{B}_p(U) \tag{3.4}$$

is zero. That is, a real analytic solution u of $p(D)u=0$ in $U\setminus K$ can be continued (uniquely) to U as a hyperfunction solution.

PROOF. We will prove that $u_j(x')=u_j^+(x')-u_j^-(x')$ all vanish on K. We can apply Theorem 2.7 to our operator $p(D)$ with respect to every micro-direction in

$$\sqrt{-1}\,(\boldsymbol{R}dx_{n_0} + \cdots + \boldsymbol{R}dx_n)\infty \in \sqrt{-1}\,\boldsymbol{S}_\infty^{*n-2}. \tag{3.5}$$

Thus the spacial trace of $\bigcup_{j=0}^{m-1} \mathrm{S.S.}u_j^{\pm}(x')$ with these directions cannot have compact intersection with $x_{n_0}=\cdots=x_n=\text{const}$. Since we have $\text{sing supp } u_j^{\pm}(x') \subset K$ by the assumption and K has compact intersection with such a level variety, we thus conclude that $\mathrm{S.S.}u_j^{\pm}(x')$, hence $\mathrm{S.S.}u_j(x')$ do not contain these directions anywhere in $x_{n_0+1}^2+\cdots+x_n^2<1$.

On these informations we now employ the method of sweeping out combined with the Holmgren type theorem. Consider the following family of real analytic hypersurfaces:

$$S: x_{n_0} = \frac{1}{1-(x_{n_0+1}^2+\cdots+x_n^2)} - \lambda.$$

By what is shown above, $u_j(x')$ are micro-analytic at the conormal elements of S_λ. For $\lambda \ll 0$, S_λ do not touch K. Thus we can sweep out $\mathrm{supp}\, u_j(x')$ to conclude that $u_j(x') \equiv 0$. This establishes the required continuation of u as a hyperfunction solution. q.e.d.

This time the interior propagation of real analyticity cannot be expected in general. Remark that on the other hand we can assert the same conclusion for the subspace of $\mathcal{B}_p(U\setminus K)$ consisting of those which are micro-analytic to the directions (3.5).

The special case $n_0=n-1$ (i.e. the consequence of Theorem 2.7) is the following which improves Theorem 2.1 of [4].

COROLLARY 3.5. *Let L be a compact subset of $\boldsymbol{R}^n$ with $n \geqslant 3$, contained in the hyperplane $\{x_1=0\}$. Put $K=L \cap \{-1<x_n<1\}$. Let U be an open neighborhood of*

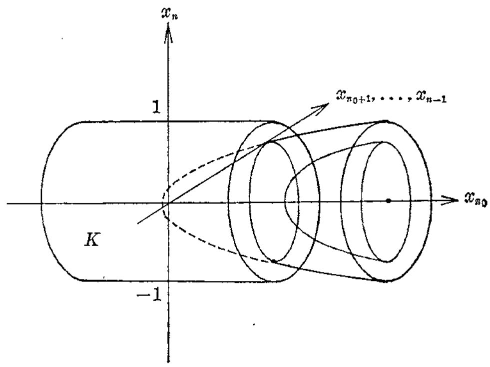

Fig. 3. (x_1, $\cdots$, x_k are not written)

K. Assume that the roots $\tau_k(\zeta')$, $k=1, \cdots, m$ *of the equation* $p(\zeta_1, \zeta')=0$ *satisfy*

(3.6) $$|\mathrm{Im}\,\tau_k(\zeta')| \leqslant \varepsilon(|\mathrm{Re}\,\zeta_{n-1}|+|\mathrm{Re}\,\zeta_n|)+b(|\mathrm{Im}\,\zeta_{n-1}|+|\mathrm{Im}\,\zeta_n|)+C_{\zeta_2,\dots,\zeta_{n-2},\varepsilon}, \quad \text{for } \zeta' \in \boldsymbol{C}^{n-1}.$$

Then the image of the natural mapping

$$\mathcal{A}_p(U\setminus K)/\mathcal{A}_p(U) \to \mathcal{B}_p(U\setminus K)/\mathcal{B}_p(U)$$

is zero.

Recall that in Theorem 2.1 of [4] we assumed the inequality

(3.6)′ $$|\mathrm{Im}\,\tau_k(\zeta')| \leqslant a(|\mathrm{Re}\,\zeta_{n-1}|^q+|\mathrm{Re}\,\zeta_n|^q)+b(|\mathrm{Im}\,\zeta_{n-1}|+|\mathrm{Im}\,\zeta_n|)+c(|\zeta_2|+\cdots+|\zeta_{n-2}|)+C \quad \text{for } \zeta' \in \boldsymbol{C}^{n-1},$$

with $q<1$ (where the constant C is forgot). This one is far stronger than (3.6). The article [4] has been our starting point to the study of the estimation of S. S. of boundary values in connection with the problem of continuation of real analytic solutions. In order to explain the result of [4], we have extracted from it the following conjecture: "Under the assumption on the roots

(3.7) $$\mathrm{Im}\,\tau_k(\zeta') \leqslant a|\mathrm{Re}\,\zeta_n|^q+b|\mathrm{Im}\,\zeta_n|+c|\zeta''|+C, \quad \text{for} \quad \mathrm{Re}\,\zeta_n>0,$$

the boundary values of the local real analytic solutions of $p(D)u=0$ on $x_1>0$ will become micro-analytic to the direction $\sqrt{-1}\,dx_n\infty$".

It is true (see Schapira [18], Kaneko [8]) that in some cases even in the glancing region, the same phenomenon as in the reflective region ($\tau_k^0(\xi') \leqslant 0$ for $\xi' \sim \nu'$) holds, that is, the propagation of micro-analyticity up to the boundary, which is stronger than the phenomenon of the propagation of micro-analyticity along the boundary. Here, however, the basic result for the conjecture is clarified by the latter phenomenon unwillingly to the author's intention. The examination of the condition (3.7) is made in [8] to some extent, but not yet satisfactory.

§ 4. Remarks concerning pseudo-differential operators with constant coefficients.

The results of § 2 as well as our similar ones in the previous papers can be translated with little modification to pseudo-differential operators with constant coefficients. Here we will shortly explain how to formulate it, especially in connection with the Fourier analysis. A more systematic treatment will be given in a forthcoming paper with the preparation of the theory of Fourier hyperfunctions with hyperfunction parameters. Also at the end of this section we will construct a solution of the boundary value problem showing that Theorem 2.5 is best possible in some sense.

The operator which we treat here is of the following form:

(4.1) $$p(D)=D_1^m+a_1(D')D_1^{m-1}+\cdots+a_m(D'),$$

where $a_j(\zeta')$ are (algebraic) functions of ζ', holomorphic on

(4.2) $$\operatorname{Re}\zeta_n\geqslant c(|\operatorname{Re}\zeta''|+1),\quad |\operatorname{Im}\zeta'|\leqslant\frac{1}{c}(|\operatorname{Re}\zeta'|+1),$$

and there satisfy

(4.3) $$|a_j(\zeta')|\leqslant M^j(|\zeta'|+1)^j.$$

It is easily seen that such $p(D)$ defines a pseudo-differential operator in the usual sense on the corresponding neighborhood of the direction $+\sqrt{-1}\,dx_n\infty$. In fact, the inequality (4.3) combined with Cauchy's furnishes the necessary estimate for the derivatives of $a_j(\zeta')$. (See e.g. S-K-K. [16]. In order to neglect the verification of the asymptotic expansion a more wide class given in Kataoka [9] will also be convenient.) Here we will introduce a simple definition of this operator via the partial Fourier transformation.

Put

(4.4) $$A_j(x')=\frac{1}{(2\pi)^{n-1}}\int_E e^{\sqrt{-1}x'\xi'}a_j(\xi')d\xi',$$

where

$$E=\{\xi'\in \boldsymbol{R}^{n-1};\ \xi_n\geqslant c(|\xi''|+1)\}.$$

Lemma 4.1. *$A_j(x')$ is a temperate distribution with the following regularity property:*

(4.5) $$\text{S.S.}A_j(x')\subset\{0\}\times\sqrt{-1}\,\Delta'^{\circ}dx'\infty\cup\boldsymbol{R}^{n-1}\times\sqrt{-1}\,\partial\Delta'^{\circ}dx'\infty,$$

where $\Delta'^\circ=\{\xi'\in \boldsymbol{R}^{n-1};\ \xi_n\geqslant c|\xi''|\}$ *is the dual cone of* $\Delta'=\{y'\in\boldsymbol{R}^{n-1};\ y_n>(1/c)|y'|\}$, *and* $\partial\Delta'^\circ$ *is its boundary.* (*We are denoting* $\Delta'^\circ dx'\infty$ *instead of* $(\Delta'^\circ\cap \boldsymbol{S}^{n-2})dx'\infty$ *etc.*)

In fact, $A_j(x')$ is the boundary value of the holomorphic function $A_j(z')\in\mathcal{O}(\boldsymbol{R}^{n-1}+\sqrt{-1}\Delta')$ which is defined by the same integral (4.4) with z' in place of x'. A routine argument by the deformation of the path at the interior of E shows that it is micro-analytic to the direction $\mathrm{Int}(\Delta'^\circ)$ outside 0. Thus the integral operator $a_j(D')=A_j(x')\underset{x'}{*}$ to the hyperfunctions with compact support gives rise to a micro-local operator in $\boldsymbol{R}^{n-1}\times\mathrm{Int}(\Delta'^\circ)$. Hence the operation of $p(D)=\sum_{j=0}^{m}a_j(D')D_1^{m-j}$ to a hyperfunction u with support compact in x' can be micro-localized in the region $\boldsymbol{R}^n\times\rho^{-1}(\mathrm{Int}(\Delta'^\circ))$, where ρ: $\boldsymbol{S}_\infty^{*n-1}\setminus\{\pm\sqrt{-1}\,dx_1\infty\}\to\boldsymbol{S}_\infty^{*n-2}$ is the projection along the meridians. It is also clear that this operation in the micro-local sense on the very neighborhood of $\boldsymbol{R}^n\times\rho^{-1}(\sqrt{-1}\,dx_n\infty)$ does not depend on the choice of the neighborhood Δ'° of $(0,\cdots,0,1)$ or the corresponding integral region E in (4.4).

DEFINITION 4.2. We say that a hyperfunction u with support in $x_1\geqslant 0$ defined on a neighborhood of 0 is a *solution of* $p(D)u=0$ *at the boundary on* $x_1>0$ if for *a* (in fact *any*) cutting-off of u into $[u]$ with support compact in x' (say $\mathrm{supp}[u]\subset\{x_1\geqslant 0\}\times\overline{U'}$, where U' is a neighborhood of $0\in\boldsymbol{R}^{n-1}$), we have

$$p(D)[u]=\sum_{j=0}^{m-1}f_j(x')D_1^{m-1-j}\delta(x_1)+v, \tag{4.6}$$

where $f_j(x')\in{}'\mathcal{B}[\overline{U'}]$, and the remainder term v is such that

$$\mathrm{supp}\,v\subset\{x_1\geqslant 0\},\quad \mathrm{S.S.}\,v|_{R\times U'}\subset\boldsymbol{R}^n\times(\boldsymbol{S}_\infty^{*n-1}\setminus\rho^{-1}(\mathrm{Int}(\Delta'^\circ))).$$

Remark that $f_j(x')|_{U'}$ gives the boundary values of u as micro-functions, that is, as hyperfunctions which are determined modulo those micro-analytic in the directions $\mathrm{Int}(\Delta'^\circ)$. The uniqueness of the boundary values in this sense follows from the proof of Lemma 1.7 in [8].

Let $W(x',\omega')$ be the component (2.15) of the decomposition of $\delta(x')$. Choose another closed conic neighborhood $\Delta_1'^\circ\Subset\Delta'^\circ$ of $(0,\cdots,0,1)$ and put $W(x',\Delta_1'^\circ)=\int_{S^{n-2}\cap\Delta_1'^\circ}W(x',\omega')d\omega'$. If we convolute $W(x',\Delta_1'^\circ)$ to both sides of (4.6), we will obtain

$$p(D)([u]\underset{x'}{*}W(x',\Delta_1'^\circ))=\sum_{j=0}^{m-1}(f_j(x')*W(x',\Delta_1'^\circ))\,D_1^{m-1-j}\delta(x_1)+v\underset{x'}{*}W(x',\Delta_1'^\circ),$$

where now

$$\mathrm{S.S.}(v*W(x',\Delta_1'^\circ))|_{R\times U'}\subset\boldsymbol{R}^n\times\{\pm\sqrt{-1}\,dx_1\infty\}.$$

Now assume that u is micro-analytic in $\{x_1>0\}\times\rho^{-1}(\mathrm{Int}(\Delta'^{\circ}))$. Then we can cut off the support of $[u]\underset{x'}{*}W(x', \Delta_1'^{\circ})$ in $x_1\leqslant\varepsilon$ in such a way that the modified hyperfunction, denoted by $[[u]]$, satisfies

$$\begin{aligned}&\mathrm{supp}\,[[u]]\subset\{0\leqslant x_1\leqslant\varepsilon\},\\&[[u]]=[u]\underset{x'}{*}W(x', \Delta_1'^{\circ}) \quad \text{in} \quad x_1<\varepsilon/2,\\&\mathrm{S.S.}\,[[u]]|_{\{x_1>0\}\times U'}\subset R^n\times\{\pm\sqrt{-1}\,dx_1\infty\}.\end{aligned}$$

In fact, in this case we have

$$\mathrm{S.S.}[u]\underset{x'}{*}W(x', \Delta_1'^{\circ})|_{\{x_1>0\}\times U'}\subset R^n\times\{\pm\sqrt{-1}\,dx_1\infty\},$$

hence, by the flabbiness of the sheaf of microfunctions we can first choose a hyperfunction v_1 such that

$$\begin{aligned}&\text{sing supp}\, v_1\subset\{0\leqslant x_1\leqslant 3\,\varepsilon/4\}\times\overline{U'},\\&\mathrm{S.S.}v_1|_{\{x_1>0\}\times U'}\subset\{x_1\leqslant 3\,\varepsilon/4\}\times\{\pm\sqrt{-1}\,dx_1\infty\},\\&[u]\underset{x'}{*}W(x', \Delta_1'^{\circ})-v_1 \quad \text{is real analytic in} \quad x_1<3\,\varepsilon/4.\end{aligned}$$

Then it suffices to choose as $[[u]]$ a modification of

$$([u]\underset{x'}{*}W(x', \Delta_1'^{\circ})-v_1)\,Y(\varepsilon/2-x_1)+v_1Y(\varepsilon-x_1)$$

with support in $\{0\leqslant x_1\leqslant\varepsilon\}\times\overline{U'}$ (employing the flabbiness of $\mathcal{B}$!).

Summing up we obtain

$$p(D)[[u]]=\sum_{j=0}^{m-1}(f_j(x')*W(x', \Delta_1'^{\circ}))D_1^{m-1-j}\delta(x_1)+w. \tag{4.7}$$

Cutting the supports again, if necessary, in $\overline{V'}\Subset U'$ in a more delicate way according to the demand of the problem (see e.g. Lemma 1.1 where the use of $\chi_{U''}(x'')$ or $Y(x_1)$ is allowed also in the present case under the same regularity assumption for the boundary values), we can apply the Fourier transformation to both sides. Thus we obtain

$$p(\zeta)\widehat{[[u]]}=\sum_{j=0}^{m-1}\widehat{f_j(x')*W(x', \Delta_1'^{\circ})}\zeta_1^{m-j-1}+\hat{w}(\zeta). \tag{4.8}$$

(Here we have conserved the same notation for the sake of simplicity.) The left hand side disappears when it is restricted to $p(\zeta)=0$. Thus we can deduce the regularity information of $f_j(x')*W(x', \Delta_1'^{\circ})$ from $\hat{w}(\zeta)|_{N(p)}$. Note that for any $W'\Subset V'$ we can decompose $w(x)$ in a way

$$w(x)=w_0(x)+\sum_{k=1}^{N}w_k(x) \tag{4.9}$$

such that

a) w_0 is a rapidly (i.e. exponentially) decreasing real analytic function outside $\{0\leqslant x_1\leqslant\varepsilon\}\times\overline{W'}$ on a conical neighborhood of the real axis, and

$$\text{S. S.}\, w_0\subset \boldsymbol{R}^n\times\{\pm\sqrt{-1}\,dx_1\infty\},$$

b) $\overline{V'}\setminus W'=\bigcup_{k=1}^{N} L'_k$, where $L'_k\subset \boldsymbol{R}^{n-1}$ is compact and w_k is a rapidly decreasing real analytic function outside $\{0\leqslant x_1\leqslant\varepsilon\}\times L'_k$. (If V' is not convex, $\bigcup_{k=1}^{N} L'_k$ may go out into a small neighborhood of $\overline{V'}$.) Here a rapidly decreasing real analytic function on a conical neighborhood of the real axis means a section of the sheaf $\underset{\approx}{\mathcal{O}}$ discussed in Kawai [12] (see pp. 494-495). The construction of this decomposition is carried out in the following way: Let $h\in\mathcal{B}(\boldsymbol{R}^n)$ be such that

$$\text{S.S. } h\subset\overline{\text{S.S. } w|_{\boldsymbol{R}\times W'}},\qquad (w-h)|_{\boldsymbol{R}\times W'}\in\mathcal{A}(\boldsymbol{R}\times W').$$

Then h can be considered as a section with compact support of the quotient sheaf $\mathcal{B}/\mathcal{A}$, which is also equal to $Q/\underset{\approx}{\mathcal{O}}$ on $\boldsymbol{R}^n$, where Q is the sheaf of Fourier hyperfunctions. By the cohomology vanishing property for $\underset{\approx}{\mathcal{O}}$, we can find a global representative in Q which we will again denote by h (see the cited part of [12]). Then h is a section of $\underset{\approx}{\mathcal{O}}$ outside $\{0\leqslant x_1\leqslant\varepsilon\}\times\overline{W'}$, $\text{S.S.}\,h\subset\boldsymbol{R}^n\times\{\pm\sqrt{-1}\,dx_1\infty\}$ and $w-h$ is a section of $\underset{\approx}{\mathcal{O}}$ outside $\{0\leqslant x_1\leqslant\varepsilon\}\times(\overline{V'}\setminus W')$. Further decomposition of $w-h$ conformed to that of $\overline{V'}\setminus W'$ is similar.

Concerning the estimate for $\hat{w}_k(\zeta)$ recall the following Lemma due to Kawai: ([12], Lemma 5.1.2. See also Remark to Lemma 2.3 of Kaneko [4].)

LEMMA 4.3. *Let $K\subset\boldsymbol{R}^n$ be convex compact. For $u\in\mathcal{B}(\boldsymbol{R}^n)$ the following are equivalent.*

a) *u extends to a section of $\underset{\approx}{\mathcal{O}}$ outside K.*

b) *The Fourier transform $\hat{u}(\zeta)$ is holomorphic on a conical neighborhood of the real axis. Further, for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that on $|\mathrm{Im}\,\zeta|\leqslant\delta(|\mathrm{Re}\,\zeta|+1)$, $\hat{u}(\zeta)$ satisfies, for every $\gamma>0$,*

$$|\hat{u}(\zeta)|\leqslant C_\gamma e^{\gamma|\zeta|+H_K(\mathrm{Im}\,\zeta)+\varepsilon|\mathrm{Im}\,\zeta|}. \tag{4.10}$$

Remark also that for any $\gamma>0$ there exist $\delta=\delta(\gamma)$, $\delta'=\delta'(\gamma)>0$ such that $\hat{w}_0(\zeta)$ satisfies

$$|\hat{w}_0(\zeta)|\leqslant C\exp(\varepsilon\max\{\mathrm{Im}\,\zeta_1, 0\}+H_{\overline{W'}}(\mathrm{Im}\,\zeta')+\gamma|\mathrm{Im}\,\zeta|-\delta|\mathrm{Re}\,\zeta|) \tag{4.11}$$

on the domain

$$|\mathrm{Im}\,\zeta|\leqslant\delta'(|\mathrm{Re}\,\zeta|+1),\quad |\mathrm{Re}\,\zeta_1|\leqslant 3M(|\mathrm{Re}\,\zeta'|+1)$$

where M is the constant in (4.3). This follows from the properties of w_0 listed up in a) above. (See the proof of Lemma 2.3 in Kaneko [6].) Since $|\zeta_1|\leqslant 2M(|\zeta'|+1)$ holds on $N(p)$, (4.11) certainly holds on $N(p)$. Remark that we can voluntarily choose $\delta'\ll\delta$.

Shortly speaking, we are pursuing the argument of §1 with the notion of support replaced by the support modulo $\underset{\approx}{\mathcal{O}}$. Accordingly, we consider the objects in the Fourier image on the following complex domain instead of the whole $\boldsymbol{C}^n$:

$$\text{(4.12)}\qquad \{\zeta\in\boldsymbol{C}^n;\ |\zeta_1|\leqslant 2M(|\zeta'|+1),\quad \mathrm{Re}\,\zeta_n\geqslant c(|\mathrm{Re}\,\zeta''|+1),\ |\mathrm{Im}\,\zeta'|\leqslant\frac{1}{c}(|\mathrm{Re}\,\zeta'|+1)\}.$$

(Here $c>0$ may have been replaced by a greater constant.) The remaining part goes parallelly and we can obtain results corresponding to Theorems 2.5–2.7 for the pseudo-differential operators $p(D)$ of the type (4.1)–(4.3). The Fundamental Principle to such a function $p(\zeta)$ on a domain of the form (4.12) will be proved in the same way as the usual one for a polynomial $p(\zeta)$ on the whole space. We need here, however, only its trivial part.

REMARK 4.4. The argument becomes even easier than §1, because we need not employ $J(D')$ in the step corresponding to Lemma 1.4. In fact the use of $\underset{\approx}{\mathcal{O}}$ instead of C_0^∞ cutting-off functions gives the factor $e^{-\delta|\mathrm{Re}\,\zeta|}$ in the estimate (4.11), which allows us to deduce the necessary estimate implying the real analyticity for the results of the substitution of the roots $\zeta_1=\tau_k(\zeta')$ of $p(\zeta_1,\zeta')=0$. This way can be considered as an alternative proof even for the differential operator $p(D)$. On the other hand, in the present case we cannot remove the final ambiguity term w as in the proof of Lemma 1.3 in [8], because our operator $p(D)$ (and even $\sum_{j=0}^{m}a_j(D')W(D',\Delta_1'^{\circ})D_1^{m-j}$) is not defined on the very neighborhood of $\pm\sqrt{-1}\,dx_1\infty$.

To conclude this section we will construct an example showing that Theorem 2.5 is best possible.

PROPOSITION 4.5. *Let $p(D)$ be a differential operator with respect to which $x_1=0$ is non-characteristic (or a pseudo-differential operator of the type (4.1)–(4.3)). Assume that it is not partially $\sqrt{-1}\,\nu'dx'\infty$-semihyperbolic to $x_1<0$. Then there exists a local hyperfunction solution u of $p(D)u=0$ at the boundary on $x_1>0$ satisfying the following conditions:*

a) *u is real analytic on $x_1>0$.*

b) *$\mathrm{S.S.}\,u_j\subset\{0\}\times\{\sqrt{-1}\,\nu'dx'\infty\}$ for $\forall j$, and the equality holds for some j,*

where u_j are the boundary values of u.
That is, in this case the propagation of micro-analyticity to the direction $\sqrt{-1}\,\nu' dx' \infty$ does not hold in any sense along the boundary.

PROOF. Choose $\nu'=(0, \cdots, 0, 1)$. The assumption implies that the equation $p_m(\zeta_1, \nu')=0$ for ζ_1 has a root $\tau_k^0(\nu')$ such that $\operatorname{Im}\tau_k^0(\nu')>0$. Grouping such roots if this one is not simple, we can obtain a pseudo-differential factor $q(\zeta)$ (say, of order $\mu\leqslant m$) of $p(\zeta)$ of the type (4.1)-(4.3) such that all the roots $\tau_k(\xi')$, $k=1, \cdots, \mu$, of $q(\zeta_1, \zeta')=0$ for ζ_1 satisfy

$$\operatorname{Im}\tau_k(\zeta')\geqslant\frac{1}{c}|\xi'| \quad \text{for } \xi'\in E=\{\xi_n\geqslant C(|\xi''|+1)\},$$

hence

$$\operatorname{Im}\tau_k(\zeta')\geqslant\frac{1}{2c}|\operatorname{Re}\zeta'| \text{ for } \operatorname{Re}\zeta_n\geqslant c(|\operatorname{Re}\zeta''|+1),\ |\operatorname{Im}\zeta'|\leqslant\delta|\operatorname{Re}\zeta'|, \tag{4.13}$$

for a suitable constant c. Put $\Delta'^{\circ}=\{\xi_n\geqslant c|\xi''|\}$. Thus $q(D)$ is an elliptic pseudo-differential operator in $\boldsymbol{R}^n\times\rho^{-1}(\operatorname{Int}(\Delta'^{\circ}))$ for which the boundary value problem is micro-locally solvable to $x_1>0$. The construction from now on is routine. We will shortly indicate it. Recall $W(x', \nu')$ as in (2.15), and consider the following Cauchy problem for the ordinary differential equation with parameters:

$$\begin{cases} q(D_{x_1}, \xi')\hat{u}=0, \\ D_{x_1}^k\hat{u}|_{x_1=0}=0, \quad 0\leqslant k\leqslant\mu-2, \\ D_{x_1}^{\mu-1}\hat{u}|_{x_1=0}=\hat{W}(\xi', \nu')\chi_E(\xi'). \end{cases}$$

Here $\hat{W}(\xi', \nu')$ is the Fourier transform of $W(x', \nu')$ and $\chi_E(\xi')$ is the characteristic function of E. The solution $\hat{u}(x_1; \xi')$ is given by the well known formula

$$\hat{u}(x_1;\ \xi')=\frac{\begin{vmatrix} 1 & \cdots & 1 \\ \tau_1(\xi') & \cdots & \tau_\mu(\xi') \\ & \cdots\cdots & \\ e^{\sqrt{-1}\tau_1(\xi')x_1} & \cdots & e^{\sqrt{-1}\tau_\mu(\xi')x_1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ \tau_1(\xi') & \cdots & \tau_\mu(\xi') \\ & \cdots\cdots & \\ \tau_1(\xi')^{\mu-1} & \cdots & \tau_\mu(\xi')^{\mu-1} \end{vmatrix}}\hat{W}(\xi', \nu')\chi_E(\xi').$$

(The right hand side can be given a suitable meaning even for ξ' giving multiple roots.) Owing to (4.13), $\hat{u}(x_1;\ \xi')$ is exponentially decaying in ξ' for $x_1>0$. Hence the inverse partial Fourier transform $[u]$ of $Y(x_1)\hat{u}(x_1;\ \xi')$ gives a real analytic function on $x_1>0$, which satisfies the following equality (even in the distribution sense):

$$q(D)[u]=a\cdot{}'\mathcal{F}^{-1}(\hat{W}(\xi',\nu')\chi_E(\xi'))\delta^{(\mu-1)}(x_1), \tag{4.14}$$

a being a non-zero constant. The technique employed in the proof of Lemma 4.1 shows that

$$\text{S. S.}'\mathcal{F}^{-1}(\hat{W}(\xi',\nu')\chi_E(\xi'))=\{0\}\times\{\sqrt{-1}\nu' dx'\infty\}.$$

Finally we let the remaining factor operate to both sides of (4.14) to obtain $p(D)$ in the left. Then to the right hand side appear the boundary values which are more complicated but anyway whose S. S. are contained in $\{0\}\times\{\sqrt{-1}\nu' dx'\infty\}$ and in total containing this set. In order to verify this, the cutting off of support with respect to x' is not necessary: We can simply make this composition in the Fourier image, where the factor $\chi_E(\xi')$ restricts the operator to the domain on which the decomposition takes place with regular coefficients.

References

[1] Ivrii, V. Ja., Propagation of singularities of the solution of the wave equation in a domain having angular points, Dokl. Acad. Nauk USSR **241** (1978), 536-539; and other papers by the same author.

[2] Kaneko, A., Representation of hyperfunctions by measures and some of its applications, J. Fac. Sci. Univ. Tokyo Sect. IA **19** (1972), 321-352.

[3] Kaneko, A., On continuation of regular solutions of partial differential equations with constant coefficients, J. Math. Soc. Japan **26** (1974), 92-123.

[4] Kaneko, A., On linear exceptional sets of solutions of linear partial differential equations with constant coefficients, Publ. Res. Inst. Math. Sci. Kyoto University **11** (1976), 441-460.

[5] Kaneko, A., Remarks on hyperfunctions with analytic parameters, J. Fac. Sci. Univ. Tokyo Sect. IA **22** (1975), 371-407.

[6] Kaneko, A., On the singular spectrum of boundary values of real analytic solutions, J. Math. Soc. Japan **29** (1977), 385-398.

[7] Kaneko, A., Singular spectrum of boundary values of solutions of partial differential equations with real analytic coefficients, Sci. Papers College Gen. Ed. Univ. Tokyo **25** (1975), 59-68.

[8] Kaneko, A., Estimation of singular spectrum of boundary values for some semihyperbolic operators, J. Fac. Sci. Univ. Tokyo Sect. IA **27** (1980), 401-461.

[9] Kataoka, K., On the theory of Radon transformations of hyperfunctions, J. Fac. Sci. Univ. Tokyo Sect. IA **28** (1981), 331-413.

[10] Kataoka, K., Micro-local theory of boundary value problems I, J. Fac. Sci. Univ. Tokyo Sect. IA **27** (1980), 355-399.
[11] Kataoka, K., Micro-local theory of boundary value problems II, J. Fac. Sci. Univ. Tokyo Sect. IA **28** (1981), 31-56.
[12] Kawai, T., On the theory of Fourier hyperfunctions and its applications to partial differential equations with constant coefficients, J. Fac. Sci. Univ. Tokyo Sect. IA **17** (1970), 467-517.
[13] Komatsu, H., A local version of Bochner's tube theorem, J. Fac. Sci. Univ. Tokyo Sect. IA **19** (1972), 201-214.
[14] Melrose, R. B. and J. Sjöstrand, Singularities of boundary value problems I, Comm. Pure Appl. Math. **31** (1978), 593-617.
[15] Nirenberg, L., Lectures on Linear Partial Differential Equations, Regional Conference Series in Mathematics 20, Amer. Math. Soc., Providence, Rhode Island (1973).
[16] Sato, M., Kawai, T. and M. Kashiwara (S-K-K), Microfunctions and pseudo-differential equations, Lecture Notes in Math. 287, Springer, 1973, pp. 263-529.
[17] Schapira, P., Propagation at the boundary and reflection of analytic singularities of solutions of linear partial differential equations I, Proc. Oji Seminar, Publ. Res. Inst. Math. Sci. Kyoto University **12** Suppl. (1977), 441-453.
[18] Schapira, P., Propagation au bord et reflexion des singularités analytiques des solutions des équations aux dérivées partielles II, Séminaire Goulaouic-Schwartz, 1976/7, Exposé N°9.
[19] Sjöstrand, J., Sur la propagation des singularités analytiques pour certains problèmes de Dirichlet, C.R. Acad. Sci. Paris **288** Sér A (1979), 673-674.
[20] Wakabayashi, S., Microlocal parametrices for hyperbolic mixed problems in the case where boundary waves appear, Publ. Res. Inst. Math. Sci. Kyoto University **14** (1978), 283-307.
[21] Andersson, K. G. and R. B. Melrose, The propagation of singularities along gliding rays, Invent. Math. **41** (1977), 197-232.
[22] Eskin, G., A parametrix for interior mixed problems for strictly hyperbolic equations, J. Analyse Math. **32** (1977), 17-62.
[23] Kaneko, A., On continuation of real analytic solutions of linear partial differential equations, Astérisque, to appear.
[24] Sjöstrand, J., Propagation of analytic singularities for second order Dirichlet problems, Comm. Partial Differential Equations **5** (1980), 41-94.
[25] Sjöstrand, J., Propagation of analytic singularities for second order Dirichlet problems II, Comm. Partial Differential Equations **5** (1980), 187-207.

(Received May 9, 1981)

Department of Mathematics
College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan