Scattering theory for Schrödinger equations with time-dependent potentials of long-range type

By Hitoshi KITADA

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§1. Main results.

In this paper we shall discuss the scattering problem by a time-dependent potential $V(t, x)$ of long-range type satisfying the following Assumption (A):

ASSUMPTION (A)

i) For each $t \in R^1$, $V(t, x)$ is a real-valued C^∞-function of $x \in R^n$.

ii) For any multi-index α, $\partial_x^\alpha V(t, x)$ is continuous in $(t, x) \in R^1 \times R^n$.

iii) There exists a positive constant ε such that for any multi-index α with $|\alpha| \neq 0$

$$|\partial_x^\alpha V(t, x)| \leqq C_\alpha \langle t\rangle^{-|\alpha|-\varepsilon}, \tag{1.1}$$

where $\langle t\rangle = \sqrt{1+t^2}$ and the constant $C_\alpha > 0$ is independent of $(t, x) \in R^1 \times R^n$. We denote by $H(t)$ the self-adjoint realization of the Schrödinger operator $-(1/2)\Delta + V(t, x)\times$ in $L^2(R^n)$, where $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$ denotes the Laplacian. According to Kato's theorem [6], [7] generalized by Kobayasi [11] and Yagi [15], [16], there exists a family of unitary operators $U(t, s)$ $(t, s \in R^1)$ in $L^2(R^n)$ satisfying the following properties a)～d):

a) $U(t, s)$ is strongly continuous in $(t, s) \in R^2$.

b) $U(t, r)U(r, s) = U(t, s)$, $\quad U(s, s) = I \quad (t, r, s \in R^1)$.

c) Let $Y = \{f \in L^2(R^n) \mid \|f\|_Y = (f, f)_Y^{1/2} < \infty\}$ be a Hilbert space with the inner product

$$(f, g)_Y = \sum_{|\alpha|+l \leq 2} \int \langle x\rangle^{2l} \partial_x^\alpha f(x) \overline{\partial_x^\alpha g(x)}\, dx, \qquad \langle x\rangle = \sqrt{1+|x|^2}. \tag{1.2}$$

Then $U(t, s) Y \subset Y$ for any $t, s \in R^1$, and $U(t, s)f$ $(f \in Y)$ is strongly continuous in $(t, s) \in R^2$ as an $L^2(R^n)$-valued function.

d) For any $f \in Y$, $U(t, s)f$ is strongly continuous differentiable in $L^2(R^n)$ in t and s, and

(1.3) $$(d/dt)U(t,s)f=-iH(t)U(t,s)f,\quad (d/ds)U(t,s)f=iU(t,s)H(s)f.$$

(The existence of such $U(t,s)$ is also proved by Fujiwara [2], [3], Kitada [8], [9] and Kitada and Kumano-go [10] for the special case of Schrödinger equations.)

To state our main results, we need to define the modified free propagator $U_D^{\pm}(t)$. For a sufficiently large $T>0$ we shall in Section 3 construct the unique solution $\phi(s,t;x,\xi)$ $(t\geqq s\geqq T, x, \xi\in R^n)$ of the Hamilton-Jacobi equation

(1.4) $$\begin{cases}\partial_t\phi(s,t;x,\xi)-H(t,\nabla_\xi\phi(s,t;x,\xi),\xi)=0,\\ \phi(s,s;x,\xi)=x\cdot\xi,\end{cases}$$

where $H(t,x,\xi)=(1/2)|\xi|^2+V(t,x)$. Using this ϕ we define the modified free propagator $U_D^{+}(t)$ by

(1.5) $$U_D^{+}(t)=\mathcal{F}^{-1}[e^{-i\phi(T,t;0,\xi)}\times]\mathcal{F},$$

where $\mathcal{F}$ denotes the Fourier transformation in $L^2(R^n)$:

(1.6) $$\mathcal{F}f(\xi)=\hat{f}(\xi)\equiv\operatorname*{s-lim}_{N\to\infty}\int_{|x|\leqq N}e^{-ix\cdot\xi}f(x)dx,\quad x\cdot\xi=\sum_{j=1}^{n}x_j\xi_j.$$

For $t\leqq s\leqq -T$ we can similarly construct $\phi(s,t;x,\xi)$ and $U_D^{-}(t)$. Then our main results are as follows.

THEOREM 1.1. *Let Assumption* (A) *be satisfied. Then there exist the limits (called "modified wave operators")*

(1.7) $$W_D^{\pm}=\operatorname*{s-lim}_{t\to\pm\infty}U(t,0)^*U_D^{\pm}(t),$$

and they are unitary operators in $L^2(R^n)$. *Thus the scattering operator* S_D *defined by* $S_D=(W_D^{+})^*W_D^{-}$ *is unitary in* $L^2(R^n)$.

THEOREM 1.2. *Let Assumption* (A) *be satisfied. Define for* $f\in L^2(R^n)$

(1.8) $$V^{\pm}(t)f(x)=(2\pi it)^{-n/2}e^{i\psi^{\pm}(t,x)}\hat{f}(x/t),\quad t\gtreqless\pm T.$$

Here $\psi^{\pm}(t,x)$ *are defined by*

(1.9) $$\psi^{\pm}(t,x)=x\cdot\xi_c^{\pm}-\phi(\pm T,t;0,\xi_c^{\pm}),\quad t\gtreqless\pm T,$$

where $\xi_c^{\pm}=\xi_c^{\pm}(t,x)$ *are the solutions of*

(1.10) $$x=\nabla_\xi\phi(\pm T,t;0,\xi_c^{\pm}(t,x)),\quad t\gtreqless\pm T.$$

Then for any $f\in L^2(R^n)$ *we have*

(1.11) $$\lim_{t\to\pm\infty}\|U(t,0)f-V^{\pm}(t)W_D^{\pm *}f\|=0.$$

Therefore the probability density of $U(t,0)f$ *converges asymptotically to* $|2\pi t|^{-n} \times |(\mathcal{F} W_D^{\pm *} f)(x/t)|^2$ *as* $t\to\pm\infty$, *hence* $U(t,0)f$ *behaves like a free state at* $t\to\pm\infty$.

Theorem 1.1 asserts that the modified wave operators are complete. Theorem 1.2 gives an asymptotic behavior as $t\to\pm\infty$ of the solution $u(t)=U(t,0)f$ of the Schrödinger equation

$$\frac{1}{i}\frac{du}{dt}(t)+H(t)u(t)=0,\ u(0)=f\ (\in L^2(R^n)). \tag{1.12}$$

Furthermore Theorem 1.2 justifies the name "modified wave operators" for $W_D^{\pm}$, together with the equation (1.4) which is the same as in Hörmander [5] used to construct the modified wave operators for time-independent long-range potentials.

As to the decay rate as $t\to\pm\infty$ of the potential, our Assumption (A) is weaker than that studied by Howland [4, §4]. Furthermore the following examples are covered by (A).

Example 1.3. Let $\chi\in C^\infty(R^n)$ satisfy $0\leqq\chi(x)\leqq 1$ and $\chi(x)=1$ ($|x|\geqq 2\delta$), $=0$ ($|x|\leqq\delta$) for some $\delta>0$. Let $b_j(t), c(t)\in C^0(R^1)$ be real-valued and satisfy $|b_j(t)|\leqq C\langle t\rangle^{-1-\varepsilon}$ ($j=1,\dots,n$) for some constant $\varepsilon>0$ and $C>0$, and put $B(t,x)=\sum_{j=1}^n b_j(t)x_j+c(t)$. Let $q(x)\in C^\infty(R^n)$ be real-valued and satisfy for any α

$$|\partial_x^\alpha q(x)|\leqq C\langle x\rangle^{-m(|\alpha|)} \tag{1.13}$$

for some sequence $\{m(k)\}_{k=0}^\infty$. Then $V(t,x)$ in the following examples satisfies Assumption (A):

i) $V(t,x)=\chi(\langle t\rangle^{-1}x)q(x)+B(t,x)$, $m(k)=k+\varepsilon_0$, $\varepsilon_0>0$.

ii) $V(t,x)=q(\langle t\rangle^{-a}x)+B(t,x)$, $a>1$, $m(k)=0$ $(k\geqq 1)$, $m(0)\in R^1$.

iii) $V(t,x)=\chi(\langle t\rangle^{-1}x)q(\langle t\rangle^{-a}x)+B(t,x)$, $a<1$, $m(k)=k+\varepsilon_0$, $\varepsilon_0>0$.

iv) $V(t,x)=\chi(x)q(\langle t\rangle^{-a}x)+B(t,x)$, $a<-1/\varepsilon_0$, $m(k)=k+\varepsilon_0$, $\varepsilon_0>0$.

v) $V(t,x)=\langle t\rangle^{-\varepsilon_0}q(\langle t\rangle^{-1}x)+B(t,x)$, $m(k)=0$ $(k\geqq 1)$, $\varepsilon_0>0$.

The case ii) covers some of the time-dependent potentials of long-range type refered to in Kuroda and Morita [14, §4] (see Remark 4.2 below).

Our proof uses the approximate fundamental solution, expressed globally in time as a Fourier integral operator, of the Schrödinger equation (1.12), and is completely time-dependent. In this respect our proof seems to be new compared with the existing proof of the completeness.

In the following we confine ourselves to considering only W_D^+, since W_D^- can be dealt with similarly.

§ 2. Classical orbits.

Let $(q(t, s; x, \xi),\ p(t, s; x, \xi))$ be the solution of the Hamilton equation

$$(2.1)\qquad \begin{cases} \dfrac{dq}{dt}(t, s)=p(t, s), \\ \dfrac{dp}{dt}(t, s)=-\nabla_x V(t, q(t, s)) \end{cases}$$

with the initial condition

$$(2.2)\qquad q(s, s)=x, \quad p(s, s)=\xi,$$

where $\nabla_x={}^t(\partial_{x_1}, \dots, \partial_{x_n})$, $\partial_{x_j}=\partial/\partial x_j$. The equation (2.1)-(2.2) is equivalent to the integral equation

$$(2.3)\qquad \begin{cases} q(t, s)=x+\displaystyle\int_s^t p(\tau, s)d\tau, \\ p(t, s)=\xi-\displaystyle\int_s^t \nabla_x V(\tau, q(\tau, s))d\tau. \end{cases}$$

By successive approximation this equation can be easily solved and we have the following proposition.

PROPOSITION 2.1. *Let Assumption* (A) *be satisfied. Then there exists a unique solution of* (2.3). *The solution* $(q, p)(t, s; x, \xi)$ *is* C^∞ *in* (x, ξ) *for each* $(t, s) \in R^2$ *and its derivative* $\partial_\xi^\alpha \partial_x^\beta (q, p)(t, s; x, \xi)$ *is* C^1 *in* $(t, s; x, \xi)$. *Furthermore there exist positive constants* T_0 *and* C_0 *such that the following estimates hold:*

i) *For any* $t \geqq s \geqq T_0$ *and* $x, \xi \in R^n$

$$(2.4)\qquad \begin{cases} |q(s, t; x, \xi)-x|+|q(t, s; x, \xi)-x| \leqq C_0(t-s)(\langle s\rangle^{-\varepsilon}+|\xi|), \\ |p(s, t; x, \xi)-\xi|+|p(t, s; x, \xi)-\xi| \leqq C_0\langle s\rangle^{-\varepsilon}; \end{cases}$$

$$(2.5)\qquad \begin{cases} |\nabla_x q(s, t; x, \xi)-I| \leqq C_0\langle s\rangle^{-\varepsilon}, \quad |\nabla_x q(t, s; x, \xi)-I| \leqq C_0(t-s)\langle s\rangle^{-1-\varepsilon}, \\ |\nabla_x p(s, t; x, \xi)|+|\nabla_x p(t, s; x, \xi)| \leqq C_0\langle s\rangle^{-1-\varepsilon}; \end{cases}$$

$$(2.6)\qquad \begin{cases} |\nabla_\xi q(s, t; x, \xi)-(s-t)I| \leqq C_0(t-s)\langle s\rangle^{-\varepsilon}, \\ |\nabla_\xi p(s, t; x, \xi)-I| \leqq C_0(t-s)\langle s\rangle^{-1-\varepsilon}; \end{cases}$$

and

$$(2.7)\qquad \begin{cases} |\nabla_\xi q(t, s; x, \xi)-(t-s)I| \leqq C_0(t-s)\langle s\rangle^{-\varepsilon}, \\ |\nabla_\xi p(t, s; x, \xi)-I| \leqq C_0\langle s\rangle^{-\varepsilon}, \end{cases}$$

where I *denotes the* $n\times n$ *identity matrix.*

ii) *For any* α, β *with* $|\alpha+\beta|\geqq 2$, *there is a constant* $C_{\alpha,\beta}$ *independent of* $t\geqq s(\geqq T_0)$ *and* x, ξ *such that*

$$\begin{cases} |\partial_\xi^\alpha \partial_x^\beta q(t, s; x, \xi)| \leqq C_{\alpha,\beta}(t-s)\langle s\rangle^{-\varepsilon}, \\ |\partial_\xi^\alpha \partial_x^\beta p(t, s; x, \xi)| \leqq C_{\alpha,\beta}\langle s\rangle^{-\varepsilon}. \end{cases} \tag{2.8}$$

iii) *More precisely we have for any* $t\geqq s\geqq T_0$

$$|\partial_\xi^\alpha(q(t, s; x, \xi)-x-(t-s)p(t, s; x, \xi))| \leqq C_0 \min\{\langle t\rangle^{1-\varepsilon},\ (t-s)\langle s\rangle^{-\varepsilon}\}\ \ for\ |\alpha|\leqq 1. \tag{2.9}$$

PROOF. The existence and smoothness of the solution $(q, p)(t, s; x, \xi)$ and the estimate (2.4) are easily seen by successive approximation from (2.3). Differentiating (2.3) with respect to x and interchanging t and s, we have for $t\geqq s$

$$\begin{cases} \nabla_x q(s,t) = I + \displaystyle\int_t^s \nabla_x p(\tau, t)d\tau, \\ \nabla_x p(s, t) = -\displaystyle\int_t^s \vec{\nabla}_x \nabla_x V(\tau, q(\tau, t))\nabla_x q(\tau, t)d\tau, \end{cases} \tag{2.10}$$

where $\vec{\nabla}_x = {}^t\nabla_x$. From this we obtain the estimate (2.5) for $(\nabla_x q, \nabla_x p)(s, t; x, \xi)$ by successive approximation if we note $t\geqq s$ and $|\vec{\nabla}_x \nabla_x V(\tau, q(\tau, t))| \leqq C\langle\tau\rangle^{-2-\varepsilon}$ by Assumption (A)-iii). Similarly we can prove (2.5)-(2.7). Then (2.8) is proved by induction similarly by virtue of the equality

$$\begin{cases} \partial_\xi^\alpha \partial_x^\beta q(t, s) = \displaystyle\int_s^t \partial_\xi^\alpha \partial_x^\beta p(\tau, s)d\tau, \\ \partial_\xi^\alpha \partial_x^\beta p(t, s) = -\displaystyle\int_s^t \partial_\xi^\alpha \partial_x^\beta \{\nabla_x V(\tau, q(\tau, s))\}d\tau, \end{cases}$$

which holds for $|\alpha+\beta|\geqq 2$. We next prove iii). Writing $p(\infty, s) = p(\infty, s; x, \xi) = \lim_{t\to\infty} p(t, s; x, \xi) = \xi - \int_s^\infty \nabla_x V(\tau, q(\tau, s))d\tau$, we have using (2.3) and Assumption (A)-iii),

$$\begin{aligned} &|q(t, s; x, \xi)-x-(t-s)p(t, s; x, \xi)| \\ &\leqq |q(t, s)-x-(t-s)p(\infty, s)| + (t-s)|p(\infty, s)-p(t, s)| \\ &\leqq \int_s^t \left|\int_\tau^\infty \nabla_x V(\sigma, q(\sigma, s))d\sigma\right| d\tau + (t-s)\left|\int_t^\infty \nabla_x V(\tau, q(\tau, s))d\tau\right| \\ &\leqq C\int_s^t \langle\tau\rangle^{-\varepsilon}d\tau + C(t-s)\langle t\rangle^{-\varepsilon} \leqq C_0 \min\{\langle t\rangle^{1-\varepsilon}, (t-s)\langle s\rangle^{-\varepsilon}\}. \end{aligned} \tag{2.11}$$

Similarly writing $\nabla_\xi p(\infty, s) = \nabla_\xi p(\infty, s; x, \xi) = \lim_{t\to\infty} \nabla_\xi p(t, s; x, \xi) = I - \int_s^\infty \vec{\nabla}_x \nabla_x V(\tau, q(\tau, s)) \times \nabla_\xi q(\tau, s)d\tau$, we have using (2.7) and Assumption (A)-iii),

$$
\begin{aligned}
&|\nabla_\xi q(t,s)-(t-s)\nabla_\xi p(t,s)|\\
&\leqq |\nabla_\xi q(t,s)-(t-s)\nabla_\xi p(\infty,s)|+(t-s)|\nabla_\xi p(\infty,s)-\nabla_\xi p(t,s)|\\
&\leqq \int_s^t |\nabla_\xi p(\tau,s)-\nabla_\xi p(\infty,s)|d\tau+(t-s)\int_t^\infty |\vec{\nabla}_x\nabla_x V(\tau,q(\tau,s))\nabla_\xi q(\tau,s)|d\tau\\
&\leqq C_0\min\{\langle t\rangle^{1-\varepsilon},(t-s)\langle s\rangle^{-\varepsilon}\}. \quad \square
\end{aligned}
\tag{2.12}
$$

From this proposition we can easily get the following important proposition.

PROPOSITION 2.2. *Let Assumption* (A) *be satisfied. Take* $T>T_0$ *so large that* $C_0\langle T\rangle^{-\varepsilon}<1/2$ *for the constant* C_0 *in Proposition* 2.1. *Then for* $t\geqq s\geqq T$ *there exist the inverse* C^∞ *diffeomorphisms* $x\mapsto y(s,t;x,\xi)$ *and* $\xi\mapsto\eta(t,s;x,\xi)$ *of the mappings* $y\mapsto x=q(s,t;y,\xi)$ *and* $\eta\mapsto\xi=p(t,s;x,\eta)$, *respectively. These mappings* y *and* η *are* C^∞ *in* (x,ξ) *for each* $t\geqq s(\geqq T)$ *and their derivatives* $\partial_\xi^\alpha\partial_x^\beta y$ *and* $\partial_\xi^\alpha\partial_x^\beta\eta$ *are* C^1 *in* $(t,s;x,\xi)$. *Furthermore* y *and* η *satisfy the following properties*:

i) $q(s,t;y(s,t;x,\xi),\xi)=x,\quad p(t,s;x,\eta(t,s;x,\eta))=\xi.$

ii) $$\begin{cases} q(t,s;x,\eta(t,s;x,\xi))=y(s,t;x,\xi),\\ p(s,t;y(s,t;x,\xi),\xi)=\eta(t,s;x,\xi).\end{cases}$$

iii) *There exists a constant* $C_1>0$ *such that for any* $t\geqq s(\geqq T)$ *and* $x,\xi\in R^n$

$$\begin{cases} |\eta(t,s;x,\xi)-\xi|\leqq C_1\langle s\rangle^{-\varepsilon},\\ |y(s,t;x,\xi)-x-(t-s)\xi|\leqq C_1\min\{\langle t\rangle^{1-\varepsilon},(t-s)\langle s\rangle^{-\varepsilon}\};\end{cases} \tag{2.13}$$

$$\begin{cases} |\nabla_x y(s,t;x,\xi)-I|\leqq C_1\langle s\rangle^{-\varepsilon},\\ |\nabla_\xi y(s,t;x,\xi)-(t-s)I|\leqq C_1\min\{\langle t\rangle^{1-\varepsilon},(t-s)\langle s\rangle^{-\varepsilon}\};\end{cases} \tag{2.14}$$

and

$$\begin{cases} |\nabla_x\eta(t,s;x,\xi)|\leqq C_1\langle s\rangle^{-1-\varepsilon},\\ |\nabla_\xi\eta(t,s;x,\xi)-I|\leqq C_1\langle s\rangle^{-\varepsilon}.\end{cases} \tag{2.15}$$

iv) *For any* α, β *with* $|\alpha+\beta|\geqq 2$, *there is a constant* $C_{\alpha,\beta}>0$ *such that for any* $t\geqq s(\geqq T)$ *and* $x,\xi\in R^n$

$$\begin{cases} |\partial_\xi^\alpha\partial_x^\beta\eta(t,s;x,\xi)|\leqq C_{\alpha,\beta}\langle s\rangle^{-\varepsilon},\\ |\partial_\xi^\alpha\partial_x^\beta y(s,t;x,\xi)|\leqq C_{\alpha,\beta}(t-s+1)\langle s\rangle^{-\varepsilon}.\end{cases} \tag{2.16}$$

PROOF. The existence of the inverse C^∞ diffeomorphisms y and η is proved from the estimates (2.5) and (2.7) by using the contraction mapping theorem (see e.g. the proof of Lemma 3.2 of Kumano-go [12]). Then i) and ii) are obvious by definition. The assertion iii) except the second estimates in (2.13) and (2.14) follows from i), ii) and Proposition 2.1-i). The second estimate of (2.13) is clear from (2.9) with $|\alpha|=0$ and i)-ii) above. Also using i)-ii), (2.15), and (2.9) with $|\alpha|=1$ we have

$$\begin{aligned}
&|\nabla_\xi y(s,t;x,\xi)-(t-s)I|\\
&\quad=|\nabla_\xi[q(t,s;x,\eta(t,s;x,\xi))-(t-s)p(t,s;x,\eta(t,s;x,\xi))]|\\
&\quad\leqq|\nabla_\xi q(t,s;x,\eta(t,s;x,\xi))-(t-s)\nabla_\xi p(t,s;x,\eta(t,s;x,\xi))||\nabla_\xi\eta(t,s;x,\xi)|\\
&\quad\leqq C\min\{\langle t\rangle^{1-\varepsilon},(t-s)\langle s\rangle^{-\varepsilon}\}.
\end{aligned}$$

We finally prove (2.16). The first estimate in (2.16) is proved by induction by using the estimate (2.8) of Proposition 2.1-ii). Then the second estimate in (2.16) follows from this and (2.8) by using the first relation in ii). □

§ 3. Approximate fundamental solution global in time.

We begin with the definition of the phase function $\phi(s,t;x,\xi)$ of the approximate fundamental solution.

DEFINITION 3.1. For $t\geqq s\geqq T$, define

$$\phi(s.t;x,\xi)=u(s,t;y(s,t;x,\xi),\xi), \tag{3.1}$$

where

$$u(s,t;y,\eta)=y\cdot\eta+\int_t^s L(\tau,q(\tau,t;y,\eta),p(\tau,t;y,\eta))d\tau \tag{3.2}$$

and

$$L(t,x,\xi)=\xi\cdot\nabla_\xi H(t,x,\xi)-H(t,x,\xi)=\frac{1}{2}|\xi|^2-V(t,x). \tag{3.3}$$

PROPOSITION 3.2. *Let Assumption* (A) *be satisfied. Let* $t\geqq s\geqq T$. *Then* $\phi(s,t;x,\xi)$ *defined above satisfies*

$$\partial_s\phi(s,t;x,\xi)+H(s,x,\nabla_x\phi(s,t;x,\xi))=0, \tag{3.4}$$

$$\partial_t\phi(s,t;x,\xi)-H(t,\nabla_\xi\phi(s,t;x,\xi),\xi)=0, \tag{3.5}$$

$$\phi(s,s;x,\xi)=x\cdot\xi, \tag{3.6}$$

and

$$\begin{cases}\nabla_x\phi(s,t;x,\xi)=\eta(t,s;x,\xi),\\ \nabla_\xi\phi(s,t;x,\xi)=y(s,t;x,\xi).\end{cases} \tag{3.7}$$

Furthermore ϕ *is uniquely determined as the solution of the equation* (3.4) *and* (3.6) (*or* (3.5) *and* (3.6)).

Proof is done by direct calculations (or see Kumano-go [12] and Kumano-go, Taniguchi and Tozaki [13]).

Before defining the approximate fundamental solution, we prepare a proposition. Let $\mathcal{B}^{k,\infty}(R^m)$ ($k\geqq 0$, integer) be the space of C^∞-functions $f(y)$ whose derivatives $\partial_y^\alpha f(y)$ are all bounded on R^m for all α with $|\alpha|\geqq k$. We often write $\mathcal{B}^\infty(R^m)=\mathcal{B}^{0,\infty}(R^m)$. Further we denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing functions on R^n.

PROPOSITION 3.3. *Let Assumption* (A) *be satisfied. Let* $t\geqq s\geqq T$, *and let* $p(\xi, y)\in\mathcal{B}^{k,\infty}(R^n\times R^n)$ *for some integer* $k\geqq 0$. *Then for any* $f\in\mathcal{S}$ *and* $\chi\in\mathcal{S}$ *with* $\chi(0)=1$, *the integral*

$$P_\varepsilon[f](x)=\iint e^{i(x\cdot\xi-\phi(s,t;y,\xi))}p(\xi,y)f(y)\chi(\varepsilon\xi)dy\,\bar{d}\xi, \tag{3.8}$$

where $\bar{d}\xi=(2\pi)^{-n}d\xi$, *has the limit* $P[f](x)$ *when* $\varepsilon\downarrow 0$, *which does not depend on any particular choice of* χ. *Moreover* P *defines a continuous linear mapping from* $\mathcal{S}$ *into* $\mathcal{S}$. *We write* $P[f]$ *as*

$$P[f](x)=\mathrm{O}_s\text{-}\iint e^{i(x\cdot\xi-\phi(s,t;y,\xi))}p(\xi,y)f(y)dy\,\bar{d}\xi. \tag{3.9}$$

PROOF. Let $t\geqq s(\geqq T)$ be fixed and write $\phi(y,\xi)=\phi(s,t;y,\xi)$. Then putting $\psi(x,\xi,y)=x\cdot\xi-\phi(y,\xi)$, we have from (3.7) and (2.13) that $C\langle\xi\rangle\leqq\langle\nabla_y\psi\rangle\leqq C'\langle\xi\rangle$ for some constants $C, C'>0$. Thus the differential operator $L=\langle\nabla_y\psi\rangle^{-2}(1-i\nabla_y\psi\cdot\nabla_y)$ is well-defined, and by integration by parts we have for any $l\geqq 0$

$$P_\varepsilon[f](x)=\iint e^{i\psi}({}^tL)^l[p(\xi,y)f(y)\chi(\varepsilon\xi)]dy\,\bar{d}\xi, \tag{3.10}$$

where ${}^tL$ denotes the transposed operator of L. Then taking $l>n+k$, noting $f\in\mathcal{S}$ and letting $\varepsilon\downarrow 0$, we have

$$P[f](x)=\iint e^{i\psi}({}^tL)^l[p(\xi,y)f(y)]dy\,\bar{d}\xi, \tag{3.11}$$

which is independent of χ. From this, taking l sufficiently large, we can easily see that $P:\mathcal{S}\to\mathcal{S}$ is continuous. $\square$

Now we can define the approximate fundamental solution.

DEFINITION 3.4. For $t\geqq s\geqq T$ and $f\in\mathcal{S}$, we define

(3.12) $$E(t, s)f(x)=\mathrm{O}_s\text{-}\iint e^{i(x\cdot\xi-\phi(s,t;y,\xi))}f(y)dy\,d\!\!{}^{-}\!\xi.$$

Then we have the following theorem.

THEOREM 3.5. *Let Assumption* (A) *be satisfied. For* $t\geqq s\geqq T$ *and* $f\in\mathcal{S}$, *define*

(3.13) $$G(t, s)f(x)=-i(D_t+H(t))E(t, s)f(x),\ D_t=-i\partial/\partial t.$$

Then:

i) *We have*

(3.14) $$E(s, s)=I$$

and

(3.15) $$G(t, s)f(x)=\mathrm{O}_s\text{-}\iint e^{i(x\cdot\xi-\phi(s,t;y,\xi))}g(t, s;\xi, y)f(y)dy\,d\!\!{}^{-}\!\xi.$$

Here $g(t, s;\xi, y)\in\mathcal{B}^\infty(R^n_\xi\times R^n_y)$ *for* $t\geqq s\geqq T$.

ii) *More precisely, we have*

(3.16) $$g(t, s;\xi, y)=\sum_{l,k=1}^{n}\int_0^1 \mathrm{O}_s\text{-}\iint e^{-iy\cdot\eta}(\partial_{x_k}\partial_{x_l}V)(t, \theta y+\tilde{\nabla}_\xi\phi(s, t;\xi, y, \xi-\eta))$$
$$\times\left(\int_0^1 r(\partial_{\xi_k}\partial_{\xi_l}\phi)(s, t;y, \xi-r\eta)dr\right)dy\,d\!\!{}^{-}\!\eta\,d\theta,$$

where

(3.17) $$\tilde{\nabla}_\xi\phi(s, t;\xi, y, \eta)=\int_0^1\nabla_\xi\phi(s, t;y, \eta+\theta(\xi-\eta))d\theta.$$

Hence we have for any α, β

(3.18) $$|\partial_\xi^\alpha\partial_y^\beta g(t, s;\xi, y)|\leqq C_{\alpha,\beta}\langle t\rangle^{-1-\varepsilon},$$

where the constant $C_{\alpha,\beta}>0$ *is independent of* $t\geqq s(\geqq T)$ *and* $\xi, y\in R^n$. *Furthermore we have for* $t\geqq s(\geqq T)$

(3.19) $$\|G(t, s)\|_{L^2\to L^2}\leqq C\langle t\rangle^{-1-\varepsilon}$$

and

(3.20) $$\|E(t, s)\|_{L^2\to L^2}\leqq C$$

for some constant $C>0$ *independent of* $t\geqq s\geqq T$.

PROOF. (3.14) is obvious by definition. We next prove (3.15) and (3.16). Noting that $H(t)$ is symmetric in $L^2(R^n)$, we obtain

$$H(t)f(x)=\mathrm{O_s}\text{-}\iint e^{i(x-y)\cdot\xi}H(t, y, \xi)f(y)dy\bar{d}\xi \tag{3.21}$$

for $f\in\mathcal{S}$. (Here $\mathrm{O_s}\text{-}\iint\cdots$ means the usual oscillatory integral. For the definition see e.g. Kumano-go [12].) So we have

$$H(t)E(t, s)f(x)= \mathrm{O_s}\text{-}\iint e^{i(x\cdot\xi-\phi(s,t;z,\xi))}h(t, s;\xi, z)f(z)dz\bar{d}\xi, \tag{3.22}$$

where

$$h(t, s;\xi, z)=\mathrm{O_s}\text{-}\iint e^{i(\eta-\xi)\cdot(y-\tilde{\nabla}_\xi\phi(s,t;\eta,z,\xi))}H(t, y, \xi)dy\bar{d}\eta. \tag{3.23}$$

Making a change of variable $\tilde{y}=y-\tilde{\nabla}_\xi\phi(s, t;\eta, z, \xi)$, we obtain

$$h(t, s;\xi, z)=\mathrm{O_s}\text{-}\iint e^{i(\eta-\xi)\cdot y}H(t, y+\tilde{\nabla}_\xi\phi(s, t;\eta, z, \xi), \xi)dy\bar{d}\eta. \tag{3.24}$$

By Taylor's expansion formula of order one we have

$$\begin{aligned} H(t, y+\tilde{\nabla}_\xi\phi(s, t;\eta, z, \xi), \xi)&=H(t, \tilde{\nabla}_\xi\phi(s, t;\eta, z, \xi), \xi)\\ &\quad+\int_0^1(\nabla_x V)(t, \theta y+\tilde{\nabla}_\xi\phi(s, t;\eta, z, \xi))d\theta\cdot y. \end{aligned} \tag{3.25}$$

Then, by Fourier's inversion formula and integration by parts, we get

$$h(t, s;\xi, z)=H(t, \nabla_\xi\phi(s, t;z, \xi), \xi)+ig(t, s;\xi, z), \tag{3.26}$$

where g is the function defined by (3.16). Therefore by the equality (3.5) we get (3.15).

The estimate (3.18) directly follows from the expression (3.16) if we use (2.14), (2.16) and Assumption (A)-iii).

We finally prove (3.19). For $f\in\mathcal{S}$ we have

$$\|G(t, s)f\|_{L^2}^2=(2\pi)^{-n}\|\mathcal{F}G(t, s)f\|_{L^2}^2=(K(t, s)f, f)_{L^2}, \tag{3.27}$$

where

$$K(t, s)f(x)=\mathrm{O_s}\text{-}\iint e^{i(\phi(s,t;x,\xi)-\phi(s,t;y,\xi))}g(t, s;\xi, y)\overline{g(t, s;\xi, x)}f(y)dy\bar{d}\xi. \tag{3.28}$$

Noting that $\phi(s, t; x, \xi)-\phi(s, t; y, \xi)=(x-y)\cdot\tilde{\nabla}_x\phi(s, t; x, \xi, y)$ and that $\xi\mapsto\eta=\tilde{\nabla}_x\phi(s, t; x, \xi, y)$ has the inverse C^∞ diffeomorphism $\eta\mapsto\phi(t, s; x, \eta, y)$, since

$|\vec{\nabla}_\xi \tilde{\nabla}_x \phi(s, t; x, \xi, y) - I| = \left| \int_0^1 (\nabla_\xi \eta(t, s; y+\theta(x-y), \xi) - I) d\theta \right| \leqq C_1 \langle s \rangle^{-\varepsilon} < 1/2$ by (2.15), we make a change of variable $\eta = \tilde{\nabla}_x \phi(s, t; x, \xi, y)$ in (3.28). Then we obtain

$$K(t, s) f(x) = \mathrm{O}_s\text{-}\iint e^{i(x-y)\cdot\eta} r(t, s; x, \eta, y) f(y) dy \bar{d}\eta \tag{3.29}$$

where

$$r(t, s; x, \eta, y) = r(x, \eta, y) = g(t, s; \phi(t, s; x, \eta, y), y) \overline{g(t, s; \phi(t, s; x, \eta, y), x)} \times |\det \nabla_\eta \phi(t, s; x, \eta, y)|. \tag{3.30}$$

Thus by Calderón-Vaillancourt theorem ([1]) we have

$$\|K(t, s) f\|_{L^2} \leqq C \max_{|\beta+\alpha+\beta'| \leqq M} \sup_{x, \eta, y} |\partial_x^\beta \partial_\eta^\alpha \partial_y^{\beta'} r(x, \eta, y)| \|f\|_{L^2}, \tag{3.31}$$

where $M = 2([n/2] + [5n/4] + 2)$. From (3.30) we have for $|\beta+\alpha+\beta'| \leqq M$

$$|\partial_x^\beta \partial_\eta^\alpha \partial_y^{\beta'} r(x, \eta, y)| \leqq C \left\{ \sup_{\substack{|\alpha+\beta| \leqq M \\ \xi, y}} |\partial_\xi^\alpha \partial_y^\beta g(t, s; \xi, y)| \right\}^2 \tag{3.32}$$

where the constant $C>0$ is independent of t, s, x, η and y, and we have used the estimates (2.15)-(2.16). Combining this with (3.31) and (3.18) proves (3.19). The proof of (3.20) is quite similar. □

Now we can prove the following proposition, which shows the appropriateness of the name "approximate fundamental solution" for $E(t, s)$ and plays a crucial role in the proof of the completeness in the next section.

Proposition 3.6. *Let Assumption* (A) *be satisfied. Define for* $t \geqq s \geqq T$

$$D(t, s) = U(t, s) - E(t, s). \tag{3.33}$$

Then we have for $t \geqq s \geqq T$

$$\|D(t, s)\|_{L^2 \to L^2} \leqq C \langle s \rangle^{-\varepsilon} \tag{3.34}$$

for some constant $C>0$ *independent of* t, s.

Proof. From (1.3) and (3.13) we have for $f \in \mathcal{S}$

$$\begin{aligned} D(t, s) f &= U(t, s)(I - U(s, t) E(t, s)) f \\ &= U(t, s) \int_t^s \frac{d}{d\theta} [U(s, \theta) E(\theta, s) f] d\theta = \int_s^t U(t, \theta) G(\theta, s) f \, d\theta. \end{aligned} \tag{3.35}$$

From this and (3.19) follows (3.34). □

§ 4. Scattering theory.

In this section we shall prove Theorems 1.1 and 1.2 for W_D^+. For $T(>T_0)$ fixed at the end of Section 2, we define the modified free propagator $U_D^+(t)$ by (1.5) for $t \geqq T$. Then the following proposition holds.

PROPOSITION 4.1. *Let Assumption* (A) *be satisfied. Then for* $V^+(t)$ *defined by* (1.8) *for* $t \geqq T$ *we have*

$$\lim_{t\to\infty} \|U_D^+(t)f - V^+(t)f\| = 0, \qquad f \in L^2(R^n). \tag{4.1}$$

PROOF. We give only the sketch, since the proof is done in a standard manner by the stationary phase method. We write

$$U_D^+(t)f(x) = \int e^{it\varphi(t,x,\xi)} \hat{f}(\xi) d\!\!{}^-\xi \tag{4.2}$$

for $f \in \mathcal{S}$, where $\varphi(t, x, \xi) = (x\cdot\xi - \phi(T, t; 0, \xi))/t$. Then the critical point $\xi_c^+ = \xi_c^+(t, x)$ of φ is given by

$$x = \nabla_\xi \phi(T, t; 0, \xi_c^+(t, x)) = y(T, t; 0, \xi_c^+(t, x)), \tag{4.3}$$

which has a unique solution ξ_c^+ by (2.14). Thus by the stationary phase method and integration by parts we have for $f \in \mathcal{D} \equiv \mathcal{F}^{-1}C_0^\infty(R^n \backslash \{0\})$

$$\lim_{t\to\infty} \|U_D^+(t)f - \tilde{V}^+(t)f\| = 0, \tag{4.4}$$

where

$$\tilde{V}^+(t)f(x) = (2\pi t)^{-n/2} e^{-\pi i n/4} |\det \vec{\nabla}_\xi \nabla_\xi \varphi(t, x, \xi_c^+)|^{-1/2} e^{it\varphi(t,x,\xi_c^+)} \hat{f}(\xi_c^+). \tag{4.5}$$

It is easy to see that $\lim_{t\to\infty} \|\tilde{V}^+(t)f - V^+(t)f\| = 0$ for $f \in \mathcal{D}$ by (4.3) and (2.13). Thus we have (4.1), since $V^+(t)$ and $U_D^+(t)$ are unitary operators in $L^2(R^n)$. □

Now Theorem 1.2 follows from this proposition and Theorem 1.1. So we prove Theorem 1.1.

PROOF OF THEOREM 1.1. The existence of the limits (1.7) is proved in a way quite similar to the proof of Theorem 3.9 of Hörmander [5] by using the stationary phase method, the relation (3.5), and the estimates in Proposition 2.2. We leave the details to the reader.

So to prove Theorem 1.1, we have only to prove the existence of the limit

$$\operatorname*{s-lim}_{t\to\infty} U_D^+(t)^* U(t, 0). \tag{4.6}$$

For this purpose it suffices to prove for $f\in L^2(R^n)$ that $\{U_D^+(t)^*U(t,0)f\}_{t\geqq T}$ forms a Cauchy net in $L^2(R^n)$. For $t\geqq r\geqq s\geqq T$ we have

$$\begin{aligned}&U_D^+(t)^*U(t,0)f-U_D^+(r)^*U(r,0)f\\ &=[U_D^+(t)^*D(t,s)-U_D^+(r)^*D(r,s)]U(s,0)f+\{U_D^+(t)^*U_D^+(t,s)\cdot U_D^+(t,s)^*E(t,s)\\ &\quad-U_D^+(r)^*U_D^+(r,s)\cdot U_D^+(r,s)^*E(r,s)\}U(s,0)f,\end{aligned}\tag{4.7}$$

where $U_D^+(t,s)\equiv\mathcal{F}^{-1}[e^{-i\phi(s,t;0,\xi)}\times]\mathcal{F}$ for $t\geqq s\geqq T$ and $D(t,s)$ is defined by (3.33). Then by (3.34) the norm of the first summand in the right hand side of (4.7) is bounded by $a\langle s\rangle^{-\varepsilon}\|f\|$ for some constant $a>0$ independent of t,r,s and f. In the following we shall fix $s(\geqq T)$ so large that $a\langle s\rangle^{-\varepsilon}\|f\|$ is sufficiently small, and show that the norm of the second summand in the right hand side of (4.7) converges to zero as $r\to\infty$ with $t\geqq r$ and s fixed. To do this it suffices to prove for $g\in\mathcal{F}^{-1}C_0^\infty(R^n)$ that the limit $\lim_{t\to\infty}U_D^+(t)^*E(t,s)g=\lim_{t\to\infty}U_D^+(t)^*U_D^+(t,s)\cdot U_D^+(t,s)^*E(t,s)g$ exists in $L^2(R^n)$, since the operator $U_D^+(t)^*E(t,s)$ is uniformly bounded in $t(\geqq s)$ by (3.20).

Define for $t\geqq s(\geqq T)$ and $g\in\mathcal{F}^{-1}C_0^\infty(R^n)$

$$T(t,s)g(\xi)\equiv e^{i\phi(s,t;0,\xi)}(\mathcal{F}E(t,s)g)(\xi)=\mathrm{O}_s\text{-}\iint e^{i[y\cdot\eta-(\phi(s,t;y,\xi)-\phi(s,t;0,\xi))]}\hat{g}(\eta)dy\,đ\eta.\tag{4.8}$$

Then we can write

$$\begin{aligned}U_D^+(t)^*E(t,s)g&=U_D^+(t)^*U_D^+(t,s)\cdot U_D^+(t,s)^*E(t,s)g\\ &=\mathcal{F}^{-1}e^{i(\phi(T,t;0,\xi)-\phi(s,t;0,\xi))}T(t,s)g.\end{aligned}\tag{4.9}$$

By (3.5) and (3.7) we have for $t\geqq r\geqq s(\geqq T)$

$$\begin{aligned}&(\phi(T,t;0,\xi)-\phi(s,t;0,\xi))-(\phi(T,r;0,\xi)-\phi(s,r;0,\xi))\\ &=(\phi(T,t;0,\xi)-\phi(T,r;0,\xi))-(\phi(s,t;0,\xi)-\phi(s,r;0,\xi))\\ &=\int_0^1[\partial_t\phi(T,r+\theta(t-r);0,\xi)-\partial_t\phi(s,r+\theta(t-r);0,\xi)]d\theta\cdot(t-r)\\ &=\int_0^1[H(\tau,y(T,\tau;0,\xi),\xi)-H(\tau,y(s,\tau;0,\xi),\xi)]_{\tau=r+\theta(t-r)}d\theta\cdot(t-r)\\ &=\int_0^1\int_0^1(\nabla_xV)(\tau,y(s,\tau;0,\xi)+\sigma(y(T,\tau;0,\xi)-y(s,\tau;0,\xi)))d\sigma\\ &\quad\times(y(T,\tau;0,\xi)-y(s,\tau;0,\xi))\Big|_{\tau=r+\theta(t-r)}d\theta\cdot(t-r).\end{aligned}\tag{4.10}$$

Here by Proposition 2.2-i), (2.5), (2.1) and (2.4) we see that $y(T,\tau;0,\xi)-y(s,\tau;0,\xi)=\int_s^T \partial_\sigma y(\sigma,\tau;0,\xi)d\sigma$ is bounded by $b(s-T)\langle\xi\rangle$ for some constant b independent of $\tau\geqq r\geqq s(\geqq T)$. Hence using Assumption (A)-iii) and the inequality

$$(4.11)\qquad (t-r)\int_0^1(1+r+\theta(t-r))^{-1-\varepsilon}d\theta\leqq\varepsilon^{-1}(1+r)^{-\varepsilon},\ t\geqq r,$$

we see that the limit

$$(4.12)\qquad \lim_{t\to\infty}(\phi(T,t;0,\xi)-\phi(s,t;0,\xi))$$

exists for each fixed $\xi\in R^n$ and $s(\geqq T)$.

On the other hand, noting $\phi(s,t;y,\xi)-\phi(s,t;0,\xi)=y\cdot\tilde{\nabla}_x\phi(s,t;y,\xi,0)$ and making a change of variable $\tilde{\eta}=\eta-\tilde{\nabla}_x\phi(s,t;y,\xi,0)$ in (4.8), we get for $g\in\mathcal{F}^{-1}C_0^\infty(R^n)$ and $t\geqq r\geqq s(\geqq T)$

$$(4.13)\qquad T(t,s)g(\xi)-T(r,s)g(\xi)=\mathrm{O}_s\text{-}\iint e^{iy\cdot\eta}G(t,r,s;y,\eta,\xi)dy\,đ\eta,$$

where

$$(4.14)\qquad \begin{aligned}G(t,r,s;y,\eta,\xi)&=\hat{g}(\eta+\tilde{\nabla}_x\phi(s,t;y,\xi,0))-\hat{g}(\eta+\tilde{\nabla}_x\phi(s,r;y,\xi,0))\\&=\int_0^1(\nabla_\xi\hat{g})(\eta+\tilde{\nabla}_x\phi(s,r+\theta(t-r);y,\xi,0))\\&\qquad\times[\partial_\tau\tilde{\nabla}_x\phi(s,\tau;y,\xi,0)]_{\tau=r+\theta(t-r)}d\theta\cdot(t-r).\end{aligned}$$

Here we have from (3.5)

$$(4.15)\qquad \begin{aligned}\partial_\tau\tilde{\nabla}_x\phi(s,\tau;y,\xi,0)&=\int_0^1\partial_\tau\nabla_x\phi(s,\tau;\theta y,\xi)d\theta=\int_0^1\nabla_x\{H(\tau,\nabla_\xi\phi(s,\tau;x,\xi),\xi)\}\Big|_{x=\theta y}d\theta\\&=\int_0^1(\nabla_x V)(\tau,\nabla_\xi\phi(s,\tau;\theta y,\xi))\cdot\nabla_\xi\eta(\tau,s;\theta y,\xi)\,d\theta.\end{aligned}$$

Therefore from Assumption (A)-iii), (2.15) and (2.16) and using the inequality (4.11), we obtain

$$(4.16)\qquad |\partial_\eta^\alpha\partial_y^\beta G(t,r,s;y,\eta,\xi)|\leqq C_{\alpha,\beta}\langle r\rangle^{-\varepsilon}$$

for some constant $C_{\alpha,\beta}>0$ independent of r,t,s,y,η and ξ. Thus using (4.16) and (4.13) and noting $\hat{g}\in C_0^\infty(R^n)$ and (2.13), we get

$$(4.17)\qquad |T(t,s)g(\xi)-T(r,s)g(\xi)|\leqq C_g\langle r\rangle^{-\varepsilon}\langle\xi\rangle^{-n-1}$$

for some constant $C_g>0$ independent of $t\geqq r\geqq s(\geqq T)$ and ξ. Thus the limit

(4.18) $$L(s)g \equiv \text{s-}\lim_{t\to\infty} T(t, s)g$$

exists.

Therefore by (4.12), (4.18) and (4.9), we easily see that the limit $\lim_{t\to\infty} U_D^+(t)^* \times E(t, s)g$ exists in $L^2(R^n)$ for $g \in L^2(R^n)$, hence the norm of the second summand of the right hand side of (4.7) converges to zero as $r\to\infty$ with $t \geqq r$ and $s(\geqq T)$ fixed. This proves the existence of the limit (4.6). □

REMARK 4.2. By (3.5) and (3.6) the phase $\phi(T, t; 0, \xi)$ of the modified free propagator $U_D^+(t)$ is written as

(4.19) $$\phi(T, t; 0, \xi) = \int_T^t \partial_\tau \phi(T, \tau; 0, \xi) d\tau = \frac{1}{2}|\xi|^2(t-T) + \int_T^t V(\tau, y(T, \tau; 0, \xi)) d\tau.$$

In Example 1.3-ii) with $B(t, x)=0$, we can use another modified free propagator of somewhat more concrete form. Applying the Taylor's expansion formula of order $N(\geqq 1)$ for $V(\tau, y) = q(\langle\tau\rangle^{-a}y)$ where $y = y(T, \tau; 0, \xi)$, we obtain

(4.20) $$\begin{aligned} V(\tau, y) &= q(\langle\tau\rangle^{-a}y) \\ &= \sum_{|\alpha|<N} \frac{y^\alpha}{\alpha!}\langle\tau\rangle^{-a|\alpha|}\partial_x^\alpha q(0) + N\sum_{|\gamma|=N}\frac{y^\gamma}{\gamma!}\int_0^1 (1-\theta)^{N-1}\langle\tau\rangle^{-aN}\partial_x^\gamma q(\theta\langle\tau\rangle^{-a}y)d\theta. \end{aligned}$$

Using (2.13) and (1.13) we see that the second term on the right side is bounded by $C_N\langle\tau\rangle^{N(1-a)}\langle\xi\rangle^N$. Hence taking N as $a > (N+1)/N > 1$ and defining

(4.21) $$W_N(t, \xi) = \frac{1}{2}|\xi|^2(t-T) + \sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_x^\alpha q(0)\int_T^t \langle\tau\rangle^{-a|\alpha|} y(T, \tau; 0, \xi)^\alpha d\tau,$$

we see that the limit

(4.22) $$\lim_{t\to\infty} (\phi(T, t; 0, \xi) - W_N(t, \xi))$$

exists for each $\xi \in R^n$. Therefore if we define $U_{D,N}^+(t)$ by

(4.23) $$U_{D,N}^+(t) = \mathcal{F}^{-1}[e^{-iW_N(t,\xi)} \times]\mathcal{F},$$

then by (4.22) and Theorem 1.1 the modified wave operator

(4.24) $$W_{D,N}^+ = \text{s-}\lim_{t\to\infty} U(t, 0)^* U_{D,N}^+(t)$$

exists and is unitary in $L^2(R^n)$. When $N=2$, i.e., $a > 3/2$, we can construct another simpler modified free propagator. Define

$$
(4.25) \qquad \begin{cases} \tilde{W}(t,\xi)=(t-T)\left(\dfrac{1}{2}|\xi|^2+q(0)\right)+\tilde{X}(t,\xi), \\ \tilde{X}(t,\xi)=\displaystyle\int_T^t (q(\langle\tau\rangle^{1-a}\xi)-q(0))d\tau. \end{cases}
$$

Then using (4.19) and (2.13) with $\varepsilon=a-1$, we see that the limit $\lim_{t\to\infty}(\phi(T,t;0,\xi)-\tilde{W}(t,\xi))$ exists. Thus the modified wave operator $W_D^+=\text{s-}\lim_{t\to\infty} U(t,0)^*\tilde{U}_D^+(t)$ exists and is unitary in $L^2(R^n)$, where $\tilde{U}_D^+(t)=\mathcal{F}^{-1}[e^{-i\tilde{W}(t,\xi)}]\mathcal{F}$. Note that $\tilde{X}(t,\xi)$ in (4.25) coincides with $\tilde{X}_t(\xi)$ given in Remark 4.1 of Kuroda and Morita [14][1]. Furthermore if $a>2$, it is easily seen from (4.25) that we can use $\tilde{U}_0(t)=\exp\{-it(H_0+q(0))\}$, $H_0=-\frac{1}{2}\Delta$, instead of $U_D^+(t)$. This result almost covers Theorem 4 of Kuroda and Morita [14].

REMARK 4.3. In the above we assumed (1.1) for all $\alpha\neq 0$. However, as can be seen by checking the above discussions, this assumption is redundant and it suffices to assume (1.1) up to a certain finite order.

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1) The formulae defining $X_t(\xi)$ and $\tilde{X}_t(\xi)$ in Remark 4.1 of [14] are incorrect so far as the factor "2" in the right sides of them is concerned. The factor "2" should be placed in front of "ξ" in the right sides in order to make the formulae correct.

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Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Meguro-ku, Tokyo
153 Japan