# *The quasi-classical approximation to Dirac equation II, scattering theory*

By Kenji YAJIMA

## § 1. Introduction, Theorem.

This paper is a direct continuation of a preceding one [9] referred to as [I] hereafter in which we studied the quasi-classical approximation for the Dirac equation in finite time. Here the same subject is discussed for the associated scattering operator.

We consider the Dirac equation with static external electromagnetic field $(A^0(x), \dots, A^3(x)) \equiv (\phi(x), \vec{A}(x))$,

$$
\begin{aligned}
(1.1) \qquad i\hbar\partial u/\partial t &= \sum_{j=1}^{3} \alpha^j(-i\hbar\partial/\partial x_j - eA^j(x))u + m\beta u + e\phi(x)u \\
&= H^\hbar u.
\end{aligned}
$$

We assume that the potential $(A^\mu(x))$ satisfies the following.

ASSUMPTION (VSR). (i) For any $\mu=0,1,2,3$, $A^\mu(x)$ is a real valued $C^\infty$-function of $x \in \boldsymbol{R}^3$. (ii) There exists a constant $\varepsilon>0$ such that for any multi-index $\alpha$

$$|(\partial/\partial x)^\alpha A^\mu(x)| \leqq C_\alpha(1+|x|)^{-2-|\alpha|-\varepsilon}, \quad x \in \boldsymbol{R}^3.$$

Under the assumption (VSR) (or milder conditions) the scattering theory for (1.1) was established as far as the existence and the completeness of the wave operators are concerned ([3], [11]). We write the free Hamiltonian as

$$H_0^\hbar = \sum_{j=1}^{3} \alpha^j(-i\hbar\partial/\partial x_j) + m\beta.$$

$H^\hbar$ and $H_0^\hbar$ are selfadjoint operators on $\mathcal{H} = L^2(\boldsymbol{R}^3, \boldsymbol{C}^4)$ with the domain $D(H_0^\hbar) = D(H^\hbar) = H^1(\boldsymbol{R}^3, \boldsymbol{C}^4)$.

THEOREM 1.1 ([3], [11]). *The wave operators*

$$\underset{t\to\pm\infty}{\text{s-lim}} \exp(itH^\hbar/\hbar)\exp(-itH_0^\hbar/\hbar) = W_\pm^\hbar$$

*exist and are complete*:

$$R(W_+^\hbar) = R(W_-^\hbar) = \mathcal{H}_{ac}(H^\hbar),$$

*where $\mathcal{H}_{ac}(H^\hbar)$ is the absolutely continuous subspace for $H^\hbar$. The scattering* *operator* $S^{\hbar}$ *is defined as* $S^{\hbar}=(W_{+}^{\hbar})^{*}W_{-}^{\hbar}$ *and is a unitary operator.*

In the quasi-classical limit, as was shown in [I], associated with the equation (1.1) are the Hamilton equations for classical relativistic particles

$$(1.2)_{\pm} \qquad \frac{dx^{\pm}}{dt}=\frac{\partial H^{\pm}}{\partial\xi}(x^{\pm},\xi^{\pm}),\quad \frac{d\xi^{\pm}}{dt}=-\frac{\partial H^{\pm}}{\partial x}(x^{\pm},\xi^{\pm})$$

with the Hamiltonians

$$(1.3)_{\pm} \qquad H^{\pm}(x,\xi)=\pm((\xi-e\vec{A}(x))^{2}+m^{2})^{1/2}+e\phi(x)$$

and the equation

$$(1.4)_{\pm} \qquad p_{\pm}^{0}(t)(df^{\pm}/dt)+\sum(i\sigma^{\mu\nu}/4)F_{\mu\nu}(x^{\pm}(t))f^{\pm}(t)=0$$

for a four vector $f^{\pm}(t)$ which may be interpreted as describing the internal degree of freedom of the particle, where $p_{\pm}^{0}(t)=H^{\pm}(x^{\pm}(t),\xi^{\pm}(t))-e\phi(x^{\pm}(t))$, $\sigma^{\mu\nu}$ is the spinor tensor and $F_{\mu\nu}(x)$ is the field strength tensor. Equations $(1.2)_{+}$ (or $(1.4)_{+}$) and $(1.2)_{-}$ (or $(1.4)_{-}$) are respectively corresponding to the positive and negative energy parts of the solution of (1.1) and they can be treated separately in the quasi-classical approximation. In this paper we restrict ourselves to study the positive energy part only and accordingly omit the suffix $+$ indicating the positivity of energy in the expressions.

The scattering theory for relativistic classical particles (1.2) is studied in [10] and the results corresponding to the existence and the completeness of wave operators in the quantum case are known. We write as $\nu(\eta)=\eta/(\eta^{2}+m^{2})^{1/2}$.

THEOREM 1.2 ([10]). (i) *For any* $(a,\eta)\in\Gamma\equiv\boldsymbol{R}^{3}\times(\boldsymbol{R}^{3}\backslash\{0\})$, *there exists a unique solution* $(x_{\pm}(t,a,\eta),\xi_{\pm}(t,a,\eta))$ *of* (1.2) *such that*

$$\lim_{t\to\pm\infty}|x_{\pm}(t,a,\eta)-tv(\eta)-a|=0 \quad \text{and} \quad \lim_{t\to\pm\infty}|\xi_{\pm}(t,a,\eta)-\eta|=0.$$

(ii) *There exists a closed null set* $e\subset\Gamma$ *and a smooth canonical mapping* $S(a,\eta)=(a_{+}(a,\eta),\eta_{+}(a,\eta))$ *defined on* $\Gamma\backslash e$ *such that*

$$x_{+}(t,a_{+}(a,\eta),\eta_{+}(a,\eta))=x_{-}(t,a,\eta) \quad \text{and} \quad \xi_{+}(t,a_{+}(a,\eta),\eta_{+}(a,\eta))=\xi_{-}(t,a,\eta).$$

(iii) *If* $K$ *is a compact subset of* $\Gamma$, *then for any multi-index* $\alpha$ *and* $\beta$ *there exists a constant* $C_{\alpha\beta}>0$ *such that*

$$(1.5) \qquad |(\partial/\partial a)^{\alpha}(\partial/\partial\eta)^{\beta}(x_{\pm}(t,a,\eta)-tv(\eta)-a)|\leqq C_{\alpha\beta}(1+|t|)^{-1-\varepsilon},$$

$$(1.6) \qquad |(\partial/\partial a)^{\alpha}(\partial/\partial\eta)^{\beta}(\xi_{\pm}(t,a,\eta)-\eta)|\leqq C_{\alpha\beta}(1+|t|)^{-2-\varepsilon}$$

*for* $(a, \eta) \in K$ *and* $\pm t > 0$.

In this paper we study the asymptotic behavior of $S^{\hbar}(\exp(ix\cdot\eta/\hbar)f)$ as $\hbar \downarrow 0$ for $\eta \in \boldsymbol{R}^3 \backslash \{0\}$ and suitable $f \in \mathcal{H}$. To state the main theorem, we introduce some terminology. We take and fix $\eta \in \boldsymbol{R}^3 \backslash \{0\}$ which we assume without loss of generality to be $\eta = (\eta_1, 0, 0)$. Since $\eta$ is fixed throughout the paper, we often omit the variable $\eta$ in the expressions. $\boldsymbol{R}^2_\eta$ is the plane containing the origin and perpendicular to $\eta$. For a set $D \subset \boldsymbol{R}^3$, $D_\eta$ is the projection of $D$ to $\boldsymbol{R}^2_\eta$. For $a \in \boldsymbol{R}^3$, $a_\eta \equiv (a_2, a_3) \in \boldsymbol{R}^2_\eta$ is its projection to $\boldsymbol{R}^2_\eta$ and $a_1 \equiv a\cdot\eta/|\eta|$; $a = (a_1, a_\eta)$. $e(\eta) \equiv \{a \in \boldsymbol{R}^3$: $(a, \eta) \in e\}$.

By the first statement of Theorem 1.2, $a + tv(\eta) \notin e(\eta)$ for all $t \in \boldsymbol{R}^1$ if $a \notin e(\eta)$ and for such $a$,

$$\eta_+(a + tv(\eta), \eta) = \eta_+(a, \eta), \quad a_+(a + tv(\eta), \eta) = a_+(a, \eta) + tv(\eta_+(a, \eta)). \tag{1.7}$$

By the conservation of energy

$$|\eta_+(a, \eta)|^2 = |\eta|^2. \tag{1.8}$$

By (1.7) and (1.8), it is natural to define a mapping $\Omega_\eta$ from $\boldsymbol{R}^2 \backslash e(\eta)_\eta$ to the unit sphere $S^2$ as

$$\Omega_\eta(a_\eta) = \eta_+((a_1, a_\eta), \eta)/|\eta|, \quad a_1 \in \boldsymbol{R}^1.$$

$\det \Omega_\eta$ is the determinant of $\Omega_\eta$ with respect to the natural surface elements (and orientations) of $\boldsymbol{R}^2_\eta$ and $S^2$. $|\det \Omega_\eta(a_\eta)|^{-1}$ is known to be the differential cross section in the scattering theory of classical particles.

$$e(\eta)^{ex} = \{(a_1, a_\eta) \in \boldsymbol{R}^3 \backslash e(\eta) \colon \det \Omega_\eta(a_\eta) = 0\} \cup e(\eta). \tag{1.9}$$

If $f \in C^\infty(\boldsymbol{R}^3 \backslash e(\eta))$ satisfies

$$D(\eta) f(a) \equiv \Big(\sum_{j=1}^{3} \alpha^j \eta^j + m\beta\Big) f(a) = \sqrt{\eta^2 + m^2} f(a), \tag{1.10}$$

we will show that the equation (1.4) (replacing $x(t)$ by $x_-(t, a, \eta)$) has a unique solution $f(t, a, \eta)$ such that $\lim_{t\to-\infty} f(t, a, \eta) = f(a)$ and that $\lim_{t\to\infty} f(t, a, \eta) = f_+(a, \eta)$ exists.

$$S(a, \eta) = \lim_{\substack{t\to\infty \\ s\to-\infty}} \left\{\int_s^t L(x_-(\sigma, a), \dot{x}_-(\sigma, a)) d\sigma + (t - s) m (1 - v(\eta)^2)^{1/2}\right\} + a\cdot\eta$$

where $L(x, \dot{x}) = -m(1 - \dot{x}^2)^{1/2} + e\vec{A}(\dot{x})\cdot\dot{x} - e\phi(x)$ is the Lagrangian. $\operatorname{Ind}\gamma(a, \eta)$ is the Keller-Maslov index of the orbit

$$\{(t, -H(x_-(t, a), \xi_-(t, a)), x_-(t, a), \xi_-(t, a)) \colon -\infty < t < \infty\} \quad \text{and} \quad \operatorname{Ind}\gamma_0(a, \eta)$$

is that of $\{(t, -(\eta^2+m^2)^{1/2}, tv(\eta_+(a, \eta))+a_+(a, \eta), \eta_+(a, \eta)) : 0 \leqq t < \infty\}$. $\|\cdot\|_m$ is the $m$-th Sobolev norm. For a subset $A$ of $\{1, 2, 3\}$, say $A=\{1, 2\}$, $x_A=(x_1, x_2)$; $A^c$ is the complement of $A$; $|A|$ is the cardinal number of $A$; $\mathcal{F}_A^\hbar$ is the partial Fourier transform:

$$(\mathcal{F}_A^\hbar f)(x_{A^c}, \xi_A)=(2\pi\hbar)^{-|A|/2}\int \exp(-ix_A\cdot\xi_A/\hbar)f(x_A, x_{A^c})dx_A.$$

For $f\in\mathcal{H}$, $f_\eta^\hbar(x)=\exp(ix\cdot\eta/\hbar)f(x)$. If $\operatorname{supp} f\subset\Omega\subset\boldsymbol{R}^3$ and the mapping $\Omega\ni a\to(a_+(a, \eta)_{A^c}, \eta_+(a, \eta)_A)$ is a diffeomorphism for some $A\subset\{1, 2, 3\}$, we set as

$$(Q_{\eta,A}^\hbar f)(x_{A^c}, \xi_A)=\begin{cases}\exp(i(S(a, \eta)-a_+(a, \eta)_A\cdot\eta_+(a, \eta)_A)/\hbar-i\pi(\operatorname{Ind}\gamma(a, \eta)-\operatorname{Ind}\gamma_0(a, \eta)+\\ \quad\operatorname{Inert}(\partial a_{+,A}/\partial\eta_{+,A})(a, \eta))/2+i\pi|A|/4)\,|\det\partial(a_+(a, \eta)_{A^c},\\ \quad\eta_+(a, \eta)_A)/\partial a|^{-1/2}f_+(a, \eta), \text{ if } (x_{A^c}, \xi_A)=(a_{+,A^c}(a, \eta), \eta_{+,A}(a, \eta))\\ 0, \text{ otherwise.}\end{cases}$$

Now we can state our main theorem in the paper.

THEOREM 1.3. *Let Assumption* (VSR) *be satisfied and let $K$ be a compact subset of $\boldsymbol{R}^3\backslash e(\eta)^{ex}$. Then there exists a finite open covering $\{\Omega_l\}$ of $K$ so that the following statements hold.*

(1) *For each $l$, $\bar{\Omega}_l\subset\boldsymbol{R}^3\backslash e(\eta)^{ex}$ and there exists a subset $A_l$ of $\{1, 2, 3\}$ such that the mapping $\Omega_l\ni a\to(a_+(a, \eta)_{A_l^c}, \eta_+(a, \eta)_{A_l})$ is a diffeomorphism.*

(2) *If $f\in C_0^\infty(\Omega_l)$ satisfies* (1.10), *then*

$$\|(\mathcal{F}_{A_l}^\hbar S^\hbar f_\eta^\hbar)(x_{A_l^c}, \xi_{A_l})-(Q_{\eta,A_l}^\hbar f)(x_{A_l^c}, \xi_{A_l})\|\leqq C\hbar^{\varepsilon/(2+\varepsilon)}\|f\|_2, \tag{1.11}$$

*where the constance $C$ is independent of $f\in C_0^\infty(\Omega_l)$ and $\varepsilon>0$ is that of* (VSR).

REMARK 1.4. (1) If $K\Subset\boldsymbol{R}^3\backslash e(\eta)^{ex}$, then by (1.8) the matrix $(\eta_+(a, \eta), \partial\eta_+(a, \eta)/\partial a_2, \partial\eta_+(a, \eta)/\partial a_3)$ is non-singular on $K$. Since

$$(\partial a_+/\partial a)(a+tv(\eta), \eta)=(v(\eta_+(a, \eta)), t(\partial v/a\eta)(\eta_+(a, \eta))\cdot(\partial\eta_+/\partial a_\eta)(a, \eta)+(\partial a_+/\partial a_\eta)(a, \eta)),$$

we see that if $K$ is replaced by $K+tv(\eta)$ with large $t>0$, we may choose $A_l=\varnothing$ for all $l$. ($\varnothing$ is the empty set.)

(2) The quasi-classical approximation for the scattering operator associated with non-relativistic Schrödinger equations was studied in [6], [7] and [8]. In [6] and [7], it was studied in the momentum space representation and the decay of the potential at the infinity was assumed only $(1+|x|)^{-1-\varepsilon}$ in [6] and $(1+|x|)^{-\varepsilon}$ in [7]. On the other hand in [8], it was studied in the configuration space and our tech-

nique required the decay $(1+|x|)^{-2-\varepsilon}$ of the potential. The technique employed here is similar to that in [8] and we assume the assumption (VSR).

We refer to [1], [4] and [5] for other approaches to the quasi-classical approximation or the classical limit of quantum scattering theory.

The rest of the paper is devoted to proving Theorem 1.3. Several lemmas will be prepared in Section 2 which will be used in Section 3 to complete the proof. The theorems, formulas and etc. of [1] will be referred to as Theorem 4.I, (2.1, I) and etc. and the notation in [I] may be used without any comment.

## § 2. Lemmas.

We collect several lemmas here which will be needed in Section 3 for proving the theorem. We assume in this section that (AK): *$K \Subset \boldsymbol{R}^3 \setminus e(\eta)^{ex}$, $K$ is convex and $\Omega_\eta$ is a diffeomorphism on $K_\eta$.* As a convention, $a_-(a, \eta)=a$, $\eta_-(a, \eta)=\eta$.

LEMMA 2.1. *There exists a constant $R_\pm>0$ such that the mapping $K \ni a \to x_-(t, a, \eta)$ and $K \ni a \to tv(\eta_\pm(a))+a_\pm(a)$ are diffeomorphisms on $K$ for all $\pm t>R_\pm$.*

PROOF. For lower signs the statement is obvious by (1.5). We prove the lemma for upper signs. By (1.7), (1.8) and (AK), the matrix

$$M(a) \equiv (\eta^2+m^2)^{1/2}(\eta_+(a), (\partial\eta_+/\partial a_2)(a), (\partial\eta_+/\partial a_3)(a))$$

is independent of $a_1=a\cdot\eta/|\eta|$ and is non-singular on $K$. Writing as $I_t$ the diagonal matrix with diagonal elements $(1, 1/t, 1/t)$, we have by (1.5) that

$$|(\partial/\partial a)^\alpha\{(\partial x_-/\partial a)(t, a, \eta)I_t - M(a)\}| \leqq Ct^{-1}. \tag{2.1}$$

It follows that for large enough $t$, $x_-(t, a, \eta)$ is a local diffeomorphism on $K$ and we have to show that $x_-(t, a)$ is one to one on $K$ for large enough $t>R_+$. If not so, there exist sequences $\{t_n\}_{n=1}^\infty$, $\{a_n\}$, $\{a_n'\} \subset K$ such that $t_n \to \infty$, $a_n \neq a_n'$ and $x_-(t_n, a_n) = x_-(t_n, a_n')$. We may assume, by passing to a subsequence if necessary, that $a_n \to a$ and $a_n' \to a'$ for some $a, a' \in K$. It is clear by Theorem 1.2 that $\eta_+(a, \eta)=\eta_+(a', \eta)$ and by (AK) $a_\eta = a_\eta'$. Hence by (1.7)

$$\int_0^1 M(a_n\theta+(1-\theta)a_n')d\theta \to M(a), \quad n\to\infty$$

and by (2.1), $\int_0^1 (\partial x_-/\partial a)(t_n, \theta a_n+(1-\theta)a_n')d\theta$ is non-singular for large enough $n$. However this cannot happen since

$$0 = x_-(t_n, a_n, \eta) - x_-(t_n, a_n', \eta) = \left(\int_0^1 (\partial x_-/\partial a)(t_n, \theta a_n+(1-\theta)a_n')d\theta\right)(a_n-a_n')$$

and $a_n - a_n' \neq 0$. This proves the lemma for $x_-(t, a, \eta)$. A similar proof clearly applies to $tv(\eta_+(a, \eta)) + a_+(a, \eta)$. Q.E.D.

Since $x_-(t, a+sv(\eta), \eta) = x_-(t+s, a, \eta)$, Lemma 2.1 implies the following.

COROLLARY 2.2. *Let $K$ and $R_\pm$ be as above. Then the mappings $x_-(t, a, \eta)$ and $tv(\eta_+(a, \eta)) + a_+(a, \eta)$ are diffeomorphisms on $K + R_+ v(\eta)$ for $t \geqq 0$.*

For $\pm t > R_\pm$, we write the inverse of $x = x_-(t, a, \eta)$ as $a = a(t, x)$. $a(t, x)$ is defined on $x_-(t, K)$.

LEMMA 2.3. *For any multi-index $\alpha$, there exists a constant $C_\alpha$ independent of $\pm t > R_\pm$ such that*

$$|(\partial^\alpha x_- / \partial a^\alpha)(t, a)| \leqq C_\alpha |t|, \quad a \in K; \tag{2.2}$$

$$|I_t^{-1} (\partial^\alpha a / \partial x^\alpha)(t, x)| \leqq C_\alpha, \quad x \in x_-(t, K); \tag{2.3}$$

$$|(\partial / \partial a)^\alpha |\det(\partial x_- / \partial a)(t, a)|^{-1/2}| \leqq C_\alpha |\det(\partial x_- / \partial a)(t, a)|^{-1/2}. \tag{2.4}$$

PROOF. We prove for $t > R_+$ only. The other case may be proved similarly. By Theorem 1.2, the estimate (2.2) is obvious. Differentiating the identity $x = x_-(t, a(t, x))$ by $x$, we have

$$I = (\partial x_- / \partial a)(t, a(t, x))(\partial a / \partial x)(t, x). \tag{2.5}$$

By (2.1) and (2.5) we have

$$|I_t^{-1} (\partial a / \partial x)(t, x) - M(a(t, x))^{-1}| \leqq C t^{-1}, \tag{2.6}$$

and (2.3) is proved for $|\alpha| = 1$. Suppose we already have (2.3) for all $\beta < \alpha$. Differentiating (2.5) by $x$, we have

$$\begin{aligned} 0 = \sum_{2 \leqq |\beta|} & (\partial^\beta x_- / \partial a^\beta)(t, a)(\partial^{\alpha_1} a / \partial x^{\alpha_1})(t, x) \cdots (\partial^{\alpha_k} a / \partial x^{\alpha_k})(t, x) \\ & + (\partial x_- / \partial a)(t, a(t, x))(\partial^\alpha a / \partial x^\alpha)(t, x), \end{aligned} \tag{2.7}$$

where summation runs over $\beta \leqq \alpha$, $\alpha_1 + \cdots + \alpha_n = \alpha$, $|\alpha_j| \geqq 1$.
Since

$$|(\partial^\beta x_- / \partial a^\beta)(t, a)(\partial^{\alpha_1} a / \partial x^{\alpha_1})(t, x(t, a))| \leqq C, \tag{2.8}$$

by (2.1) and the assumption of the induction, we obtain (2.3) for $\alpha$ by (2.7). To show (2.4), we note that any derivative of $|\det(\partial x_- / \partial a)|^{-1/2}$ can be written as $|\det(\partial x_- / \partial a)|^{-1/2}$ times a polynomial of the derivatives of $ln|\det(\partial x_- / \partial a)(t, a)|$. Since $ln|\det(\partial x_- / \partial a)(t, a)|$ has uniformly bounded derivatives by virtue of (2.1) and

$\det M(a) \neq 0$, we obtain (2.4). Q.E.D.

COROLLARY 2.4. *For any* $k \geqq 0$ *and* $\alpha$,

$$|(\partial/\partial t)^k (\partial/\partial x)^\alpha a(t, x)| \leqq C_{\alpha k}, \quad \pm t > R_\pm, \ x \in x_-(t, K). \tag{2.9}$$

PROOF. For $k=0$, (2.9) is proved in Lemma 2.3.
Differentiating the identity $x = x_-(t, a(t, x))$ by $t$ and using the equations $\dot{x}_-(t, a) = v(\xi_-(t, a) - \vec{A}(x_-(t, a)))$ and $\dot{\xi}_-(t, a) = -(\partial H/\partial x)(x_-(t, a), \xi_-(t, a))$, we see that the derivative $(\partial^k a/\partial t^k)(t, x)$ may be expressed as a polynomial of $(\partial/\partial t)^l (\partial/\partial x)^\alpha a(t, x)$, $(\partial^\beta \xi_-/\partial a^\beta)(t, a)$, $(\partial^\gamma A^\mu/\partial x^\gamma)(x)$ and $((\xi_-(t, a) - e\vec{A}(x_-(t, a)))^2 + m^2)^{-1/2}$ with $l \leqq k-1$, $|\alpha| \leqq k$, $|\beta| \leqq k-1$, $|\gamma| \leqq k-1$ $(k \geqq 1)$. Thus if we have (2.9) for all $l \leqq k-1$, we obtain (2.9) for $k$ by the assumption (VSR), Theorem 1.2 and Lemma 2.3. Q.E.D.

Let us write the inverse of $x = x_{0,\pm}(t, a) \equiv tv(\eta_\pm(a, \eta)) + a_\pm(a, \eta)$ as $a = a_\pm^0(t, x)$ for $\pm t > R_\pm$.

COROLLARY 2.5. *Let* $K_1 \Subset K$. *Then for large enough* $\pm t > R_\pm$,

$$x_-(t, K_1) \cup x_{0,\pm}(t, K_1) \subset x_-(t, K) \cap x_{0,\pm}(t, K)$$

*and*

$$|(\partial/\partial x)^\alpha (a(t, x) - a_\pm(t, x))| \leqq C(1+|t|)^{-1-\varepsilon}, \quad x \in x_-(t, K) \cap x_{0,\pm}(t, K). \tag{2.10}$$

PROOF. Let us write $a = a(t, x)$ and $\tilde{a} = a_\pm^0(t, x)$.
Since $x_-(t, a(t, x)) = x_{0,\pm}(t, a_\pm^0(t, x))$, we have by (1.5) and Lemma 2.3 that

$$|(\partial/\partial x)^\alpha \{t(v(\eta_\pm(a)) - v(\eta_\pm(\tilde{a}))) + a_\pm(a) - a_\pm(\tilde{a})\}| \leqq C_\alpha (1+|t|)^{-1-\varepsilon}. \tag{2.11}$$

Hence as $|t| \to \infty$, $|\eta_\pm(a) - \eta_\pm(\tilde{a})| \to 0$ and hence $|a_\eta - \tilde{a}_\eta| \to 0$. Since

$$\begin{aligned} &t(v(\eta_\pm(a)) - v(\eta_\pm(\tilde{a}))) + a_\pm(a) - a_\pm(\tilde{a}) \\ &\qquad = \left\{ \int_0^1 M(\theta a + (1-\theta)\tilde{a}) d\theta + 0(t^{-1}) \right\} I_t^{-1}(a - \tilde{a}) \end{aligned}$$

and $M(a)^{-1}$ is bounded on $K$, we obtain (2.10) for $\alpha = 0$. The case $\alpha \neq 0$ is obtained by using (2.11) and Lemma 2.3. Q.E.D.

For $s \leqq -R_-$ and $t < -R_-$ or $t > R_+$ we define as

$$S(t, s, x) = \int_s^t L(x_-(\sigma, a), \dot{x}_-(\sigma, a)) d\sigma + a \cdot \eta - sm(1 - v(\eta)^2)^{1/2} \tag{2.12}$$

for $x = x_-(t, a, \eta) \in x_-(t, K, \eta)$.

LEMMA 2.6. *Let* $S(t, s, x)$ *be defined as above. Then*

(1) $\lim_{s\to-\infty} S(t, s, x) \equiv S(t, x)$ *exists in the* $C^\infty$*-topology.*

(2) $|S(t, x)+tm(1-v(\eta)^2)^{1/2}-a(t, x)\cdot\eta| \leqq C(1+|t|)^{-1-\varepsilon}$ *for* $t<-R_-$, $x \in x_-(t, K, \eta)$.

(3) $\lim_{t\to\infty} \{S(t, x_-(t, a))+t(1-v(\eta)^2)^{1/2}\} \equiv S(a, \eta)$ *exists in the* $C^\infty$*-topology.*

(4) *For* $t>R_+$, $|S(a, \eta)-t(1-v(\eta)^2)^{1/2}-S(t, x_-(t, a))| \leqq C(1+t)^{-1-\varepsilon}$.

PROOF. We first prove (1) and (2). Since $\dot{x}_-(t, a, \eta)=(\partial H/\partial\xi)(x_-(t, a), \xi_-(t, a))$, (1.5) implies

$$|(\partial/\partial a)^\alpha\{\dot{x}_-(t, a)-v(\eta)\}| \leqq C(1+|t|)^{-2-\varepsilon}, \quad t<-R_-. \tag{2.13}$$

Writing explicitly, we have

$$\begin{aligned} S(t, s, x_-(t, a)) &= -t(1-v(\eta)^2)^{1/2}+a\cdot\eta+m\int_s^t \{(1-v(\eta)^2)^{1/2}-(1-\dot{x}_-(\sigma, a)^2)^{1/2}\}d\sigma \\ &\quad +e\int_s^t \{\vec{A}(x_-(\sigma, a))\cdot\dot{x}_-(\sigma, a)-\phi(x_-(\sigma, a))\}d\sigma. \end{aligned} \tag{2.14}$$

Hence the assumption (VSR), (1.5), (2.2), (2.13) and the Lebesgue's dominated convergence theorem show that $\lim_{s\to-\infty} (\partial/\partial a)^\alpha S(t, s, x_-(t, a, \eta))$ exists uniformly on $K$ and the estimate (2) holds. By (1.5), we have

$$|(\partial/\partial a)^\alpha\{\dot{x}_-(t, a)-v(\eta_+(a))\}| \leqq C(1+t)^{-1-\varepsilon}. \tag{2.15}$$

Since $\eta^2=\eta_+(a, \eta)^2$, we see that

$$\begin{aligned} S(t, x_-(t, a))+t(1-v(\eta)^2)^{1/2} &= S(R_+, x_-(R_+, a))+R_+(1-v(\eta)^2)^{1/2} \\ &\quad +\int_{R_+}^t \{m(1-v(\eta_+(a))^2)^{1/2}-m(1-\dot{x}_-(\sigma, a)^2)^{1/2}+e\vec{A}(x_-(\sigma, a))\cdot\dot{x}_-(\sigma, a) \\ &\quad -e\phi(x_-(\sigma, a))\}d\sigma. \end{aligned}$$

Thus (VSR), (1.5), (2.2) and (2.15) show that the limit

$$\begin{aligned} \lim_{t\to\infty} (S(t, x_-(t, a))+t(1-v(\eta)^2)^{1/2}) &= S(R_+, x_-(R_+, a))+R_+(1-v(\eta)^2)^{1/2} \\ &\quad +\int_{R_+}^\infty \{m(1-v(\eta)^2)^{1/2}-m(1-\dot{x}_-(\sigma, a)^2)^{1/2}+e\vec{A}(x_-(\sigma, a))\cdot\dot{x}_-(\sigma, a)-e\phi(x_-(\sigma, a))\}d\sigma \end{aligned}$$

exists in the $C^\infty$-topology on $K$ and

$$|S(a, \eta)-t(1-v(\eta)^2)^{1/2}-S(t, x_-(t, a))| \leqq \int_t^\infty C(1+\sigma)^{-2-\varepsilon}d\sigma = C(1+t)^{-1-\varepsilon}.$$

Q.E.D.

LEMMA 2.7. *For any* $f_-(a) \in C_0^\infty(K)$ *such that* $D(\eta)f_-(a)=(\eta^2+m^2)^{1/2}f(a)$ *there exists a unique solution* $f(t, a)$ *of the transport equation*

(2.16) $$p_{0,-}(t, a)(df/dt)(t, a)+\sum i(\sigma^{\mu\nu}/4)F_{\mu\nu}(x_-(t, a))f(t, a)=0$$

*satisfying* $\lim_{t\to-\infty} f(t, a)=f_-(a)$, *where* $p_{0,-}(t, a)=((\xi_-(t, a)-e\vec{A}(x(t, a)))^2+m^2)^{1/2}$. $f(t, a)$ *satisfies the following properties.*

(1) $\lim_{t\to\infty} f(t, a)=f_+(a)$ *exists in the* $C^\infty$*-topology and*

(2.17) $$|(\partial/\partial a)^\alpha(f(t, a)-f_\pm(a))|\leqq C_\alpha(1+|t|)^{-2-\varepsilon}\|f\|_{\mathscr{B}^{|\alpha|}}, \quad \pm t>0.$$

(2) $|(\partial/\partial t)^k(\partial/\partial a)^\alpha f(t, a)|\leqq C_{\alpha k}\|f\|_{\mathscr{B}^{k+|\alpha|}}$.

(3) $D(x_-(t, a), \xi_-(t, a))f(t, a)=p_{0,-}(t, a)f(t, a)$.

PROOF. Let $B(t, a)=\sum i(\sigma^{\mu\nu}/4p_{0,-}(t, a))F_{\mu\nu}(x_-(t, a))$. Then by (1.5) and the condition (VSR),

(2.18) $$|(\partial/\partial a)^\alpha B(t, a)|\leqq C_\alpha(1+|t|)^{-3-\varepsilon}.$$

(2.16) can be written as

$$df/dt+B(t, a)f(t, a)=0.$$

Thus from a well-known theorem ([2], Theorem 8.1) the existence and uniqueness of $f(t, a)$ follows; statements (1) and (2) are obvious consequences of (2.18). By Lemma 2.8.I, $P_+(x_-(t, a, \eta), \xi_-(t, a, \eta))f(t, a)$ is also a solution of (2.16). Since $\lim_{t\to-\infty} P_+(x_-(t, a, \eta), \xi_-(t, a, \eta))f(t, a)=f_-(a)$, the uniqueness part of the lemma shows that $P_+(x_-(t, a, \eta), \xi_-(t, a, \eta))f(t, a)=f(t, a)$. This implies (3). Q.E.D.

COROLLARY 2.8. *Let* $g(t, x)\equiv f(t, a(t, x))$ *on* $x\in x_-(t, K, \eta)$, $\pm t>R_\pm$. *Then*

(2.19) $$|(\partial/\partial t)^k(\partial/\partial x)^\alpha g(t, x)|\leqq C_{k\alpha}, \quad \pm t>R_\pm.$$

The following lemmas are concerned about the propagators and have different characters from the previous ones.

LEMMA 2.9. *Let* $S(x)$ *be a real-valued* $C^\infty$*-function on* $\boldsymbol{R}^3$ *satisfying*

(1) $|(\partial/\partial x)^\alpha S(x)|\leqq C_\alpha$ *for* $|\alpha|\geqq 2$;

(2) *there exists a constant* $\delta>0$ *such that* $|\mathrm{grad}\, S(x)|\geqq\delta$ *for all* $x\in\boldsymbol{R}^3$.

*Suppose* $\chi\in C^\infty(\boldsymbol{R}^3)$ *be such that* $\chi(\xi)=1$ *for* $|\xi|\geqq\frac{3}{4}\delta$ *and* $\chi(\xi)=0$ *for* $|\xi|\leqq\delta/2$. *Then for any* $n\geqq 0$,

(2.20) $$\|(\chi(\xi)-1)\mathscr{F}^{\hbar}(\exp(iS(x)/\hbar)f)\|\leqq C_n\hbar^n\|f\|_n.$$

PROOF. By conditions (1) and (2),

$$\phi(x,\xi)\equiv(1-\chi(\xi))(\xi\text{-grad } S(x))(\xi\text{-grad } S(x))^{-2}\in\mathcal{B}(\boldsymbol{R}^3\times\boldsymbol{R}^3).$$

By integration by parts, we have for any integer $n$ that

$$\begin{aligned}&(1-\chi(\xi))\mathcal{F}^{\hbar}(\exp(iS(x)/\hbar)f)(\xi)\\&=(2\pi\hbar)^{-3/2}(i\hbar)^n\int\exp(i(S(x)-x\cdot\xi)/\hbar)[(\phi(x,\xi)\cdot\partial/\partial x)^*]^n f(x)dx.\end{aligned}$$

Applying Theorem A.1.I, we obtain (2.20) for any integer $n\geqq 0$. For non-integral $n$, (2.20) is obtained by the interpolation theorem. Q.E.D.

We write as $V(x)\equiv H^{\hbar}-H_0^{\hbar}=-e\sum_{j=1}^{3}\alpha^j A_j(x)+e\phi(x)$.

LEMMA 2.10. *Let $S(x)$ and $\chi(\xi)$ be as in Lemma 2.9. Then*

$$\|V(x)\exp(-itH_0^{\hbar}/\hbar)\mathcal{F}^{\hbar *}\chi(\xi)\mathcal{F}^{\hbar}(\exp(iS(x)/\hbar)f)\|\leqq C(1+|t|)^{-2-\varepsilon}\|(1+|x|)^{2+\varepsilon}f\|. \tag{2.21}$$

PROOF. Let $w(y)\in C_0^\infty(\boldsymbol{R}^3)$ be such that $w(y)=1$ for $|y|\leqq 1$ and $w(y)=0$ for $|y|\geqq 2$. Write $\tilde{v}(s)=s/(m^2+s^2)^{1/2}$, $\tilde{w}(y)=1-w(y)$, $f_t(y)=w(8y/t\tilde{v}(\delta/2))f(y)$ and $\tilde{f}_t(y)=f(y)-f_t(y)$. It is clear that

$$\begin{aligned}&\text{LHS of (2.21)}\\&\leqq\|V(x)\exp(-itH_0^{\hbar}/\hbar)\mathcal{F}^{\hbar *}\chi(\xi)\mathcal{F}^{\hbar}(\exp(iS(x)/\hbar)\tilde{f}_t)\|\\&\quad+\|w(4x/t\tilde{v}(\delta/2))V(x)\exp(-itH_0^{\hbar}/\hbar)\mathcal{F}^{\hbar *}\chi(\xi)\mathcal{F}^{\hbar}(\exp(iS(x)/\hbar)f_t\|\\&\quad+\|\tilde{w}(4x/t\tilde{v}(\delta/2))V(x)\exp(-itH_0^{\hbar}/\hbar)\mathcal{F}^{\hbar *}\chi(\xi)\mathcal{F}^{\hbar}(\exp(iS(x)/\hbar)f_t\|\\&\equiv I_1+I_2+I_3\end{aligned}\tag{2.22}$$

and that

$$\begin{aligned}I_1&\leqq C\|\tilde{f}_t\|\leqq C\|(1+|y|)^{-2-\varepsilon}\tilde{w}(8y/t\tilde{v}(\delta/2))\|_{L^\infty}\|(1+|y|)^{2+\varepsilon}f\|\\&\leqq C(1+|t|)^{-2-\varepsilon}\|(1+|x|)^{2+\varepsilon}f\|,\end{aligned}\tag{2.23}$$

$$I_3\leqq\|\tilde{w}(4x/t\tilde{v}(\delta/2))V(x)\|_{L^\infty}\|f_t\|\leqq C(1+|t|)^{-2-\varepsilon}\|f\|. \tag{2.24}$$

We estimate $I_2$. $P_\pm(\xi)$ are the orthogonal projections to the eigenspaces of $D(\xi)$ with eigenvalues $\pm(\xi^2+m^2)^{1/2}$. As an oscillatory integral,

$$\begin{aligned}\boldsymbol{I}(t,x)&\equiv t^{3/2}w(4x/\tilde{v}(\delta/2))(\exp(-itH_0^{\hbar}/\hbar)\mathcal{F}^{\hbar *}\chi(\xi)\mathcal{F}^{\hbar}(\exp(iS(x)/\hbar)f_t))(tx)\\&=\sum_{\pm}(2\pi/\nu)^{-3}\int\exp\{i\nu(x\cdot\xi-y\cdot\xi\mp(\xi^2+m^2)^{1/2}+t^{-1}S(ty))\}\\&\quad\times\chi(\xi)w(4x/\tilde{v}(\delta/2))w(8y/\tilde{v}(\delta/2))P_\pm(\xi)t^{3/2}f(ty)dyd\xi,\end{aligned}$$

where $\nu=t/\hbar$. Since $a_\pm(x,\xi,y)\equiv\chi(\xi)w(4x/\tilde{v}(\delta/2))w(8y/\tilde{v}(\delta/2))P_\pm(\xi)\in C_0^\infty(\boldsymbol{R}^3\times\boldsymbol{R}^3\times\boldsymbol{R}^3)$ and on supp $a_\pm$

$$|\mathrm{grad}_\xi(x\cdot\xi-y\cdot\xi\mp\sqrt{\xi^2+m^2})|\geqq v(\delta/2)/4>0,$$

the integration by parts with respect to $\xi$ and Theorem A.1.I show that

$$\|I(t,\cdot)\|\leqq C_l\nu^{-l}\|f\|=C_l(\hbar^l/t^l)\|f\|. \tag{2.25}$$

Since $I_2\leqq C\|I(t,\cdot)\|$, (2.22), (2.23), (2.24) and (2.25) imply (2.21). Q.E.D.

LEMMA 2.11. *Let* $S(x)$ *be as in Lemma* 2.9. $\beta=2/(2+\varepsilon)$ *and* $\gamma=\varepsilon/(2+\varepsilon)$. *Then for any* $f\in C_0^\infty(\boldsymbol{R}^3)$,

$$\begin{aligned}\sup_{\pm t>\hbar^{-\beta}}&\|W_\pm^\hbar(\exp(iS(x)/\hbar)f)(x)-\exp(itH^\hbar/\hbar)\exp(-itH_0^\hbar/\hbar)(\exp(iS(x)/\hbar)f)(x)\|\\&\leqq C\hbar^\gamma(\|(1+|x|)^{(2+\varepsilon)}f\|+\|f\|_\gamma),\end{aligned} \tag{2.26}$$

*where the constant does not depend on* $f\in C_0^\infty(\boldsymbol{R}^3)$.

PROOF. By virtue of Lemma 2.9 and the isometry property of $W_\pm^\hbar$ and $\exp(itH^\hbar/\hbar)\exp(-itH_0^\hbar/\hbar)$, it suffices to show (2.26) for

$$u^\hbar(x)\equiv\mathcal{F}^{\hbar*}\chi(\xi)\mathcal{F}^\hbar(\exp(iS(x)/\hbar)f)$$

in place of $\exp(iS(x)/\hbar)f(x)$. Since $u^\hbar\in D(H^\hbar)=D(H_0^\hbar)$, we obtain by (2.21) that

$$\begin{aligned}\sup_{\pm t\geqq\hbar^{-\beta}}&\|W_\pm^\hbar u^\hbar-\exp(itH^\hbar/\hbar)\exp(-itH_0^\hbar/\hbar)u^\hbar\|\\&\leqq\pm\hbar^{-1}\int_{\pm\hbar^{-\beta}}^{\pm\infty}\|V(x)\exp(-itH_0^\hbar/\hbar)u^\hbar\|dt\\&\leqq C\hbar^{-1}\int_{\hbar^{-\beta}}^{\infty}(1+t)^{-2-\varepsilon}\|(1+|x|)^{2+\varepsilon}f\|dt\\&\leqq C\hbar^\gamma\|(1+|x|)^{2+\varepsilon}f\|.\end{aligned}$$

Q.E.D.

## § 3. Proof of Theorem 1.3.

Since the mapping $\Omega_\eta:\boldsymbol{R}_\eta^2\backslash e(\eta)_\eta^{ex}\to S^2$ is a local diffeomorphism, for any $K\Subset\boldsymbol{R}^3\backslash e(\eta)^{ex}$ we can find a finite open covering $\{K_j\}_{j=1}^l$ of $K$ such that $K_j\Subset\boldsymbol{R}^3\backslash e(\eta)^{ex}$, $K_j$ is convex and $\Omega_\eta$ is a diffeomorphism on $(K_j)_\eta$ for each $j=1,\dots,l$. Clearly it suffices to show the theorem for each $K_j$. We take and fix one of the $K_j$'s and denote it as $K$. $R_\pm>0$ are the constants appeared in Lemma 2.1 associated with $K$. By the intertwining property of the $S$-matrix,

$$S^\hbar f_\eta^\hbar=\exp(iR_+H_0^\hbar/\hbar)S^\hbar\exp(-iR_+H_0^\hbar/\hbar)$$

and by Theorem 4.I we have

$$\|\exp(-iR_+H_0^\hbar/\hbar)f_\eta^\hbar(x)-\exp(-iR_+m/(1-v(\eta)^2)^{1/2}\hbar)\exp(ix\cdot\eta/\hbar)f(x-R_+v(\eta))\| \\ \leqq C\hbar\|f\|_2.$$

Hence by virtue of Theorem 4.I, it suffices to show the theorem for $f\in C_0^\infty(K+R_+v(\eta))$, and therefore by replacing $K$ by $K+R_+v(\eta)$, we may assume that the mappings $x_-(t,a,\eta)$ and $tv(\eta_+(a,\eta))+a_+(a,\eta)$ are diffeomorphisms on $K$ for all $t\geqq 0$ by Corollary 2.2. We write for $f\in C_0^\infty(K)$ with $D(\eta)f(a)=(\eta^2+m^2)^{1/2}f(a)$ as

$$Z_-^\hbar(t)f(x)=\exp(-itm/(1-v(\eta)^2)^{1/2}\hbar)\exp(ix\cdot\eta/\hbar)f(x-tv(\eta)) \text{ for } t<0;$$

$$Z^\hbar(t)f(x)=\begin{cases}\exp(iS(t,x_-(t,a))/\hbar)|\det(\partial x_-(t,a)/\partial a)|^{-1/2}(p_{0,-}(t,a)/(\eta^2+m^2)^{1/2})^{-1/2}f(t,a), \\ \qquad \text{if } x=x_-(t,a),\ a\in K,\ t<-R_-, \\ 0, \text{ otherwise, for } t<-R_-;\end{cases}$$

$$Z^\hbar(t)f(x)=\begin{cases}\exp(iS(t,x_-(t,a))/\hbar-i\pi\operatorname{Ind}\gamma(a,\eta)/2)|\det(\partial x_-(t,a)/\partial a)|^{-1/2} \\ \qquad \times(p_{0,-}(t,a)/(\eta^2+m^2)^{1/2})^{-1/2}f(t,a), \text{ if } x=x_-(t,a),\ a\in K,\ t>0, \\ 0, \text{ otherwise, for } t>0;\end{cases}$$

$$Z_+^\hbar(t)f(x)=\begin{cases}\exp(i(S(a,\eta)-tm(1-v(\eta)^2)^{1/2})/\hbar-i\pi\operatorname{Ind}\gamma(a,\eta)/2) \\ \qquad \times|\det(\partial/\partial a)(tv(\eta_+(a,\eta))+a_+(a,\eta))|^{-1/2}f_+(a), \\ \qquad \text{if } x=tv(\eta_+(a,\eta))+a_+(a,\eta),\ a\in K, \\ 0, \text{ for } t>0;\end{cases}$$

$$Q_\eta^\hbar f(x)=\begin{cases}\exp(iS(a,\eta)/\hbar-i\pi\operatorname{Ind}\gamma(a,\eta)/2)|\det\partial a_+/\partial a(a,\eta)|^{-1/2}f_+(a), \\ \qquad \text{if } x=a_+(a,\eta),\ a\in K, \\ 0, \text{ otherwise.}\end{cases}$$

LEMMA 3.1. *Let $K$ be as above and $f\in C_0^\infty(K)$. Then the following estimates hold.*

$$\|\exp(-itH_0^\hbar/\hbar)f_\eta^\hbar(x)-Z_-^\hbar(t)f(x)\|\leqq C\hbar|t|\|f\|_2,\ t\leqq -1; \tag{3.1}$$

$$\|\exp(-itH^\hbar/\hbar)Z^\hbar(s)f-Z^\hbar(s+t)f\|\leqq C\hbar|t|\|f\|_2,\ s\leqq -R_- \text{ and } t\leqq -1; \tag{3.2}$$

$$\|\exp(-itH^\hbar/\hbar)Z^\hbar(s)f-Z^\hbar(s+t)f\|\leqq C\hbar|t|\|f\|_2,\ s\geqq 0 \text{ and } t\geqq 1; \tag{3.3}$$

$$\|\exp(-itH_0^\hbar/\hbar)Q_\eta^\hbar f(x)-Z_+(t)f(x)\|\leqq C\hbar|t|\|f\|_2,\ t\geqq 1, \tag{3.4}$$

*where the constants in* (3.1)-(3.4) *do not depend on $t,\hbar$ and $f\in C_0^\infty(K)$.*

PROOF. Since the proofs of (3.1)-(3.4) are similar, we only prove (3.3) with $s=0$ as a prototype. We write $Z^\hbar(t)f(x)=\exp(iS(t,x)/\hbar)f_1(t,x)$ and $f_2(t,x)=$

$i\beta(2p_{0,-}(t, a(t, x)))^{-1}\times\tilde{\partial} f_1(t, x)$. The calculations of §2. I show that

$$(3.5)\qquad \begin{aligned}&(i\hbar\partial/\partial t-H^{\hbar})\{\exp(iS(t, x)/\hbar)(f_1(t, x)+\hbar f_2(t, x))\}\\&=i\beta\hbar^2\exp(iS(t, x)/\hbar)\tilde{\partial} f_2(t, x).\end{aligned}$$

Since the functions appearing in (3.5) are $\mathcal{H}$-valued smooth functions of $t\geqq 0$, du Hamel's principle implies,

$$\begin{aligned}&\exp(iS(t, x)/\hbar)(f_1(t, x)+\hbar f_2(t, x))\\&=\exp(-itH^{\hbar}/\hbar)\{\exp(iS(0, x)/\hbar)(f_1(0, x)+\hbar f_2(0, x))\}\\&\quad+\hbar\int_0^t\exp(-i(t-\sigma)H^{\hbar}/\hbar)\{\beta\exp(iS(\sigma, \cdot)/\hbar)\tilde{\partial} f_2(\sigma, \cdot)\}d\sigma.\end{aligned}$$

Hence by the unitarity of the propagator, we have

$$(3.6)\qquad \begin{aligned}&\|\exp(-itH^{\hbar}/\hbar)Z^{\hbar}(0)f-Z^{\hbar}(t)f\|\\&\leqq\hbar\left(\|f_2(0, \cdot)\|+\|f_2(t, \cdot)\|+\int_0^t\|\tilde{\partial} f_2(\sigma, \cdot)\|d\sigma\right).\end{aligned}$$

By Lemma 2.3, Corollary 2.4 and Lemma 2.7, $\|f_2(t, \cdot)\|\leqq C\|f\|_1$ and $\|\tilde{\partial} f_2(t, \cdot)\|\leqq C\|f\|_2$ for $t\geqq 0$. Hence the right hand side of (3.6) is bounded by $C\hbar|t|\|f\|_2$ for large $t$.

Q.E.D.

LEMMA 3.2. *Let $0<\pm t<\hbar^{-1}$ be sufficiently large. Then*

$$(3.7)\qquad \|Z_{\pm}^{\hbar}(t)f-Z^{\hbar}(t)f\|\leqq C(1/t^{1+\varepsilon}\hbar)\|f\|_1.$$

PROOF. Let us write $f_0^{\pm}(t, a)=|\det(\partial/\partial a)(tv(\eta_{\pm}(a, \eta))+a_{\pm}(a, \eta))|^{-1/2}\times f_{\pm}(a)$, and $f_1(t, x)$ is as in the proof of Lemma 3.1 with $\exp(-i\pi\operatorname{Ind}\gamma/2)$ being removed for $t>0$. We prove $+$ case only and omit unnecessary $+$ signs in the expressions. The other case may be proved similarly. Clearly

$$(3.8)\qquad \begin{aligned}&\|Z_+^{\hbar}(t)f(x)-Z^{\hbar}(t)f(x)\|\\&=\|\exp(i(S(t, x)-S(a_+^0(t, x), \eta)+tm(1-v(\eta)^2)^{1/2})/\hbar)\times f_1(t, x)-f_0(t, a_+^0(t, x))\|\end{aligned}$$

(recall $a_+^0(t, x)$ is the inverse of $x=tv(\eta_+(a))+a_+(a))$. Write $x_{0,+}(t, a)=tv(\eta_+(a, \eta))+a_+(a, \eta)$. By (1.5) and (2.1)

$$(3.9)\qquad ||\det(\partial x_{0,+}(t, a)/\partial a)|^{-1/2}|\det(\partial x_-(t, a)/\partial a)|^{1/2}-1|\leqq C(1+t)^{-1-\varepsilon},$$

$$(3.10)\qquad |p_{0,-}(t, a)/(\eta^2+m^2)^{1/2}-1|\leqq C(1+t)^{-1-\varepsilon},\ \text{for } a\in K,\ t>0,$$

and by Corollary 2.5 and Lemma 2.6,

(3.11)
$$|S(t,x)-S(a_+^0(t,x),\eta)+tm(1-v(\eta)^2)^{1/2}|$$
$$\leqq |S(t,x)-S(a(t,x),\eta)+tm(1-v(\eta)^2)^{1/2}|+|S(a(t,x),\eta)-S(a_+^0(t,x),\eta)|$$
$$\leqq C(1+t)^{-1-\varepsilon}$$

for $t>0$ on $\operatorname{supp} f_1(t,\cdot)\cup\operatorname{supp} f_0(t,\cdot)$. By (3.8) and (3.11),

(3.12)
$$\|Z_+^{\hbar}(t)f(x)-Z^{\hbar}(t)f(x)\|\leqq C(1+t)^{-1-\varepsilon}\hbar^{-1}\|f\|+\|f_1(t,x)-f_0(t,a_+^0(t,x))\|$$
$$\leqq C(1+t)^{-1-\varepsilon}\hbar^{-1}\|f\|+\|f_1(t,x)-f_0(t,a(t,x))\|+\|f_0(t,a(t,x))-f_0(t,a_+^0(t,x))\|.$$

Using a change of variables, Lemma 2.7, (3.9) and (3.10), we see

(3.13)
$$\|f_1(t,x)-f_0(t,a(t,x))\|$$
$$=\|(p_{0,-}(t,a)/(\eta^2+m^2)^{1/2})^{-1/2}f(t,a)$$
$$-|\det(\partial x_-(t,a)/\partial a)|^{1/2}|\det(\partial x_{+,0}(t,a)/\partial a)|^{-1/2}f_+(a)\|$$
$$\leqq\|\{(p_{0,-}(t,a)/(\eta^2+m^2)^{1/2})^{-1/2}-1\}f(t,a)\|+\|f(t,a)-f_+(a)\|$$
$$+\|\{|\det(\partial x_-(t,a)/\partial a)|^{1/2}|\det(\partial x_{+,0}(t,a)/\partial a)|^{-1/2}-1\}f_+(a)\|$$
$$\leqq C(1+t)^{-1-\varepsilon}\|f\|.$$

By Corollary 2.5, we have

(3.14)
$$\|f_0(t,a(t,x))-f_0(t,a_+^0(t,x))\|$$
$$\leqq C(1+t)^{-1-\varepsilon}\int_0^1\|(\partial f_0/\partial a)(t,\theta a(t,x)+(1-\theta)a_+^0(t,x))\|d\theta.$$

For large $t>0$, $a_+^0(t,x)$ is a diffeomorphism and by Lemma 2.3 and Corollary 2.5 $|(\partial a_+^0/\partial x)(t,x)^{-1}(\partial a(t,x)/\partial x-(\partial a_+^0/\partial x)(t,x))|\leqq C(1+t)^{-\varepsilon}$. Therefore for any $0\leqq\theta\leqq1$ the mapping $y=\theta a(t,x)+(1-\theta)a_+^0(t,x)$ is a diffeomorphism and

(3.15)
$$\|(\partial f_0/\partial a)(t,\theta a(t,x)+(1-\theta)a_+^0(t,x))\|\leqq C\|f\|_1,$$

since $\partial/\partial x(\theta a(t,x)+(1-\theta)a_+^0(t,x))=(\partial a_+^0(t,x)/\partial x)(1+\theta(\partial a_+^0(t,x)/\partial x)^{-1}(\partial a(t,x)/\partial x-\partial a_+^0(t,x)/\partial x))$,

$$|(\partial f_0/\partial a)(t,a)|\leqq C|\det(\partial x_{+,0}(t,a)/\partial a)|^{-1/2}(|f_+(a)|+|\partial f_+/\partial a|)$$

by Lemma 2.3 and

$$|\det\{\partial/\partial x(\theta a(t,x)+(1-\theta)a_+(t,x))\}|=|\det(\partial a_+(t,x)/\partial x)|(1+t^{-\varepsilon}).$$

Summarizing (3.12), (3.13), (3.14), (3.15) and (3.16), we have the desired result (3.7). Q.E.D.

LEMMA 3.3. *Let $R_-$ be as in Lemma 2.1. Let $f\in C_0^\infty(K)$. Then*

(3.17) $$\|\exp(iR_-H^{\hbar}/\hbar)W_-^{\hbar}f_\eta^{\hbar}(x)-Z^{\hbar}(-R_-)f(x)\|\leqq C\hbar^\gamma\|f\|_2,\ \gamma=\varepsilon/(2+\varepsilon).$$

PROOF. By (3.1), (3.2) and (3.7), we have for $-t>0$ large enough

$$\begin{aligned}&\|\exp(-i(R_-+t)H^{\hbar}/\hbar)Z^{\hbar}(-R_-)f(x)-\exp(-itH_0^{\hbar}/\hbar)f_\eta^{\hbar}(x)\|\\&\leqq C(\hbar|t|+1/\hbar|t|^{1+\varepsilon})\|f\|_2.\end{aligned}$$

In particular, setting $t=-\hbar^{-\beta}$, $\beta=2/(2+\varepsilon)$, we have

(3.18) $$\|Z^{\hbar}(-R_-)f(x)-\exp(i(R_--\hbar^{-\beta})H^{\hbar}/\hbar)\exp(i\hbar^{-\beta}H_0^{\hbar}/\hbar)f_\eta^{\hbar}(x)\|\leqq C\hbar^\gamma\|f\|_2.$$

Estimates (3.18) and (2.26) clearly imply (3.17). Q.E.D.

By a similar argument, we obtain

LEMMA 3.4. *Let $R_+\geqq 0$ and $f\in C_0^\infty(K)$. Then*

(3.19) $$\|\exp(-iR_+H^{\hbar}/\hbar)W_+^{\hbar}Q_\eta^{\hbar}f(x)-Z^{\hbar}(R_+)f\|\leqq C\hbar^\gamma\|f\|_2.$$

*Completion of the proof of Theorem* 1.3.

By Theorem 4.I and the definition of $Z^{\hbar}(t)$, we have

(3.20) $$\|\exp(-(R_++R_-)H^{\hbar}/\hbar)Z^{\hbar}(-R_-)f(x)-Z^{\hbar}(R_+)f(x)\|\leqq C\hbar\|f\|_2.$$

Combining (3.17) and (3.19) with (3.20), we obtain

(3.21) $$\|W_+^{\hbar}Q_\eta^{\hbar}f(x)-W_-^{\hbar}f_\eta^{\hbar}(x)\|\leqq C\hbar^\gamma\|f\|_2.$$

Since $(W_+^{\hbar})^*W_+^{\hbar}=I$ and $(W_+^{\hbar})^*W_-^{\hbar}=S^{\hbar}$, it follows from (3.21) that

$$\|Q_\eta^{\hbar}f(x)-S^{\hbar}f(x)\|\leqq C\hbar^\gamma\|f\|_2$$

which is our desired estimate. Q.E.D.

## References

[1] Antŏnetz, A. M., The classical limit of Cauchy problem for Heisenberg-Weyl evolution equation, Bestn. Moskow Univ. Ser. Math-Mech. **6** (1977), 26-33 (in Russian).
[2] Coddington, A. and N. Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955.
[3] Eckardt, K. J., Scattering theory for Dirac operators, Math. Z. **139** (1974), 105-131.
[4] Hagedorn, G., Semi-classical quantum mechanics, I, The $h\to 0$ limit for coherent states, Comm. Math. Phys. **71** (1980), 77-93.
[5] Ginibre, J. and G. Velo, The classical field limit of scattering theory for non-relativistic many boson systems, I, Comm. Math. Phys. **66** (1979), 37.
[6] Yajima, K., The quasi-classical limit of quantum scattering theory, Comm. Math. Phys. **69** (1979), 101-129.

[7] Yajima, K., The quasi-classical limit of quantum scattering theory II, long range scattering, Duke J. Math. **48** (1981), 1-22.
[8] Yajima, K., Towards the quasi-classical limit of quantum scattering operator, Kagaku **50** (1980), 614-620 (in Japanese).
[9] Yajima, K., The quasi-classical approximation to Dirac equation, I, J. Fac. Sci. Univ. Tokyo Sect. IA **29** (1982), 161-194.
[10] Yajima, K., Classical scattering for relativistic particles, preprint, Princeton Univ., 1979.
[11] Yamada, O., Eigenfunction expansions and scattering theory for Dirac operators, Publ. Res. Inst. Math. Sci. Kyoto University **11** (1976), 651-689.

(Received March 23, 1981)

Department of Mathematics
Princeton University
Princeton, N. J. 08544
U.S.A.

and

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan

Address after September 1, 1982
Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan