The support property of a Gaussian white noise and its applications

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1. Introduction

A Gaussian white noise is the probability measure μ on $\mathcal{S}'(\boldsymbol{R}^d)$ such that

$$\int_{\mathcal{S}'(\boldsymbol{R}^d)} \exp(i_S\langle f, w\rangle_{S'})\mu(dw) = \exp\left(-\frac{1}{2}\int_{\boldsymbol{R}^d} |f(x)|^2 dx\right)$$

for any $f \in \mathcal{S}(\boldsymbol{R}^d)$, where $\mathcal{S}(\boldsymbol{R}^d)$ is a space of real valued rapidly decreasing smooth functions defined in a d-dimensional Euclidean space $\boldsymbol{R}^d$ and $\mathcal{S}'(\boldsymbol{R}^d)$ is a space of real valued tempered distributions defined in $\boldsymbol{R}^d$. $\mathcal{S}'(\boldsymbol{R}^d)$ is, however, so large that continuous nonlinear functionals on it are rather poor, and thus $\mathcal{S}'(\boldsymbol{R}^d)$ is not suitable to study nonlinear transformations of white noise.

Therefore it is useful to find support linear subspaces with much continuous nonlinear functionals. Here we say that E is a support linear subspace, when E is a Borel subset of $\mathcal{S}'(\boldsymbol{R}^d)$ as a set, E itself is a topological vector space, and $\mu(E)=1$. The purpose of the present paper is to introduce several weighted Sobolev spaces easy to analyze, and to study as to when they become support linear subspaces.

Let us shortly summarize the content of our paper. We will introduce some weighted Sobolev spaces and give the necessary and sufficient condition for them to become support linear subspaces in Section 3. We will show in Proposition 3.3 that the set

$$(1-\Delta)^{d/4}\{\log(2-\Delta)\}^{t/2}(1+|x|^2)^{d/2p}\{\log(2+|x|^2)\}^{s/p}L^p(\boldsymbol{R}^d),$$

$1<p<\infty$, has μ-measure one (resp. zero) if and only if $s>1$ and $t>1$ (resp. $s\leq 1$ or $t\leq 1$), where Δ is the Laplacian in $\boldsymbol{R}^d$ and $|x|^2=x_1^2+\cdots+x_d^2$.

We will also show in Proposition 3.4 that the set

$$(1-\Delta_{d-1})^{d/4-1/2}\{\log(2-\Delta_{d-1})\}^{t/2}(1+|x|^2)^{d/2p}\{\log(2+|x|^2)\}^{s/p}L^p(\boldsymbol{R}^d),$$

$1<p<\infty$ and $d\geq 2$, has ν-measure one (resp. zero) if and only if $s>1$ and $t>1$

*) Research partially supported by the SAKKOKAI FOUNDATION.

(resp. $s\leq 1$ or $t\leq 1$), where ν is the free measure for Boson fields and Δ_{d-1} is the Laplacian in any $d-1$ dimensions. These results have been proved by Reed and Rosen [7] in the case that $p=2$.

We will study in Section 4 about the relations between weighted Sobolev spaces and pseudo-differential operators. We will consider in Section 5 the following nonlinear stochastic pseudo-differential equation

$$p(D_x)X-b(q_1(D_x)X,\dots,q_n(D_x)X)=W,$$

where W is a Gaussian white noise, $b: \boldsymbol{R}^n\to\boldsymbol{R}$ is a smooth function, and $p(D_x)$ and $q_i(D_x)$, $i=1,\dots,n$ are pseudo-differential operators. We will show the existence and uniqueness of a solution X under some conditions.

The author wishes to thank Professor H. Kumano-go for his valuable advice, and also wishes to thank Professor Y. Okabe for useful conversation and his hearty encouragement.

2. Preliminaries

DEFINITION 2.1. We say that a complex-valued smooth function $p(x,\xi)$ defined in $\boldsymbol{R}^d\times\boldsymbol{R}^d$ belongs to a class $\tilde{S}^m$, $m\in\boldsymbol{R}$, if $p(x,\xi)$ satisfies the following two conditions:

(S-1) $\bar{p}(x,\xi)=p(x,-\xi)$ for any $(x,\xi)\in\boldsymbol{R}^d\times\boldsymbol{R}^d$, and

(S-2) for any multi-indices α,β, there exists a constant $C_{\alpha,\beta}$ such that $|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}$.

Here $\bar{z}$ denotes a conjugate number of a complex number z, $\alpha=(\alpha_1,\dots,\alpha_d)$ and $\beta=(\beta_1,\dots,\beta_d)$ are multi-indices whose elements are non-negative integers, $|\beta|=\beta_1+\cdots+\beta_d$, $\langle\xi\rangle=(1+|\xi|^2)^{1/2}$, $|\xi|=(\xi_1^2+\cdots+\xi_d^2)^{1/2}$, and

$$\partial_{x_j}=\frac{\partial}{\partial x_j},\qquad \partial_{\xi_j}=\frac{\partial}{\partial\xi_j},\qquad j=1,\dots,d,\qquad \partial_x^\alpha=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d},\qquad \partial_\xi^\beta=\partial_{\xi_1}^{\beta_1}\cdots\partial_{\xi_d}^{\beta_d}.$$

REMARK 2.1. $\tilde{S}^m$ is a Fréchet space with semi-norms $|\ |_n^{(m)}$, $n=0,1,2,\dots$, defined by

$$|p|_n^{(m)}=\sup\{|\partial_x^\alpha\partial_\xi^\beta p(x,\xi)|\langle\xi\rangle^{-m+|\beta|};\ |\alpha|+|\beta|\leq n,\ x,\xi\in\boldsymbol{R}^d\}.$$

DEFINITION 2.2. A smooth function $k(x)$ defined in $\boldsymbol{R}^d$ will be called a smooth tempered weight function, if $k(x)$ satisfies the following three conditions:

(K–1) there exist positive constants m and C such that $C^{-1}\langle x\rangle^{-m}\leq k(x)$ and $k(x+y)\leq Ck(x)\langle y\rangle^{m}$ for any $x, y\in \boldsymbol{R}^{d}$,

(K–2) $k(x)=k(-x)$ for any $x\in \boldsymbol{R}^{d}$, and

(K–3) there exist real constants m_{α} and C_{α} for any multi-index α such that

$$|\partial_{x}^{\alpha}k(x)|\leq C_{\alpha}\langle x\rangle^{m_{\alpha}} \qquad \text{for any} \quad x\in \boldsymbol{R}^{d}.$$

The set of all smooth tempered weight functions will be denoted by $\mathcal{K}$.

REMARK 2.2. If $k(x)$ belongs to $\mathcal{K}$, $k(x)^{-1}$ also belongs to $\mathcal{K}$.

For any $p(x, \xi)\in\tilde{\mathcal{S}}^{m}$, we get the pseudo-differential operator $p(X, D_{x})$, a continuous linear operator from $\mathcal{S}(\boldsymbol{R}^{d})$ into $\mathcal{S}(\boldsymbol{R}^{d})$ defined by

$$(p(X, D_{x})u)(x)=\left(\frac{1}{2\pi}\right)^{d}\int_{Rd}e^{ix\cdot\xi}p(x, \xi)\hat{u}(\xi)d\xi$$

for each $u\in\tilde{\mathcal{S}}(\boldsymbol{R}^{d})$, where $x\cdot\xi=x_{1}\xi_{1}+\cdots+x_{d}\xi_{d}$ and $\hat{u}(\xi)=\int_{Rd}e^{-x\cdot\xi}u(x)dx$ is a Fourier transform of u.

We can also get pseudo-differential operators $k(X)$ and $k(D_{x})$ for any $k\in\mathcal{K}$ defined by

$$(k(X)u)(x)=k(x)u(x) \qquad \text{and} \qquad (k(D_{x})u)(x)=\left(\frac{1}{2\pi}\right)^{d}\int_{Rd}e^{ix\cdot\xi}k(\xi)\hat{u}(\xi)d\xi$$

for each $u\in\mathcal{S}(\boldsymbol{R}^{d})$. It is easy to see that $k(X)$ and $k(D_{x})$ are extensible to continuous linear maps from $\mathcal{S}'(\boldsymbol{R}^{d})$ into $\mathcal{S}'(\boldsymbol{R}^{d})$.

DEFINITION 2.3. We say that a complex-valued smooth function $a(x, \xi)$ defined in $\boldsymbol{R}^{d}\times\boldsymbol{R}^{d}$ belongs to $\mathcal{A}^{m}$, $m\in\boldsymbol{R}$, if for any multi-indices α and β, there exists a constant $C_{\alpha,\beta}$ such that $|\partial_{x}^{\alpha}\partial_{\xi}^{\beta}a(x, \xi)|\leq C_{\alpha,\beta}\langle x\rangle^{m}\langle\xi\rangle^{m}$ for any $x, \xi\in\boldsymbol{R}^{d}$. For any $a(x, \xi)\in\mathcal{A}^{m}$, we define the oscillatory integral $Os[a]$ by

$$\begin{aligned} Os[a]&=Os-\iint e^{-ix\cdot\xi}a(x, \xi)dxd\xi \\ &=\lim_{\varepsilon\downarrow 0}\iint_{Rd\times Rd}e^{-ix\cdot\xi}\chi_{\varepsilon}(x, \xi)a(x, \xi)dxd\xi, \end{aligned}$$

where $\chi_{\varepsilon}(x, \xi)=\chi(\varepsilon x, \varepsilon\xi)$, $\varepsilon\in(0,1)$, for a $\chi(x, \xi)\in\mathcal{S}(\boldsymbol{R}^{2d})$ such that $\chi(0,0)=1$.

By virtue of Kumano-go and Taniguchi [6], $Os[a]$ is well-defined and does not depend on $\chi(x, \xi)$.

REMARK 2.3. $\mathcal{A}^{m}$ is a Fréchet space with semi-norms $|\ |_{\mathcal{A}^{m},n}$, $n=0, 1, 2, \ldots$, defined by

$$|a|_{\mathcal{A}^m,n}=\sup\{|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\langle x\rangle^{-m}\langle\xi\rangle^{-m};\ |\alpha|+|\beta|\leq n,\ x,\xi\in \boldsymbol{R}^d\}.$$

It has been proved by Kumano-go and Taniguchi [6] that there exist an integer n and a constant C for any m such that $|Os[a]|\leq C|a|_{\mathcal{A}^m,n}$ for any $a\in\mathcal{A}^m$.

3. Weighted Sobolev spaces and support of a Gaussian white noise

DEFINITION 3.1. Let σ and ρ belong to $\mathcal{K}$. For $u\in\mathcal{S}'(\boldsymbol{R}^d)$, we say that u belongs to $W_p^{\sigma,\rho}$, $1<p<\infty$, if $\rho(X)\sigma(D_x)u$ belongs to $L^p(\boldsymbol{R}^d)$. $W_p^{\sigma,\rho}$ is a Banach space with a norm $\|\ \|_{\sigma,\rho,p}$ defined by $\|u\|_{\sigma,\rho,p}=\|\rho(X)\sigma(D_x)u\|_{L_p}$.

PROPOSITION 3.1. (1) *$\mathcal{S}(\boldsymbol{R}^d)$ is dense in $W_p^{\sigma,\rho}$.*

(2) *$(W_p^{\sigma,\rho})'=W_q^{\sigma^{-1},\rho^{-1}}$, where E' denotes the dual Banach space of a Banach space E and $1/p+1/q=1$.*

(3) *$W_p^{\sigma,\rho}$ is a Borel subset of $\mathcal{S}'(\boldsymbol{R}^d)$.*

PROOF. By Remark 2.2, we see that $\rho(X)$ and $\sigma(D_x)$ are homeomorphisms of $\mathcal{S}(\boldsymbol{R}^d)$ and those of $\mathcal{S}'(\boldsymbol{R}^d)$. It is known that $\mathcal{S}(\boldsymbol{R}^d)$ is dense in $L^p(\boldsymbol{R}^d)$, so we get (1). It is easy to see that $(u,v)_{L^2}=(\rho(X)\sigma(D_x)u,\ \rho^{-1}(X)\sigma^{-1}(D_x)v)_{L^2}$ for any $u,v\in\mathcal{S}(\boldsymbol{R}^d)$, which implies (2). In order to prove (3), let V be a countable dense subset of $\mathcal{S}(\boldsymbol{R}^d)$. Then $u\in\mathcal{S}'(\boldsymbol{R}^d)$ belongs to $W_p^{\sigma,\rho}$ if and only if

$$\sup\{|{}_{S'}\langle u,v\rangle_S|/\|v\|_{\sigma^{-1},\rho^{-1},q};\ v\in V,\ v\neq 0\}<\infty.$$

This implies (3). Thus our proposition has been proved.

Let μ be a probability measure on $\mathcal{S}'(\boldsymbol{R}^d)$ such that

$$\int_{S'(R^d)}\exp(i_S\langle f,w\rangle_{S'})\mu(dw)=\exp\left(-\frac{1}{2}\int_{R^d}|f(x)|^2dx\right)$$

for any $f\in\mathcal{S}(\boldsymbol{R}^d)$. The existence and uniqueness of such a probability measure μ is guaranteed by Minlos-Sazanov-Kolmogorov's theorem.

The following is our main theorem.

THEOREM 1. *Let $\sigma,\rho\in\mathcal{K}$ and let $1<p<\infty$. Then $\mu(W_p^{\sigma,\rho})=1$, if and only if $\sigma\in L^2(\boldsymbol{R}^d)$ and $\rho\in L^p(\boldsymbol{R}^d)$.*

REMARK 3.2. Since μ is $\mathcal{S}(\boldsymbol{R}^d)$-ergodic, $\mu(W_p^{\sigma,\rho})=0$ unless $\mu(W_p^{\sigma,\rho})=1$.

Before proving Theorem 1, we will prepare the following.

PROPOSITION 3.2. *Let $\phi\in\mathcal{S}(\boldsymbol{R}^d)$. Then there exist constants C and m such that*

$$\|\phi(D_x)u\|_{\sigma,\rho,p}\leq C\|u\|_{\sigma,\rho,p}\int_{R^d}\langle x\rangle^m|\hat{\phi}(x)|dx$$

for any $u\in\mathcal{S}(\boldsymbol{R}^d)$. *Here C and m depend only on* ρ.

PROOF. Since $\rho^{-1}\in\mathcal{K}$ by Remark 2.2, there exist constants C and m such that $\rho^{-1}(x+y)\leq C\rho^{-1}(x)\langle y\rangle^m$. Let $v=\rho(X)\sigma(D_x)u$. Then we obtain

$$\begin{aligned}\|\phi(D_x)u\|_{\sigma,\rho,p}^p&=\|\rho(X)\phi(D_x)\sigma(D_x)u\|_{L^p}^p\\&=\int_{R^d}\rho(x)^p|(\phi(D_x)\rho^{-1}(X)v)(x)|^p dx\\&=\int_{R^d}\rho(x)^p dx\left|\int_{R^d}\rho^{-1}(x-y)v(x-y)\hat{\phi}(-y)dy\right|^p\\&\leq C^p\int_{R^d}dx\left|\int_{R^d}v(x-y)\langle -y\rangle^m\hat{\phi}(-y)dy\right|^p.\end{aligned}$$

The Young inequality proves that

$$\begin{aligned}\|\phi(D_x)u\|_{\sigma,\rho,p}&\leq C\|v\|_{L^p}\int_{R^d}\langle y\rangle^m|\hat{\phi}(y)|dy\\&=C\|u\|_{\sigma,\rho,p}\int_{R^d}\langle y\rangle^m|\hat{\phi}(y)|dy.\end{aligned}$$

This completes the proof.

Now we prove Theorem 1.

Let $\phi(x)=\exp\left(-\frac{1}{2}|x|^2\right)$ and $\phi_\varepsilon(x)=\phi(\varepsilon x)$ for $\varepsilon>0$.

First we will prove the 'only if' part. Assume that $\mu(W_p^{\sigma,\rho})=1$. Then we get $\int_{W_p^{\sigma,\rho}}\|w\|_{\sigma,\rho,p}^p\mu(dw)<\infty$ by Fernique [1]. It follows from Proposition 3.2 that

$$\begin{aligned}&\sup_{0<\varepsilon<1}\int_{W_p^{\sigma,\rho}}\|\rho(X)\sigma(D_x)\phi_\varepsilon(D_x)w\|_{L^p}^p\mu(dw)\\&=\sup_{0<\varepsilon<1}\int_{W_p^{\sigma,\rho}}\|\phi_\varepsilon(D_x)w\|_{\sigma,\rho,p}^p\mu(dw)<\infty.\end{aligned}\tag{3.1}$$

Since $(\sigma(D_x)\phi_\varepsilon(D_x)w)(x)=\langle(\widehat{\sigma\phi_\varepsilon})(\cdot-x),w(\cdot)\rangle$ is a smooth function in x for any $w\in\mathcal{S}'(\boldsymbol{R}^d)$, we can define $X_\varepsilon(x,w)$ by $X_\varepsilon(x,w)=(\sigma(D_x)\phi_\varepsilon(D_x)w)(x)$ for any $x\in\boldsymbol{R}^d$ and $w\in\mathcal{S}'(\boldsymbol{R}^d)$. Then $\{X_\varepsilon(x,w);\ x\in\boldsymbol{R}^d\}$ is a stationary Gaussian random field with mean 0 and a spectral measure $\left(\frac{1}{2\pi}\right)^d\sigma(\xi)^2|\phi_\varepsilon(\xi)|^2d\xi$ under the probability measure $\mu(dw)$. Therefore we obtain

$$\int_{W_p^{\sigma,\rho}} \|\rho(X)\sigma(D_x)\phi_\varepsilon(D_x)w\|_{L^p}^p \mu(dw) = \int_{W_p^{\sigma,\rho}} \left(\int_{R^d} \rho(x)^p |X_\varepsilon(x,w)|^p dx \right) \mu(dw)$$
$$= C_p \int_{R^d} \rho(x)^p dx \left(\left(\frac{1}{2\pi} \right)^d \int_{R^d} \sigma(\xi)^2 |\phi_\varepsilon(\xi)|^2 d\xi \right)^{p/2},$$

where $C_p = \left(\frac{1}{2\pi} \right)^{1/2} \int_R |t|^p \exp\left(-\frac{1}{2} t^2 \right) dt$. Thus (2.1) implies that

$$\lim_{\varepsilon \downarrow 0} \int_{W_p^{\sigma,\rho}} \|\rho(X)\sigma(D_x)\phi_\varepsilon(D_x)w\|_{L^p}^p \mu(dw) = C_p \int_{R^d} \rho(x)^p dx \left(\left(\frac{1}{2\pi} \right)^d \int_{R^d} \sigma(\xi)^2 d\xi \right)^{p/2} < \infty.$$

This shows that $\rho \in L^p(\boldsymbol{R}^d)$ and $\sigma \in L^2(\boldsymbol{R}^d)$.

Conversely we will show the 'if' part. Suppose that $\rho \in L^p(\boldsymbol{R}^d)$ and $\sigma \in L^2(\boldsymbol{R}^d)$. Since $\sigma(D_x)\phi_\varepsilon(D_x)w$ is a smooth function for each $w \in \mathcal{S}'(\boldsymbol{R}^d)$, we can define $X_\varepsilon(x, w)$ by $X_\varepsilon(x, w) = (\sigma(D_x)\phi_\varepsilon(D_x)w)(x)$ for any $x \in \boldsymbol{R}^d$ and $w \in \mathcal{S}'(\boldsymbol{R}^d)$. Then $\{X_\varepsilon(x, w); x \in \boldsymbol{R}^d\}$ is a stationary Gaussian random field with mean 0 and a spectral measure $\left(\frac{1}{2\pi} \right)^d \sigma(\xi)^2 |\phi_\varepsilon(\xi)|^2 d\xi$ under the probability measure $\mu(dw)$. So we get

$$\int_{S'(R^d)} \|\phi_\varepsilon(D_x)w\|_{\sigma,\rho,p}^p \mu(dw) = C_p \int_{R^d} |\rho(x)|^p dx \left(\left(\frac{1}{2\pi} \right)^d \int_{R^d} \sigma(\xi)^2 |\phi(\varepsilon\xi)|^2 d\xi \right)^{p/2}$$

where we consider $\|u\|_{\sigma,\rho,p} = \infty$ unless u belongs to $W_P^{\sigma,\rho}$. Therefore Fatou's lemma proves that

$$\int_{S'(R^d)} \varliminf_{m \to \infty} \|\phi_{1/m}(D_x)w\|_{\sigma,\rho,p}^p \mu(dw) \le \varliminf_{m \to \infty} \int_{S'(R^d)} \|\phi_{1/m}(D_x)w\|_{\sigma,\rho,p}^p \mu(dw).$$

Let $\Omega = \{w \in \mathcal{S}'(\boldsymbol{R}^d); \varliminf_{m \to \infty} \|\phi_{1/m}(D_x)w\|_{\sigma,\rho,p} < \infty\}$. Since $W_p^{\sigma,\rho}$ is a separable reflexive Banach space by Proposition 3.1, $\{\phi_{1/m}(D_x)w;\ \mathrm{m}=1, 2, \dots\}$ contains a subsequence which is convergent in $W_p^{\sigma,\rho}$ for any $w \in \Omega$. On the other hand, $\phi_{1/m}(D_x)w \to w$, $m \to \infty$, in $\mathcal{S}'(\boldsymbol{R}^d)$, which shows that $\Omega \subset W_p^{\sigma,\rho}$. Since $\mu(\Omega) = 1$, we obtain $\mu(W_p^{\sigma,\rho}) = 1$. This completes the proof.

Now we give some examples.

Example 1. Let $1 < p < \infty$, $\sigma_t(x) = \langle x \rangle^{-d/2} \{\log (1 + \langle x \rangle^2)\}^{-t/2}$, and

$$\rho_s(x) = \langle x \rangle^{-d/p} \{\log (1 + \langle x \rangle^2)\}^{-s/p}, \qquad s, t \in \boldsymbol{R}.$$

Then it is clear that σ_t and ρ_s belong to $\mathcal{K}$ for any $s, t \in \boldsymbol{R}$, and it is easy to see that $\sigma_t \in L^2(\boldsymbol{R}^d)$ and $\rho_s \in L^p(\boldsymbol{R}^d)$ if and only if $t > 1$ and $s > 1$. So we get the following by Theorem 1 and Remark 2.1.

PROPOSITION 3.3. *$\mu(W_p^{\sigma_t,\rho_s}) = 1$ (resp. 0) if and only if $t > 1$ and $s > 1$ (resp. $t \le 1$ or $s \le 1$).*

Example 2. Assume that $d \geq 2$. Let $1 < p < \infty$,

$$\tilde{\sigma}_t(x) = \langle x \rangle^{-1} \left(1 + \sum_{i=1}^{d-1} x_i^2\right)^{-d/4+1/2} \left\{\log\left(2 + \sum_{i=1}^{d-1} x_i^2\right)\right\}^{-t/2},$$

and $\rho_s(x) = \langle x \rangle^{-d/p} \{\log(1 + \langle x \rangle^2)\}^{-s/p}$, $s, t \in \boldsymbol{R}$. Then it is clear that $\tilde{\sigma}_t$ and ρ_s belong to $\mathcal{K}$ for any $s, t \in \boldsymbol{R}$. It is easy to see that $\tilde{\sigma}_t \in L^2(\boldsymbol{R}^d)$ and $\rho_s \in L^p(\boldsymbol{R}^d)$ if and only if $t > 1$ and $s > 1$. So Theorem 1 implies that $\mu(W_p^{\tilde{\sigma}_t, \rho_s}) = 1$ if and only if $s > 1$ and $t > 1$.

Let $\sigma_t(x) = \langle x \rangle \tilde{\sigma}_t(x)$. Then $\langle D_x \rangle^{-1}$ is an isomorphism from $W_p^{\tilde{\sigma}_t, \rho_s}$ onto $W_p^{\sigma_t, \rho_s}$. Let $\nu = \langle D_x \rangle^{-1} \mu$ be the image measure induced by μ through $\langle D_x \rangle^{-1}$: $\mathcal{S}'(\boldsymbol{R}^d) \to \mathcal{S}'(\boldsymbol{R}^d)$. Then $\nu(W_p^{\sigma_t, \rho_s}) = 1$ if and only if $t > 1$ and $s > 1$. However, ν is the free measure for Boson fields.

Therefore we get the following.

PROPOSITION 3.4. *Let ν be the free measure for Boson fields. Then $\nu(W_p^{\sigma_t, \rho_s}) = 1$ (resp. 0) if and only if $t > 1$ and $s > 1$ (resp. $t \leq 1$ or $s \leq 1$). Here*

$$\sigma_t(x) = \left(1 + \sum_{i=1}^{d-1} x_i^2\right)^{-d/4+1/2} \left\{\log\left(2 + \sum_{i=1}^{d-1} x_i^2\right)\right\}^{-t/2}$$

and

$$\rho_s(x) = \langle x \rangle^{-d/p} \{\log(1 + \langle x \rangle^2)\}^{-s/p}, \qquad s, t \in \boldsymbol{R}.$$

This fact has been proved by Reed and Rosen [7] in the case when $p = 2$.

4. Weighted Sobolev spaces and pseudo-differential operators

It is not easy to analyze $W_p^{\sigma, \rho}$ in general, $\sigma, \rho \in \mathcal{K}$. Hence we will introduce a better subclass $\mathcal{W}$ of smooth tempered weight functions in this section.

DEFINITION 4.1. We say that a smooth function $h(x)$ defined in $\boldsymbol{R}^d$ belongs to a class $\mathcal{W}_{m_1, m_2}$, $-\infty < m_2 \leq m_1 < \infty$, if $h(x)$ satisfies the following four conditions

(W-1) there exist positive constants C_1 and C_2 such that $C_1 \langle x \rangle^{m_2} \leq h(x) \leq C_2 \langle x \rangle^{m_1}$ for any $x \in \boldsymbol{R}^d$,

(W-2) $h(-x) = h(x)$ for any $x \in \boldsymbol{R}^d$,

(W-3) there exist positive constants m_3 and C_3 such that $h(x+y) \leq C_3 h(x) \langle y \rangle^{m_3}$ for any $x, y \in \boldsymbol{R}^d$, and

(W-4) for any multi-index α, there exists a constant C_α such that $|\partial_x^\alpha h(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|} h(x)$ for any $x \in \boldsymbol{R}^d$.

We also define a class $\mathcal{W}$ by

$$\mathcal{W}=\cup\{\mathcal{W}_{m_1,m_2};\ 0\leq m_1-m_2\leq 1\}.$$

REMARK 4.1. (1) It is clear that $\mathcal{W}_{m_1,m_2}\subset\mathcal{K}$.

(2) Let $h(x)\in\mathcal{W}_{m_1,m_2}$. Then $h(\xi)\in\tilde{\mathcal{S}}^{m_1}$, $h(x)^{-1}\in\mathcal{W}_{-m_2,-m_1}$, and $\langle x\rangle^m h(x)\in\mathcal{W}_{m_1+m,m_2+m}$, $m\in\boldsymbol{R}$.

(3) Let $h\in\mathcal{W}_{m_1,m_2}$ and h_λ, $\lambda>0$, be a function given by $h_\lambda(x)=h(\lambda x)$ for any $x\in\boldsymbol{R}^d$. Then $h_\lambda\in\mathcal{W}_{m_1,m_2}$.

The following two propositions are due to Kumano-go [3].

PROPOSITION 4.1. *Let $p(x,\xi)\in\tilde{\mathcal{S}}^m$, $m\in\boldsymbol{R}$. Then the continuous linear map $p(X,D_x)$ from $\mathcal{S}(\boldsymbol{R}^d)$ into $\mathcal{S}(\boldsymbol{R}^d)$ is extensible to a continuous linear map from $\mathcal{S}'(\boldsymbol{R}^d)$ into $\mathcal{S}'(\boldsymbol{R}^d)$.*

PROPOSITION 4.2. *Let $p_j(x,\xi)\in\tilde{\mathcal{S}}^{m_j}$, $j=1,2$. Then a smooth function $q(x,\xi)$ given by*

$$q(x,\xi)=\left(\frac{1}{2\pi}\right)^d Os-\iint e^{-iy\cdot\eta}p_1(x,\xi+\eta)p_2(x+y,\xi)dyd\eta$$

belongs to $\tilde{\mathcal{S}}^{m_1+m_2}$, and $q(X,D_x)=p_1(X,D_x)p_2(X,D_x)$. Furthermore a smooth function $r(x,\xi)$ given by

$$r(x,\xi)=q(x,\xi)-p_1(x,\xi)p_2(x,\xi)$$

belongs to $\tilde{\mathcal{S}}^{m_1+m_2-1}$ and the mapping from $\tilde{\mathcal{S}}^{m_1}\times\tilde{\mathcal{S}}^{m_2}$ into $\tilde{\mathcal{S}}^{m_1+m_2-1}$ by which (p_1,p_2) corresponds to r is continuous.

The following proposition is due to Kagan [2] and Kumano-go and Nagase [5].

PROPOSITION 4.3. *Let $1<q<\infty$. Then there exist a constant C and a positive integer n such that*

$$\|p(X,D_x)u\|_{L^q}\leq C|p|_n^{(0)}\|u\|_{L^q}$$

for any $p(x,\xi)\in\tilde{\mathcal{S}}^0$ and any $u\in\mathcal{S}(\boldsymbol{R}^d)$.

The following shows an advantage of $\mathcal{W}$.

LEMMA 4.1. *Let $p(x,\xi)\in\tilde{\mathcal{S}}^{m_0}$, $m_0\in\boldsymbol{R}$, and $h(x)\in\mathcal{W}_{m_1,m_2}$, $-\infty<m_2\leq m_1<\infty$. Let h_λ, $0<\lambda\leq 1$, be smooth functions given by $h_\lambda(x)=h(\lambda x)$, and q_λ: $\boldsymbol{R}^d\times\boldsymbol{R}^d\to\boldsymbol{R}$ be smooth functions given by*

$$q_\lambda(x,\xi)=\left(\frac{1}{2\pi}\right)^d h_\lambda(x)^{-1}Os-\iint e^{-iy\cdot\eta}p(x,\xi+\eta)h_\lambda(x+y)dyd\eta$$

for any $x,\xi\in\boldsymbol{R}^d$. Then $q_\lambda(x,\xi)\in\tilde{\mathcal{S}}^{m_0}$ and $q_\lambda(X,D_x)=h_\lambda(X)^{-1}p(X,D_x)h_\lambda(X)$ for any

$\lambda\in(0,1]$. *Moreover* $r_\lambda(x,\xi)=q_\lambda(x,\xi)-p(x,\xi)\in\tilde{\mathcal{S}}^{m_0-1}$, *and for any integer* n, *there exist a constant* C *and an integer* n' *depending only on* h, m_0 *and* n, *not on* p *or* λ, *such that* $|r_\lambda|_n^{(\tilde{m_0-1})}\leq C\lambda|p|_{n'}^{(m_0)}$.

PROOF. It is clear that

$$p(x,\xi+\eta)=p(x,\xi)+\sum_{j=1}^{d}\xi_j\int_0^1\partial_{\xi_j}p(x,\xi+\theta\eta)d\theta.$$

Let

$$\tilde{q}_\lambda(x,\xi)=\left(\frac{1}{2\pi}\right)^d Os-\iint e^{-iy\cdot\eta}p(x,\xi+\eta)h_\lambda(x+y)dyd\eta.$$

Then we obtain

$$\tilde{q}_\lambda(x,\xi)=p(x,\xi)h_\lambda(x)+\left(\frac{1}{2\pi}\right)^d\sum_{j=1}^{d}\int_0^1 d\theta Os-\iint e^{-iy\cdot\eta}r_{\theta,\lambda,j}(y,\eta;\ x,\xi)dyd\eta, \tag{4.1}$$

where

$$r_{\theta,\lambda,j}(y,\eta;\ x,\xi)=\partial_{\xi_j}p(x,\xi+\theta\eta)\partial_{y_j}h_\lambda(x+y),\qquad j=i,\dots,d$$

and $0\leq\theta\leq1$. It is easy to see that

$$\begin{aligned}\partial_y^{\alpha'}\partial_\eta^{\beta'}\partial_x^{\alpha}\partial_\xi^{\beta}r_{\theta,\lambda,j}(y,\eta;\ x,\xi)&=\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\theta^{|\beta'|}\partial_x^{\alpha-\gamma}\partial_\xi^{\beta+\beta'}p(x,\xi+\theta\eta)\partial_y^{\alpha+\alpha'}\partial_{y_j}h_\lambda(x+y)\\&=\lambda\sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}\lambda^{|\alpha+\alpha'|}\theta^{|\beta'|}\partial_x^{\alpha-\gamma}\partial_\xi^{\beta+\beta'}\partial_{\xi_j}p(x,\xi+\theta\eta)\\&\quad\times(\partial_y^{\alpha+\alpha'}\partial_{y_j}h)(\lambda x+\lambda y),\end{aligned}$$

where $\binom{\alpha}{\gamma}=\binom{\alpha_1}{\gamma_1}\cdots\binom{\alpha_d}{\gamma_d}$.

The definition of $\tilde{\mathcal{S}}^{m_0}$ implies that

$$\begin{aligned}|\partial_x^{\alpha-\gamma}\partial_\xi^{\beta+\beta'}\partial_{\xi_j}p(x,\xi+\theta\eta)|&\leq\langle\xi+\theta\eta\rangle^{m_0-|\beta+\beta'|-1}|p|_{|\alpha+\beta+\beta'|+1}^{(m_0)}\\&\leq\langle\xi\rangle^{m_0-|\beta|-1}\langle\eta\rangle^{|m_0|+|\beta|+1}|p|_{|\alpha+\beta+\beta'|+1}^{(m_0)}.\end{aligned}$$

The definition of $\mathcal{W}_{m_1,m_2}$ also implies that

$$|(\partial_y^{\alpha+\alpha'}\partial_{y_j}h)(\lambda x+\lambda y)|\leq C_{\alpha+\alpha',j}h_\lambda(x)\langle y\rangle^{m_3}$$

for some constant $C_{\alpha+\alpha',j}$ independent of λ, where m_3 is as in the condition (W-3) in Definition 3.1.

Therefore we get

$$|\partial_y^{\alpha'}\partial_\eta^{\beta'}\partial_x^{\alpha}\partial_\xi^{\beta}r_{\theta,\lambda,j}(y,\eta;\ x,\xi)|\leq\lambda C_{\alpha',\beta',\alpha,\beta,j}\langle y\rangle^{\tilde{m}}\langle\eta\rangle^{\tilde{m}}h_\lambda(x)|p|_{|\alpha+\beta+\alpha'+\beta'|}^{(m_0)}, \tag{4.2}$$

$m=|m_0|+m_3+|\beta|+1$, for some constant $C_{\alpha',\beta',\alpha,\beta,j}$. Observe that $C_{\alpha',\beta',\alpha,\beta,j}$ depends only on $\alpha',\beta',\alpha,\beta,j,m_0$ and h, not on λ or p.

(4.2) shows that $\partial_x^\alpha \partial_\xi^\beta r_{\theta,\lambda,j}(y, \eta;\ x, \xi) \in \mathcal{A}^{\tilde{m}}$ as a function of (y, η) for any α, β and $x, \xi \in \boldsymbol{R}^d$. Thus it follows from (4.1) and Remark 2.3 that for any multi-indices α and β, there exist an integer n' and a constant $C_{\alpha,\beta}$ depending only on α, β, m_0 and h, not on p or λ, such that

$$|\partial_x^\alpha \partial_\xi^\beta (q_\lambda(x, \xi) - p(x, \xi))| \leq \lambda C_{\alpha,\beta} |p|_{n'}^{(m_0)} \langle \xi \rangle^{m_0 - 1 - |\beta|} \qquad \text{for any} \quad x, \xi \in \boldsymbol{R}^d.$$

This proves $r_\lambda(x, \xi) \in \tilde{\mathcal{S}}^{m_0 - 1}$ and $q_\lambda(x, \xi) \in \tilde{\mathcal{S}}^{m_0}$. We can prove that $q_\lambda(X, D_x) = h_\lambda(X)^{-1} p(X, D_x) h_\lambda(X)$ similarly as in Kumano-go [3] Theorem 1.1. This completes the proof.

Let $\tilde{S}^m$ denote the set of all pseudo-differential operators $p(X, D_x)$'s induced by $p(x, \xi) \in \tilde{\mathcal{S}}^m$.

COROLLARY TO LEMMA 4.1. *Let $h, k \in \mathcal{S}$ and $P \in \tilde{S}^m$, $m \in \boldsymbol{R}$. Then*

$$k(D_x)^{-1} h(X)^{-1} P k(D_x) h(X), \quad h(X)^{-1} k(D_x)^{-1} P h(X) k(D_x),$$
$$k(D_x)^{-1} h(X)^{-1} P h(X) k(D_x) \quad \text{and} \quad h(X)^{-1} k(D_x)^{-1} P k(D_x) h(X)$$

belong to $\tilde{S}^m$.

PROOF. By the definition of $\mathcal{W}$, there exist real numbers m_1 and m_2 such that $0 \leq m_1 - m_2 \leq 1$ and $k \in \mathcal{W}_{m_1, m_2}$. Using Proposition 4.2, we get $Pk(D_x) \in \tilde{S}^{m+m_1}$ and $Pk(D_x) - k(D_x)P \in \tilde{S}^{m+m_1-1}$. This and Lemma 4.1 imply that $h(X)^{-1} Pk(D_x) h(X) - Pk(D_x) \in \tilde{S}^{m+m_1-1}$.

On the other hand, it follows from Remark 4.1 that $k(x)^{-1} \in \mathcal{W}_{-m_2, -m_1} \subset \tilde{\mathcal{S}}^{-m_2}$, which implies that $k(D_x)^{-1} \in \tilde{S}^{-m_2}$. Observe that

$$Pk(D_x) - k(D_x)P,\ h(X)^{-1} Pk(D_x) h(X) - Pk(D_x) \in \tilde{S}^{m+m_1-1} \subset \tilde{S}^{m+m_2}$$

and

$$k(D_x)^{-1} h(X)^{-1} Pk(D_x) h(X) - P = k(D_x)^{-1} (h(X)^{-1} Pk(D_x) h(X) - Pk(D_x)) + k(D_x)^{-1} (Pk(D_x) - k(D_x)P).$$

Then we obtain $k(D_x)^{-1} h(X)^{-1} Pk(D_x) h(X) - P \in \tilde{S}^m$ by Proposition 4.2. So we have got $k(D_x)^{-1} h(X)^{-1} Pk(D_x) h(X) \in \tilde{S}^m$.

The proofs of the other are similar.

THEOREM 2. *Let $\sigma, \rho \in \mathcal{W}$, $p(x, \xi) \in \tilde{\mathcal{S}}^m$, $m \in \boldsymbol{R}$, and $1 < q < \infty$, and let $\tilde{\sigma}(x) = \sigma(x) \langle x \rangle^{-m}$. Then there exist a constant C and a positive integer n such that*

$$\|p(X, D_x) u\|_{W_q^{\tilde{\sigma}, \rho}} \leq C |p|_n^{(m)} \|u\|_{W_q^{\sigma, \rho}} \qquad \textit{for any} \quad u \in \mathcal{S}(\boldsymbol{R}^d).$$

Here C and n depend only on σ, ρ, m and q. Hence we can regard $p(X, D_x)$ as a *bounded linear operator from* $W_q^{\sigma,\rho}$ *into* $W_q^{\tilde{\sigma},\rho}$.

PROOF. It is easy to see that

$$\|p(X, D_x)u\|_{W_q^{\tilde{\sigma},\rho}} = \|\rho(X)\sigma(D_x)\langle D_x\rangle^{-m}p(X, D_x)\sigma(D_x)^{-1}\rho(X)^{-1}\rho(X)\sigma(D_x)u\|_{L^q}.$$

Since $\langle D_x\rangle^{-m}p(X, D_x) \in \tilde{S}^0$, we get $\rho(X)\sigma(D_x)\langle D_x\rangle^{-m}p(X, D_x)\sigma(D_x)^{-1}\rho(X)^{-1} \in \tilde{S}^0$ by Corollary to Lemma 4.1. It follows from Proposition 4.1, Proposition 4.3 and Lemma 4.1 that there exist a constant and a positive integer n such that

$$\|p(X, D_x)u\|_{W_q^{\tilde{\sigma},\rho}} \leq C|p|_n^{(m)}\|\rho(X)\sigma(D_x)u\|_{L^q}$$

for any $u \in \mathcal{S}(\boldsymbol{R}^d)$. This completes the proof.

LEMMA 4.2. *Let* $\rho \in \mathcal{W}$ *and* $k(\xi) \in \tilde{\mathcal{S}}^0$. *Let* ρ_λ, $\lambda \in (0, 1]$, *be elements of* $\mathcal{W}$ *given by* $\rho_\lambda(x) = \rho(\lambda x)$ *for any* $x \in \boldsymbol{R}^d$. *Then there exists a constant* C *such that*

$$\|k(D_x)u\|_{W_2^{1,\rho_\lambda}} \leq (\|k\|_{L^\infty} + \lambda \cdot C)\|u\|_{W_2^{1,\rho_\lambda}}$$

for any $\lambda \in (0, 1]$ *and* $u \in W_2^{1,\rho_\lambda}$.

PROOF. It is clear that

$$\begin{aligned}\|k(D_x)u\|_{W_2^{1,\rho_\lambda}} &= \|\rho_\lambda(X)k(D_x)u\|_{L^2} \\ &\leq \|k(D_x)\rho_\lambda(X)u\|_{L^2} + \|(\rho_\lambda(X)k(D_x)\rho_\lambda(X)^{-1} - k(D_x))\rho_\lambda(X)u\|_{L^2}.\end{aligned}$$

Thus it follows from Lemma 4.1 and Proposition 4.3 that there exists a constant C such that

$$\|k(D_x)u\|_{W_2^{1,\rho_\lambda}} \leq (\|k\|_{L^\infty} + \lambda \cdot C)\|\rho_\lambda(X)u\|_{L^2},$$

which completes the proof.

5. The unique existence of a solution of a certain stochastic pseudo-differential equation

Let $b(y_1, \ldots, y_n)$ be a bounded smooth function defined in $\boldsymbol{R}^n$ with bounded first partial derivatives, i.e.

$$\left\|\frac{\partial b}{\partial y_j}\right\|_\infty = \sup\left\{\left|\frac{\partial b}{\partial y_j}(y)\right|;\ y \in \boldsymbol{R}^n\right\}, \qquad j = 1, \ldots, n.$$

Let $q_j(\xi)$, $j = 1, \ldots, n$, be elements of $\tilde{\mathcal{S}}^r$, $r \in \boldsymbol{R}$, and $p(\xi)$ be an element of $\tilde{\mathcal{S}}^m$, $m \in \boldsymbol{R}$, such that $|p(\xi)| > 0$ for any $\xi \in \boldsymbol{R}^d$ and $p(\xi)^{-1} \in \tilde{\mathcal{S}}^{-m}$.

Now let us consider the following stochastic pseudo-differential equation

(5.1) $$p(D_x)X - b(q_1(D_x)X, \dots, q_n(D_x)X) = W,$$

where W is a Gaussian white noise. We say that a Borel map X from $\mathcal{S}'(\boldsymbol{R}^d)$ into $\mathcal{S}'(\boldsymbol{R}^d)$ is a solution of the equation (5.1) if X satisfies the following two conditions:

(C-1) $q_j(D_x)X(w)$, $j=1,\dots,n$, are able to be identified with locally L^1 measurable functions on $\boldsymbol{R}^d$ as elements of $\mathcal{S}'(\boldsymbol{R}^d)$ for μ-a.e. w. Then we can define a bounded function $b(q_1(D_x)X(w),\dots,q_n(D_x)X(w))$ defined in $\boldsymbol{R}^d$ for μ-a.e. w.

(C-2) $p(D_x)X(w) - b(q_1(D_x)X(w),\dots,q_n(D_x)X(w)) = w$ in $\mathcal{S}'(\boldsymbol{R}^d)$ for μ-a.e. w.

THEOREM 3. *Assume that* $m > r + \dfrac{d}{2}$ *and* $\sum_{j=1}^{n} \left\| \dfrac{\partial b}{\partial y_j} \right\|_\infty \cdot \|q_j \cdot p^{-1}\|_{L^\infty} < 1$. *Then there exists a solution of the equation* (5.1), *and if* X *and* $\tilde{X}$ *are solutions of the equation* (5.1), *then* $X(w) = \tilde{X}(w)$ *for* μ-*a.e.* w.

PROOF. Let $\sigma_\eta(x) = \langle \eta x \rangle^{-t}$, $\eta \in (0,1]$ and $t = \dfrac{1}{2}\left(m - r + \dfrac{d}{2}\right)$, and $\rho_\lambda(x) = \langle \lambda x \rangle^{-d/2-1}$, $\lambda \in (0,1]$, for any $x \in \boldsymbol{R}^d$. Then it is clear that $\sigma_\eta, \rho_\lambda \in \mathcal{W} \cap L^2(\boldsymbol{R}^d)$, and so $\mu(W_2^{\sigma_\eta,\rho_\lambda}) = 1$ for any $\eta, \lambda \in (0,1]$. Observe that $q_j(D_x)p(D_x)^{-1}$, $j=1,\dots,n$, are considered as bounded linear operators from $W_2^{\sigma_\eta,\rho_\lambda}$ into W_2^{1,ρ_λ} by virtue of Theorem 2, and that every bounded measurable function belongs to $W_2^{\sigma_\eta,\rho_\lambda}$ for any $\eta, \lambda \in (0,1]$.

Let $B_{\eta,\lambda}$ be an operator from $W_2^{\sigma_\eta,\rho_\lambda}$ into $W_2^{\sigma_\eta,\rho_\lambda}$ given by

$$(B_{\eta,\lambda}u)(x) = b(q_1(D_x)p(D_x)^{-1}u(x), \dots, q_n(D_x)p(D_x)^{-1}u(x))$$

for any $x \in \boldsymbol{R}^d$. Let $q_{\eta,j}(\xi) = q_j(\xi)p(\xi)^{-1}\sigma_\eta(\xi)^{-1}$. Then it is easy to see that

$$\lim_{\eta \downarrow 0} \|q_{\eta,j}\|_{L^\infty} = \|q_j \cdot p^{-1}\|_{L^\infty}, \qquad j=1,\dots,n.$$

So there exists some $\eta_0 \in (0,1]$ such that

$$\sum_{j=1}^{d} \left\| \frac{\partial b}{\partial y_j} \right\|_\infty \cdot \|q_{\eta_0,j}\|_{L^\infty} < 1.$$

Lemma 4.2 implies that there exists constants C_j, $j=0,1,\dots,n$, such that

(5.2) $$\|v\|_{W_2^{\sigma_{\eta_0},\rho_\lambda}} = \|\sigma_{\eta_0}(D_x)v\|_{W_2^{1,\rho_\lambda}} \le (1+\lambda \cdot C_0)\|v\|_{W_2^{1,\rho_\lambda}},$$

and

(5.3) $$\|q_{\eta_0,j}(D_x)v\|_{W_2^{1,\rho_\lambda}} \le (\|q_{\eta_0,j}\|_{L^\infty} + \lambda \cdot C_j)\|v\|_{W_2^{1,\rho_\lambda}}$$

for any $v \in W_2^{1,\rho_\lambda}$.

Then for any $u_1, u_2 \in W_2^{\sigma_{\eta_0},\rho_\lambda}$, we obtain

$$\begin{aligned}
\|B_{\eta_0,\lambda}u_1 - B_{\eta_0,\lambda}u_2\|_{W_2^{\sigma_{\eta_0},\rho_\lambda}} &\leq (1+\lambda\cdot C_0)\|B_{\eta_0,\lambda}u_1 - B_{\eta_0,\lambda}u_2\|_{W_2^{1,\rho_\lambda}} \\
&\leq (1+\lambda\cdot C_0)\left[\int_{R^d} \rho_\lambda(x)^2 \|(B_{\eta_0,\lambda}u_1)(x) - (B_{\eta_0,\lambda}u_2)(x)\|^2 dx\right]^{1/2} \\
&\leq (1+\lambda\cdot C_0)\left[\int_{R^d} \rho_\lambda(x) \sum_{j=1}^{n} \left\|\frac{\partial b}{\partial y_j}\right\|_\infty \right. \\
&\qquad \left. \times \|q_{\eta_0,j}(D_x)\sigma_{\eta_0}(D_x)(u_1-u_2)(x)\|^2 dx\right]^{1/2} \\
&\leq (1+\lambda\cdot C_0)\sum_{j=1}^{n}\left\|\frac{\partial b}{\partial y_j}\right\|_\infty \|q_{\eta_0,j}(D_x)\sigma_{\eta_0}(D_x)(u_1-u_2)\|_{W_2^{1,\rho_\lambda}} \\
&\leq (1+\lambda\cdot C_0)\left[\sum_{j=1}^{n}\left\|\frac{\partial b}{\partial y_j}\right\|_\infty (\|q_{\eta_0,j}\|_{L^\infty} + \lambda\cdot C_j)\right]\|u_1-u_2\|_{W_2^{\sigma_{\eta_0},\rho_\lambda}}.
\end{aligned}$$

Thus there exist $\lambda_0 \in (0,1]$ and $C \in (0,1)$ such that

$$\|B_{\eta_0,\lambda_0}u_1 - B_{\eta_0,\lambda_0}u_2\|_{W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}} \leq C\|u_1-u_2\|_{W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}}$$

for any $u_1, u_2 \in W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}$.

By virtue of the fixed point theorem for contraction maps, we see that $I-B_{\eta_0,\lambda_0}$: $W_2^{\sigma_{\eta_0},\rho_{\lambda_0}} \to W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}$ is a bijective bicontinuous map, where I denotes the identity map on $W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}$.

Let X: $\mathcal{S}'(\boldsymbol{R}^d) \to \mathcal{S}'(\boldsymbol{R}^d)$ be a Borel map given by

$$X(w) = p(D_x)^{-1}(I-B_{\eta_0,\lambda_0})^{-1}w \qquad \text{if} \qquad w \in W_2^{\sigma_{\eta_0},\rho_{\lambda_0}},$$

and

$$X(w) = 0 \qquad \text{if} \qquad w \in \mathcal{S}'(\boldsymbol{R}^d) \backslash W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}.$$

Then it is obvious that X is a solution of the equation (5.1).

On the other hand, assume that $\tilde{X}$ is another solution of the equation (5.1). Then we get

$$p(D_x)\tilde{X}(w) = w + b(q_1(D_x)\tilde{X}(w), \ldots, q_n(D_x)\tilde{X}(w)) \qquad \text{for } \mu\text{-a.e. } w.$$

Since $\mu(W_2^{\sigma_{\eta_0},\rho_{\lambda_0}})=1$ and every bounded measurable function defined in $\boldsymbol{R}^d$ belongs to $W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}$, we know that $p(D_x)\tilde{X}(w) \in W_2^{\sigma_{\eta_0},\rho_{\lambda_0}}$ for μ-a.e. w. Thus we see that $(I-B_{\eta_0,\lambda_0})(p(D_x)\tilde{X}(w)) = w$ for μ-a.e. w, which shows that $\tilde{X}(w) = X(w)$ for μ-a.e. w. This completes the proof.

The property of the solution of the equation (5.1) will be investigated in the forthcoming paper.

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(Received July 31, 1981)

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