Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation[*)]

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1. Introduction.

Let $w=w(x)$ be a real-valued function on the interval $\bar{I}=[0,1]$. We say w is piecewise monotone if $\bar{I}$ can be divided into a finite number of non-overlapping sub-intervals $J_1, J_2, \cdots, J_m\left(\bigcup_{i=1}^{m} J_i=\bar{I}\right)$ on each of which w is monotone. Naturally such a division of $\bar{I}$ is not unique, but there is the least value of the numbers m for which we can find a division $\{J_i\}$ as above. This value is called the *lap-number* of w, and we shall denote it by $l(w)$. As an exceptional case, we set $l(w)=0$ when w is a constant function. And, for convenience sake, we understand that $l(w)=\infty$ if w is not a piecewise monotone function. Thus $l(w)$ can be regarded as a kind of functional whose value ranges over nonnegative integers (possibly infinity), and $l(w)=0$ if and only if w is a constant function.

Another equivalent definition of $l(w)$ is as follows: Suppose w is not a constant function. Then we can choose distinct points $0=x_0<x_1<\cdots<x_k=1$ such that $w(x_i)\neq w(x_{i+1})$ for $i=0,1,\cdots,k-1$ and that the sign of each $w(x_{i+1})-w(x_i)$ is opposite to that of $w(x_i)-w(x_{i-1})$. It is clear that the number k above cannot be arbitrarily large when w is piecewise monotone; and we define $l(w)$ by the supremum of the possible numbers k. This new definition of lap-number is easily seen to be equivalent to the former one so long as w is continuous. And when w is discontinuous, $l(w)$ will be understood in the latter sense throughout this paper. For example, let $w(x)=x$ for $0\leqq x\leqq 1/2$ and $w(x)=x-1$ for $1/2<x\leqq 1$. Then we have $l(w)=3$ (while $l(w)=2$ in the former definition; see Remark 6.1 and Example 6.2 for further details).

If $u=u(x,t)$ is a function of two variables $x\in\bar{I}$ and $t\in\boldsymbol{R}$, we can define the lap-number of u for each fixed value of t, and then $l(u(\cdot,t))$ becomes an integer-valued function of t. In this paper we shall show that the lap-number of any solution for the one-dimensional semilinear parabolic equation

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$$(1)\qquad \frac{\partial u}{\partial t}=a(x,t)\frac{\partial^2 u}{\partial x^2}+b(x,t)\frac{\partial u}{\partial x}+f\left(t,u,\frac{\partial u}{\partial x}\right)$$

will not increase as time passes; moreover, what is important, the non-increasing property is "stable" under a small perturbation to the initial data (Section 5). (Note that the nonlinear term f in the above equation does not depend on x explicitly. This is an essential requirement, and we shall impose no further specific assumption on f.) As a corollary to this result, it holds that if the solution $u(x,t)$ converges to some equilibrium state $v(x)$ as $t\to\infty$, then

$$(2)\qquad l(v)\leqq l(u_0),$$

where u_0 is the initial data for u. Since the lap-number $l(w)$ roughly evaluates the complexity of the shape of the graph of $w(x)$, the above inequality can be interpreted that the final state always exhibits a simpler spatial pattern than the initial state. This fact is in marked contrast to the case of parabolic systems (see Fife [2] for instance).

The basic method to be employed in this paper is the maximum principle; and some elementary topological discussions in the xt-plane based on the Jordan curve theorem (Lemmas 2.5 and 2.6). One finds a similar discussion in Serrin [12; Theorem 4, p. 92]; and also in Matano [6], in which the dynamical structure of one-dimensional parabolic equations has been studied.

First we shall discuss the problem under the Neumann boundary conditions (Section 2). This is the simplest case. Then we consider the cases of the Dirichlet boundary conditions and the third boundary conditions (Section 3). As a matter of fact, the lap-number of a solution is not always nonincreasing under the Dirichlet or the third boundary conditions. In certain circumstances, the lap-number may grow fairly larger (see Example 6.5). In order to exclude such a situation, we shall put in Section 3 the additional assumption $f(t,0)\equiv 0$ or the assumption $u\geqq 0, u_0\not\equiv 0$. These additional assumptions, however, do not reduce the applicability of our results considerably, for they are fulfilled practically in many important equations.

In Section 4, we study the autonomous equation

$$(3)\qquad \frac{\partial u}{\partial t}=a(x)\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial u}{\partial x}+f(u)$$

as a special case of (1). The results for the Dirichlet and the third boundary conditions are then improved somewhat. That is, without putting any assumption such as $f(0)=0$ or as $u\geqq 0$, we shall show that the increase of the lap-number $l(u(\cdot, t))$ does not exceed 2 in value. Consequently we always have

$$l(v)\leqq l(u_0)+2,$$

where v, u_0 are the final and the initial state respectively. Of course $l(u(\cdot, t))$ never increases under the condition $f(0)=0$ or the condition $u\geqq 0, u_0\not\equiv 0$, or if the boundary condition is homogeneous Neumann; hence in these situations the stronger assertion (2) holds.

In Section 5, which forms the most crucial part of this paper, we shall investigate the long time behavior of the lap-number of a solution of (3) when its initial state u_0 gets a spatially inhomogeneous small perturbation. Such a perturbation may cause a sudden increase of the lap-number, but in the long run this increase will be reduced entirely, except when u_0 is a constant function. More precisely, we shall show that for any nonconstant function $u_0(x)$ and for any $t_2>t_1>0$ there exist positive numbers δ and t_0 with $t_2>t_0>t_1$ such that

$$l(w(\cdot, t_0))\leqq l(u(\cdot, t_0))$$

holds for any solution $w(x, t)$ of (3) whose initial data w_0 satisfies

$$\sup_{0\leqq x\leqq 1} |w_0(x)-u_0(x)|<\delta.$$

In particular, we have

$$l\left(\lim_{t\to\infty} w(\cdot, t)\right)\leqq l(u_0) \tag{4}$$

either in the Neumann case or in the Dirichlet case with $f(0)=0$, if the limit on the left-hand side exists (it exists if w is uniformly bounded; see [6]). As a matter of fact, the assertion (4) remains true even if u_0 is a constant function, provided that it is not an unstable constant equilibrium solution of (3). (When u_0 is an unstable constant equilibrium solution, we have a counterexample as shown at the beginning of Section 5.)

In Sections 2 to 4, we shall also prove that if $l(u_0)>0$, then $l(u(\cdot, t))>0$ for all $t\geqq 0$. This is a consequence of the uniqueness theorems for backward parabolic equations. Note that the above assertion does not exclude the possibility that $u(x, t)$ converge to a constant equilibrium state as $t\to\infty$.

As for one-dimensional single equations of the form (3), any equilibrium solution $v(x)$ with $l(v)\geqq 1$ (i.e. nonconstant solution) is unstable in the case of the Neumann boundary conditions (see Chafee [1] for instance), while any equilibrium solution with $l(v)\geqq 3$ is unstable in the case of the Dirichlet boundary conditions (see Maginu [5] for instance), provided that the coefficients $a(x)$, $b(x)$ are constants in either of the cases. These results imply that, in one-dimensional single equations, there virtually (i.e. experimentally) occurs no complicated spatial pattern formation, while such phenomena are often observed in parabolic systems. However, these assertions do not apply to the situation where a, b depend on x; and in fact, in this case there sometimes exists a stable equilibrium solution with a large lap-number (Example 6.6). Nonetheless our results in the present paper still apply to this case, showing that simple initial data never yield complicated spatial patterns.

Some of the results in this paper, namely Theorems 1 and 3, were first obtained by M. Tabata [13] through the method of finite difference approximation. He first established similar theorems on the solutions of difference equations which approximate (1), then he proved the original assertions by the limit process. His proof therefore reveals that the nonincrease of lap-number is a property that can be inherited to approximate solutions. This fact is of importance from a numerical standpoint. We, however, treat the equation (1) directly, which makes our discussions fairly shorter. And our work covers many topics and results that have not been discussed in [13].

Lastly, in Section 6, we give some examples and counterexamples. Most of them are elementary, but they will probably be instructive.

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2. Case of the Neumann boundary conditions.

With $w(x)$ being a piecewise monotone function on $\bar{I}$ and with $l(w)$ as defined above, we introduce some more notation: Let $0=x_0<x_1<\cdots<x_k=1$ $(k=l(w))$ be points such that

$$\{w(x_i)-w(x_{i-1})\}\{w(x_{i+1})-w(x_i)\}<0$$

for $i=1, 2, \cdots, k-1$, and set

$$l^+(w)=\text{the cardinal number of the set } \{i\in N;\ 1\leqq i\leqq k,\ w(x_i)-w(x_{i-1})>0\};$$

$l^-(w)=$the cardinal number of the set
$$\{i\in N;\ 1\leqq i\leqq k,\ w(x_i)-w(x_{i-1})<0\},$$

where N is the set of all natural numbers.

Since we are assuming $k=l(w)$, such points $\{x_i\}$ really exist by the definition, and the two sets on the right-hand sides do not depend on the choice of the points $\{x_i\}$; so both $l^+(w)$ and $l^-(w)$ are well-defined. And we understand that $l^+(w)=l^-(w)=0$ when w is a constant function. Note that if w is continuous, $l^+(w)$ (resp. $l^-(w)$) equals the number of maximal sub-intervals of $\bar{I}$ on which w is monotone nondecreasing (resp. nonincreasing) and is not identically a constant. (By a maximal sub-interval we mean an interval $J\subset\bar{I}$ such that w is monotone on J but no longer monotone on any interval $J'\supsetneqq J$.) For example, if $w_n(x)=\sin n\pi x$, we have

$$l^+(w_{2m})=m+1,\ l^-(w_{2m})=m$$

and

$$l^+(w_{2m-1})=l^-(w_{2m-1})=m$$

for $m=1, 2, \cdots$.

From the definition it immediately follows that

$$l^-(w)=l^+(-w),$$
$$|l^+(w)-l^-(w)|\leqq 1$$

and

$$l(w)=l^+(w)+l^-(w).$$

Let us now consider the equation

$$\frac{\partial u}{\partial t}=a(x,t)\frac{\partial^2 u}{\partial x^2}+b(x,t)\frac{\partial u}{\partial x}+f(t,u)\quad\text{in}\quad I\times(0,T) \tag{5}$$

together with the initial and the boundary conditions

$$u(x,0)=u_0(x)\quad\text{in}\quad I, \tag{6}$$

$$\frac{\partial u}{\partial x}=0\quad\text{on}\quad \partial I\times(0,T), \tag{7}$$

where $I=(0,1)$, $\partial I=\{0\}\cup\{1\}$ and $0<T\leqq+\infty$. All the coefficients and the solution are real-valued, and we assume

(A.1) $a(x,t)\geqq\delta$ in $\bar{I}\times[0,T)$ for some constant $\delta>0$;

(A.2) $a\in C^1(\bar{I}\times[0,T))$;

(A.3) $b\in C^\alpha(\bar{I}\times[0,T))$ for some $0<\alpha<1$;

(A.4) $\qquad f \in C^1([0, T) \times \boldsymbol{R})$.

Assumption (A.1) implies the uniform parabolicity of the equation (5). Smoothness conditions on the coefficients may be relaxed somewhat, for which we do not care in this paper. By a "solution" we mean a classical solution; therefore u belongs to $C^1(\bar{I} \times (0, T)) \cap C^2(I \times (0, T))$ and satisfies (5), (7) in the classical sense. The initial condition (6) is understood in the sense that

(6′) $$\lim_{t \searrow 0} u(x, t) = u_0(x) \quad \text{a.e.} \quad x \in \bar{I} \quad \text{(bounded convergence)}.$$

We shall always assume that u_0 is piecewise continuous and bounded; so that the initial-boundary value problem (5), (6), (7) has a local solution. We do not discuss how far this local solution can be continued. Instead, we always understand that the value of T is taken appropriately so that the local solution can be continued until $t=T$. Note that the convergence in (6′) is automatically a uniform convergence if u_0 is continuous on $\bar{I}$.

Now we are ready to state our main results in this section:

THEOREM 1. *Let* (A.1)~(A.4) *hold, and let* $u_0(x)$ *be a bounded and piecewise continuous function on* $\bar{I}$ *(not necessarily piecewise monotone). Then the solution* $u(x, t)$ *of the initial-boundary value problem* (5), (6), (7) *satisfies*

(8a) $$l^+(u(\cdot, t_1)) \geqq l^+(u(\cdot, t_2)),$$

(8b) $$l^-(u(\cdot, t_1)) \geqq l^-(u(\cdot, t_2))$$

for any $0 \leqq t_1 \leqq t_2 < T$.

THEOREM 2. *Let the conditions in Theorem* 1 *hold. Suppose*

$$l(u(\cdot, t_0)) = 0$$

for some $t_0, 0 \leqq t_0 < T$. *Then*

$$l(u(\cdot, t)) = 0$$

for all $t, 0 < t < T$. *Consequently* $u_0(x)$ *is equal to a constant for* a.e. $x \in I$.

REMARK 2.1. Note that the nonlinear term f in (5) does not depend on x explicitly. This is an essential requirement. But our results can quickly be extended to the case where $f = f\left(t, u, \dfrac{\partial u}{\partial x}\right)$.

REMARK 2.2. In the statement of the above theorems (and throughout this paper) we understand that $l(w) = l^+(w) = l^-(w) = \infty$ if $w(x)$ is not piecewise monotone.

REMARK 2.3. The conclusions of Theorems 1 and 2 remain true if the bounded interval I is replaced by the entire real line $(-\infty, +\infty)$, so long as bounded solutions are concerned. The proof will be almost the same as that of Theorems 1 and 2.

REMARK 2.4. If $w_n(x)\to w(x)$ as $n\to\infty$ for all $x\in\bar{I}$, then, as is easily seen,

$$\liminf_{n\to\infty} l^+(w_n)\geqq l^+(w),$$
$$\liminf_{n\to\infty} l^-(w_n)\geqq l^-(w).$$

Consequently, given any continuous function $h(x,t)$ on $\bar{I}\times(0,T)$, both $l^+(h(\cdot,t))$ and $l^-(h(\cdot,t))$ are lower semi-continuous in $0<t<T$. In view of this fact and Theorem 1, we see that both $l^+(u(\cdot,t))$ and $l^-(u(\cdot,t))$ are monotone nonincreasing in $0\leqq t<T$ and right continuous in $0<t<T$. (If, in addition, u_0 is right or left continuous in I and continuous at $x=0,1$, then $l^+(u(\cdot,t))$ and $l^-(u(\cdot,t))$ are continuous at $t=0$. In fact, in this case, from (6′) follows

$$\liminf_{t\searrow 0} l^\pm(u(\cdot,t))\geqq l^\pm(u_0);$$

hence

$$\lim_{t\searrow 0} l^\pm(u(\cdot,t))=l^\pm(u_0).)$$

Before setting about the proof of Theorems 1 and 2, we present some notation and preliminary lemmas, which will be needed in this and the next sections.

For any function w on $\bar{I}\times[0,T)$, we set

(9a) $$A^+(w)=\{(x,t)\in\bar{I}\times[0,T);\ w(x,t)>0\},$$

(9b) $$A^-(w)=\{(x,t)\in\bar{I}\times[0,T);\ w(x,t)<0\}.$$

We shall also use the notation

(10a) $$\Omega_{t_1,t_2}=\{(x,t)\in\bar{I}\times[0,T);\ t_1\leqq t\leqq t_2\},$$

(10b) $$I_t=I\times\{t\},\ \ \bar{I}_t=\bar{I}\times\{t\}(=\Omega_{t,t}),$$

(10c) $$\Sigma_{t_1,t_2}=\partial\Omega_{t_1,t_2}\setminus I_{t_2},$$

(10d) $$S_0=\{0\}\times(0,T),\ \ S_1=\{1\}\times(0,T).$$

Σ_{t_1,t_2} consists of two vertical segments $\{0\}\times[t_1,t_2]$, $\{1\}\times[t_1,t_2]$ and one horizontal segment I_{t_1}. For convenience sake we regard Σ_{t_1,t_2} as an ordered set: Given $P,Q\in\Sigma_{t_1,t_2}$, we say $P\leq Q$ if either P coincides with Q, or P and the point $(0,t_2)$ belong to the same component of $\Sigma_{t_1,t_2}\setminus\{Q\}$. Clearly the above relation $\leq$

is an order relation, and in a similar manner we introduce into $\bar{I}_t$ an order relation, which we again denote by $\leq$. That is, two points P, Q on the line segment $\bar{I}_t$ satisfy the relation $P\leq Q$ if and only if P either coincides with Q or lies on the left-hand side of it. Finally we introduce the notation

$$P<Q, \tag{11}$$

which represents $P\leq Q$ and $P\neq Q$. It will be clear from the context whether $P<Q$ represents the usual order relation between two real numbers or the relation just defined in (11).

LEMMA 2.5. *Let $w=w(x,t)$ be a continuous function in $\bar{I}\times[0,T)$ having continuous derivatives $\partial w/\partial x$, $\partial^2 w/\partial x^2$ and $\partial w/\partial t$ in $I\times(0,T)$. And suppose w satisfies the parabolic equation*

$$\frac{\partial w}{\partial t}=a_1(x,t)\frac{\partial^2 w}{\partial x^2}+b_1(x,t)\frac{\partial w}{\partial x}+c_1(x,t)w \quad in \quad I\times(0,T), \tag{12}$$

where the coefficients a_1, b_1, c_1 are bounded in $I\times(0,T)$ and $a_1>\delta_1$ for some constant $\delta_1>0$. Then for any $0\leqq t_1<t_2<T$ and for any (nonempty) connected component C of $A^+(w)\cap\Omega_{t_1,t_2}$, it holds that

$$C\cap\Sigma_{t_1,t_2}\neq\emptyset.$$

The same is true for any connected component of $A^-(w)\cap\Omega_{t_1,t_2}$.

PROOF. Suppose $C\cap\Sigma_{t_1,t_2}=\emptyset$. Then w vanishes on $\partial C\setminus I_{t_2}$. Hence, by the maximum principle (see for instance [9; Chapter 3]), w should vanish in C, which obviously contradicts the supposition $C\subset A^+(w)$. This contradiction proves the lemma.

The above lemma, though elementary, plays a fundamental role throughout this section, and in the next section as well. As a corollary, we have

LEMMA 2.6. *Let the assumptions in Lemma 2.5 hold. And let $P_1, P_2, \cdots, P_m$ be any points on I_{t_2} such that*

$$P_1<P_2<\cdots<P_m \quad (see\ (11))$$

and that the sign of $w(P_{i+1})$ is opposite to that of $w(P_i)$ for $i=1,2,\cdots,m$ (in other words, $P_1, P_2, \cdots, P_m$ belong to $A^+(w)$ and $A^-(w)$ alternately). Then there exist points

$$Q_1<Q_2<\cdots<Q_m$$

on Σ_{t_1,t_2} such that for each $i=1,2,\cdots,m$ the pair P_i, Q_i are contained in the same *connected component of* $A^+(w)\cap\Omega_{t_1,t_2}$ *or of* $A^-(w)\cap\Omega_{t_1,t_2}$. *In particular,* $w(Q_i)$ *has the same sign as* $w(P_i)$.

PROOF. Suppose $P_i\in A^+(w)$. Then, by Lemma 2.5, P_i can be connected to some point $Q_i\in\Sigma_{t_1,t_2}$ by a path Γ_i in $A^+(w)\cap\Omega_{t_1,t_2}$. Similarly, if $P_j\in A^-(w)$, then it can be connected to some point $Q_j\in\Sigma_{t_1,t_2}$ by a path Γ_j in $A^-(w)\cap\Omega_{t_1,t_2}$. Thus, for each $i=1,2,\cdots,m$, we get a path Γ_i lying in the rectangular region Ω_{t_1,t_2} and connecting the point P_i to some point $Q_i\in\Sigma_{t_1,t_2}$; and that w does not change sign on each Γ_i. Since $\Gamma_1,\cdots,\Gamma_m$ are contained in $A^+(w)$ and in $A^-(w)$ alternately, Γ_i and Γ_{i+1} do not intersect, where $i=1,2,\cdots,m-1$. Using the Jordan curve theorem we easily get

$$Q_1<Q_2<\cdots<Q_m. \quad \text{(See Fig. 1.)}$$

This proves the lemma.

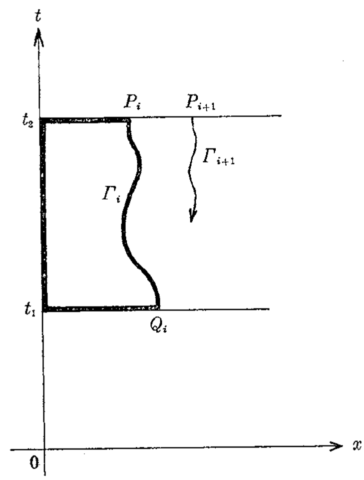

Fig. 1. Apply the Jordan curve theorem to the closed curve indicated above.

The following lemmas are needed for Theorem 2 and also for Theorem 6 in Section 3.

LEMMA 2.7. *Let* $w=w(x,t)$ *be a continuous function in* $\bar{I}\times(0,T)$ *having continuous derivatives* $\partial w/\partial x$, $\partial^2 w/\partial x^2$ *and* $\partial w/\partial t$ *in* $\bar{I}\times(0,T)$, *in* $I\times(0,T)$ *and in* $I\times(0,T)$ *respectively. Suppose* w *satisfies the equation* (12) *together with the boundary condition*

$$\frac{\partial w}{\partial x}=0 \quad on \quad \partial I\times(0,T),$$

where we assume $a_1\in C^1(\bar{I}\times[0,T))$, $a_1>\delta_1$ *for some constant* $\delta_1>0$ *and that* b_1, c_1 *are bounded on* $\bar{I}\times(0,T)$. *Suppose finally that* w *vanishes everywhere on* I_{t^*} *for some* $0<t^*<T$. *Then* w *vanishes identically in* $\bar{I}\times(0,T)$.

LEMMA 2.8. *The conclusion of Lemma* 2.7 *remains true if the boundary condition* $\partial w/\partial x=0$ *is replaced by*

$$w=0 \quad \text{on} \quad \partial I\times(0,T).$$

Lemmas 2.7 and 2.8 are consequences of the uniqueness theorems for parabolic equations and backward parabolic equations; see Lees and Protter [4] and

Friedman [3; Chapter 6].

PROOF OF THEOREM 1. For a while we assume that $\partial a/\partial x$, $\partial b/\partial x \in C^{\alpha}(\bar{I}\times[0, T))$, that $\partial f/\partial u \in C^{\alpha}([0, T)\times \boldsymbol{R})$ and that $\partial u_0/\partial x$ is a continuous function vanishing on ∂I. These additional assumptions will be removed soon later.

By the assumption above, $\partial u/\partial x$ is continuous in $\bar{I}\times[0, T)$. The continuity at $t=0$ follows from the fact that $w=\partial u/\partial x$ is a solution of the equation (obtained by differentiating (5) by x)

$$\frac{\partial w}{\partial t}=a\frac{\partial^2 w}{\partial x^2}+b_1\frac{\partial w}{\partial x}+cw \quad \text{in} \quad I\times(0, T),$$

under the conditions $w(0, t)=w(1, t)=0$ and $w(x, 0)=\partial u_0/\partial x$, where $b_1=\partial a/\partial x+b$ and $c=\partial b/\partial x+\partial f/\partial u$. (The differentiability of (5) follows from [3; Theorem 10, p. 72].) Let $P_1, P_2, \cdots, P_m$ be any points on I_{t_2} (here we assume $t_2>t_1$ since in the case $t_2=t_1$ the theorem is trivial) such that

$$P_1<P_2<\cdots<P_m \qquad \text{(see (11))}$$

and that the sign of $w(P_{i+1})$ is opposite to that of $w(P_i)$ for $i=1, 2, \cdots, m-1$. Then, by Lemma 2.6, there exist points

$$Q_1<Q_2<\cdots<Q_m$$

on Σ_{t_1,t_2} such that $w(Q_i)$ and $w(P_i)$ have the same sign for each $i=1, 2, \cdots, m$. Furthermore, as $w(=\partial u/\partial x)$ vanishes on $\partial I\times[0, T)$, the points $Q_1, \cdots, Q_m$ must lie on the line segment I_{t_1}. Thus, for any system of points $\{P_1, \cdots, P_m\}\subset I_{t_2}$ on which $\partial u/\partial x$ changes sign alternately, we can find a similar system of points $\{Q_i\}$ on I_{t_1}. This fact clearly shows that the inequalities (8a), (8b) hold.

It remains to remove the additional smoothness assumptions on a, b and u_0. Let u_0 be piecewise continuous and bounded on $\bar{I}$, and let a, b, f be as in (A.1)$\sim$(A.4). Choose sequences of smooth functions $\{a_n(x, t)\}$, $\{b_n(x, t)\}$, $\{f_n(t, u)\}$ converging to a, b, f respectively as $n\to\infty$ (in the topology of $C^1(\bar{I}\times[0, T))$, $C^{\beta}(\bar{I}\times[0, T))$ for some $0<\beta\leqq\alpha$ and $C^1([0, T)\times\boldsymbol{R})$ respectively). And let $\{u_{0,n}(x)\}$ be a sequence of uniformly bounded smooth functions satisfying

$$\frac{\partial u_{0,n}}{\partial x}=0 \quad \text{on} \quad \partial I,$$

$$l^+(u_{0,n})=l^+(u_0) \quad \text{and} \quad l^-(u_{0,n})=l^-(u_0)$$

for $n=1, 2, \cdots$ and converging to $u_0(x)$ for a.e. $x\in I$. (One easily finds that such a sequence $u_{0,n}$ really exists.) To such approximating sequences of coefficients and initial data, there corresponds a sequence of solutions $\{u_n(x, t)\}$ that converge to the original solution $u(x, t)$; more precisely, given any T_1 with $0<T_1<T$, u_n exists for $0\leqq t<T_1$ if n is sufficiently large, and it converges to u in $\bar{I}\times(0, T_1)$ as $n\to\infty$. (The convergence of u_n to u follows from the uniform boundedness of $\{u_n\}$ (for large n) in $\bar{I}\times(0, T_1)$ and *a priori* estimates for derivatives of u_n; see [3; Theorem 5, p. 64 and Theorem 15, p. 80].) Therefore

$$\liminf_{n\to\infty} l^+(u_n(\cdot, t))\geqq l^+(u(\cdot, t))$$

for $0\leqq t<T_1$, and hence, applying (8a) to u_n, we get

$$\begin{aligned} l^+(u_0) &= \liminf l^+(u_{0,n}) \\ &\geqq \liminf l^+(u_n(\cdot, t)) \\ &\geqq l^+(u(\cdot, t)) \end{aligned}$$

for $0\leqq t<T_1$. Since T_1 is any number between 0 and T,

$$l^+(u_0)\geqq l^+(u(\cdot, t))$$

holds for $0\leqq t<T$. In quite a similar manner we can show

$$l^-(u_0)\geqq l^-(u(\cdot, t))$$

for $0\leqq t<T$. Thus (8a) and (8b) are verified for the case $t_1=0$ without putting any additional assumptions on a, b, f and u_0. The general case $t_1\geqq 0$ now follows immediately. This completes the proof of Theorem 1.

Proof of Theorem 2. First, note that if $u_0(x)$ is equal to a constant for a.e. $x\in I$, then $l(u(\cdot, t))=0$ for all $0<t<T$, in other words, u does not depend on x. This is an immediate consequence of the uniqueness theorem for parabolic equations.

Now set

$$t^*=\inf\{t;\ 0\leqq t<T,\ l(u(\cdot, t))=0\}.$$

If $t^*=0$, then $u_0(x)$ is equal to a constant for a.e. $x\in I$; hence, as is remarked above, the conclusion of Theorem 2 follows. Supposing $t^*>0$, we shall derive a contradiction.

By the continuity of u, there exists a constant ξ such that $u(x, t^*)=\xi$ for all $x\in\bar{I}$. Let $\phi(t)$ be the solution of the ordinary differential equation

$$\frac{d\phi}{dt}=f(t, \phi)$$

under the condition $\phi(t^*)=\xi$. If $\delta>0$ is small enough, the solution $\phi(t)$ exists in the interval $[t^*-\delta, t^*+\delta]$. Then $\tilde{u}(x, t)=u(x, t)-\phi(t)$ satisfies

$$\frac{\partial \tilde{u}}{\partial t}=a\frac{\partial^2 \tilde{u}}{\partial x^2}+b\frac{\partial \tilde{u}}{\partial x}+g(t, \tilde{u}) \quad \text{in} \quad I\times[t^*-\delta, t^*+\delta]$$

where

$$g(t, \tilde{u})=f(t, \tilde{u}+\phi(t))-\phi'(t).$$

As $g(t, 0)=0$, the function g can be expressed in the form $g(t, \tilde{u})=\tilde{u}g_1(t, \tilde{u})$ with g_1 being continuous. Therefore $\tilde{u}$ satisfies a linear parabolic equation of the form (12) with $c(x, t)=g_1(t, \tilde{u}(x, t))$. Besides it satisfies the boundary condition $\partial\tilde{u}/\partial x=0$ and the condition $\tilde{u}(x, t^*)=0$ on $\bar{I}$. Applying Lemma 2.7 to $w=\tilde{u}$, we get

$$\tilde{u}(x, t^*-\delta)=0 \quad \text{for} \quad x\in\bar{I},$$

which implies $l(u(\cdot, t^*-\delta))=0$, an impossibility by the definition of t^*. Thus the assertion of Theorem 2 is verified.

3. Case of the Dirichlet boundary conditions.

Now we consider the equation (5) under the Dirichlet boundary condition

(13) $$u=0 \quad \text{on} \quad \partial I\times(0, T).$$

In this case, inequalities of the type (8a), (8b) no longer hold in general; the lap-number of a solution may grow arbitrarily large (Example 6.5). Therefore we should consider the problem in some restricted situation.

Also considered is the case of the third boundary condition

(14) $$\frac{\partial u}{\partial x}+\sigma(x, t)u=0 \quad \text{on} \quad \partial I\times(0, T),$$

where $\sigma(0, t)<0$ and $\sigma(1, t)>0$ for $0<t<T$. Since this case can be discussed quite analogously to the Dirichlet case, we shall only mention it briefly.

The notation given in the previous section is freely used; and we further make the following assumptions (only one of them will be assumed in each occasion):

(B.1) $$f(t, 0)=0 \quad \text{for} \quad 0\leqq t<T;$$

(B.2) $$f(t, 0)\geqq 0 \quad \text{for} \quad 0\leqq t<T;$$

(B.3) $$u\geqq 0 \quad \text{in} \quad \bar{I}\times[0, T).$$

Symmetrically we set

(B.2′) $$f(t,0)\leqq 0 \quad \text{for} \quad 0\leqq t<T;$$

(B.3′) $$u\leqq 0 \quad \text{in} \quad \bar{I}\times[0,T).$$

THEOREM 3. *Suppose* (A.1)$\sim$(A.4) *and* (B.1) *hold, and let* u_0 *be a bounded and piecewise continuous function on* $\bar{I}$ *vanishing on* ∂I. *Then the solution* $u(x,t)$ *of the initial-boundary value problem* (5), (6), (13) (*if it exists*) *satisfies the inequalities* (8a), (8b) *for any* $0\leqq t_1\leqq t_2<T$.

THEOREM 4. *Let* (A.1)$\sim$(A.4) *hold, and let either* (B.2) *or* (B.2′) *hold. Finally let* u_0 *be as in Theorem* 3. *Then the solution* $u(x,t)$ *of* (5), (6), (13) *satisfies*

(15a) $$l^+(u(\cdot,t_1))+1\geqq l^+(u(\cdot,t_2)),$$

(15b) $$l^-(u(\cdot,t_1))+1\geqq l^-(u(\cdot,t_2))$$

for any $0\leqq t_1\leqq t_2<T$.

THEOREM 5. *Let* (A.1)$\sim$(A.4) *hold, and let* u_0 *be as in Theorem* 3. *Suppose the solution* u *of* (5), (6), (13) *satisfies one of the conditions* (B.3), (B.3′). *Then* $l^+(u(\cdot,t))=l^-(u(\cdot,t))$ *for* $0\leqq t<T$. *Furthermore, for any* $0\leqq t_1\leqq t_2<T$,

(i) *the inequalities* (8a), (8b) *hold provided that* $l(u(\cdot,t_1))\neq 0$;

(ii) $l^+(u(\cdot,t_2))=l^-(u(\cdot,t_2))\leqq 1$ *if* $l(u(\cdot,t_1))=0$.

THEOREM 6. *Let the conditions in Theorem* 3 *hold* (*except that* u_0 *need not vanish on* ∂I), *and suppose*

$$l(u(\cdot,t_0))=0$$

for some t_0, $0<t_0<T$. *Then*

$$l(u(\cdot,t))=0$$

for all t, $0<t<T$. *Consequently* $u_0(x)=0$ *for* a.e. $x\in\bar{I}$.

REMARK 3.1. The assumption that u_0 vanishes on ∂I cannot be removed from Theorems 3, 4 and 5, as we see in the following simple example:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \qquad u_0\equiv 1.$$

In this case $l(u_0)=0$, while $l(u(\cdot,t))=2$ for $t>0$. In order that Theorem 3 or Theorem 5 (i) may hold, we must modify u_0 slightly as follows: $u_0(x)=1$ for $0<x<1$, $u_0(0)=u_0(1)=0$. This modification, of course, does not change the solution u for $t>0$. More generally, the same modification (at the boundary) should be done to any initial data that does not vanish at the boundary. (Note that this modification may increase $l(u_0)$ by up to 2.)

REMARK 3.2. The conclusion of Theorem 6 does not remain true if the assumption (B.1) is removed (see Example 6.4).

REMARK 3.3. In Theorem 5 (i), the assumption $l(u(\cdot, t_1))\neq 0$ is essential as we see from the simple example

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+1, \quad u_0\equiv 0.$$

In this case $l(u_0)=0$, while $l(u(\cdot, t))=2$ for $t>0$.

PROOF OF THEOREM 6. The proof of this theorem can be carried out in the same manner as the proof of Theorem 2, except that we use Lemma 2.8 instead of Lemma 2.7. Note that the function $\phi(t)$ defined in the proof of Theorem 2 is identically equal to zero in the present case.

PROOF OF THEOREM 5. It suffices to consider the case (B.3), for the case (B.3′) follows by applying the first case to $-u$. The assertion $l^+(u)=l^-(u)$ is obvious; therefore we begin with the proof of the assertion (i).

Let $\{a_n(x, t)\}$, $\{b_n(x, t)\}$, $\{f_n(t, u)\}$ and $\{u_{0,n}(x)\}$ be sequences of sufficiently smooth functions satisfying

$$\begin{aligned}
&\lim_{n\to\infty} a_n=a,\ \lim_{n\to\infty} b_n=b \ \text{ in } C^\beta(\bar{I}\times[0, T)) \text{ for some } \beta,\ 0<\beta\leqq\alpha,\\
&\lim_{n\to\infty} f_n=f \quad \text{in } C^1([0, T)\times R),\\
&\lim_{n\to\infty} u_{0,n}(x)=u_0(x) \quad \text{for } x\in\bar{I} \text{ (bounded convergence)},\\
&u_{0,n}=0 \text{ on } \partial I,\ u_{0,n}>0 \text{ in } I,\\
&l^+(u_{0,n})=l^+(u_0) \ \text{(hence } l^-(u_{0,n})=l^-(u_0)),\\
&a_n\frac{\partial^2 u_{0,n}}{\partial x^2}+b_n\frac{\partial u_{0,n}}{\partial x}+f_n(t, 0)=0 \text{ for } x\in\partial I, \quad t=0.
\end{aligned}$$

These approximating sequences of the coefficients and the initial data (such sequences really exist since $u_0\not\equiv 0$) yield a sequence of approximate solutions $\{u_n(x, t)\}$ which converge to $u(x, t)$ as $n\to\infty$. Note that each u_n is smooth at $t=0$; and it is not difficult to show that $u_n>0$ in $I\times[0, t_2]$ and $\partial u_n/\partial x\neq 0$ on $\partial I\times[0, t_2]$ for all large n, provided that the sequence $\{u_{0,n}\}$ is chosen appropriately. (The sequence $\{u_{0,n}\}$ can be constructed, for example, as follows: Let $\{\varepsilon_n\}$ be a sequence of positive numbers converging to 0, and let $\tilde{u}_n$ be a solution of (5), (6), (13) with initial data $u_0+\varepsilon_n$. By the comparison theorem, $\tilde{u}_n>u\geqq 0$ in $I\times[0, t_2]$. ($\tilde{u}_n$ exists on

$\bar{I}\times[0, t_2]$ at least for large n.) More precisely,

$$\tilde{u}_n \geqq \delta_n x(1-x) \quad \text{on} \quad \bar{I}\times[0, t_2]$$

for some $\delta_n>0$. Therefore, if $\{a_n\}$, $\{b_n\}$, $\{f_n\}$ converge to a, b, f very rapidly, then

$$\tilde{\tilde{u}}_n \geqq \frac{\delta_n}{2} x(1-x) \quad \text{on} \quad \bar{I}\times[0, t_2]$$

for all large n, where $\tilde{\tilde{u}}_n$ is a solution of the approximate equation with initial data $u_0+\varepsilon_n$. Now approximate $u_0+\varepsilon_n$ by a smooth function $u_{0,n}$ satisfying the last three of the above six conditions (and that $\partial u_{0,n}/\partial x \neq 0$ on ∂I). By virtue of the above inequality for $\tilde{\tilde{u}}_n$, it is clear that u_n possesses the required property if $u_{0,n}$ is sufficiently "close" to $u_0+\varepsilon_n$.) In view of this fact, and arguing as in the proof of Theorem 1, we see that it suffices to prove the statement under the assumptions that a, b, u are smooth in $\bar{I}\times[0, T)$ and that

$$u>0 \quad \text{in} \quad I\times[0, t_2],$$

$$\frac{\partial u}{\partial x} \neq 0 \quad \text{on} \quad \partial I\times[0, t_2].$$

Differentiating the equation (5) by x, we obtain

$$\frac{\partial w}{\partial t} = a\frac{\partial^2 w}{\partial x^2} + b_1\frac{\partial w}{\partial x} + cw, \tag{16}$$

where $w=\partial u/\partial x$, $b_1=\partial a/\partial x+b$ and $c=\partial b/\partial x+\partial f/\partial u$.

Let $P_1, P_2, \cdots, P_m$ be any points on I_{t_2} such that

$$P_1<P_2<\cdots<P_m \qquad \text{(see (11))}$$

and that they belong to $A^+(\partial u/\partial x)$ and $A^-(\partial u/\partial x)$ alternately; i.e. $\partial u/\partial x$ has the opposite sign on each pair of the points P_i, P_{i+1}. Then, by virtue of Lemma 2.5, there exist distinct points

$$Q_1<Q_2<\cdots<Q_m$$

on Σ_{t_1,t_2} such that $\partial u/\partial x$ takes the same sign on each pair of the points P_i, Q_i. On the other hand, as u vanishes on $S_0\cup S_1$ and is supposed to be positive in $I\times[0, t_2]$, we have

$$A^+\left(\frac{\partial u}{\partial x}\right)\cap S_1=\varnothing,$$

$$A^-\left(\frac{\partial u}{\partial x}\right)\cap S_0=\varnothing,$$

where $S_0=\{0\}\times(0, T)$ and $S_1=\{1\}\times(0, T)$. Therefore the following alternatives hold:

(a) $$\{Q_1, \cdots, Q_m\}\cap S_0=\varnothing;$$

(b) $$\{Q_1, \cdots, Q_m\}\cap S_0=\{Q_1\} \text{ and } Q_1\in A^+(\partial u/\partial x).$$

Similarly we have the alternatives

(a′) $$\{Q_1, \cdots, Q_m\}\cap S_1=\varnothing;$$

(b′) $$\{Q_1, \cdots, Q_m\}\cap S_1=\{Q_m\} \text{ and } Q_m\in A^-(\partial u/\partial x).$$

Suppose (b) holds. Since we are assuming $\partial u/\partial x\neq 0$ on $\partial I\times[0, t_2]$, $\partial u/\partial x$ is positive in a neighborhood of $\{0\}\times[0, t_2]$. Therefore we can find a point $Q_1'\in A^+(\partial u/\partial x)\cap I_{t_1}$ satisfying $Q_1'<Q_2$. Thus the case (b) reduces to the case (a). In the same manner the case (b′) reduces to the case (a′). That is, for any points $P_1<\cdots<P_m$ on I_{t_2} as above, we can find a similar system of points $Q_1<\cdots<Q_m$ on I_{t_1}. This clearly implies that the inequalities (8a), (8b) hold; hence the assertion (i) is verified.

Next we prove the assertion (ii). Without loss of generality we may assume $t_1=0$ (hence $u_0\equiv 0$). Take a sequence of uniformly bounded smooth functions $\{u_{0,n}(x)\}$ satisfying

$$\begin{aligned}&u_{0,n}=0 \quad \text{on} \quad \partial I,\ u_{0,n}>0 \quad \text{in} \quad I,\\ &l^+(u_{0,n})=l^-(u_{0,n})=1,\\ &\lim_{n\to\infty} u_{0,n}(x)=0 \quad \text{in} \quad \bar{I}.\end{aligned}$$

Then, given any $t_2\in(0, T)$, the solution u_n of (5), (13) with initial data $u_{0,n}(x)$ exists for $0\leqq t\leqq t_2$ if n is sufficiently large, and it satisfies $u_n\geqq u\geqq 0$ by the comparison theorem. Applying the statement (i) to u_n and taking $n\to\infty$, we get

$$l^+(u(\cdot, t_2))\leqq 1 \quad \text{and} \quad l^-(u(\cdot, t_2))\leqq 1,$$

which proves the assertion (ii) for the case $t_1=0$. The case $t_1>0$ now follows immediately. This completes the proof of Theorem 5.

PROOF OF THEOREM 4. It suffices to consider the case (B.2), for the case (B.2′) follows by applying the case (B.2) to $-u$.

First, suppose $u_0\equiv 0$. Then $u\geqq 0$ in $\bar{I}\times[0, T)$ by virtue of (B.2) and the maximum principle. Hence the assertion follows immediately from Theorem 5 (ii).

Next, suppose $u_0\not\equiv 0$. Then, as in the proof of Theorem 5 (i), we may assume that a, b and u are smooth in $\bar{I}\times[0, T)$. Therefore $w=\partial u/\partial x$ satisfies a linear parabolic equation of the form (16) and is continuous in $\bar{I}\times[0,T)$. We now use the following lemmas:

LEMMA 3.4. *Let the assumptions in Theorem 4 (case (B.2)) hold, and let a, b and u be suitably smooth in $\bar{I}\times[0,T)$. Suppose $\partial u/\partial x$ is positive on a point $(x_0,t_0)\in I\times[0,T)$ and is nonnegative on the line segment $\{0\leqq x\leqq x_0, t=t_0\}$. Then there exists a $\delta>0$ such that*

$$\{(x,t);\ 0\leqq x\leqq x_0,\ t_0<t<t_0+\delta\}\subset A^+\left(\frac{\partial u}{\partial x}\right);$$

that is, $\partial u/\partial x>0$ in this rectangular region.

LEMMA 3.5. *Let the assumptions in Theorem 4 (case (B.2)) hold, and suppose a, b and u are suitably smooth in $\bar{I}\times[0,T)$. Let C be any connected component of $A^+(\partial u/\partial x)\cap\Omega_{t_1,t_2}$ $(0\leqq t_1<t_2<T)$ such that*

$$C\cap S_0\neq\emptyset,\ C\cap I_{t_2}\neq\emptyset.$$

Then $C\cap S_0$ is connected and it contains the point $P=(0,t_2)$.

The proof of these lemmas will be given later in this section. Now we return to the proof of Theorem 4.

Let $P_1<P_2<\cdots<P_m$ be any points on I_{t_2} that belong to $A^+(\partial u/\partial x)$ and $A^-(\partial u/\partial x)$ alternately; i.e. $\partial u/\partial x$ has the opposite sign on each pair of the points P_i, P_{i+1} $(i=1,2,\cdots,m-1)$. By Lemma 2.6, there exist distinct points

$$Q_1<Q_2<\cdots<Q_m$$

on Σ_{t_1,t_2} such that $\partial u/\partial x$ takes the same sign on each pair of the points P_i, Q_i $(i=1,2,\cdots,m)$. (For the sake of the later argument, we choose Q_i appropriately so that the pair P_i, Q_i are contained in the same connected component of $A^+(\partial u/\partial x)\cap\Omega_{t_1,t_2}$ or $A^-(\partial u/\partial x)\cap\Omega_{t_1,t_2}$.) In view of Lemma 3.5, we easily see that one of the following four cases holds:

$$\{Q_1,\cdots,Q_m\}\cap S_0=\emptyset; \tag{17a}$$

$$\{Q_1,\cdots,Q_m\}\cap S_0=\{Q_1\}\quad\text{and}\quad Q_1\in A^+\left(\frac{\partial u}{\partial x}\right); \tag{17b}$$

$$\{Q_1,\cdots,Q_m\}\cap S_0=\{Q_1\}\quad\text{and}\quad Q_1\in A^-\left(\frac{\partial u}{\partial x}\right); \tag{17c}$$

$$\{Q_1,\cdots,Q_m\}\cap S_0=\{Q_1,Q_2\},\ Q_1\in A^+\left(\frac{\partial u}{\partial x}\right)\quad\text{and}\quad Q_2\in A^-\left(\frac{\partial u}{\partial x}\right). \tag{17d}$$

Suppose (17c) holds. Then there exists a point Q_1' on I_{t_1} satisfying $Q_1' < Q_2$ and $Q_1' \in A^-(\partial u/\partial x)$. In fact, if there were no such point, we should have either

$$u \equiv 0 \quad \text{on} \quad I_{t_1}$$

or

$$Q_2 \in I_{t_1}, \ \frac{\partial u}{\partial x} \geqq 0 \quad \text{on the line segment } RQ \ (R=(0, t_1)).$$

By virtue of (B.2), the former case implies $u \geqq 0$ in Ω_{t_1, t_2}, which clearly contradicts (17c). And in the latter case, by Lemmas 3.4 and 3.5, the connected component of $A^+(\partial u/\partial x) \cap \Omega_{t_1, t_2}$ containing the pair P_2, Q_2 should also contain the line segment $\{x=0, t_1 < t \leqq t_2\}$ entirely, which again is incompatible with (17c). Thus the existence of a point Q_1' as above is verified. Therefore the case (17c) reduces to the case (17a). In the same manner the case (17d) reduces to the case (17b). Similarly we have the alternatives

(17a′) $\qquad \{Q_1, \cdots, Q_m\} \cap S_1 = \varnothing;$

(17b′) $\qquad \{Q_1, \cdots, Q_m\} \cap S_1 = \{Q_m\} \quad$ and $\quad Q_m \in A^-\left(\dfrac{\partial u}{\partial x}\right),$

in the sense that all other possible cases reduce to these two cases. The discussion here shows that the inequalities (15a), (15b) hold. This completes the proof of Theorem 4.

PROOF OF THEOREM 3. The assumption (B.1) implies that both (B.2) and (B.2′) hold. In the case $u_0 \equiv 0$, we have $u \equiv 0$ by the uniqueness theorem; hence the statement is trivial.

Now suppose $u_0 \not\equiv 0$. Then, as in the proof of Theorems 4 and 5, we may assume that a, b, u are smooth in $\bar{I} \times [0, T)$. The conclusion of Lemma 3.5 remains true even if C is a connected component of $A^-(\partial u/\partial x) \cap \Omega_{t_1, t_2}$, since $-u$ also satisfies (B.2). It follows that the case (17d) never occurs. Arguing as in the proof of Theorem 4, we see the cases (17b), (17c) reduce to the case (17a). Similarly the case (17b′) and other possible cases reduce to the case (17a′). Hence we may assume that the points $Q_1, \cdots, Q_m$ all lie on the line segment I_{t_1}. This implies that (8a) and (8b) hold, completing the proof of Theorem 3.

We now prove Lemmas 3.4 and 3.5.

PROOF OF LEMMA 3.4. Since u vanishes at the point $R=(0, t_0)$, we have

$$u > 0 \quad \text{at} \quad Q \quad \text{and} \quad u \geqq 0 \quad \text{on} \quad RQ,$$

where $Q=(x_0, t_0)$. Let $\delta > 0$ be small enough that $u > 0$ and $\partial u/\partial x > 0$ on the closed line segment QQ', where $Q'=(x_0, t_0+\delta)$. Then

$$u \geqq 0 \quad \text{and} \quad u \not\equiv 0 \quad \text{on} \quad R'R \cup RQ \cup QQ',$$

where $R'=(0, t_0+\delta)$. Consequently, by (B.2) and the maximum principle,

$$u>0 \quad \text{for} \quad 0<x<x_0, \ t_0<t<t_0+\delta.$$

Considering that u vanishes on $S_0=\{x=0, 0<t<T\}$ and applying the maximum principle again (on the boundary), we find $\partial u/\partial x>0$ on the open line segment $R'R \setminus \{R', R\}$. Therefore

$$\frac{\partial u}{\partial x} \geqq 0 \quad \text{and} \quad \frac{\partial u}{\partial x} \not\equiv 0 \quad \text{on} \quad R'R \cup RQ \cup QQ'.$$

Recalling that $\partial u/\partial x$ satisfies the linear parabolic equation (16) and applying the maximum principle to $\partial u/\partial x$, we obtain

$$\frac{\partial u}{\partial x}>0 \quad \text{for} \quad 0<x<x_0, \ t_0<t<t_0+\delta.$$

Since $\partial u/\partial x$ is also positive on $R'R \setminus \{R', R\}$ and on QQ', the conclusion of the lemma follows.

PROOF OF LEMMA 3.5. Let $R^*=(0, t^*)$ be any point of $C \cap S_0$, and let J be the connected component of $C \cap S_0$ containing R^*. It suffices to show that J contains the point $P=(0, t_2)$. Assuming that $P \notin J$, we shall derive a contradiction.

Naturally J is a vertical line segment; let $R=(0, t_0)$ be the upper end of J. From the assumption $P \notin J$ follows $R \notin J$, hence

$$R \notin A^+\left(\frac{\partial u}{\partial x}\right). \tag{18}$$

Clearly we have $t_1 \leqq t^* < t_0 \leqq t_2$. Since C intersects I_{t_2}, R^* can be connected to I_{t_2} by a path in C, and therefore, R^* can be connected to some point $Q=(x_0, t_0) \in I_{t_0} \cap A^+(\partial u/\partial x)$ by a path Γ in $C \cap \Omega_{t_1, t_0}$ (see Fig. 2). Without loss of generarity we may assume that Γ is a simple curve, and that $\Gamma \cap I_{t_0}=\{Q\}$. Denote by Ω the region in $\bar{I} \times [0, T)$ whose boundary $\partial\Omega$ consists of Γ, R^*R and RQ. As

$$\frac{\partial u}{\partial x}>0 \quad \text{on} \quad (\partial\Omega \setminus RQ) \cup \{Q\},$$

it follows from the maximum principle that

$$\frac{\partial u}{\partial x}>0 \quad \text{in} \quad \Omega \quad \text{and on} \quad RQ \setminus \{R\}.$$

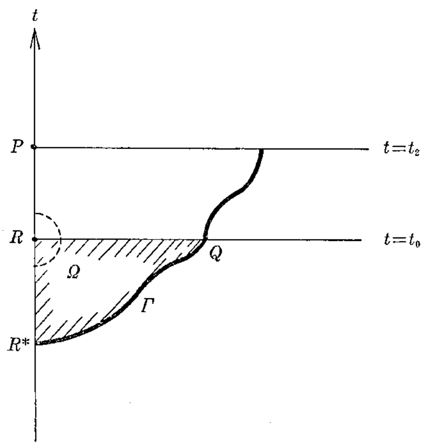

Fig. 2

Combining this fact and Lemma 3.4, we get

$$\frac{\partial u}{\partial x}>0 \quad \text{in} \quad (V\setminus\{R\})\cap \bar{I}\times[0, T),$$

where V is some small open circle with center R. Consequently

$$u>0 \quad \text{in} \quad V\cap I\times[0, T),$$

since u vanishes on S_0. Applying the maximum principle to u on the boundary (recall (B.2)), we see

$$\frac{\partial u}{\partial x}>0 \quad \text{on} \quad V\cap S_0,$$

which contradicts (18). This contradiction proves the lemma.

In the rest of this section we consider the case of the third boundary condition (14). This case can be discussed in a manner quite parallel to the case of the Dirichlet boundary condition (13). Requirement on the initial data u_0 is now reformulated as follows:

(C) There exists a sequence of smooth functions $\{u_{0,n}(x)\}$ converging to $u_0(x)$ for a.e. $x\in\bar{I}$ and satisfying

$$l^+(u_{0,n})\leqq l^+(u_0),\ l^-(u_{0,n})\leqq l^-(u_0),$$
$$\frac{\partial u_{0,n}}{\partial x}+\sigma(x, 0)u_{0,n}=0 \quad \text{on} \quad \partial I.$$

Note that if u_0 vanishes on ∂I and is bounded and piecewise continuous on $\bar{I}$, then the condition (C) is fulfilled. Another situation where (C) holds is that u_0 is piecewise continuous and bounded on $\bar{I}$ and that $u_0(+0)u_0(x)$ is monotone nondecreasing (not constant) near $x=0$ while $u_0(1-0)u_0(x)$ is monotone nonincresing (not constant) near $x=1$.

THEOREM 7. *The conclusions of Theorems* 3, 4 *and* 5 *remain true for the solution u of* (5), (6), (14) *provided that the condition* $u_0|_{\partial I}=0$ *is replaced by* (C).

THEOREM 8. *The conclusion of Theorem* 6 *remains true even if u is the solution of* (5), (6), (14).

We omit the proof of these theorems.

4. Autonomous problems and the stable sets.

In this section we discuss the case where the equation (5) is autonomous: The problem under the Neumann boundary condition then takes the form

$$(\mathrm{E_N}) \qquad \begin{cases} \dfrac{\partial u}{\partial t}=a(x)\dfrac{\partial^2 u}{\partial x^2}+b(x)\dfrac{\partial u}{\partial x}+f(u) \quad \text{in} \quad I\times(0,T), \\ u(x,0)=u_0(x) \quad \text{in} \quad I, \\ \dfrac{\partial u}{\partial x}=0 \quad \text{on} \quad \partial I\times(0,T), \end{cases}$$

which we simply denote by ($\mathrm{E_N}$). The problem under the Dirichlet boundary condition

$$u=0 \quad \text{on} \quad \partial I\times(0,T)$$

will be denoted by ($\mathrm{E_D}$). The assumptions (A.1)$\sim$(A.4) given in Section 2 remain unchanged and are restated as follows:

(A.1′) $\qquad a(x)>0$ for all $x\in\bar{I}$;

(A.2′) $\qquad a\in C^1(\bar{I}),\ b\in C^\alpha(\bar{I})$ for some $0<\alpha<1$ and $f\in C^1(R)$.

For convenience sake we introduce semigroups $U_N(t)$ and $U_D(t)$ generated by ($\mathrm{E_N}$) and ($\mathrm{E_D}$) respectively. More precisely, given a bounded piecewise continuous function $\phi(x)$ on $\bar{I}$, we set

$$U_N(t)\phi=u_\phi(\cdot,t) \quad (\text{resp. } U_D(t)\phi=u_\phi(\cdot,t)),$$

where $u_\phi(x,t)$ is the solution of ($\mathrm{E_N}$) (resp. ($\mathrm{E_D}$)) with initial data $u_0=\phi$. Strictly speaking, $U_N(t)$ and $U_D(t)$ depend on a,b,f, but we shall regard a,b,f as fixed functions and shall not express them explicitly. Set

$$(19) \qquad \begin{aligned} &T_N(\phi)=\sup\{T;\ U_N(t)\phi \text{ can be continued till } t=T\}; \\ &T_D(\phi)=\sup\{T;\ U_D(t)\phi \text{ can be continued till } t=T\}. \end{aligned}$$

In other words, if $U_N(t)\phi$ (resp. $U_D(t)\phi$) blows up in a finite time, then it blows up at $t=T_N(\phi)$ (resp. $T_D(\phi)$); otherwise $T_N(\phi)=\infty$ (resp. $T_D(\phi)=\infty$).

Combining the results in the preceding sections, we get the following proposition and theorems:

PROPOSITION 9. *Suppose* (A.1′), (A.2′) *hold, and let* $\phi(x)$ *be a bounded and piecewise continuous function on* $\bar{I}$. *Then* $l^+(U_N(t)\phi)$, $l^-(U_N(t)\phi)$, $l^+(U_D(t)\phi)$ *and* $l^-(U_D(t)\phi)$ *are lower semicontinuous in* $t>0$. *If, in addition,* ϕ *is continuous on*

$\bar{I}$, the former two are continuous at $t=0$, and the latter two are lower semicontinuous at $t=0$.

THEOREM 10. *Suppose* (A.1′), (A.2′) *hold, and let $\phi(x)$ be a bounded and piecewise continuous function on $\bar{I}$. Then $l^+(U_N(t)\phi)$ and $l^-(U_N(t)\phi)$ are monotone nonincreasing in $0\leqq t<T_N(\phi)$. Moreover, if $l(U_N(t_0)\phi)=0$ for some $t_0\in[0, T_N(\phi))$, then $l(U_N(t)\phi)=0$ for all $0<t<T_N(\phi)$.*

THEOREM 11. *Suppose* (A.1′), (A.2′) *hold, and let $\phi(x)$ be a bounded and piecewise continuous function on $\bar{I}$ satisfying $\phi(0)=\phi(1)=0$. Then*

$$l^+(U_D(t_1)\phi)+1\geqq l^+(U_D(t_2)\phi),$$
$$l^-(U_D(t_1)\phi)+1\geqq l^-(U_D(t_2)\phi)$$

for any $0\leqq t_1\leqq t_2<T_D(\phi)$.

THEOREM 12. *Let the conditions in Theorem* 11 *hold, and assume further that $f(0)=0$. Then $l^+(U_D(t)\phi)$ and $l^-(U_D(t)\phi)$ are monotone nonincreasing in $0\leqq t<T_D(\phi)$.*

THEOREM 13. *Let the conditions in Theorem* 11 *hold, and let T' be any number with $0<T'\leqq T_D(\phi)$. Suppose*

$$\phi\not\equiv 0,$$
$$U_D(t)\phi\geqq 0 \quad \text{for} \quad 0\leqq t<T'.$$

Then $l^+(U_D(t)\phi)$ and $l^-(U_D(t)\phi)$ are monotone nonincreasing in $0\leqq t<T'$.

THEOREM 14. *Let the conditions in Theorem* 11 *hold, and suppose $l(U_D(t_0)\phi)=0$ for some t_0, $0<t_0<T_D(\phi)$. Then $l(U_D(t)\phi)=0$ for all $0<t<T_D(\phi)$. Consequently $f(0)=0$ and $\phi(x)=0$* (a.e. $x\in I$).

See Remark 2.4 for the proof of Proposition 9. Theorem 10 is simply a restatement of Theorems 1 and 2. Theorem 11 follows from Theorem 4, for at least one of the conditions (B.2), (B.2′) is satisfied automatically. Theorem 12 is a restatement of Theorem 3. Theorem 13 follows immediately from Theorems 5 and 14. Lastly, Theorem 14 follows from Theorem 6. In fact, substituting $t=t_0$ and letting $x\to 0$ in the equation ($\mathrm{E_D}$) yield $f(0)=0$; hence, by applying Theorem 6, we get to the conclusion of Theorem 14.

Next we consider stationary equations corresponding to ($\mathrm{E_N}$) and ($\mathrm{E_D}$):

$$(\tilde{\mathrm{E}}_\mathrm{N}) \qquad a(x)v''(x)+b(x)v'(x)+f(v(x))=0 \quad \text{in} \quad I,\ v'(0)=v'(1)=0;$$

$$(\tilde{\mathrm{E}}_\mathrm{D}) \qquad a(x)v''(x)+b(x)v'(x)+f(v(x))=0 \quad \text{in} \quad I,\ v(0)=v(1)=0,$$

where $v'=dv/dx$. A solution of ($\tilde{\mathrm{E}}_{\mathrm{N}}$) or ($\tilde{\mathrm{E}}_{\mathrm{D}}$) is called an equilibrium solution of (E_{N}) or (E_{D}) respectively.

DEFINITION. Let $v=v(x)$ be a solution of ($\tilde{\mathrm{E}}_{\mathrm{N}}$). The *stable set* of v (under the Neumann boundary condition) is the set of all $\phi\in C(\bar{I})$ satisfying $T_N(\phi)=\infty$ and

$$\lim_{t\to\infty} U_N(t)\phi=v \quad \text{in} \quad C(\bar{I}).$$

Similarly, if v is a solution of ($\tilde{\mathrm{E}}_{\mathrm{D}}$), the stable set of v (under the Dirichlet boundary condition) is the set of all $\phi\in C(\bar{I})$ satisfying $T_D(\phi)=\infty$ and

$$\lim_{t\to\infty} U_D(t)\phi=v \quad \text{in} \quad C(\bar{I}).$$

A simple compactness argument shows that if $U_N(t)\phi$ or $U_D(t)\phi$ converges to v in the topology of $C(\bar{I})$, it also converges to v in $C^2(\bar{I})$. We can also prove that a $\phi\in C(\bar{I})$ belongs to the stable set of some solution of ($\tilde{\mathrm{E}}_{\mathrm{N}}$) (resp. ($\tilde{\mathrm{E}}_{\mathrm{D}}$)) if $U_N(t)\phi$ (resp. $U_D(t)\phi$) remains bounded in $C(\bar{I})$ as $t\to\infty$; see [6], where formally self-adjoint equations are discussed; but any one-dimensional parabolic equation can be reduced to a formally self-adjoint one by the coordinate transformation

$$x\longmapsto\tilde{x}=\int_0^x \exp\left(\int_0^y \frac{b(z)-a'(z)}{a(z)}dz\right)dy.$$

COROLLARY 15. *Suppose* (A.1′), (A.2′) *hold, and let* v *be a solution of* ($\tilde{\mathrm{E}}_{\mathrm{N}}$). *If* $\phi\in C(\bar{I})$ *belongs to the stable set of* v (*under the Neumann boundary condition*), *then*

$$l^+(\phi)\geqq l^+(v),\ l^-(\phi)\geqq l^-(v). \tag{20}$$

COROLLARY 16. *Suppose* (A.1′), (A.2′) *hold, and let* v *be a solution of* ($\tilde{\mathrm{E}}_{\mathrm{D}}$). *If* $\phi\in C(\bar{I})$ *satisfies* $\phi(0)=\phi(1)=0$ *and belongs to the stable set of* v (*under the Dirichlet boundary condition*), *then*

$$l^+(\phi)+1\geqq l^+(v),\ l^-(\phi)+1\geqq l^-(v). \tag{21}$$

If, in addition, $f(0)=0$, *then the inequalities* (20) *hold.*

These are immediate consequences of Theorems 10, 11 and 12; see also Remark 2.4.

5. Spatially inhomogeneous perturbations.

Consider, for example, the following equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+ku(1-u^2) \tag{22}$$

under the Neumann boundary condition $\partial u/\partial x=0$ at $x=0$, $x=1$; here k is a positive constant. Let m be any positive integer. As is easily seen, if $k>m^2\pi^2$, there exists an equilibrium solution v_m of (22) such that

$$l(v_m)=m$$

and that the constant equilibrium solution $v\equiv 0$ lies in the closure of the stable set of v_m. This implies that a very small perturbation to $v\equiv 0$ may cause a drastic increase of lap-number that cannot be reduced even in the final state as $t\to\infty$.

The aim of this section is to show that the situation described above is rather an exceptional case and that the increase of the lap-number caused by a very small perturbation will usually be reduced entirely in the long run.

Although this claim is true for non-autonomous equations, we shall only consider autonomous ones for simplicity. The notation in the previous section will be used freely. Smoothness assumptions are slightly strengthened; namely

$$a, b\in C^{1+\alpha}(\bar{I}) \quad \text{and} \quad f\in C^{1+\alpha}(\boldsymbol{R}). \tag{A.2$''$}$$

THEOREM 17. *Assume* (A.1′) *and* (A.2″). *Let $\phi(x)$ be a nonconstant continuous function on $\bar{I}$, and let t_1, t_2 be any numbers with $0<t_1<t_2<T_N(\phi)$, where $T_N(\phi)$ is as defined in* (19). *Then there exist positive numbers δ and t_0 with $t_1<t_0<t_2$ such that*

$$l^+(U_N(t_0)w)\leqq l^+(U_N(t_0)\phi), \tag{23a}$$

$$l^-(U_N(t_0)w)\leqq l^-(U_N(t_0)\phi) \tag{23b}$$

hold for all functions $w\in C(\bar{I})$ satisfying

$$\max_{0\leqq x\leqq 1}|w(x)-\phi(x)|\leqq\delta. \tag{24}$$

THEOREM 18. *Assume* (A.1′) *and* (A.2″). *Let $\phi(x)$ be a nonconstant continuous function on $\bar{I}$, and let t_1, t_2 be any number with $0<t_1<t_2<T_D(\phi)$. Then there exist positive numbers δ and t_0, $t_1<t_0<t_2$, such that*

$$l^+(U_D(t_0)w)\leqq l^+(U_D(t_0)\phi)+1, \tag{25a}$$

$$l^-(U_D(t_0)w)\leqq l^-(U_D(t_0)\phi)+1 \tag{25b}$$

for all function $w\in C(\bar{I})$ satisfying (24). *If, in addition, $f(0)=0$, then δ and t_0*

can be so chosen that

$$l^+(U_D(t_0)w) \leqq l^+(U_D(t_0)\phi), \tag{26a}$$

$$l^-(U_D(t_0)w) \leqq l^-(U_D(t_0)\phi) \tag{26b}$$

hold.

REMARK 5.1. The assertions (26a), (26b) in Theorem 18 hold true even if $f(0) \neq 0$, provided that ϕ is *not* an equilibrium solution of (E_D) satisfying

$$\phi'(0)=0 \quad \text{or} \quad \phi'(1)=0.$$

Before presenting the following theorems, we give the definition of stability with respect to the usual $C(\bar{I})$-topology: An equilibrium solution v of (E_N) (i.e. a solution of ($\tilde{E}_N$)) or of (E_D) is said to be *stable* if, given any $\varepsilon>0$, there exists a $\delta>0$ such that

$$\|U_N(t)w-v\|_{C(\bar{I})}<\varepsilon \quad (\text{resp. } \|U_D(t)w-v\|_{C(\bar{I})}<\varepsilon)$$

for all $t>0$ and any $w \in C(\bar{I})$ satisfying $\|w-v\|_{C(\bar{I})}<\delta$. We say v is *unstable* if it is not stable.

THEOREM 19. *Let* (A.1′), (A.2″) *hold, and let* $\phi(x)$ *be a continuous function on* $\bar{I}$. *Suppose* ϕ *is not an unstable equilibrium solution of* (E_N); *here the term "unstable" is with respect to the usual* $C(\bar{I})$-*topology. Then there exists a* $\delta>0$ *such that*

$$l^+\Big(\lim_{t\to\infty} U_N(t)w\Big) \leqq l^+(\phi),$$

$$l^-\Big(\lim_{t\to\infty} U_N(t)w\Big) \leqq l^-(\phi)$$

hold for any $w \in C(\bar{I})$ *satisfying* (24) *and the condition*

$$\lim_{t\to\infty} U_N(t)w \quad \textit{exists in } C(\bar{I})\textit{-topology}. \tag{27}$$

THEOREM 20. *Let* (A.1′), (A.2″) *hold, and let* $\phi(x)$ *be a continuous function on* $\bar{I}$ *satisfying* $\phi(0)=\phi(1)=0$. *Suppose* ϕ *is not an unstable constant equilibrium solution of* (E_D). *Then there exists a* $\delta>0$ *such that*

$$l^+\Big(\lim_{t\to\infty} U_D(t)w\Big) \leqq l^+(\phi)+1, \tag{28a}$$

$$l^-\Big(\lim_{t\to\infty} U_D(t)w\Big) \leqq l^-(\phi)+1 \tag{28b}$$

hold for any $w \in C(\bar{I})$ *satisfying* (24) *and the condition*

(29) $$\lim_{t\to\infty} U_D(t)w \quad \text{exists in } C(\bar{I})\text{-topology}.$$

If, in addition, $\phi(x)\not\equiv 0$ *and* $f(0)=0$, *then* $\delta>0$ *can be so chosen that the inequalities* (28a), (28b) *hold strictly.*

REMARK 5.2. As has been mentioned in Section 4, the condition (27) (resp. (29)) is satisfied if and only if $U_N(t)w$ (resp. $U_D(t)w$) remains bounded in $C(\bar{I})$ as $t\to\infty$.

The following are analogues of Theorems 19 and 20.

COROLLARY 21. *Let* (A.1′), (A.2″) *hold, and let* ϕ *belong to* $C(\bar{I})$. *Suppose that* $U_N(t)\phi$ *converges as* $t\to\infty$ *in* $C(\bar{I})$*-topology and that* $\lim_{t\to\infty} U_N(t)\phi$ *is not an unstable constant solution of* ($\tilde{\mathrm{E}}_{\mathrm{N}}$). *Then there exists a* $\delta>0$ *such that*

$$l^+\Big(\lim_{t\to\infty} U_N(t)w\Big)\leqq l^+\Big(\lim_{t\to\infty} U_N(t)\phi\Big),$$
$$l^-\Big(\lim_{t\to\infty} U_N(t)w\Big)\leqq l^-\Big(\lim_{t\to\infty} U_N(t)\phi\Big)$$

for any $w \in C(\bar{I})$ *satisfying* (24), (27).

COROLLARY 22. *Let* (A.1′), (A.2″) *hold, and let* ϕ *belong to* $C(\bar{I})$. *Suppose* $U_D(t)\phi$ *converges as* $t\to\infty$ *in the topology of* $C(\bar{I})$ *and that* $\lim_{t\to\infty} U_D(t)\phi$ *is not an unstable constant solution of* ($\tilde{\mathrm{E}}_{\mathrm{D}}$). *Then there exists a* $\delta>0$ *such that*

$$l^+\Big(\lim_{t\to\infty} U_D(t)w\Big)\leqq l^+\Big(\lim_{t\to\infty} U_D(t)\phi\Big)+1,$$
$$l^-\Big(\lim_{t\to\infty} U_D(t)w\Big)\leqq l^-\Big(\lim_{t\to\infty} U_D(t)\phi\Big)+1$$

for any $w \in C(\bar{I})$ *satisfying* (24), (29).

Now we set about the proof of these results. The most fundamental ones among them are Theorems 17 and 18, for which we need the following lemmas:

LEMMA 5.3. *Let* w *be of class* $C^2(\bar{I})$ *and assume*

(30) $$|w''(x)|+|w'(x)|>0$$

for all $x \in \bar{I}$. *Suppose there exists a sequence of functions* $w_n \in C^2(\bar{I})$ *converging to* w *as* $n\to\infty$ *in the topology of* $C^2(\bar{I})$. *Then there exists a positive integer* n_0 *such that*

$$l(w_n) \leqq l(w)+2$$

for all $n \geqq n_0$. *If, in addition, either*

$$w'(0) \neq 0, \ w'(1) \neq 0$$

or

$$w_n'(0)=w_n'(0)=0 \quad \textit{for} \quad n=1, 2, \cdots,$$

then

$$l^+(w_n)=l^+(w) \quad \textit{and} \quad l^-(w_n)=l^-(w)$$

for all $n \geqq n_0$.

LEMMA 5.4. *Let the assumptions* (A.1′), (A.2″) *hold, and let* $\phi(x)$ *be a non-constant, continuous and piecewise monotone function on* $\bar{I}$. *Then there exists a dense open subset* G *of the interval* $(0, T_N(\phi))$ *such that*

$$\left| \frac{\partial^2}{\partial x^2} u_\phi(x, t) \right| + \left| \frac{\partial}{\partial x} u_\phi(x, t) \right| > 0 \tag{31}$$

for all $x \in \bar{I}$ *and* $t \in G$, *where* $u_\phi(\cdot, t)$ *stands for* $U_N(t)\phi$. *The same assertion holds true if we replace* $T_N(\phi)$ *by* $T_D(\phi)$ *and* $U_N(t)\phi$ *by* $U_D(t)\phi$.

PROOF OF LEMMA 5.3. We only consider the case $w'(0) \neq 0$, $w'(1) \neq 0$. The remaining part of the lemma can be proved in a similar manner. By the assumption, there exists points

$$0=x_0<x_1<\cdots<x_k=1 \quad (k=l(w))$$

such that $w'(x)$ changes sign at each of the points $x=x_1, x=x_2, \cdots, x=x_{k-1}$ and that

$$w'(x) \neq 0 \quad \text{in} \quad \bar{I} \setminus \{x_1, x_2, \cdots, x_{k-1}\}.$$

Let δ be any positive number such that the intervals $J_{\delta,i}=(x_i-\delta, x_i+\delta)$, $i=1, 2, \cdots, k-1$, are disjoint. Then there exists a positive integer n_δ such that

$$\frac{w_n'(x)}{w'(x)}>0 \quad \text{in} \quad \bar{I} \setminus (J_{\delta,1} \cup \cdots \cup J_{\delta,k-1}) \tag{32}$$

for all $n \geqq n_\delta$. It follows that

$$l^+(w_n) \geqq l^+(w) \quad \text{and} \quad l^-(w_n) \geqq l^-(w) \tag{33}$$

for all $n \geqq n_\delta$.

Now suppose that the conclusion of the theorem were false. Without loss of generality we may assume that

$$l^+(w_n)>l^+(w)$$

for all $n=1, 2, \cdots$. Then, by (33), $l(w_n)>l(w)$ for all $n \geqq n_\delta$. In view of this and (32), we see that for each $n \geqq n_\delta$ the function $w_n'(x)$ changes sign at least twice in one of the intervals $J_{\delta,i}$ $(1 \leqq i \leqq k-1)$. Letting $\delta \to 0$, we find that at least one of $w''(x_1), \cdots, w''(x_{k-1})$ vanishes. However, this is impossible by virtue of (30). This contradiction proves the assertion of the lemma.

PROOF OF LEMMA 5.4. First we consider the case of the Neumann boundary conditions. From the assumption and Theorem 10 follows

$$0<l(u(\cdot, t)) \leqq l(\phi)<\infty$$

for all $0 \leqq t<T_N(\phi)$, where, for simplicity, we denote $U_N(t)\phi$ by $u(\cdot, t)$. Let G be the set of all t, $0<t<T_N(\phi)$, such that (31) holds for all $x \in \bar{I}$ with $u=u_\phi$. By the continuity of $\partial^2 u/\partial x^2$ and $\partial u/\partial x$, G is an open subset of the interval $(0, T_N(\phi))$. Let t_1, t_2 be any numbers satisfying $0<t_1<t_2<T_N(\phi)$. What we have to show is that the interval $[t_1, t_2]$ contains a point of G, i.e.

$$G \cap [t_1, t_2] \neq \emptyset.$$

Since $l(u(\cdot, t))$ is monotone nonincreasing and bounded, its discontinuity points in $0 \leqq t<T_N(\phi)$ are finite in number. We can therefore choose $\delta>0$ and t^* with $t_1<t^*-\delta<t^*+\delta<t_2$ such that

$$l(u(\cdot, t))=\text{constant} \quad (t^*-\delta \leqq t \leqq t^*+\delta).$$

Let

$$0<x_1<\cdots<x_k<1 \quad (k=l(u(\cdot, t^*)))$$

be such points that

$$\frac{\partial u}{\partial x}(x_i, t^*)\frac{\partial u}{\partial x}(x_{i+1}, t^*)<0$$

for $i=1, \cdots, k-1$. The existence of such points is guaranteed by the fact $l(u(\cdot, t^*)) \neq 0$. Taking $\delta>0$ small enough if necessary, we may assume that

$$\frac{\partial u}{\partial x}(x_i, t)\frac{\partial u}{\partial x}(x_{i+1}, t)<0$$

for all $t \in [t^*-\delta, t^*+\delta]$ and $i=1, \cdots, k-1$. Considering that $l(u(\cdot, t)) \equiv k$ in $[t^*-\delta, t^*+\delta]$, we see that for each $t \in [t^*-\delta, t^*+\delta]$ and for each $i=1, \cdots, k-1$ the function $\dfrac{\partial u}{\partial x}(x, t)$ changes sign only once as x varies from x_i to x_{i+1} and that it does not change sign in either of the intervals $0 \leqq x \leqq x_1$, $x_k \leqq x \leqq 1$.

Now suppose

$$G \cap [t_1, t_2]=\emptyset. \tag{34}$$

Let F_i $(i=0, 1, \cdots, k)$ be the set of all $t \in [t^*-\delta, t^*+\delta]$ such that there exists an $x' \in [x_i, x_{i+1}]$ satisfying

$$\frac{\partial^2 u}{\partial x^2}(x', t) = \frac{\partial u}{\partial x}(x', t) = 0,$$

where we set $x_0=0$ and $x_{k+1}=1$. Obviously each F_i is closed, and (34) implies that

$$F_0 \cup F_1 \cup \cdots \cup F_k = [t^*-\delta, t^*+\delta].$$

Therefore at least one of $F_0, \cdots, F_k$ contains an open sub-interval of $[t^*-\delta, t^*+\delta]$. Here we have two cases:

Case 1. Suppose that F_0 or F_k contains an open interval. Recalling that $w=\partial u/\partial x$ satisfies a parabolic equation of the form (16) and vanishes when $x=0$ or $x=1$, and applying the maximum principle to $\partial u/\partial x$, we get

$$\frac{\partial u}{\partial x} \neq 0 \quad \text{in} \quad (0, x_1] \times (t^*-\delta, t^*+\delta],\ [x_k, 1) \times (t^*-\delta, t^*+\delta]$$

and

$$\frac{\partial^2 u}{\partial x^2} \neq 0 \quad \text{on} \quad \{0\} \times (t^*-\delta, t^*+\delta],\ \{1\} \times (t^*-\delta, t^*+\delta].$$

It follows that F_0 and F_k contain, if any, at most one point (namely $t^*-\delta$); this contradicts the supposition.

Case 2. Suppose one of $F_1, \cdots, F_{k-1}$ contains an open interval; say $(t', t'') \subset F_j$. Denote by Γ_ε the curve

$$\Gamma_\varepsilon = \{(x, t);\ x = \varepsilon(t''-t)(t-t') + x_j,\ t' \leqq t \leqq t''\}.$$

If $\varepsilon=0$, then Γ_ε is a vertical line segment connecting the points (x_j, t') and (x_j, t''). Set

$$\varepsilon_0 = \inf\left\{\varepsilon > 0;\ \frac{\partial u}{\partial x} \text{ vanishes somewhere on } \Gamma_\varepsilon\right\}.$$

Clearly ε_0 is positive, and $\partial u/\partial x$ vanishes at some point $(\tilde{x}, \tilde{t})$ of Γ_{ε_0}. As is easily seen, we have

$$x_j < \tilde{x} < x_{j+1},\ t' < \tilde{t} < t''$$

and, by the maximum principle,

$$\frac{\partial^2 u}{\partial x^2}(\tilde{x}, \tilde{t}) \neq 0.$$

Therefore, by virtue of the implicit function theorem, there exists a $\tilde{\delta}>0$ with $t'<\tilde{t}-\tilde{\delta}<\tilde{t}+\tilde{\delta}<t''$ and a C^2-function $x=x(t)$ defined on the interval $(\tilde{t}-\tilde{\delta}, \tilde{t}+\tilde{\delta})$ such that

$$\frac{\partial u}{\partial x}(x(t), t)=0,$$

$$\frac{\partial^2 u}{\partial x^2}(x(t), t)\neq 0$$

for all $t\in(\tilde{t}-\tilde{\delta}, \tilde{t}+\tilde{\delta})$. Recall that $\partial u/\partial x$ changes sign only once in the interval $[x_j, x_{j+1}]$ for each fixed $t\in[t^*-\delta, t^*+\delta]$. Using the maximum principle, we easily find that $\partial u/\partial x$ does not vanish in the rectangular region $[x_j, x_{j+1}]\times(\tilde{t}-\tilde{\delta}, \tilde{t}+\tilde{\delta})$ except on the curve $x=x(t)$. It follows that

$$F_j\cap(\tilde{t}-\tilde{\delta}, \tilde{t}+\tilde{\delta})=\varnothing,$$

contradicting the supposition.

Thus, in either case, we get to a contradiction, which implies that the supposition (34) is false. This completes the proof of Lemma 5.4 in the case of the Neumann boundary conditions.

In the case of the Dirichlet boundary conditions, $l(U_D(t)\phi)$ is not necessarily monotone nonincreasing. However it is lower semicontinuous by Proposition 9, and hence there is a dense open subset W of the interval $(0, T_D(\phi))$ such that $l(U_D(t)\phi)$ is constant in each connected component of W. Therefore there is no loss of generality in assuming that $l(U_D(t)\phi)$ is constant on the interval $t_1<t<t_2$. Seeing this, and following the same argument as in the case of the Neumann boundary conditions (with a slight modification at $x=0, 1$), we easily get the desired conclusion. Thus the proof of Lemma 5.4 is completed.

PROOF OF THEOREM 17. We have only to consider the case where $l(U_N(t)\phi)<\infty$ for $t_1<t<t_2$, for otherwise the assertion is trivial.

By Lemma 5.4 there exists a number t_0 with $t_1<t_0<t_2$ such that

$$\left|\frac{\partial^2 u_\phi}{\partial x^2}(x, t_0)\right|+\left|\frac{\partial u_\phi}{\partial x}(x, t_0)\right|>0 \quad (u_\phi(\cdot, t)=U_N(t)\phi) \tag{35}$$

for all $x\in\bar{I}$. Let $\{w_n(x)\}$ be a sequence of continuous functions converging to ϕ in $C(\bar{I})$ as $n\to\infty$. A standard compactness argument shows that

$$U_N(t_0)w_n\longrightarrow U_N(t_0)\phi \quad \text{in} \quad C^2(\bar{I}) \quad \text{as} \quad n\to\infty. \tag{36}$$

Combining (35), (36) and Lemma 5.3, we see that

$$l^+(U_N(t_0)w_n)=l^+(U_N(t_0)\phi),$$
$$l^-(U_N(t_0)w_n)=l^-(U_N(t_0)\phi)$$

for all n sufficiently large. This proves Theorem 17.

PROOF OF THEOREM 18. The proof of this theorem is quite similar to that of Theorem 17. Consider, for example, the case $f(0)=0$. In order to apply Lemma 5.3 (together with Lemma 5.4), it is sufficient to show that

$$\frac{\partial}{\partial x}u_\phi(0, t)\neq 0, \quad \frac{\partial}{\partial x}u_\phi(1, t)\neq 0 \tag{37}$$

for any point $t>0$ at which $l(u_\phi(\cdot, t))$ is continuous, provided that $l(\phi)<\infty$. This can easily be shown by applying the maximum principle to u_ϕ (not to $\partial u_\phi/\partial x$), which satisfies the linear equation

$$\frac{\partial u}{\partial t}=a(x)\frac{\partial^2 u}{\partial x^2}+b(x)\frac{\partial u}{\partial x}+c(x, t)u,$$

where $c(x, t)=g(u_\phi(x, t))$ and $f(u)=ug(u)$.

REMARK 5.5. Note that (37) holds for generic t even if $l(\phi)=\infty$. This is a consequence of the unique continuation theorem established by S. Mizohata [8].

PROOF OF REMARK 5.1. We only give the outline. By virtue of Lemma 5.3, what we have to show is that if

$$\frac{\partial u_\phi}{\partial x}(0, t)=0 \text{ or } \frac{\partial u_\phi}{\partial x}(1, t)=0 \quad (u_\phi(\cdot, t)=U_D(t)\phi)$$

holds in some interval $t_1<t<t_2$, then ϕ is an equilibrium solution of $(\mathrm{E_D})$. This can easily be proved by applying the result of [8] to the function $U_D(t+t_0)\phi-U_D(t+t_1)\phi$, where t_0 is any number with $t_1<t_0<t_2$.

PROOF OF THEOREM 19. If ϕ is not a constant function, then the assertion follows directly from Theorems 10 and 17 and Remark 2.4.

Suppose ϕ is a constant function but not a solution of $(\tilde{\mathrm{E}}_\mathrm{N})$. Then, either

$$\lim_{t\to T_N(\phi)}\|U_N(t)\phi\|_{C(I)}=\infty$$

or $T_N(\phi)=\infty$ and $U_N(t)\phi$ converges to some constant solution of $(\tilde{\mathrm{E}}_\mathrm{N})$ as $t\to\infty$. In the first case, as is easily seen, if δ is sufficiently small, then

$$\lim_{t\to T_N(w)}\|U_N(t)w\|_{C(I)}=\infty$$

holds for all $w \in C(\bar{I})$ satisfying (24). Hence the statement of the theorem is trivial. In the second case, if δ is sufficiently small, $U_N(t)w$ converges to the same constant as $U_N(t)\phi$ as $t \to \infty$; hence

$$l^+\left(\lim_{t\to\infty} U_N(t)w\right) = l^-\left(\lim_{t\to\infty} U_N(t)w\right) = 0$$

as required.

Next suppose ϕ is a stable constant solution of $(\tilde{\mathrm{E}}_\mathrm{N})$; hence $\phi(x) \equiv \kappa$ for some constant κ and that $f(\kappa)=0$. Since ϕ is stable, given any $w \in C(\bar{I})$ close to ϕ, we have $T_N(w)=\infty$, and $U_N(t)w$ converges to some solution of $(\tilde{\mathrm{E}}_\mathrm{N})$ that is again close to ϕ. Therefore it suffices to show that there is no such sequence of nonconstant solutions of $(\tilde{\mathrm{E}}_\mathrm{N})$ that converges to ϕ uniformly in $\bar{I}$.

Assume that the assertion above were not true. Then there exist nonconstant solutions $v_n(x)$ $(n=1, 2, \cdots)$ of $(\tilde{\mathrm{E}}_\mathrm{N})$ converging to κ uniformly in $\bar{I}$ as $n \to \infty$. As is easily seen, there exist constants κ_n $(n=1, 2, \cdots)$ satisfying $f(\kappa_n)=0$ and

$$\min_{x\in I} v_n(x) < \kappa_n < \max_{x\in I} v_n(x).$$

Each $\bar{v}_n(x) = v_n(x) - \kappa_n$ is the solution of

$$\text{(38)} \qquad \begin{cases} a(x)v'' + b(x)v' + c_n(x)v = 0 & \text{in } I, \\ w'(0) = w'(1) = 0, \end{cases}$$

where $c_n(x) = g_n(v_n(x))$ and $(u-\kappa_n)g_n(u) = f(u)$. Since $\phi = k$ is an accumulation point of the set of solutions of $(\tilde{\mathrm{E}}_\mathrm{N})$, κ is not asymptotically stable; hence

$$f'(\kappa) = 0.$$

It follows that

$$\text{(39)} \qquad \lim_{n\to\infty} c_n(x) = 0 \text{ uniformly in } x \in \bar{I}.$$

In view of (39), we easily find that the boundary value problem (38) does not possess a solution that changes sign in I, if n is sufficiently large. On the other hand, $\bar{v}_n(x) \equiv v_n(x) - \kappa_n$ changes sign in I for each $n=1, 2, \cdots$. This contradiction proves that ϕ cannot be accumulated by a sequence of nonconstant solutions of $(\tilde{\mathrm{E}}_\mathrm{N})$. Thus we get to the completion of the proof of Theorem 19.

PROOF OF THEOREM 20. First we consider the case where ϕ is not a constant function. Let $t_0 > 0$ and $\delta > 0$ be such that (25a) and (31) (for $t = t_0$) hold (δ will later be chosen smaller if necessary). By virtue of Theorem 11 and Remark 2.4, we have

$$l^+\left(\lim_{t\to\infty} U_D(t)w\right) \leqq \liminf_{t\to\infty} l^+(U_D(t)w),$$

(40) $$\liminf_{t\to\infty} l^+(U_D(t)w) \leqq l^+(U_D(t_0)w)+1,$$

(41) $$l^+(U_D(t_0)\phi) \leqq l^+(\phi)+1.$$

Let us now show that neither of the pairs of the equalities in (25a), (40), (41) hold simultaneously. Without loss of generality we may assume $f(0) \geqq 0$.

Suppose the equality in (41) holds. Then, reading the proof of Theorem 4 carefully, we find that

$$\frac{\partial}{\partial x}\phi(0) \leqq 0, \quad \frac{\partial}{\partial x}u_\phi(0, t_0) > 0. \quad (u_\phi(\cdot, t) = U_D(t)\phi.)$$

Therefore, if $\delta > 0$ is sufficiently small,

$$\frac{\partial}{\partial x}u_w(0, t_0) > 0$$

holds for all $w \in C(\bar{I})$ satisfying (24). It follows that the equality in (40) does not hold (see the proof of Theorem 4). Furthermore, substituting $x=1$ in (E_D) and using (31), we have either

$$\frac{\partial}{\partial x}u_\phi(1, t_0) \neq 0$$

or

$$\frac{\partial}{\partial x}u_\phi(1, t_0) = 0, \quad \frac{\partial^2}{\partial x^2}u_\phi(1, t_0) < 0.$$

In the former case, it follows from the latter part of Lemma 5.3 that $l^+(U_D(t_0)w) = l^+(U_D(t_0)\phi)$ for all $w \in C(\bar{I})$ close to ϕ. One can easily check that the same assertion holds for the latter case as well. It implies that, if δ is sufficiently small, the equality in (25a) does not hold (provided that the equality in (41) holds). Consequently, the pair of the equalities in (41), (25a) do not hold simultaneously, and neither do the pair of those in (41), (40).

It still remains to show that the equalities in (25a), (40) do not hold simultaneously. Let $\delta > 0$ be sufficiently small and let $w \in C(\bar{I})$ satisfy (24) and

$$l^+(U_D(t_0)w) = l^+(U_D(t_0)\phi) + 1.$$

Then, as is easily seen,

$$\frac{\partial}{\partial x}u_w(0, t_0) > 0.$$

It follows that the equality in (40) does not hold (see the proof of Theorem 4). Hence the equalities in (25a), (40) do not hold simultaneously if δ is sufficiently small.

Combining these observations, we get the desired inequality

$$l^{+}\left(\lim_{t\to\infty} U_D(t)w\right) \leqq l^{+}(\phi)+1.$$

The other inequality for l^{-} can be verified in the same manner.

Next we consider the case where $\phi(x)\equiv 0$ and 0 is not a solution of ($\tilde{\mathrm{E}}_{\mathrm{D}}$) (i.e. $f(0)\neq 0$). In this case, as is easily seen, either

$$\lim_{t\to\infty} U_D(t)\phi = \lim_{t\to\infty} U_D(t)w = v \quad \text{in} \quad C(\bar{I})$$

for some solution v of ($\tilde{\mathrm{E}}_{\mathrm{D}}$), or

$$\lim_{t\to T_D(\phi)} \|U_D(t)\phi\|_{C(I)} = \lim_{t\to T_D(w)} \|U_D(t)w\|_{C(I)} = \infty,$$

provided that the constant δ in (24) is sufficiently small. By virtue of Corollary 16, v satisfies the inequalities

$$l^{+}(v) \leqq 1 \quad \text{and} \quad l^{-}(v) \leqq 1,$$

as required.

Lastly, suppose $\phi\equiv 0$ and that 0 is a stable solution of ($\tilde{\mathrm{E}}_{\mathrm{D}}$). As in the proof of Theorem 19, it suffices to prove that if a sequence of solutions $\{v_n(x)\}$ of ($\tilde{\mathrm{E}}_{\mathrm{D}}$) satisfies

$$\lim_{n\to\infty} v_n(x) = 0 \quad \text{uniformly in} \quad x\in\bar{I},$$

then

$$l(v_n) \leqq 2 \tag{42}$$

for all n sufficiently large. Let v be any solution of ($\tilde{\mathrm{E}}_{\mathrm{D}}$) with $l(v)\geqq 3$. Then there exists a constant κ such that $f(\kappa)=0$ and that

$$\min_{x\in I} v(x) < \kappa < \max_{x\in I} v(x).$$

Seeing this, and following the same argument as in the proof of Theorem 19, we find that (42) should hold for large n.

Thus, in each of the cases, we have verified the assertions (28a) and (28b). The proof of the latter part of the theorem (i.e. the case $\phi\not\equiv 0$ and $f(0)=0$) is much easier, and therefore we omit it. This completes the proof of Theorem 20.

Corollaries 21 and 22 follow immediately from Theorems 19 and 20; in fact, we have only to apply these theorems to $\lim_{t\to\infty} U_N(t)\phi$ or $\lim_{t\to\infty} U_D(t)\phi$ instead of ϕ itself.

6. Remark and examples.

REMARK 6.1. In Section 1, we have presented two different definitions of lap-number that are equivalent so far as continuous functions are concerned. In general we have the relation

$$l(w) \geqq \tilde{l}(w),$$

where $\tilde{l}(w)$ denotes the "lap-number" of w in the sense defined at the beginning of Section 1. Example 6.2 shows that Theorem 1 would be no longer true if we replace $l(u)$ by $\tilde{l}(u)$.

Example 6.2. Let $u(x,t)$ be a solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

together with the Neumann boundary conditions $\partial u/\partial x = 0$ at $x=0, 1$ and with the initial condition

$$u(x,0) \equiv u_0(x) = \begin{cases} x & \left(0 \leqq x < \frac{1}{2}\right) \\ x-1 & \left(\frac{1}{2} \leqq x \leqq 1\right). \end{cases}$$

Then $l^+(u_0)=2$, $l^-(u_0)=1$ and

$$l^+(u(\cdot,t)) = \begin{cases} 2 & (0 \leqq t < t^*) \\ 0 & (t^* \leqq t < \infty), \end{cases}$$

$$l^-(u(\cdot,t)) = 1 \qquad (0 \leqq t < \infty),$$

where t^* is a positive number determined by the condition

$$\frac{\partial u}{\partial t}(0, t^*) = 0.$$

Therefore we have

$$l(u(\cdot,t)) = \begin{cases} 3 & (0 \leqq t < t^*) \\ 1 & (t^* \leqq t < \infty), \end{cases}$$

while

$$\tilde{l}(u(\cdot,t)) = \begin{cases} 2 & (t=0) \\ 3 & (0 < t < t^*) \\ 1 & (t^* \leqq t < \infty), \end{cases}$$

where $\tilde{l}(u)$ is as in Remark 6.1.

Example 6.3. Consider the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-k \tag{43}$$

under the Dirichlet boundary condition

$$u(0, t)=u(1, t)=0,$$

where k is a positive constant. Let

$$\begin{aligned} u(x, 0)\equiv u_0(x)=1 \quad &\text{in} \quad I, \\ u_0(0)=u_0(1)=0 \quad &\text{on} \quad \partial I. \end{aligned}$$

Then there exist positive numbers $t'<t''$ (depending on k) such that

$$l^+(u(\cdot, t))=l^-(u(\cdot, t))=\begin{cases} 1 & (0\leqq t\leqq t') \\ 2 & (t'<t<t'') \\ 1 & (t''\leqq t<\infty). \end{cases} \tag{44}$$

(PROOF.) Since u_0 is an upper solution of (43), u is monotone decreasing in t and converges to the equilibrium solution

$$v(x)=-kx(1-x)$$

as $t\to\infty$ (see Sattinger [10]). Therefore there exists a positive number t' such that

$$\frac{\partial u}{\partial x}(0, t)=-\frac{\partial u}{\partial x}(1, t)\begin{cases} >0 & (0<t<t') \\ <0 & (t'<t<\infty). \end{cases}$$

A careful reading of the proof of Theorem 5 shows that $l^+(u(\cdot, t))$ and $l^-(u(\cdot, t))$ are monotone nonincreasing both in $0\leqq t<t'$ and in $t'<t<\infty$. Moreover we have

$$1\leqq l^+(u(\cdot, t))=l^-(u(\cdot, t))\leqq 2$$

for all $t\geqq 0$ by virtue of Theorems 11 and 14. In particular, we have

$$l^+(u(\cdot, t))=l^-(u(\cdot, t))=1$$

for $0\leqq t\leqq t'$ (see Proposition 9). Considering that $\partial u/\partial x(0, t)$ and $\partial u/\partial x(1, t)$ change sign at $t=t'$, and arguing as in the proof of Theorem 5, we easily find that

$$l^+(u(\cdot, t))=l^-(u(\cdot, t))\geqq 2$$

in some interval $t'<t<t''$; for otherwise $u(x, t')$ should vanish everywhere in $\bar{I}$, which is impossible by virtue of the fact $l(u(\cdot, t'))=2$. It remains to show that

$$\lim_{t\to\infty} l^{+}(u(\cdot, t)) = \lim_{t\to\infty} l^{-}(u(\cdot, t)) = 1;$$

but this quickly follows from Remark 5.1. Thus the proof of (44) is completed.

This example shows that the inequalities in Theorems 4 and 11 cannot be improved without setting further assumptions.

Example 6.4. Let t^* be a positive number and let $\phi(t)$ be a function defined on $[0, t^*]$ satisfying

$$\phi(t) \not\equiv 0,$$

$$\int_0^{t^*} e^{n^2\pi^2 s}\phi(s)ds = 0 \tag{45}$$

for $n=0, 1, 2, \cdots$. Since the series $\sum(1/n^2\pi^2)$ is convergent, such a function $\phi \in L^2(0, t^*)$ exists by the theorem of Müntz (see Schwartz [11; Théorème I, p. 54]). Moreover, as is easily seen, we may assume that $\phi(t)$ is smooth and has a compact support in $(0, t^*)$. Setting $\phi(t)=0$ for $t \geqq t^*$, we extend the domain of $\phi(t)$ to the half line $[0, \infty)$.

Now consider the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + \phi(t) & \text{in} \quad I \times (0, \infty), \\ u = (x, 0) = 0 & \text{on} \quad \bar{I}, \\ u = 0 \quad \text{on} & \partial I \times (0, \infty). \end{cases} \tag{46}$$

The solution of (46) can be expressed in the form

$$u(x, t) = 2\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin(n\pi x) e^{-n^2\pi^2 t} \int_0^t \phi(s) e^{n^2\pi^2 s} ds.$$

Therefore, by virtue of (45),

$$l(u(\cdot, t)) = 0 \quad \text{for all} \quad t \geqq t^*,$$

though $l(u(\cdot, t))$ does not vanish identically in $[0, \infty)$. This example shows that Theorem 14 is not valid for non-autonomous problems.

Example 6.5. Consider the problem (46) again; but replace the condition (45) by

$$\int_0^{t^*} e^{n^2\pi^2 s}\phi(s)ds = \begin{cases} 0 & (n \neq \mathrm{m}) \\ 1 & (n = \mathrm{m}), \end{cases}$$

where m is a positive integer. The existence of such a function ϕ is ensured by the fact that the functions $e^{n^2\pi^2 t}$ $(n=1, 2, \cdots)$ are (topologically) linearly independent in $L^2(0, t^*)$; see [11] for details. Using the expression in Example 6.4, we get

$$l(u_0)=0, \quad l(u(\cdot, t^*))=l(\sin m\pi x)=m+1.$$

This example shows that the lap-number of a solution may grow fairly larger in the case of a non-autonomous equation under the Dirichlet boundary conditions (cf. Theorems 1 and 11).

Example 6.6. The following is an example of the case where $(\tilde{\mathrm{E}}_{\mathrm{N}})$ has a stable solution with a very large lap-number, although its nonlinear term is spatially homogeneous. Consider the boundary value problem

$$\left\{\begin{array}{l} (a_m(x)v')'+kv(1-v^2)=0 \quad \text{in} \quad I, \\ v'(0)=v'(1)=0, \end{array}\right. \tag{47}$$

where k is a positive constant and $a_m(x)$ is a positive smooth function on $\bar{I}$ satisfying

$$\begin{aligned} &a_m(x)\geqq 1 \quad \text{in} \quad I_i \quad (i=0, \cdots, m), \\ &a_m(x)\leqq \delta \quad \text{in} \quad I_i' \quad (i=1, \cdots, m) \end{aligned}$$

for some constant $\delta>0$. Here

$$I_0, I_1', I_1, I_2', \cdots, I_m', I_m$$

are disjoint open sub-intervals of I lying in this order; and we assume

$$\begin{aligned} &|I_i|=h>0 \quad (i=0, 1, \cdots, m), \\ &|I_i'|=h' \quad (i=1, 2, \cdots, m), \\ &h'\geqq\frac{h}{2m}, \end{aligned}$$

where $|K|$ denotes the length of an interval K. Under these hypotheses, (47) has a stable solution v with $l(v)\geqq m$, if the following condition is satisfied:

$$0<\delta\leqq\frac{k}{32m^2}\left\{\min\left(h, \frac{\pi^2}{kh}\right)\right\}^2. \tag{48}$$

(PROOF.) We merely give the outline of the proof. The argument here is a modification of that in [7; Section 6].

Let $U_N(t)$ be as defined in Section 4; with

$$a(x)=a_m(x), \quad b(x)=a_m'(x), \quad f(u)=ku(1-u^2).$$

Set

$$J(u)=\int_I\left\{\frac{1}{2}a_m(x)\left(\frac{\partial u}{\partial x}\right)^2-kG(u)\right\}dx,$$

$$J_i(u)=\int_{I_i}\left\{\frac{1}{2}a_m(x)\left(\frac{\partial u}{\partial x}\right)^2-kG(u)\right\}dx,$$

where

$$G(u)=\int_0^u u(1-u^2)du=\frac{1}{2}u^2-\frac{1}{4}u^4.$$

We also use the notation

$$R_m=\left\{w\in C^1(\bar{I});\quad (-1)^i\int_{I_i}w(x)dx>0\quad (i=0,1,\cdots,m)\right\},$$

$$V_\varepsilon=\{w\in C^1(\bar{I});\quad 0\leqq w(x)\leqq 1,\ J(w)<\varepsilon-kG(1)\}.$$

Here ε is a positive constant and $G(1)=1/4$. As is well known, we have

$$\frac{d}{dt}J(U_N(t)\phi)=-\int_I\left(\frac{\partial}{\partial t}U_N(t)\phi\right)^2dx,$$

hence $J(U_N(t)\phi)$ is monotone nonincreasing in t. In view of this, and with the aid of the comparison theorem, we see that

$$U_N(t)V_\varepsilon\subset V_\varepsilon\quad\text{for all}\quad t\geqq 0.$$

By virtue of the Poincaré-Friedrichs inequality,

$$\int_{I_i}(w')^2dx\geqq\frac{\pi^2}{h^2}\int_{I_i}w^2dx$$

holds for any $w\in C^1(\bar{I})$ satisfying

$$\int_{I_i}w(x)dx=0. \tag{49}$$

Hence

$$J_i(w)\geqq -hkG(1)+\min\left\{\frac{\pi^2}{h},hk\right\}G(1)$$

for any $w\in C^1(\bar{I})$ satisfying (49). It follows that

$$U_N(t)(R_m\cap V_\varepsilon)\subset R_m\cap V_\varepsilon \tag{50}$$

for all $t\geqq 0$, provided that

$$\varepsilon\leqq\min\left\{\frac{\pi^2}{h},hk\right\}G(1). \tag{51}$$

Therefore, if ε satisfies (51) and $R_m \cap V_\varepsilon$ is not empty, there exists a stable solution of (47) contained in $R_m \cap V_\varepsilon$ (see [7; Theorems 4.2 and 6.2]).

Let $w_0(x)$ be a piecewise linear continuous function on $\bar{I}$ such that

$$w_0(x) = (-1)^i \quad \text{in } I_i \quad (i=0, \cdots, m),$$

$$w_0'(x) = \begin{cases} (-1)^i \dfrac{2}{h''} & \text{in } I_i'' \quad (i=1, \cdots, m) \\ 0 & \text{in } \bar{I} \setminus (\bar{I}_1'' \cup \cdots \cup \overline{I_m''}), \end{cases}$$

where I_i'' is a sub-interval of I_i' with length

$$|I_i''| = h'' = \frac{1}{2m} \min\left\{h, \frac{\pi^2}{hk}\right\}.$$

A simple calculation yields

$$J(w_0) < -kG(1) + m\left\{\frac{2\delta}{h''} + kG(1)h''\right\},$$

Therefore

$$U_N(t) w_0 \in R_m \cap V_\varepsilon$$

for all $t>0$, provided that

$$m\left\{\frac{2\delta}{h''} + kG(1)h''\right\} \leq \min\left\{\frac{\pi^2}{h}, hk\right\} G(1). \tag{52}$$

In other words, $R_m \cap V_\varepsilon$ is nonempty if (52) is satisfied. Hence (48), which is equivalent to (52), is a sufficient condition for the existence of a stable solution of (47) contained in R_m. This completes the proof.

References

[1] Chafee, N., Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, J. Differential Equations **18** (1975), 111–134.

[2] Fife, P. C., On modelling pattern formation by activator-inhivitor systems, J. Math. Biol. **4** (1977), 353-362.

[3] Friedmann, A., Partial differential equations of parabolic type, Prentice-Hall, 1963.

[4] Lees, M. and M. H. Protter, Unique continuation for parabolic differential equations and inequalities, Duke Math. J. **28** (1961), 369–382.

[5] Maginu, K., Stability of stationary solutions of a semilinear parabolic partial differential equation, J. Math. Anal. Appl. **63** (1978), 224-243.

[6] Matano, H., Convergence of solutions of one-dimensional semilinear parabolic equations, J. Math. Kyoto Univ. **18** (1978), 221–227.

[7] Matano, H., Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci. Kyoto Univ. **15** (1979), 401–454.

[8] Mizohata, S., Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques, Mem. Coll. Sci. Univ. Kyôto, **31** (1958), 219–239.
[9] Protter, M. H. and H. F. Weinberger, Maximum principles in differential equations, Prentice-Hall, 1967.
[10] Sattinger, D. H., Monotone methods in nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. **21** (1972), 979–1000.
[11] Schwartz, L., Étude des sommes d'exponentielles (deuxième édition), Hermann, Paris, 1959.
[12] Serrin, J., Mathematical aspects of boundary layer theory, Lecture Notes, Univ. Minnesota, 1962.
[13] Tabata, M., A finite difference approach to the number of peaks of solutions for semilinear parabolic problems, J. Math. Soc. Japan, **32** (1980), 171–191.

Note added in proofs: After completing this work, the author was pointed out the papers

Karlin, S., Total positivity, absorption probabilities, and applications, Trans. Amer. Math. Soc. 111 (1964), 33–107;

Sattinger, D. H., On the total variation of solutions of parabolic equations, Math. Ann. 183 (1969), 78–92.

These papers contain some results similar to Theorem 1 of the present paper, with the proof along similar lines. But their results do not cover the more complicated cases as discussed in Sections 3 and 4 of the present paper; and, of course, the stability of the nonincreasing property of lap-numbers, which is the most important point of discussion in the present paper (Section 5), is not at all argued in the above papers.

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