Bifurcation and stability of stationary solutions of nonlinear evolution equations

By Tatsuo Itoh

(Communicated by H. Fujita)

Introduction.

In this paper we consider the semilinear evolution equation in a real Banach space X with a real n-dimensional parameter λ of the form

$$\text{(E)} \qquad \frac{du}{dt}=Lu+N(u, \lambda)\equiv F(u, \lambda), \ t>0,$$

with initial value $u(0)=x_0$. Here, L is the generator of a holomorphic semigroup $\{e^{tL}\}_{t\geq 0}$. We assume that 0 is an isolated simple eigenvalue of L and that the remaining part of the spectrum of L is contained in the half-(complex) plane $\{\mathrm{Re}\, z < -c\}$, $c>0$. $N(x, \lambda)$ is a nonlinear operator of class C^2 with $N(0,0)=0$, $D_xN(0,0)=0$. ($D_xN(0,0)$ denotes the Fréchet derivative of $N(x, \lambda)$ with respect to x at $(x, \lambda)=(0,0)$). Clearly $(0,0)$ is a stationary solution of (E). (A pair (v, λ) is by definition a stationary solution of (E) if (v, λ) satisfies $F(v, \lambda)=0$.)

In the present paper we are concerned, on the one hand, with the stability problem and the bifurcation problem of stationary solutions of (E), and, on the other hand, with the asymptotic behavior of solutions of (E). These problems are closely related. In fact the stability problem may be regarded as a part of the study of the asymptotic behavior and the study of the latter is effectively used in the detailed study of the stability problem. The type of the results we want to obtain about the asymptotic behavior is to characterize it by some relation between the position of the initial value and the stable manifolds of the stationary solutions of (E).

In the study of these problems it is necessary to use locally invariant manifolds such as the center manifold and the stable manifolds. These are main tools in the present paper. An important role is also played by a function which is called a bifurcation function.

We shall explain some relations between the stability and a bifurcation function.

As is well known (e.g., G. Iooss [6], H. Kielhöfer [10, 11], D.H. Sattinger [18], K. Kirchgässner and H. Kielhöfer [12], the stability of a stationary solution (v, λ) near 0 is usually determined by the sign of the eigenvalue κ near 0 of the Fréchet derivative $D_xF(v, \lambda)$ of $F(x, \lambda)$ with respect to x at $(x, \lambda)=(v, \lambda)$. That is, if $\kappa<0$, then (v, λ) is stable, while if $\kappa>0$, it is unstable. If $\kappa=0$, however, we can not decide its stability by the knowledge of the eigenvalue κ. To decide the stability of stationary solutions completely, we have to introduce a bifurcation function. A function $h(s, \lambda)$, defined in a neighborhood of 0 in $R^1\times R^n$, is called a bifurcation function if there is a one-to-one correspondence between stationary solutions (v, λ) of (E) and solutions (s, λ) of $h(s, \lambda)=0$ through $s=\langle v, \phi^*\rangle$, where ϕ^* is some non-trivial continuous linear functional on X annihilating the range of L. Then the stationary solution is characterized by the solution of $h(s, \lambda)=0$. Hence we denote the stationary solution (v, λ) by $(v(s, \lambda), \lambda)$. A well-known example of a bifurcation function is a function $g(s, \lambda)$ constructed by the method of Ljapunov-Schmidt.

The bifurcation function f we use in this paper is defined by using the center manifold theorem (see (1.5)). As we shall see below, this bifurcation function f is very useful in the detailed study of the problems mentioned above.

Using the above g, H. Weinberger (1978) showed the identity

$$\operatorname{sign} \kappa(s, \lambda)=\operatorname{sign}\left(\frac{\partial}{\partial s} g\right)(s, \lambda),$$

where $\kappa(s, \lambda)$ is the eigenvalue of $D_xF(v(s, \lambda), \lambda)$. Using our f we can show the identity

$$\kappa(s, \lambda)=\left(\frac{\partial}{\partial s} f\right)(s, \lambda).$$

Our stability theorem can be formulated as follows.

Stability Theorem. *The stationary solution* $(v(s, \lambda), \lambda)$ *near* 0 *is stable (unstable, asymptotically stable) if and only if the stationary solution* (s, λ) *of a scalar ordinary differential equation*

$$\frac{ds}{dt}=f(s, \lambda), \qquad t>0$$

is stable (unstable, asymptotically stable).

Our bifurcation theorem follows from this theorem at once and has the following form.

BIFURCATION THEOREM. *Suppose that the parameter λ is in R^1 and that $N(0,\lambda)=0$ for λ near 0. If $(0,\lambda)$ is stable for $\lambda<0$, and unstable for $\lambda>0$, then non-trivial stationary solutions of* (E) *bifurcate from the trivial solution* $(0,0)$.

It may be emphasized that in the bifurcation theorem no assumptions other than the change of the stability of $(0,\lambda)$ at $\lambda=0$ are needed.

We now explain the asymptotic behavior of solutions. We first recall the stable manifold theorem. "For each stationary solution (v,λ) near 0, there exists a manifold $\mathcal{M}(v,\lambda)$ of codimension 1 with the following property. If $x_0\in\mathcal{M}(v,\lambda)$ then a solution $u(t,x_0,\lambda)$ of (E) with initial value x_0 converges to v as $t\to\infty$ at an exponential rate." Since each manifold $\mathcal{M}(v,\lambda)$ is of codimension 1, we can define the upper (lower) side of $\mathcal{M}(v,\lambda)$. We denote it by $\mathcal{M}_+(v,\lambda)$ ($\mathcal{M}_-(v,\lambda)$).

We can now state the main result on the asymptotic behavior. Suppose that for some λ there are at least two stationary solutions (v_1,λ), (v_2,λ) of (E) with $\langle v_1,\phi^*\rangle<\langle v_2,\phi^*\rangle$. If $x_0\in\mathcal{M}_+(v_1,\lambda)\cap\mathcal{M}_-(v_2,\lambda)$ then $u(t,x_0,\lambda)$ converges to a stationary solution (v,λ). The limit (v,λ) is determined by the position of the initial value x_0 and the behavior of the bifurcation function $f(s,\lambda)$ in $\{s:\langle v_1,\phi^*\rangle<s<\langle v_2,\phi^*\rangle\}$ (Theorem 3.11).

In Section 1 we shall state assumptions made throughout the present paper. The relation between the eigenvalue near zero of the linearized operator and the bifurcation function will be given in Section 2 (Theorem 2.5). In Section 3 we shall show our main results (Theorems 3.1, 3.2 and 3.11).

Acknowledgement.

The author expresses his sincere gratitude to Professor K. Masuda for his generous help in innumerable cases and for his unceasing encouragements. Without these, this work would not have been possible. He also expresses his hearty gratitude to Professor S. T. Kuroda for his valuable advice and important suggestions.

Section 1. Preliminaries.

§1.1. Assumptions.

X is a real Banach space. L is a linear operator in X and $N(\cdot,\lambda)$ is a non-linear operator in X with a real n-dimensional parameter λ. On L and N we assume the following two assumptions throughout the present paper.

ASSUMPTION 1.

(i) L generates a holomorphic semigroup $\{e^{tL}\}_{t\geq 0}$ in X.

(ii) 0 is an eigenvalue of L with the algebraic multiplicity one.

(iii) There exists a positive constant c_0 such that

$$\sup_{\mu\in\sigma(L)\setminus\{0\}} \mathrm{Re}\,\mu < -c_0,$$

where $\sigma(L)$ is the spectrum of L.

In these conditions (i), (ii) and (iii) L is regarded as a unique linear extension of L to the complexification X_c of X. A similar remark will not be repeated in the sequel.

Throughout the present paper we fix α such that $0\leq\alpha<1$. All the results will be valid for any fixed α within this range. We denote by X_0 the Banach space consisting of all elements in the domain of $(-L+1)^{\alpha}$. The norm of X_0 is the graph norm of $(-L+1)^{\alpha}$.

ASSUMPTION 2. The nonlinear operator N is a C^2-mapping of some neighborhood of $(0, 0)$ in $X_0\times R^n$ into X such that $N(0, 0)=0$, $D_xN(0, 0)=0$.

By Assumption 1 (ii), the kernel $\mathrm{Ker}\,L$ of L is spanned by some $\phi\in X$. Since 0 is an isolated eigenvalue with the algebraic multiplicity one, the range $R(L)$ of L is closed, its codimension is one, and ϕ is not in $R(L)$. Hence there exists a continuous linear functional ϕ^* on X such that $\langle x, \phi^*\rangle=0$ for $x\in R(L)$ and $\langle\phi, \phi^*\rangle\neq 0$: We may assume that $\langle\phi, \phi^*\rangle=1$ and $\|\phi\|_{X_0}=1$ ($\|\ \|_{X_0}$ is the norm on X_0). We fix such ϕ and ϕ^*.

We decompose X and X_0 into direct sum:

$$X=\mathrm{Ker}\,L\oplus Z,$$

$$X_0=\mathrm{Ker}\,L\oplus Z_0,$$

where

$$Z=\{x\in X:\ \langle x, \phi^*\rangle=0\},$$

$$Z_0=\{x\in X_0:\ \langle x, \phi^*|X_0\rangle=0\}$$

($\phi^*|X_0$ is the restriction of ϕ^* to X_0).

We define the projection P and Q from X onto $\mathrm{Ker}L$ and Z by

$$Px=\langle x, \phi^*\rangle\phi, \tag{1.1}$$

$$Qx=x-Px \tag{1.2}$$

respectively. The restriction $P|X_0$ (resp. $Q|X_0$) of P (resp. Q) to X_0 is also the projection from X_0 onto $\operatorname{Ker} L$ (resp. Z_0). For simplicity we also denote $P|X_0$ (resp. $Q|X_0$) by P (resp. Q).

For later convenience we replace the norm $\| \ \|_{X_0}$ on X_0 by the equivalent norm:

$$\|x\| = \max\{\|Px\|_{X_0}, \ \|Qx\|_{X_0}\}.$$

Then we have

$$\|x\| = \max\{\|Px\|, \|Qx\|\}.$$

§ 1.2. Summary of known results.

In this section we summarize the known results to be used later.

1.2.1. The existence theorem.

We begin with the definition of a solution of (E).

DEFINITION 1.1. Let λ be fixed and let $T>0$. A function $u(t)$ is called *a solution of* (E) *on* $[0, T)$ if

(i) $u(t) \in C([0, T), X_0)$,

(ii) $\dfrac{du(t)}{dt} \in C((0, T), X)$,

(iii) $u(t)$ is in the domain of L and N for $0<t<T$,

(iv) u satisfies (E) for $0<t<T$ and $u(0)=x_0$.

We shall denote $u(t)$ by $u(t, x_0, \lambda)$ or by $S_\lambda(t)x_0$. Then the following existence theorem holds.

THEOREM 1.2. *Let Assumptions* 1 *and* 2 *be satisfied. Then, for any* T, $0<T<\infty$, *there exists a neighborhood* V *of* 0 *in* $X_0 \times R^n$ *such that for each* $(x_0, \lambda) \in V$, (E) *has a unique solution* $u(t, x_0, \lambda)$ *on* $[0, T)$, *and if we regard* $u(t, x_0, \lambda)$ *as a mapping of* V *to* $C([0, T), X_0)$, *then* $u(t, \cdot, \cdot) \in C^2(V, C([0, T), X_0))$ *and* $D_{x_0}u(t, 0, 0) = e^{tL}$.

For the proof of Theorem 1.2, we refer the reader to M. A. Krasnosel'skii et al. [13; § 23] and also to M. G. Crandall and P. H. Rabinowitz [3; Lemma 1.7 and Lemma 1.12].

1.2.2. The center manifold theorem.

We next state the center manifold theorem, which enables us to reduce a stability problem for stationary solutions of (E) in a (possibly infinite dimensional) Banach space to that in one-dimensional space.

THEOREM 1.3. *Let Assumptions* 1 *and* 2 *be satisfied. Then there exist* $d_1>0$ *and a* C^1*-mapping* z_0 *of* $\{(s,\lambda)\in R^1\times R^n: |s|<d_1, |\lambda|<d_1\}$ *into* $\{z\in Z_0: \|z\|<d_1\}$ *with* $z_0(0,0)=0$, $D_s z_0(0,0)=0$ *and* $D_\lambda z_0(0,0)=0$ *such that for each* λ, $|\lambda|<d_1$, *and* $t>0$ *the following statements hold.*

(i) *Local Invariance: If* $x_0\in\mathcal{C}_\lambda$ *and* $\|u(\tau,x_0,\lambda)\|<d_1$ *for all* τ, $0\le\tau\le t$, *then* $u(\tau,x_0,\lambda)\in\mathcal{C}_\lambda$, $0\le\tau\le t$, *where* $\mathcal{C}_\lambda$ *is the one-dimensional curve defined by*

$$\mathcal{C}_\lambda=\{s\phi+z_0(s,\lambda):\ |s|<d_1\}. \tag{1.3}$$

(ii) *Local Attractivity: If* $\|u(\tau,x_0,\lambda)\|<d_1$ *for all* τ, $0\le\tau\le t$, *then*

$$\|z_0(s(\tau),\lambda)-Qu(\tau,x_0,\lambda)\|\le K_1e^{-c_0\tau}\|z_0(s(0),\lambda)-Qx_0\|,\ 0\le\tau\le t, \tag{1.4}$$

where $s(\tau)=\langle u(\tau,x_0,\lambda),\phi^*\rangle$, c_0 *is as in Assumption* 1 *and* K_1 *is a positive constant independent of* t,λ.

(iii) *If* $N\in C^p$ $(p\ge3)$, *then* $z_0\in C^{p-1}$.

(See K. Masuda and T. Itoh [15] and also J. E. Marsden and M. McCracken [14 Section 2]; see also a recent book by D. Henry [4].)

We now define the bifurcation function f by

$$f(s,\lambda)=\langle N(s\phi+z_0(s,\lambda),\lambda),\phi^*\rangle. \tag{1.5}$$

This function f has been introduced by D. Ruelle [16].

Section 2. The relation between eigenvalues of the linearized operator and the bifurcation function.

§2.1. Reduction of the evolution equation (E) to a scalar equation.

Let d_1 be fixed. We fix λ, $|\lambda|<d_1$, and write simply $u(t,x_0)$ for $u(t,x_0,\lambda)$. Using the decomposition of X_0: $X_0=\operatorname{Ker}L\oplus Z_0$, we decompose $u(t,x_0)$ in the form

$$u(t,x_0)=\langle u(t,x_0),\phi^*\rangle\phi+Qu(t,x_0)$$

(Q is the projection onto Z_0). Then the equation (E) is equivalent to a system of differential equations

$$(2.1)\qquad \begin{cases} \dfrac{ds(t)}{dt} = \langle N(s(t)\phi + v(t), \lambda), \phi^* \rangle \\ \dfrac{dv(t)}{dt} = QF(s(t)\phi + v(t), \lambda) \end{cases}$$

with the initial value

$$(2.2)\qquad s(0) = \langle x_0, \phi^* \rangle; \quad v(0) = Qx_0.$$

Since $(\langle u(t, x_0), \phi^* \rangle, Qu(t, x_0))$ satisfies (2.1)-(2.2), by the uniqueness of solutions, we have

$$s(t) = \langle u(t, x_0), \phi^* \rangle; \quad v(t) = Qu(t, x_0).$$

In the case that $x_0 \in \mathcal{C}_\lambda$, (2.1)-(2.2) can be further reduced to the initial value problem of a one-dimensional ordinary differential equation

$$(2.3)\qquad \frac{ds(t)}{dt} = f(s(t), \lambda);$$

$$(2.4)\qquad s(0) = s_0$$

$(s_0 = \langle x_0, \phi^* \rangle)$ where f is the bifurcation function given by (1.5). Indeed, we have

PROPOSITION 2.1. *Let Assumptions* 1 *and* 2 *be satisfied. Let* $x_0 \in \mathcal{C}_\lambda$. *If* $u(t, x_0)$ *is a solution of* (E) *with* $\|u(t, x_0)\| < d_1$, *then* $s(t) = \langle u(t, x_0), \phi^* \rangle$ *is a solution of* (2.3)-(2.4) *with* $s_0 = \langle x_0, \phi^* \rangle$. *Conversely, if* $s(t)$ *is a solution of* (2.3)-(2.4) *with* $|s(t)| < d_1$, *then* $u(t) = s(t)\phi + z_0(s(t), \lambda)$ *is a solution of* (E) *with the initial value* $x_0 = s_0\phi + z_0(s_0, \lambda)$.

PROOF. Suppose that $u(t, x_0)$ is a solution of (E). Since $x_0 \in \mathcal{C}_\lambda$, the local invariance of $\mathcal{C}_\lambda$ implies $u(t, x_0) \in \mathcal{C}_\lambda$, i.e.,

$$(2.5)\qquad Qu(t, x_0) = z_0(s(t), \lambda).$$

Hence, by (2.1), $s(t)$ is a solution of (2.3)-(2.4) with $s_0 = \langle x_0, \phi^* \rangle$. Conversely, we suppose that $s(t)$ is a solution of (2.3)-(2.4). If $u(t, x_0)$ is a solution of (E) with $x_0 = s_0\phi + z_0(s_0, \lambda)$, then we see that $\langle u(t, x_0), \phi^* \rangle$ is a solution of (2.3)-(2.4). Hence by the uniqueness of the solution of (2.3)-(2.4), we have $s(t) = \langle u(t, x_0), \phi^* \rangle$. By (2.5), $Qu(t, x_0) = z_0(s(t), \lambda)$. This completes the proof of the proposition.

Concerning stationary solutions of (E), we have

PROPOSITION 2.2. *Let Assumptions* 1 *and* 2 *be satisfied. If* v *is a stationary solution of* (E) *with* $\|v\| < d_1$, *then* $s = \langle v, \phi^* \rangle$ *is a stationary solution of* (2.3): $f(s, \lambda) = 0$, *and* v *satisfies* $v = s\phi + z_0(s, \lambda)$. *Conversely, if* s *is a stationary solution of* (2.3), *i.e.*, $f(s, \lambda) = 0$, *with* $|s| < d_1$, *then* $v = s\phi + z_0(s, \lambda)$ *is a stationary solution of* (E).

PROOF. The proof easily follows from Proposition 2.1.

We next consider the stability of a stationary solution s of the ordinary differential equation (2.3). Since (2.3) is a scalar ordinary differential equation, we easily see that its stability is determined by the behavior of $f(\sigma, \lambda)$ near $\sigma=s$. To state this more precisely, we introduce the following terminology, which describes completely the behavior of $f(\sigma, \lambda)$ near $\sigma=s$.

(i): A function $g(\sigma, \lambda)$ is said to be *right- (resp. left-) positive at* $(\sigma, \lambda)=(s, \lambda)$ if $g(\sigma, \lambda)>0$ for all $\sigma>s$ (resp. $\sigma<s$) near s.

(ii): A function $g(\sigma, \lambda)$ is said to be *right- (resp. left-) oscillatory at* $(\sigma, \lambda)=(s, \lambda)$ if there exists a sequence $\{s_n\}_{n=1}^{\infty}$ with $s_n \downarrow s$ (resp. $s_n \uparrow s$) and $g(s_n, \lambda)=0$.

(iii): A function $g(\sigma, \lambda)$ is said to be *right- (resp. left-) negative at* $(\sigma, \lambda)=(s, \lambda)$ if $g(\sigma, \lambda)<0$ for all $\sigma>s$ (resp. $\sigma<s$) near s.

Then we have

PROPOSITION 2.3. *A stationary solution s of* (2.3) *is unstable if and only if $f(\sigma, \lambda)$ is right-positive or left-negative at* $(\sigma, \lambda)=(s, \lambda)$. *Moreover it is asymptotically stable if and only if $f(\sigma, \lambda)$ is right-negative and left-positive at* $(\sigma, \lambda)=(s, \lambda)$.

REMARK 2.4. The result of Proposition 2.3 can be summarized in the following table:

left \ right	negative	oscillatory	positive
positive	asymptotically stable	stable	unstable
oscillatory	stable	stable	unstable
negative	unstable	unstable	unstable

We call a stationary solution (v, λ) of (E) is *right- (resp. left-) positive, oscillatory* and *negative* if $f(\sigma, \lambda)$ is right- (resp. left-) positive, oscillatory and negative at $(\sigma, \lambda)=(\langle v, \phi^* \rangle, \lambda)$ respectively. In Section 3 we shall show that the table is valid also for the stability of a stationary solution of (E).

§ 2.2. Linearized eigenvalue problem.

Let v be a stationary solution of (E). Then by Proposition 2.2, v takes the form; $v=s\phi+z_0(s, \lambda)$. Consider the eigenvalue problem for the Fréchet derivative of F at (v, λ);

$$D_x F(v, \lambda)[y] = (L + D_x N(s\phi + z_0(s, \lambda), \lambda))[y] = \kappa y \tag{2.6}$$

$(s = \langle v, \phi^* \rangle)$. Since zero is a simple eigenvalue of L and since $D_x N(0, 0) = 0$, there exists a unique eigenvalue of (2.6) near 0 provided that v and λ are sufficiently close to 0. We shall denote it by $\kappa(s, \lambda)$. (See T. Kato [9].)

§ 2.3. The relation between the eigenvalue $\kappa(s, \lambda)$ and the bifurcation function $f(s, \lambda)$.

In this section we show the following fundamental identity between $\kappa(s, \lambda)$ and $f(s, \lambda)$.

THEOREM 2.5. *Let Assumptions* 1 *and* 2 *be satisfied. If* (v, λ) *is a stationary solution of* (E), *then*

$$\kappa(s, \lambda) = \left(\frac{\partial}{\partial s} f\right)(s, \lambda), \quad s = \langle v, \phi^* \rangle. \tag{2.7}$$

REMARK 2.6. Under somewhat different assumptions from ours, H. Weinberger [19; Proposition 1] showed the identity:

$$\operatorname{sign} \kappa(s, \lambda) = \operatorname{sign}\left(\frac{\partial}{\partial s} g\right)(s, \lambda),$$

$g(s, \lambda)$ being the bifurcation function constructed by the method of Ljapunov-Schmidt.

PROOF OF THEOREM 2.5. For any σ near s, we set $x_0 = \sigma\phi + z_0(\sigma, \lambda)$. Then $x_0 \in C_\lambda$. If $s(t)$ is a solution of (2.3) with $s(0) = \sigma$, then, $s(t)\phi + z_0(s(t), \lambda)$ is a solution of (E). Hence, by (2.1), we have

$$\lim_{t \downarrow 0} \frac{dz_0(s(t), \lambda)}{dt} = \lim_{t \downarrow 0} QF(s(t)\phi + z_0(s(t), \lambda), \lambda) = QF(\sigma\phi + z_0(\sigma, \lambda), \lambda). \tag{2.8}$$

On the other hand, by (2.3),

$$\lim_{t \downarrow 0} \frac{dz_0(s(t), \lambda)}{dt} = \lim_{t \downarrow 0} D_s z_0(s(t), \lambda) \frac{ds(t)}{dt} = D_s z_0(\sigma, \lambda) f(\sigma, \lambda). \tag{2.9}$$

Hence, by (2.8) and (2.9),

$$QF(\sigma\phi + z_0(\sigma, \lambda), \lambda) = D_s z_0(\sigma, \lambda) f(\sigma, \lambda).$$

Differentiating the above equality with respect to σ at $\sigma = s$, and using $f(s, \lambda) = 0$, we have

$$(2.10)\qquad (QL+QD_xN(s\phi+z_0(s,\lambda),\lambda))(\phi+D_sz_0(s,\lambda))=D_sz_0(s,\lambda)\frac{\partial f(s,\lambda)}{\partial s}.$$

Adding $\partial f(s,\lambda)\phi/\partial s$ to both sides of (2.10), we get

$$(2.11)\qquad Q(L+D_xN(s\phi+z_0(s,\lambda),\lambda))(\phi+D_sz_0(s,\lambda))+\frac{\partial f(s,\lambda)}{\partial s}\phi=\frac{\partial f(s,\lambda)}{\partial s}(\phi+D_sz_0(s,\lambda)).$$

On the other hand, differentiating (1.5) in s, we have

$$(2.12)\qquad \frac{\partial}{\partial s}f(s,\lambda)=\langle (L+D_xN(s\phi+z_0(s,\lambda),\lambda))(\phi+D_sz_0(s,\lambda)),\phi^*\rangle.$$

Putting this identity into the second term of the left hand side of (2.11), we obtain

$$(2.13)\qquad (L+D_xN(s\phi+z_0(s,\lambda),\lambda))(\phi+D_sz_0(s,\lambda))=\frac{\partial f(s,\lambda)}{\partial s}(\phi+D_sz_0(s,\lambda)),$$

which shows that $\partial f(s,\lambda)/\partial s$ is an eigenvalue of $L+D_xN(s\phi+z_0(s,\lambda),\lambda)$. On the other hand, by setting $s=0$, $\lambda=0$ in (2.12), we have

$$\frac{\partial f(0,0)}{\partial s}=\langle (L+D_xN(0,0))(\phi+D_sz_0(0,0)),\ \phi^*\rangle=\langle L\phi,\phi^*\rangle=0,$$

which shows that $\partial f(s,\lambda)/\partial s$ is near zero if $|s|$ and $|\lambda|$ are sufficiently small. Hence by the uniqueness of $\kappa(s,\lambda)$ we have the desired identity (2.7).

§2.4. Factorization theorem.

As a corollary of Theorem 2.5, we have

THEOREM 2.7. *Let Assumptions* 1 *and* 2 *be satisfied. Suppose that there exists an* R^n*-valued* C^1*-function* $\lambda(s)=(\lambda_j(s))_{j=1,\dots,n}$ *on an open interval* I *with*

$$(2.14)\qquad F(v(s),\lambda(s))=0,\ \ v(s)=s\phi+z_0(s,\lambda(s)),\ \ s\in I.$$

Then

$$(2.15)\qquad \kappa(s,\lambda(s))=-\sum_{j=1}^{n}\hat{\kappa}_j(s)\frac{d\lambda_j(s)}{ds},\ \ s\in I,$$

where

$$\kappa_j(s)=\langle D_xN(v(s),\lambda(s))\frac{\partial}{\partial\lambda_j}z_0(s,\lambda(s))+\frac{\partial}{\partial\lambda_j}N(v(s),\lambda(s)),\phi^*\rangle.$$

PROOF OF THEOREM 2.7. Applying ϕ^* to both sides of (2.14), we have

$$\langle N(s\phi+z_0(s, \lambda(s)), \lambda(s)), \phi^*\rangle=0.$$

Differentiating with respect to s, we get

$$(2.16)\quad \langle D_xN(v(s), \lambda(s))[\phi+D_sz_0(s, \lambda(s))], \phi^*\rangle + \left\langle \sum_{j=1}^{n}\left\{D_xN(v(s), \lambda(s)) \frac{\partial}{\partial\lambda_j}z_0(s, \lambda(s))+\frac{\partial}{\partial\lambda_j} N(v(s), \lambda(s))\right\}\frac{d\lambda_j(s)}{ds}, \phi^*\right\rangle=0.$$

By (2.7) and (2.12), the first term of the left hand side of (2.16) is equal to $\kappa(s, \lambda(s))$. Hence we obtain (2.15).

REMARK 2.8. In the case where $n=1$ and $N(x, \lambda)$ is analytic in (x, λ), D. Joseph and D. Nield [8, Theorem 3] showed a version of (2.15), which they call the factorization theorem.

Section 3. Stability and asymptotic behavior.

§3.1. Stability and bifurcation.

Now we state our main results on stability and bifurcation. In the following theorems, "stable", "unstable", or "asymptotically stable" means that in the topology of X_0.

THEOREM 3.1 (STABILITY THEOREM). *Let Assumptions* 1 *and* 2 *be satisfied. A stationary solution of* (E) *near* 0 *is unstable if and only if it is right-positive or left-negative. Moreover it is asymptotically stable if and only if it is right-negative and left-positive.*

THEOREM 3.2 (BIFURCATION THEOREM). *Let Assumptions* 1 *and* 2 *be satisfied. Let λ be a real number. Assume that $N(0, \lambda)=0$. Then, if a stationary solution $(0, \lambda)$ of* (E) *is stable* (*resp. unstable*) *for each $\lambda<0$ and unstable* (*resp. stable*) *for each $\lambda>0$, then non-trivial stationary solutions of* (E) *bifurcate from $(v, \lambda)=(0, 0)$.*

REMARK 3.3. The result of Theorem 3.1 can be summarized in the table of Remark 2.4.

REMARK 3.4. On the basis of the formula mentioned in Remark 2.6, H. Weinberger [19, Corollary 3] showed that if $\kappa(0, \lambda)$ changes its sign at $\lambda=0$, then $(v, \lambda)=(0, 0)$ is a bifurcation point. In Theorem 3.2 we assumed only the change of the stability, which is weaker than the change of the sign of $\kappa(0, \lambda)$. (Note Theorem 2.5.) However, our other assumptions are somewhat different from those assumed in [19].

We shall illustrate Theorem 3.2 by a simple example. Let us consider a scalar ordinary differential equation with a real parameter λ

$$\frac{ds}{dt}=\lambda s^3-s^4.$$

Since a stationary solution $(0, \lambda)$ is asymptotically stable for $\lambda<0$ and unstable for $\lambda>0$, we can apply Theorem 3.2 to conclude that $(0,0)$ is a bifurcation point. Actually, a direct computation gives all the stationary solutions: $(0, \lambda)$, (λ, λ), $\lambda \in R^1$. Note that, as is easily seen, $\kappa(0, \lambda)=0$, $\lambda \in R^1$, in this example.

In the next section we shall show that the proof of Theorem 3.2 follows immediately from Theorem 3.1. The proof of Theorem 3.1 will be given in §3.4.

§3.2. Reduction of Theorem 3.2 from Theorem 3.1.

It is sufficient to show Theorem 3.2 in the case that $(0, \lambda)$ is stable for $\lambda<0$ and unstable for $\lambda>0$. We claim that there exists a sequence $\{(s_n, \lambda_n)\}_{n=1}^{\infty}$ such that $(s_n, \lambda_n)\to(0,0)$ $(s_n\neq 0)$ as $n\to\infty$ and $f(s_n, \lambda_n)=0$. If this is proved, then, by Proposition 2.2, $(s_n\phi+z_0(s_n, \lambda_n), \lambda_n)$ is a non-trivial stationary solution of (E). Hence $(v, \lambda)=(0,0)$ is a bifurcation point.

We now prove the above claim. Suppose that there does not exist such a sequence. Then, by Theorem 3.1 and our assumption with $(v, \lambda)=(0, \lambda)$ there exists $\varepsilon>0$ such that for $\lambda \in (0, \varepsilon)$, $(0, -\lambda)$ is right-negative and left-positive, and $(0, \lambda)$ is right-positive or left-negative. Hence, for each $\lambda \in (0, \varepsilon)$ there exists $s(\lambda)$ $\neq 0$ such that the sign of $f(s(\lambda), \lambda)$ is opposite to that of $f(s(\lambda), -\lambda)$ and $s(\lambda)\to 0$ as $\lambda\to 0$. Hence, by the mean value theorem, there exists $\lambda' \in (-\lambda, \lambda)$ such that $f(s(\lambda), \lambda')=0$. Thus we reach a contradiction. This completes the proof of Theorem 3.2.

§3.3. Asymptotic behavior.

In this section we shall consider the asymptotic behavior of $u(t, x_0, \lambda)$ as $t\to\infty$ when for some λ there are at least two stationary solutions (E) near 0. Before we state our result, we recall the stable manifold theorem.

3.3.1. The stable manifold theorem.

We first state the definition of the stable manifold of a stationary solution (v, λ) of (E). To this end, we consider the following integral equation in the Banach space $C([0,\infty), X_0)$ with the sup norm: for $(z, x, \lambda) \in Z_0\times X_0\times R^n$

$$
(3.1)\qquad \begin{cases} v(t)=e^{t(L+c_0)}z+\displaystyle\int_0^t e^{(t-s)(L+c_0)}Qe^{c_0 s}\{N(x+e^{-c_0 s}v(s),\lambda)-N(x,\lambda)\}ds \\ \qquad -\displaystyle\int_t^\infty e^{(t-s)(L+c_0)}Pe^{c_0 s}\{N(x+e^{-c_0 s}v(s),\lambda)-N(x,\lambda)\}ds, \end{cases}
$$

where c_0 is a constant in Assumption 1 and P, Q are projections given by (1.1), (1.2).

PROPOSITION 3.5. *Let Assumptions* 1 *and* 2 *be satisfied. Then there exists* $d_2>0$ *such that for* $(z, x, \lambda)\in V(d_2)\equiv\{(z, x, \lambda)\in Z_0\times X_0\times R^n: \|z\|<d_2, \|x\|<d_2, |\lambda|<d_2\}$ (3.1) *has a unique solution* $v(t;z,x,\lambda)$ *near* 0 *in* $C([0,\infty), X_0)$. *This solution satisfies*

$$
v(0;\ z,x,\lambda)=\langle v(0;\ z,x,\lambda),\phi^*\rangle\phi+z.
$$

Furthermore $v(0;z,x,\lambda)$ *is a* C^1*-mapping of* $V(d_2)$ *into* X_0 *such that* $v(0;0,x,\lambda)=0$ *and* $\langle D_z v(0;0,0,0),\phi^*\rangle=0$.

For the proof of Proposition 3.5, see, e.g., Krasnosel'skii et al. [13] and Kielhöfer [11].

Let (v,λ) be a stationary solution of (E) with $\|v\|<d_2$, $|\lambda|<d_2$. We set

$$
(3.2)\qquad s_0(z;v,\lambda)=\langle v(0;z,v,\lambda),\phi^*\rangle.
$$

We define the stable manifold $\mathcal{M}(v,\lambda)$ of a stationary solution (v,λ) of (E) by

$$
(3.3)\qquad \mathcal{M}(v,\lambda)=\{v+s_0(z;v,\lambda)\phi+z:\ z\in Z_0,\ \|z\|<d_2\}.
$$

Then we have

THEOREM 3.6 (STABLE MANIFOLD THEOREM). *Let Assumptions* 1 *and* 2 *be satisfied. Let* (v,λ) *be a stationary solution of* (E) *with* $\|v\|<d_2$, $|\lambda|<d_2$. *Then the following statements hold.*

(i) *If* $x_0\in\mathcal{M}(v,\lambda)$, *then a solution* $u(t,x_0,\lambda)$ *of* (E) *exists on* $[0,\infty)$ *and satisfies the estimate*

$$
(3.4)\qquad \|u(t,x_0,\lambda)-v\|\le K_2e^{-c_0 t}\|x_0-v\|,\ t\ge 0,
$$

K_2 *being a positive constant independent of* t, v, λ.

(ii) *If a solution* $u(t,x_0,\lambda)$ *of* (E) *satisfies*

$$
(3.5)\qquad \|u(t,x_0,\lambda)-v\|\le K_3e^{-c_0 t},\ t\ge 0,
$$

then $x_0\in\mathcal{M}(v,\lambda)$, K_3 *being a positive constant independent of* t, v, λ.

For the proof of Theorem 3.6, see M. Ito [7], G. Iooss [6] and Kielhöfer [11].

Since $\mathcal{M}(v,\lambda)$ is a manifold of codimension one, we can define locally the upper and lower sides of $\mathcal{M}(v, \lambda)$. For this purpose and for some later convenience, we choose from the beginning d_1, d_2 so small that

$$\sup_{|s|<d_1, |\lambda|<d_1} \|D_s z_0(s, \lambda)\| < \frac{1}{4}, \quad \sup_{|s|<d_1, |\lambda|<d_1} \|D_\lambda z_0(s, \lambda)\| < \frac{1}{4}, \tag{3.6}$$

$$\sup_{\substack{\|z\|<d_2, \|x\|<d_2 \\ |\lambda|<d_2}} \|D_z s_0(z; x, \lambda)\| < \frac{1}{2}$$

are satisfied. We set

$$d_3 = \min\left\{\frac{d_1}{2}, \frac{d_2}{2}, \frac{1}{2}K_2^{-1}K_3\right\}. \tag{3.7}$$

In what follows we fix such a d_3 and are exclusively concerned with a stationary solution (v, λ) of (E) satisfying $\|v\|<d_3$, $|\lambda|<d_3$.

DEFINITION 3.7. Let (v, λ) be a stationary solution of (E) with $\|v\|<d_3$, $|\lambda|<d_3$. Let $x \in X_0$ be such that $\|Px\|<d_1$, $\|Qx\|<d_3$. Then x is said to be *on the upper side of* $\mathcal{M}(v, \lambda)$ if $s>s_0(z; v, \lambda)$, where $s=\langle x-v, \phi^*\rangle$, $z=x-v-s\phi$. The lower side of $\mathcal{M}(v, \lambda)$ can be defined similarly.

Let $\mathcal{M}_+(v, \lambda)$ ($\mathcal{M}_-(v, \lambda)$) denote the set of all points on the upper (lower) side of $\mathcal{M}(v, \lambda)$. Then $(v=s\phi+z_0(s, \lambda))$

$$\mathcal{M}_+(v, \lambda)=\{x=\sigma\phi+z:\ s+s_0(z-z_0(s, \lambda);\ v, \lambda)<\sigma<d_1,\ z \in Z_0,\ \|z\|<d_3\}. \tag{3.8}$$

By (3.6) and (3.7), we see that any x satisfying $\|Px\|<d_1$ and $\|Qx\|<d_3$, belongs to $\mathcal{M}(v, \lambda)$, $\mathcal{M}_+(v, \lambda)$, or $\mathcal{M}_-(v, \lambda)$ for any stationary solution (v, λ) of (E) with $\|v\|<d_3$, $|\lambda|<d_3$.

We state here lemmas on stable manifolds for later use.

LEMMA 3.8. *Let λ, $|\lambda|<d_3$, be fixed and let (v_i, λ) be stationary solutions of* (E) *which satisfy $\|v_i\|<d_3$ and $s_1<s_2$, $s_i=\langle v_i, \phi^*\rangle$, $i=1, 2$. Then for all z with $\|z\|<d_3$*

$$s_1+s_0(z-z_0(s_1, \lambda); v_1, \lambda)<s_2+s_0(z-z_0(s_2, \lambda); v_2, \lambda). \tag{3.9}$$

REMARK 3.9. By Lemma 3.8, $\mathcal{M}_+(v_1, \lambda) \cap \mathcal{M}_-(v_2, \lambda) \neq \emptyset$. More precisely

$$\mathcal{M}_+(v_1, \lambda) \cap \mathcal{M}_-(v_2, \lambda)=\{x=\sigma\phi+z:\ z \in Z_0,\ \|z\|<d_3,$$
$$s_1+s_0(z-z_0(s_1, \lambda); v_1, \lambda)<\sigma<s_2+s_0(z-z_0(s_2, \lambda); v_2, \lambda)\}.$$

LEMMA 3.10. *Let (v, λ) be a stationary solution of* (E) *with $\|v\|<d_3$, $|\lambda|<d_3$. If $x_0 \in \mathcal{M}_+(v, \lambda)$ (resp. $x_0 \in \mathcal{M}_-(v, \lambda)$) and if for some $t>0$,*

$$\|Pu(\tau, x_0, \lambda)\|<d_1,\ \|Qu(\tau, x_0, \lambda)\|<d_3,\ 0\leq\tau\leq t,$$

then for $0\leq\tau\leq t$

$$u(\tau, x_0, \lambda)\in\mathcal{M}_+(v, \lambda)\ (resp.\ u(\tau, x_0, \lambda)\in\mathcal{M}_-(v, \lambda)). \tag{3.10}$$

PROOF OF LEMMA 3.8. First we show that (3.9) holds for $z=z_0(s_2, \lambda)$. By (3.6) we have

$$s_2-s_1-s_0(z_0(s_2, \lambda)-z_0(s_1, \lambda); v_1, \lambda)>s_2-s_1-\frac{1}{8}(s_2-s_1)>0.$$

Hence (3.9) holds for $z=z_0(s_2, \lambda)$.

Now we show that (3.9) holds for all z with $\|z\|<d_3$. Assume the contrary. Then, since (3.9) holds for $z=z_0(s_2, \lambda)$, there exists z', $\|z'\|<d_3$, such that

$$s_1+s_0(z'-z_0(s_1, \lambda); v_1, \lambda)=s_2+s_0(z'-z_0(s_2, \lambda); v_2, \lambda)\ (=s). \tag{3.11}$$

Set $x=s\phi+z'$. Then, by (3.11), we have $x\in\mathcal{M}(v_1, \lambda)$ and $x\in\mathcal{M}(v_2, \lambda)$. Hence, by Theorem 3.6 (i), $u(t, x_0, \lambda)$ converges to v_1 and v_2 as $t\to\infty$, which is a contradiction. [q. e. d.]

PROOF OF LEMMA 3.10. We shall prove only in the case that $x_0\in\mathcal{M}_+(v, \lambda)$. Suppose that (3.10) does not hold. Set

$$T=\sup\{\tau':\ u(\tau, x_0, \lambda)\in\mathcal{M}_+(v, \lambda), 0\leq\tau\leq\tau'\}. \tag{3.12}$$

Then

$$u(T, x_0, \lambda)\in\mathcal{M}(v, \lambda).$$

Hence

$$u(T, x_0, \lambda)-v=s_0(Qu(T, x_0, \lambda)-Qv; v, \lambda)\phi+Qu(T, x_0, \lambda)-Qv.$$

By (3.6) we have

$$\begin{aligned}\|u(T, x_0, \lambda)-v\|&=\max\{|s_0(Qu(T, x_0, \lambda)-Qv; v, \lambda)|,\ \|Qu(T, x_0, \lambda)-Qv\|\}\\&=\|Qu(T, x_0, \lambda)-Qv\|<2d_3.\end{aligned}$$

Hence, by Theorem 3.6 (i) and (3.7), we have

$$\begin{aligned}\|u(\tau, x_0, \lambda)-v\|&=\|u(\tau-T, u(T, x_0, \lambda), \lambda)-v\|\\&\leq K_2e^{-c_0(\tau-T)}\|u(T, x_0, \lambda)-v\|<2K_2d_3e^{-c_0(\tau-T)}\leq K_3e^{-c_0(\tau-T)}\end{aligned}$$

for $\tau\geq T$. From the continuity of $u(\cdot, x_0, \lambda)$ there exists T', $0<T'<T$, such that

$$\|u(\tau, x_0, \lambda)-v\|<K_3e^{-c_0(\tau-T')},\ \tau\geq T'.$$

Since

$$u(\tau, x_0, \lambda)=u(\tau-T', u(T', x_0, \lambda), \lambda),$$

we have by Theorem 3.6 (ii) $u(T', x_0, \lambda) \in \mathcal{M}(v, \lambda)$, which is a contradiction; note (3.12) and $T'<T$. [q.e.d.]

3.3.2. Asymptotic behavior.

We next state our main result on asymptotic behavior of $u(t, x_0, \lambda)$ in t.

THEOREM 3.11. *Let Assumptions* 1 *and* 2 *be satisfied. Let* $\lambda, |\lambda|<d_3$, *be fixed. Let* (v_i, λ) *be two stationary solutions of* (E) *with* $\|v_i\|<d_3$, $i=1,2$. *Suppose that* $\langle v_1, \phi^*\rangle<\langle v_2, \phi^*\rangle$. *Then if* $x_0 \in \mathcal{M}_+(v_1, \lambda)\cap\mathcal{M}_-(v_2, \lambda)$, *and if*

$$\|Qx_0-z_0(\langle x_0, \phi^*\rangle, \lambda)\|<\frac{1}{4}d_3K_1^{-1}$$

(K_1: *the constant which appears in Theorem* 1.3),

then either the following statement (i) *or* (ii) *holds.*

(i) *There exists a stationary solution* (v, λ) *with* $x_0 \in \mathcal{M}(v, \lambda)$ *such that*

$$\langle v_1, \phi^*\rangle<\langle v, \phi^*\rangle<\langle v_2, \phi^*\rangle;$$

in this case $u(t, x_0, \lambda)$ *converges to* v *as* $t\to\infty$.

(ii) *There exist stationary solutions* $(v_3, \lambda), (v_4, \lambda)$ *with* $x_0 \in \mathcal{M}_+(v_3, \lambda)\cap\mathcal{M}_-(v_4, \lambda)$ *such that*

$$\langle v_1, \phi^*\rangle\leq\langle v_3, \phi^*\rangle<\langle v_4, \phi^*\rangle\leq\langle v_2, \phi^*\rangle, \quad f(\sigma, \lambda)\neq0 \quad (\langle v_3, \phi^*\rangle<\sigma<\langle v_4, \phi^*\rangle);$$

in this case $u(t, x_0, \lambda)$ *converges to* v_3 *(resp.* v_4*) as* $t\to\infty$ *when* $f(\sigma, \lambda)<0$ *(resp.* $f(\sigma, \lambda)>0$*) for* σ, $\langle v_3, \phi^*\rangle<\sigma<\langle v_4, \phi^*\rangle$.

REMARK 3.12. Consider the case that λ is a real number and the bifurcation function f is given by

$$f(s, \lambda)=a\lambda s-bs^2+o(|\lambda s|+s^2),$$

where a, b are positive constants. In this case we know (see [2]) that for each $\lambda>0$ there exists a unique $s=s(\lambda)$ (>0) near 0 such that $f(s(\lambda), \lambda)=0$ and that $\kappa(0, \lambda)>0, \kappa(s(\lambda), \lambda)<0$. Using the above fact Iooss [6, IX; Théorème 3] showed that if $\lambda>0$ is sufficiently small and if $x_0 \in \mathcal{M}_+(0, \lambda)$ and $\|x_0\|<c|\lambda|$ (c is a constant independent of λ), then $u(t, x_0, \lambda)$ converges to $v(\lambda)$ $(=s(\lambda)\phi+z_0(s(\lambda), \lambda))$ as $t\to\infty$.

This result can be obtained from Theorem 3.11 and Proposition 3.13 below, by showing that we can choose positive numbers δ' and ε independent of λ, which are obtained by applying Proposition 3.13 to the right-negative stationary solution $(v(\lambda), \lambda)$.

§ 3.4. Proofs of Theorems 3.1 and 3.11.

In this section we shall give the proofs of Theorems 3.1 and 3.11. In what follows we use the following notation. For a stationary solution (v, λ) of (E) we set

$$\begin{aligned}&s=\langle v, \phi^*\rangle,\\ &B(\delta, \varepsilon)=\{x=\sigma\phi+z:\ |\sigma-s|<\delta,\ \ z\in Z_0,\ \|z-z_0(\sigma, \lambda)\|<\varepsilon\},\\ &B(d)=\{x=\sigma\phi+z:\ |\sigma|<d_1, z\in Z_0,\ \|z-z_0(\sigma, \lambda)\|<d\}.\end{aligned}$$

Here we note $u(t, x_0, \lambda)\in \mathrm{B}(\delta, \varepsilon)$ if and only if $|s(t)-s|<\delta$, $\|z(t)\|<\varepsilon$, where

$$\begin{aligned}s(t)=\langle u(t, x_0, \lambda), \phi^*\rangle,\ \ z(t)=Qu(t, x_0, \lambda)-z_0(s(t), \lambda);\\ u(t, x_0, \lambda)=s(t)\phi+z_0(s(t), \lambda)+z(t).\end{aligned}$$

3.4.1. The right-negative or left-positive case.

PROPOSITION 3.13. *Let (v, λ) be a right-negative (resp. left-positive) stationary solution of* (E) *with $\|v\|<d_3$, $|\lambda|<d_3$. Let δ', $0<\delta'<d_3$, be such that*

$$f(\sigma, \lambda)=\langle N(\sigma\phi+z_0(\sigma, \lambda), \lambda), \phi^*\rangle<0,\ \ s<\sigma\leq s+\delta' \quad (3.13)$$
$$(\textit{resp. } f(\sigma, \lambda)>0,\ \ s-\delta'\leq\sigma<s).$$

Then there exists $\varepsilon>0$ such that if

$$x_0\in \mathcal{M}_+(v, \lambda)\cap B(\delta', \varepsilon)\quad (\textit{resp. } x_0\in\mathcal{M}_-(v, \lambda)\cap B(\delta', \varepsilon)),$$

then the solution $u(t, x_0, \lambda)$ of (E) *converges to v as $t\to\infty$.*

PROOF. We shall prove the proposition only for the case that (v, λ) is right-negative. We first show the following

LEMMA 3.14. *For any $\delta\in(0, \delta']$ there exists $\varepsilon>0$ such that if $x_0\in\mathcal{M}_+(v, \lambda)\cap B(\delta, \varepsilon)$, then*

$$u(t, x_0, \lambda)\in\mathcal{M}_+(v, \lambda)\cap B(\delta, K_1\varepsilon) \quad (3.14)$$

for $t\geq 0$.

PROOF OF LEMMA 3.14. Choose $\varepsilon>0$ so small that

$$\varepsilon<\min\left\{\frac{1}{2}(d_3-\delta)K_1^{-1},\ \delta K_1^{-1}\right\} \quad (3.15)$$

and that

$$\langle N((s+\delta)\phi+z_0(s+\delta, \lambda)+z, \lambda), \phi^*\rangle<0 \tag{3.16}$$

holds for all z with $\|z\|<K_1\varepsilon$. Let $x_0\in\mathcal{M}_+(v, \lambda)\cap B(\delta, \varepsilon)$ and set

$$T=\sup\{\tau: \text{(3.14) holds for all } t\in[0, \tau]\}. \tag{3.17}$$

Suppose that $T<\infty$. Then we have at least one of the following (a), (b), (c):

(a) $|s(T)-s|=\delta$,

(b) $\|z(T)\|=K_1\varepsilon$,

(c) $u(T, x_0, \lambda)\notin\mathcal{M}_+(v, \lambda)$.

Since

$$|s(t)-s|\leq\delta, \quad 0\leq t\leq T$$

and

$$\|z(t)\|\leq K_1\varepsilon, \quad 0\leq t\leq T,$$

we have

$$|s(t)|<\delta+d_3<d_1, \quad 0\leq t\leq T,$$

and by (3.6), (3.15)

$$\begin{aligned}\|Qu(t, x_0, \lambda)\|&=\|z(t)\|+\|z_0(s(t), \lambda)-z_0(0, \lambda)\|+\|z_0(0, \lambda)\|\\&<K_1\varepsilon+\frac{1}{4}(\delta+d_3)+\frac{1}{4}d_3<d_3, \quad 0\leq t\leq T.\end{aligned}$$

Hence, by Lemma 3.10

$$u(T, x_0, \lambda)\in\mathcal{M}_+(v, \lambda), \tag{3.18}$$

showing that (c) does not hold. Since $x_0\in B(\delta, \varepsilon)$, we have by Theorem 1.3 (ii)

$$\|z(t)\|<K_1\varepsilon e^{-c_0t}, \quad 0\leq t\leq T, \tag{3.19}$$

showing that (b) does not hold. Hence we must have

$$|s(T)-s|=\delta. \tag{3.20}$$

We show that $s(T)=s+\delta$. Since $\varepsilon<\delta K_1^{-1}$, we have by (3.6)

$$-\delta<s_0(z_0(s-\delta, \lambda)+z-z_0(s, \lambda); v, \lambda), \quad \|z\|<K_1\varepsilon,$$

and so

$$(s-\delta)\phi+z_0(s-\delta, \lambda)+z\in\mathcal{M}_-(v, \lambda), \quad \|z\|<K_1\varepsilon.$$

Therefore, by (3.18), (3.19), we have $s(T)\neq s-\delta$. Thus we get by (3.20)

(3.21) $$s(T)=s+\delta.$$

On the other hand, $s(t)$ satisfies

$$\frac{ds(t)}{dt}=\langle N(s(t)\phi+z_0(s(t),\lambda)+z(t),\lambda),\phi^*\rangle.$$

By (3.16), (3.19) and (3.21)

$$\frac{ds(T)}{dt}<0.$$

Therefore there exists $T'(<T)$ such that $s(T')>s(T)=s+\delta$, contradicting (3.17). Thus Lemma 3.14 is proved.

We now turn to the proof of Proposition 3.13. In order to show that $u(t, x_0, \lambda)$, $x_0\in\mathcal{M}_+(v,\lambda)\cap B(\delta',\varepsilon)$, converges to v as $t\to\infty$, where ε is as in Lemma 3.14 with $\delta=\delta'$, it is sufficient to show that $s(t)$ converges to s. This can be seen from (3.19) with $T=\infty$ and the inequality

$$\|u(t,x_0,\lambda)-v\|\leq|s(t)-s|+\|z_0(s(t),\lambda)-z_0(s,\lambda)\|+\|z(t)\|.$$

Suppose now that $s(t)$ does not converge to s. Then, since $s(t)$ is bounded by Lemma 3.14, there exists $\hat{s}$, $s<\hat{s}\leq s+\delta'$, and a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_k\uparrow\infty$ and $s(t_k)\to\hat{s}$ as $k\to\infty$. Hence, by (1.4), we have

(3.22) $$u(t_k,x_0,\lambda)=s(t_k)\phi+z_0(s(t_k),\lambda)+z(t_k)\to\hat{s}\phi+z_0(\hat{s},\lambda)$$

as $k\to\infty$. Choosing $\delta''>0$ such that

(3.23) $$s+\delta''<\hat{s},$$

we apply Lemma 3.14 with $\delta=\delta''$. Then there exists $\varepsilon'>0$ such that if $x\in\mathcal{M}_+(v,\lambda)\cap B(\delta'',\varepsilon')$ then

(3.24) $$u(t,x,\lambda)\in\mathcal{M}_+(v,\lambda)\cap B(\delta'',K_1\varepsilon'),\ t\in[0,\infty).$$

Set $\hat{x}=\hat{s}\phi+z_0(\hat{s},\lambda)$. Since $\hat{x}\in C_\lambda$, we apply Proposition 2.1. Then, by (3.13), $\langle u(t,\hat{x},\lambda),\phi^*\rangle$ is monotone decreasing in t. Therefore there exist $\hat{T}>0$ and a neighborhood V of $\hat{x}$ such that

(3.25) $$u(\hat{T},x,\lambda)\in\mathcal{M}_+(v,\lambda)\cap B(\delta'',\varepsilon')$$

for any $x\in V$. From (3.22) there exists k' such that

(3.26) $$u(t_{k'},x_0,\lambda)\in V.$$

Since

$$S_\lambda(t)x_0 = S_\lambda(t-\hat{T}-t_{k'})S_\lambda(\hat{T}+t_{k'})x_0$$
$$= S_\lambda(t-\hat{T}-t_{k'})S_\lambda(\hat{T})S_\lambda(t_{k'})x_0 \quad (S_\lambda(t)x_0 = u(t, x_0, \lambda))$$

holds for $t \geq \hat{T}+t_{k'}$, we have by (3.24), (3.25) and (3.26)

$$S_\lambda(t)x_0 \in \mathcal{M}_+(v, \lambda) \cap B(\delta'', K_1\varepsilon'), \ t \geq t_{k'}+T.$$

This contradicts (3.22) and (3.23), proving Proposition 3.13. [q.e.d.]

From Proposition 3.13 we can immediately see

PROPOSITION 3.15. *Under the assumptions of Theorem 3.1, a stationary solution (v, λ) of* (E) *with $\|v\| < d_3 |\lambda| < d_3$ is asymptotically stable if it is right-negative and left-positive.*

3.4.2. The right-positive or left-negative case.

PROPOSITION 3.16. *Let (v, λ) be a right-positive (resp. left-negative) stationary solution of* (E) *with $\|v\| < d_3$, $|\lambda| < d_3$. Let δ, $0 < \delta < d_3$, be such that*

$$f(\sigma, \lambda) > 0, \ s < \sigma \leq s+\delta \qquad (\textit{resp. } f(\sigma, \lambda) < 0, \ s-\delta \leq \sigma < s). \tag{3.27}$$

Then there exists $\varepsilon > 0$ such that if

$$x_0 \in \mathcal{M}_+(v, \lambda) \cap B(\delta, \varepsilon) \quad (\textit{resp. } x_0 \in \mathcal{M}_-(v, \lambda) \cap B(\delta, \varepsilon)),$$

then for some $t' > 0$ (depending on x_0)

$$\langle u(t', x_0, \lambda), \phi^* \rangle = s+\delta \quad (\textit{resp. } \langle u(t', x_0, \lambda), \phi^* \rangle = s-\delta).$$

PROOF. We only consider the case that (v, λ) is right-positive. Let $\varepsilon > 0$ be such that

$$\varepsilon = \min\left\{\delta K_1^{-1}, \ \frac{1}{2}(d_3-\delta)K_1^{-1}\right\}. \tag{3.28}$$

Let $x_0 \in \mathcal{M}_+(v, \lambda) \cap B(\delta, \varepsilon)$. We set

$$T = \sup\{\tau : \ u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap B(\delta, K_1\varepsilon), \ t \in [0, \tau]\}.$$

If $T < \infty$, then by the same arguments as in Lemma 3.14, we have $s(T) = s+\delta$. If $T = \infty$, then, by the same arguments as in the proof of Proposition 3.13, we can show that $u(t, x_0, \lambda)$ converges to v as $t \to \infty$. As can be seen from the following lemma, this is impossible, which proves Proposition 3.16.

LEMMA 3.17. *Let (v, λ) be right-positive. Then there exists $\varepsilon' > 0$ such that if*

$$u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap B(\delta, \varepsilon'),\ \ t \geq 0,$$

then $u(t, x_0, \lambda)$ *does not converge to* v *as* $t \to \infty$, *where* $\delta > 0$ *is as in* (3.27).

PROOF OF LEMMA 3.17. Set

$$K = \sup_{\|x\| < d_1, |\lambda| < d_1} \|D_x N(x, \lambda)\|. \tag{3.29}$$

Let $C > 2$ and $\varepsilon' > 0$ be such that

$$e^{2K_1 K C^{-1} c_0^{-1}} < 2 \tag{3.30}$$

and

$$\varepsilon' = \min\left\{K_3 C^{-1} K_1^{-1},\ \frac{d_3}{2}\right\}. \tag{3.31}$$

First step. We show that for some $\tau \geq 0$

$$s(\tau) - s \geq C\|z(\tau)\|. \tag{3.32}$$

Suppose that (3.32) does not hold. Then

$$s(t) - s < C\|z(t)\|.$$

Since

$$u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda),$$

we have

$$s(t) - s > s_0(z(t) + z_0(s(t), \lambda) - z_0(s, \lambda); v, \lambda)$$

and so we have

$$-C\|z(t)\| < s(t) - s$$

by (3.6) and $C > 2$. Thus we get

$$|s(t) - s| < C\|z(t)\|. \tag{3.33}$$

By (3.6) and (3.33)

$$\|z(t) + z_0(s(t), \lambda) - z_0(s, \lambda)\| < \left(1 + \frac{C}{4}\right)\|z(t)\|.$$

Hence, by (3.33) and $C > 2$

$$\|u(t, x_0, \lambda) - v\| = \max\{|s(t) - s|,\ \|z(t) + z_0(s(t), \lambda) - z_0(s, \lambda)\|\} < C\|z(t)\|. \tag{3.34}$$

Since

$$u(t, x_0, \lambda) \in B(\delta, \varepsilon'),$$

we have by (3.6) and (3.31)

$$\|u(t, x_0, \lambda)\| < 2d_3 < d_1.$$

Hence, since $x_0 \in B(\delta, \varepsilon')$, we have by Theorem 1.3 (ii)

$$\|z(t)\| \leq K_1 e^{-c_0 t}\|z(0)\| < K_1 e^{-c_0 t}\varepsilon'.$$

Thus, by (3.34) we get

$$\|u(t, x_0, \lambda) - v\| < CK_1 e^{-c_0 t}\varepsilon'.$$

On the other hand, since $x_0 \notin \mathcal{M}(v, \lambda)$, it follows from Theorem 3.6 (ii) that there exists $\tau' > 0$ such that

$$\|u(\tau', x_0, \lambda) - v\| > K_3 e^{-c_0 \tau'}.$$

Therefore we get

$$K_3 e^{-c_0 \tau'} < CK_1 \varepsilon' e^{-c_0 \tau'},$$

and so

$$K_3 < CK_1 \varepsilon'.$$

This contradicts (3.31).

Second step. By the first step stated above, it suffices for the proof of Lemma 3.17 to show that $u(t, x_0, \lambda)$ does not converge to v if x_0 satisfies

$$\langle x_0, \phi^* \rangle - s \geq C\|Qx_0 - z_0(\langle x_0, \phi^* \rangle, \lambda)\|, \tag{3.35}$$

i.e.,

$$s(0) - s \geq C\|z(0)\|.$$

By (3.27), (3.29) and (3.31) we have

$$\begin{aligned} \frac{ds(t)}{dt} &= \langle N(s(t)\phi + z_0(s(t), \lambda) + z(t), \lambda), \phi^* \rangle \\ &= \langle N(s(t)\phi + z_0(s(t), \lambda), \lambda), \phi^* \rangle \\ &\quad + \int_0^1 \langle D_x N(s(t)\phi + z_0(s(t), \lambda) + rz(t), \lambda)[z(t)], \phi^* \rangle dr \geq -K\|z(t)\|. \end{aligned} \tag{3.36}$$

We show that

$$s(t) - s \geq \frac{1}{2} K_1^{-1} C e^{c_0 t}\|z(t)\|, \quad t \geq 0. \tag{3.37}$$

Since

$$u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap B(\delta, \varepsilon')$$

and (3.27) holds, we have by Proposition 2.1

$$u(t, x_0, \lambda) \notin \mathcal{C}_\lambda.$$

And so we have

$$z(t) \neq 0. \tag{3.38}$$

Hence, noticing that $K_1 \geq 1$, we have, by (3.35), that (3.37) holds for small $t \geq 0$. Set

$$T = \sup\{\tau: \text{(3.37) holds for all } t \in [0, \tau]\}. \tag{3.39}$$

Suppose that $T < \infty$. Then, by (3.36) and (3.39)

$$\frac{ds(t)}{dt} \geq -2C^{-1}K_1 K e^{-c_0 t}(s(t)-s), \quad 0 < t < T.$$

Thus we have

$$s(t) - s \geq e^{-2c_0^{-1}C^{-1}K_1 K}(s(0)-s), \quad 0 \leq t \leq T. \tag{3.40}$$

Hence, by (1.4), (3.30), (3.35) and (3.38)

$$\begin{aligned} s(t)-s &\geq e^{-2c_0^{-1}C^{-1}K_1 K} C\|z(0)\| \geq e^{-2c_0^{-1}C^{-1}K_1 K} C(K_1^{-1} e^{c_0 t}\|z(t)\|) \\ &> \frac{1}{2} C K_1^{-1} e^{c_0 t}\|z(t)\|, \quad 0 \leq t \leq T, \end{aligned}$$

contradicting (3.39). Therefore (3.37) holds for all $t \geq 0$. Hence (3.40) holds for all $t \geq 0$. This implies that $s(t)$ does not converge to s as $t \to \infty$. Thus the lemma is proved. [q. e. d.]

From Proposition 3.16 we can immediately see

PROPOSITION 3.18. *Under the assumptions of Theorem 3.1, a stationary solution (v, λ) of (E) with $\|v\| < d_3$, $|\lambda| < d_3$ is unstable if it is right-positive or left-negative.*

3.4.3. The right-oscillatory or left-oscillatory case.

PROPOSITION 3.19. *Let (v, λ) be a right-oscillatory (resp. left-oscillatory) stationary solution of (E) with $\|v\| < d_3$, $|\lambda| < d_3$. Then for any $d' > 0$, there exists $d'' > 0$ such that if $x_0 \in \mathcal{M}_+(v, \lambda)$ (resp. $x_0 \in \mathcal{M}_-(v, \lambda)$) with $\|x_0 - v\| < d''$, then*

$$\|u(t, x_0, \lambda) - v\| < d', \quad t \geq 0.$$

For the proof of Proposition 3.19 we first consider the case that there are at least two stationary solutions of (E) for some λ.

LEMMA 3.20. *Let λ, $|\lambda| < d_3$, be fixed. Let (v, λ) and $(\hat{v}, \lambda)$ be stationary solutions of (E) with $\|v\| < d_3$, $\|\hat{v}\| < d_3$ such that $s < \hat{s}$ ($s = \langle v, \phi^* \rangle$, $\hat{s} = \langle \hat{v}, \phi^* \rangle$). Then for any $d \in \left(0, \frac{1}{4} d_3 K_1^{-1}\right]$*

(3.41) $$u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap \mathcal{M}_-(\hat{v}, \lambda) \cap B(k_1 d), \quad t \geq 0$$

if $x_0 \in \mathcal{M}_+(v, \lambda) \cap \mathcal{M}_-(\hat{v}, \lambda) \cap B(d)$.

The proof of the lemma will be given later.

PROOF OF PROPOSITION 3.19. We give the proof for the case that (v, λ) is right-oscillatory. Let $0 < d' < d_3$. We set

(3.42) $$d = \min\left\{\frac{1}{2} d' K_1^{-1}, \ \frac{1}{4} d_3 K_1^{-1}\right\}.$$

Since (v, λ) is right-oscillatory, there exists a stationary solution $(\hat{v}, \lambda)$ of (E) with $\|\hat{v}\| < d_3$ such that

(3.43) $$0 < \hat{s} - s < \frac{d'}{2} \quad (s = \langle v, \phi^* \rangle, \ \hat{s} = \langle \hat{v}, \phi^* \rangle).$$

Set

(3.44) $$d'' = \min\left\{\frac{1}{2}(\hat{s} - s), \ \frac{4}{5} d, \ d_3 - \|Qv\|\right\}.$$

First we show that

(3.45) $$\{x : \|x - v\| < d''\} \subset \mathcal{M}_-(\hat{v}, \lambda) \cap B(d).$$

Since $\langle x, \phi^* \rangle < s + d''$, we have by (3.44)

(3.46) $$\langle x, \phi^* \rangle - \hat{s} < s - \hat{s} + d'' \leq \frac{1}{2}(s - \hat{s}).$$

By (3.6) and (3.44)

$$\begin{aligned}\|Qx - Q\hat{v}\| &\leq \|Qx - Qv\| + \|Qv - Q\hat{v}\| \\ &= \|Qx - Qv\| + \|z_0(s, \lambda) - z_0(\hat{s}, \lambda)\| \\ &< d'' + \frac{1}{4}(\hat{s} - s) < \hat{s} - s.\end{aligned}$$

Hence, by (3.6) and (3.46)

$$s_0(Qx - Q\hat{v}; \hat{v}, \lambda) > -\frac{1}{2}\|Qx - Q\hat{v}\| > \frac{1}{2}(s - \hat{s}) > \langle x, \phi^* \rangle - \hat{s}.$$

Since, by (3.44), $\|Qx\| < d_3$, we have

(3.47) $$x \in \mathcal{M}_-(\hat{v}, \lambda).$$

By (3.6) and (3.44) we get

$$\begin{aligned}\|Qx - z_0(\langle x, \phi^* \rangle, \lambda)\| &\leq \|Qx - Qv\| + \|z_0(\langle x, \phi^* \rangle, \lambda) - z_0(s, \lambda)\| \\ &< \|Qx - Qv\| + \frac{1}{4}|\langle x, \phi^* \rangle - s| \\ &< \frac{5}{4} d'' \leq d,\end{aligned}$$

showing that $x\in B(d)$, which together with (3.47) gives (3.45).

Next we show that

$$\mathcal{M}_+(v,\lambda)\cap\mathcal{M}_-(\hat{v},\lambda)\cap B(K_1d)\subset\{x:\ \|x-v\|<d'\}. \tag{3.48}$$

If

$$x\in\mathcal{M}_+(v,\lambda)\cap\mathcal{M}_-(\hat{v},\lambda)\cap B(K_1d),$$

then we have

$$s+s_0(Qx-Qv;v,\lambda)<\langle x,\phi^*\rangle<\hat{s}+s_0(Qx-Q\hat{v};\hat{v},\lambda) \tag{3.49}$$

and

$$\|Qx-z_0(\langle x,\phi^*\rangle,\lambda)\|<K_1d. \tag{3.50}$$

By (3.49), (3.50) and (3.6) we get

$$\begin{aligned}\langle x,\phi^*\rangle-s&<\hat{s}-s+s_0(Qx-Q\hat{v};\hat{v},\lambda)\\&<\hat{s}-s+\frac{1}{2}\|Qx-Q\hat{v}\|\\&\leq\hat{s}-s+\frac{1}{2}\|Qx-z_0(\langle x,\phi^*\rangle,\lambda)\|\\&\quad+\frac{1}{2}\|z_0(\langle x,\ \phi^*\rangle,\lambda)-z_0(s,\lambda)\|+\frac{1}{2}\|z_0(s,\lambda)-z_0(\hat{s},\lambda)\|\\&<\frac{9}{8}(\hat{s}-s)+\frac{1}{2}K_1d+\frac{1}{8}|\langle x,\phi^*\rangle-s|.\end{aligned}$$

Hence, by (3.42) and (3.43) we have $\langle x,\phi^*\rangle-s<(9/7)(\hat{s}-s)+(4/7)K_1d<d'$. Similarly we have $\langle x,\phi^*\rangle-s>-d'$. Therefore we have $|\langle x,\phi^*\rangle-s|<d'$. Hence, by (3.42), (3.50) and (3.6)

$$\|Qx-Qv\|\leq\|Qx-z_0(\langle x,\phi^*\rangle,\lambda)\|+\|z_0(\langle x,\phi^*\rangle,\lambda)-z_0(s,\lambda)\|<K_1d+d'/4<d'.$$

Thus we have (3.48).

Now let x_0 be such that $x_0\in\mathcal{M}_+(v,\lambda)$ and $\|x_0-v\|<d''$. Then, by (3.45)

$$x_0\in\mathcal{M}_+(v,\lambda)\cap\mathcal{M}_-(\hat{v},\lambda)\cap B(d).$$

Hence, by Lemma 3.20

$$u(t,x_0,\lambda)\in\mathcal{M}_+(v,\lambda)\cap\mathcal{M}_-(\hat{v},\lambda)\cap B(K_1d),\quad t\geq0.$$

Hence, by (3.48), we have

$$\|u(t,x_0,\lambda)-v\|<d',\ t\geq0.$$ [q. e. d.]

Proof of Lemma 3.20. Let

(3.51) $$x_0 \in \mathcal{M}_+(v, \lambda) \cap \mathcal{M}_-(\hat{v}, \lambda) \cap B(d).$$

Set

(3.52) $$T = \sup\{\tau:\ u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap \mathcal{M}_-(\hat{v}, \lambda) \cap B(K_1 d),\ t \in [0, \tau]\}.$$

Suppose that $T<\infty$. Then we have

(3.53) $$u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap \mathcal{M}_-(\hat{v}, \lambda), \quad 0 \leq t < T$$

and

(3.54) $$u(t, x_0, \lambda) \in B(K_1 d), \quad 0 \leq t < T.$$

By (3.53) and (3.6) we have for $t \in [0, T)$

$$\begin{aligned} s(t) &< \hat{s} + s_0(Qu - Q\hat{v}; \hat{v}, \lambda) \\ &< \hat{s} + \frac{1}{2}\|Qu - Q\hat{v}\| \\ &\leq \hat{s} + \frac{1}{2}\|z(t)\| + \frac{1}{2}\|z_0(s(t), \lambda) - z_0(\hat{s}, \lambda)\| \\ &< \hat{s} + \frac{1}{2}\|z(t)\| + \frac{1}{8}|s(t) - \hat{s}|. \end{aligned}$$

Hence, by (3.54)

$$s(t) - \hat{s} < \frac{4}{7}\|z(t)\| < \frac{4}{7}K_1 d, \quad 0 \leq t < T.$$

Similarly we have

$$s(t) - s > -\frac{4}{7}K_1 d, \quad 0 \leq t < T.$$

Since $d \leq \frac{1}{4} d_3 K_1^{-1}$, we have

(3.55) $$\begin{aligned} |s(t)| &\leq \max\left\{\hat{s} + \frac{4}{7}K_1 d,\ -s + \frac{4}{7}K_1 d\right\} \\ &< d_3 + \frac{4}{7}K_1 d < d_1, \quad 0 \leq t \leq T. \end{aligned}$$

Hence, by (3.54), (3.55) and (3.6)

(3.56) $$\begin{aligned} \|Qu(t, x_0, \lambda)\| &\leq \|z_0(s(t), \lambda) - z_0(0, \lambda)\| + \|z_0(0, \lambda)\| + \|z(t)\| \\ &< \frac{1}{4}\left(d_3 + \frac{4}{7}K_1 d\right) + \frac{1}{4}d_3 + K_1 d < d_3, \quad 0 \leq t \leq T. \end{aligned}$$

Hence, since $x_0 \in B(d)$, we have by (1.4)

$$\|z(t)\| < K_1 e^{-c_0 t} d \leq K_1 d, \quad 0 \leq t \leq T,$$

and by Lemma 3.10 and (3.51)

$$u(t, x_0, \lambda) \in \mathcal{M}_+(v, \lambda) \cap \mathcal{M}_-(\hat{v}, \lambda), \quad 0 \leq t \leq T.$$

This contradicts (3.52). [q.e.d.]

PROOF OF THEOREM 3.1.

The proof of Theorem 3.1 follows immediately from Propositions 3.13, 3.15, 3.18 and 3.19.

3.4.4. PROOF OF THEOREM 3.11.

Let

$$s_i = \langle v_i, \phi^* \rangle, \quad i=1, 2.$$

Set

$$s_3 = \sup_{v \in S_3} \langle v, \phi^* \rangle, \tag{3.57}$$

$$s_4 = \inf_{v \in S_4} \langle v, \phi^* \rangle, \tag{3.58}$$

where

$$S_3 = \{v\colon\ x_0 \in \mathcal{M}_+(v, \lambda) \text{ or } x_0 \in \mathcal{M}(v, \lambda),\ \langle v, \phi^* \rangle \in [s_1, s_2)\},$$
$$S_4 = \{v\colon\ x_0 \in \mathcal{M}_-(v, \lambda) \text{ or } x_0 \in \mathcal{M}(v, \lambda),\ \langle v, \phi^* \rangle \in (s_1, s_2]\}.$$

Then

$$s_1 \leq s_3 \leq s_4 \leq s_2.$$

By the continuity of $f(\cdot, \lambda)$ we have

$$f(s_i, \lambda) = 0, \quad i=3, 4.$$

Set

$$v_i = s_i\phi + z_0(s_i, \lambda), \quad i=3, 4.$$

Then, by Proposition 2.2, (v_i, λ), $i=3, 4$, is a stationary solution of (E). If $x_0 \in \mathcal{M}(v_3, \lambda)$ (resp. $x_0 \in \mathcal{M}(v_4, \lambda)$), then by Theorem 3.6 (i), $u(t, x_0, \lambda)$ converges to v_3 (resp. v_4) as $t \to \infty$. If $x_0 \notin \mathcal{M}(v_3, \lambda) \cup \mathcal{M}(v_4, \lambda)$, then by (3.57) and (3.58) we have

$$s_3 < s_4, \tag{a}$$

$$x_0 \in \mathcal{M}_+(v_3, \lambda) \cap \mathcal{M}_-(v_4, \lambda) \tag{b}$$

and

$$f(s, \lambda) \neq 0, \qquad s_3 < s < s_4. \tag{c}$$

Proof of (ii). We give the proof only in the case that

$$f(s, \lambda)>0, \qquad s_3<s<s_4.$$

First step. Set

$$B(v_i, \delta, \varepsilon)=\{x=s\phi+z_0(s, \lambda)+z: \ |s-s_i|<\delta, z\in Z_0, \ \|z\|<\varepsilon\}, \quad i=3, \ 4.$$

Let δ be such that $\frac{1}{2}(s_4-s_3)<\delta<\min\{d_3, s_4-s_3\}$. Then, by Proposition 3.13, there exists $\varepsilon_4>0$ such that if

$$x\in \mathcal{M}_-(v_4, \lambda)\cap B(v_4, \delta, \varepsilon_4),$$

then $u(t, x, \lambda)$ converges to v_4 as $t\to\infty$. On the other hand, by Proposition 3.16, there exists $\varepsilon_3>0$ such that if

$$x\in \mathcal{M}_+(v_3, \lambda)\cap B(v_3, \delta, \varepsilon_3),$$

then

$$\langle u(\tau, x, \lambda), \phi^*\rangle=s_3+\delta>s_4-\delta$$

holds for some $\tau>0$. Hence, if

$$x\in \mathcal{M}_+(v_3, \lambda)\cap \mathcal{M}_-(v_4, \lambda)\cap B(\varepsilon), \quad \varepsilon=\min\{\varepsilon_3, \varepsilon_4\},$$

then $u(t, x, \lambda)$ converges to v_4 as $t\to\infty$.

Second step. Let

$$x_0\in \mathcal{M}_+(v_3, \lambda)\cap \mathcal{M}_-(v_4, \lambda)\cap B(d), \quad d=\frac{1}{4}d_3K_1^{-1}.$$

Then we have by Lemma 3.20

$$u(t, x_0, \lambda)\in \mathcal{M}_+(v_3, \lambda)\cap \mathcal{M}_-(v_4, \lambda), \quad t\geq 0$$

and by (1.4)

$$\|z(t)\|<K_1e^{-c_0t}d, \ t\geq 0.$$

Hence there exists $\tau>0$ such that

$$\|z(\tau)\|<K_1e^{-c_0\tau}d<\varepsilon,$$

i.e.,

$$u(\tau, x_0, \lambda)\in B(\varepsilon).$$

Hence, by the first step, $u(t, x_0, \lambda)$ converges to v_4 as $t\to\infty$. [q.e.d.]

Note. After completing this manuscript, a book by J. Carr was published: "Applications of Centre Manifolds Theory", Applied Mathematical Sciences 35, Springer-Verlag (1981). The results presented in this book are closely related to ours. In particular the stability theorem is also given and proved by using the invariance of domain theorem.

Professor J. K. Hale kindly informed the author of the following two papers: J. C. deOliveira and J. K. Hale, "Dynamic behavior from bifurcation equations", Tôhoku Math. J. 32 (1980), 577-592;
J. K. Hale, "Stability from the bifurcation function", in Differential Equations, edited by S. Ahmad, M. Keener, and A. C. Lazer, Academic Press (1980), 23-30. The first paper concerns Hopf bifurcation of differential equations in R^n. They investigate the relation between the stability of zeros of the bifurcation function constructed by the Ljapunov-Schmidt method and that defined by using a center manifold. In the latter paper the stability theorem is also stated in a situation similar to ours. (See also recent lecture notes by J. K. Hale: "Topics in Dynamic Bifurcation Theory", C. B. M. S. No. 47, A. M. S. (1981).)

References

[1] Crandall, M. G. and P. H. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Anal. **8** (1971), 321-340.

[2] Crandall, M. G. and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues, and linearized stability, Arch. Rational Mech. Anal. **52** (1973), 161-180.

[3] Crandall, M. G. and P. H. Rabinowitz, The Hopf bifurcation theorem in infinite dimensions, Arch. Rational Mech. Anal. **67** (1977), 53-72.

[4] Henry, D., Geometric theory of semilinear parabolic equations, Lecture Notes in Math. No. 840, Springer-Verlag, 1981.

[5] Homes, P. J. and J. E. Marsden, Bifurcations of dynamical systems and nonlinear oscillations in engineering systems, Lecture Notes in Math. No. 648, Springer-Verlag, 1978, pp. 163-206.

[6] Iooss, G., Bifurcation et Stabilité, Cours de 3ième cycle 1972-1974, Publ. Math. d'Orsay, no. 31, 1974.

[7] Ito, M., The conditional stability of stationary solutions for semilinear parabolic differential equations, J. Fac. Sci. Univ. Tokyo Sect. IA **25** (1979), 263-275.

[8] Joseph, D. D. and D. A. Nield, Stability of bifurcating time periodic and steady solutions of arbitrary amplitude, Arch. Rational Mech. Anal. **58** (1975), 369-380.

[9] Kato, T., Perturbation theory for linear operators, Springer-Verlag, 2nd edition, 1976.

[10] Kielhöfer, H., Stability and semilinear evolution equations in Hilbert space, Arch. Rational Mech. Anal. **57** (1974), 150-165.

[11] Kielhöfer, H., On the Lyapounov-stability of stationary solutions of semilinear parabolic differential equations, J. Differential Equations **22** (1976), 193-208.

[12] Kirchgässner, K. and H. Kielhöfer, Stability and bifurcation in fluid dynamics, Rocky Mountain J. Math. **3** (1973), 275-318.

[13] Krasnosel'skii, M. A., Zabreiko, P. P., Pustylnik, E. I. and P. E. Sbolevskii, Integral operators in spaces of summable functions, Noordhoff International Publishing, 1976.

[14] Marsden, J. E. and M. McCracken, The Hopf Bifurcation and its Applications, Applied Mathematical Sciences 19, Springer-Verlag, 1976.

[15] Masuda, K. and T. Itoh, On the center manifold theorem, to appear.

[16] Ruelle, D., Bifurcations in the presence of a symmetry group, Arch. Rational Mech. Anal. **51** (1973), 136-152.
[17] Ruelle, D. and F. Takens, On the nature of turbulence, Comm. Math. Phys. **20** (1971), 167-192.
[18] Sattinger, D. H., The mathematical problem of hydrodynamic stability, J. Math. Mech. **9** (1970), 797-817.
[19] Weinberger, H., On the stability of bifurcating solutions, in "Nonlinear Analysis", Academic Press, New York, 1978, pp. 219-233.

(Received February 2, 1981)

Department of Pure and Applied Sciences
College of General Education
University of Tokyo
Komaba, Tokyo
153 Japan