Remarks on continuation problems of Calabi's diastatic functions

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Introduction

In 1953 E. Calabi discussed a holomorphic isometric imbedding of a kähler manifold in a complex euclidian space ([1]). He introduced a concept of a "diastatic function" for a real analytic kähler metric and by using a resolvability condition on a diastatic function, he gave a necessary and sufficient condition for existence of a holomorphic isometric imbedding. At the end of Chapter 2 in [1], he states that it is an interesting theme to formulate continuation problems of diastatic functions. It seems to the author that these problems have not considered till now.

In this paper we shall formulate continuation problems and discuss them by constructing some examples. We restrict our consideration to a simply connected kähler manifold with a real analytic kähler metric.

In [1], it is proved that a diastatic function $D(p, p_0)$ with respect to a point p_0 can be always defined on some neighborhood of p_0. $D(p, p_0)$ is equal to a square of a distance function $d(p, p_0)$ modulo small order. But $D(p, p_0)$ is not always continued to the whole manifold. If it can be continued to the whole manifold, it is an strongly pseudoconvex function, because its hessian is the original kähler form. Hence continuation problems will be very interesting in complex geometry. Here we remark that a diastatic function on a compact manifold can not be continued to the whole manifold. For example, we take a complex projective space P^n with the canonical Fubini-Study metric. Then a diastatic function can be continued to a domain which is biholomorphic to C^n in P^n. Hence we see that a diastatic function can be continued to the total manifold minus an analytic set in this case. When the paper [1] appeared, it seems that only kähler manifolds were known such that diastatic functions can be continued except (at most) an analytic set of codimension 1. Therefore we can formulate the following problem:

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Problem 1. Can a diastatic function be continued to the whole manifold except (at most) some analytic sets?

In §1 we shall give an example of a kähler metric on C whose diastatic function with respect to the origin can be continued only in some disk (see Theorem 1).

A sufficient condition for the continuation problem is given in [1]: If a diastatic function satisfies a resolvability condition at some point, then $D(p, p_0)$ can be continued everywhere for every p_0 (Corollary to Theorem 5 in [1]). It is easily seen that if a kähler metric satisfies the above condition, the holomorphic bisectional curvature is non-positive. Referring to the positivity of holomorphic bisectional curvature of P^n, we can formulate the following problem:

Problem 2. If the holomorphic bisectional curvature of a kähler metric is non-positive, can a diastatic function be continued everywhere?

This problem has a deep relationship with Greene-Wu Conjecture for a complete kähler manifold of non-positive curvature ([2]). We shall give an example of a complete kähler metric on a disk of negative sectional curvature whose diastatic function at the origin can not be continued to the total space. Hence we may say that Calabi diastatics will not contribute to solve Greene-Wu Conjecture. At the end of §1 we see that a continuation problem has no relationships with the sign of curvature.

§1. On Problems 1 and 2.

We set the following metric on C:

$$ds^2 = h_1 dz d\bar{z}, \quad h_1 = \frac{2}{(|\cos z|^2 + |\sin z|^2)^2}. \tag{1.1}$$

This is a metric of positive constant curvature $K=1$, where the curvature is defined by

$$K = -\frac{1}{h_1} \frac{\partial^2 \log h_1}{\partial z \partial \bar{z}}.$$

Hence this is not a complete metric. It is easily seen that the diastatic function at the origin is given by

$$D_1(z, 0) = 2 \log (1 + |\tan z|^2). \tag{1.2}$$

Hence $D_1(z_0, 0) = +\infty$, if $\tan z_0 = \pm\infty$. For example $D_1(\pi/2, 0) = \infty$. In this case the diastatic function at 0 can not be continued to the whole $\boldsymbol{C}$.

By using this example, we can construct a kähler metric on $\boldsymbol{C}$ whose diastatic function at the origin can not be continued beyond a certain disk $\boldsymbol{D}_c$ in $\boldsymbol{C}$. Let $\{\alpha_n\}$ be a dense set in $\partial\boldsymbol{D}_c=\{|z|=c\}$, where $c=\pi/2$. We write

$$\alpha_n=ce^{i\theta_n} \quad (\theta_n\in\boldsymbol{R}).$$

We write the rotation operation of angle θ_n by $\gamma_n: z\to ze^{i\theta_n}$. We set $ds_n^2=\gamma_n^*(ds_C^2)$. Then we see that

$$ds_n^2=h_n dz d\bar{z}, \quad h_n=\frac{2}{(|\cos(\gamma_n z)|^2+|\sin(\gamma_n z)|^2)^2}.$$

We see easily that the diastatic function Φ_n with respect to the origin is $\gamma_n^* D_1(z, 0)$ and can not be continued at α_n. We choose a sequence of positive numbers $\{\varepsilon_n\}_{n=1}^{\infty}$ with $\sum \varepsilon_n<+\infty$. We set

$$ds^2=\sum \varepsilon_n ds_n^2. \tag{1.3}$$

From $h_n\leqq 1$, we see that ds^2 gives a kähler metric on $\boldsymbol{C}$. If we choose ε_n smaller, if necessary, then

$$\Phi=\sum \varepsilon_n \Phi_n$$

convergents uniformly on any compact set in $\boldsymbol{D}_c$. Hence Φ gives a diastatic function with respect to the origin. By construction, we find that $\Phi(\alpha_n)=+\infty$ for every α_n. Hence Φ can not be continued across the boundary of $\boldsymbol{D}_c$. Hence we have proved

THEOREM 1. *There exists a kähler metric on $\boldsymbol{C}$ whose diastatic function with respect to the origin can not be continued across the boundary of a disk in $\boldsymbol{C}$.*

REMARK. The metric constructed above is not a complete metric. If we set

$$ds^2=ds_1^2+ds_R^2 \quad (R>\pi/2),$$

where ds_1^2 is given in (1.3) and ds_R^2 is a Poincaré metric on a disk $\boldsymbol{D}_R$ of radius R. Then we can obtain a complete metric on $\boldsymbol{D}_R$ with the property as stated in Theorem 1.

We write ds_R^2 as follows:

$$ds_R^2=h_2 dz d\bar{z}, \quad h_2=R^2/A(R^2-|z|^2). \tag{1.4}$$

This is a complete kähler metric on $\boldsymbol{D}_R$ of constant curvature $K=-A$. The diastatic function at the origin is given by

$$D_2(z, 0)=-\frac{1}{A}\log(1-|z|^2/R^2). \tag{1.5}$$

Hence this can be continued on $\boldsymbol{D}_R$.

Now we proceed to consider Problem 2. By using (1.1) and (1.4), we set

$$ds^2 = h dz d\bar{z}, \quad h = \varepsilon h_1 + h_2 \text{ on } \boldsymbol{D}_R, \tag{1.6}$$

where ε is a positive constant. Then the diastatic function with respect to the origin is given by

$$\Phi = \varepsilon D_1(z, 0) + D_2(z, 0),$$

where $D_1(z, 0)$ and $D_2(z, 0)$ are given in (1.2) and (1.5) respectively. Hence Φ will be $+\infty$ at $\pi/2$. From simple calculation of the curvature K of (1.6), we find that

$$K = K_1 + \varepsilon\alpha,$$

where K_1 is the curvature of the metric (1.4) and α is a bounded function on $\boldsymbol{D}_R$, i.e., $|\alpha| \leqq M_0$ holds on $\boldsymbol{D}_R$ with some constant M_0. Therefore, choosing ε smaller, we see that the curvature K is negative in $\boldsymbol{D}_R$. Hence we have proved

THEOREM 2. *There exists a complete kähler metric on $\boldsymbol{D}_R$ of strictly negative curvature (i.e., $K \leqq -c < 0$) whose diastatic function with respect to the origin can not be continued to the whole $\boldsymbol{D}_R$.*

We finish our paper by adding one more example:

Example 3. We consider the following metric

$$ds^2 = \frac{dz d\bar{z}}{(1 + |z|^2)} \quad \text{on } \boldsymbol{C}.$$

This is a complete kähler metric on $\boldsymbol{C}$ of positive curvature. The diastatic function $D_3(z, 0)$ at the origin is given by

$$D_3(z, 0) = 2\int_0^r \frac{\log(1+t^2)}{t} dt \quad (r = |z|).$$

Hence the diastatic function can be continued to $\boldsymbol{C}$. Therefore from the examples above given, we see that the sign of curvature has no relationship with our continuation problem.

References

[1] Calabi, E., Isometric imbedding of complex manifolds, Ann. of Math. 58 (1953), 1-23.

[2] Greene, R. E. and H. Wu, Analysis on non-compact kähler manifolds in several complex variables, Proc. Symp. Pure Math. vol. 30, Part 2, Amer. Math. Soc., Providence, R.I., 1977, 69-100.

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