Dirichlet forms and diffusion processes on Banach spaces

By Shigeo KUSUOKA

Introduction

Dirichlet forms and diffusion processes on infinite dimensional vector spaces have been studied by S. Albeverio and R. Høegh-Krohn [1], [2] in order to give a new approach to the constructive field theory. In [1] they gave some conditions for the closability of the form, and in [2] they constructed an associated diffusion process on a Hilbert space.

Our purpose in this paper is to extend their works with a new presentation of some part of their argument. In particular we introduce more directly an infinite dimensional analogue of the classical Sobolev space of order one.

Let us shortly summarize the content of our paper. Let B be a Banach space, H be a Hilbert space densely and continuously contained in B, and μ be a quasi-invariant probability measure on B.

In Section 1, we give some definitions on the regularity of a measurable function on B, say u, and introduce the notion of the gradient Du of u, a measurable map from B into H. Furthermore we prove some basic lemmas concerning the regularity for later use.

In Section 2, we introduce the form $\mathscr{E}$ on $L^2(B; d\mu)$ defined by

$$\mathscr{E}(u, v)=\int_B (A(z)Du(z), Du(z))_H\mu(dz),$$

where $A(z)$ is a strictly positive definite symmetric bounded operator on H for μ-a.e. $z \in B$. By virtue of a lemma in Section 1 we see that this form $\mathscr{E}$ is a Dirichlet form (a Markovian closed form) under the assumption that μ is strictly positive. The notion of the strict positivity of μ was introduced by S. Albeverio and R. Høegh-Krohn [1]. When $A(z)$ is the identity on H, this form can be regarded as a natural infinite dimensional analogue of the classical Sobolev space of order one. S. Albeverio and R. Høegh-Krohn studied the case that $A(z)$ is the identity on H, but they did not assume the strict positivity of μ, instead they imposed several regularity assumptions on μ to ensure the closability of the form $\mathscr{E}$.

In Section 3, we construct a diffusion process on B associated with the

Dirichlet form $\mathscr{E}$ under some additional conditions by using those methods by M. Fukushima [3] and M. L. Silverstein [9].

Our strategy is to realize the form $\mathscr{E}$ as a regular Dirichlet form on $L^2(M; d\mu)$, M being a certain locally compact separable metric space in which B is embedded. We then show that the difference $M\backslash B$ is of zero capacity. Thus the associated diffusion process lives on the original space B. We also show the continuity of the sample paths in the topology of B.

In Section 4, we show that the so-called abstract Wiener space (μ, H, B) fits our setting, and that the Ornstein-Uhlenbeck process is a typical example of our process.

We hope that the present Dirichlet form and diffusion process would serve as a useful basis to carry out Malliavin's program [6]. Two slightly different approaches to Malliavin's calculus have already appeared (I. Shigekawa [8], D. Stroock [10]).

We are grateful to Professor M. Fukushima for his hearty advice and valuable conversation.

Notation.

For any Banach spaces E and F,
E^* denotes the dual Banach space of E,
I_E denotes the identity on E,
$\mathscr{L}^\infty(E, F)$ denotes the Banach space consisting of all bounded linear operators from E into F with the operator norm, and
$\mathscr{B}(E)$ denotes the topological Borel field of E.

1. Basic lemmas on SGD and RAC functions

Let B be a separable real Banach space, H be a separable real Hilbert space densely and continuously embedded in B, and K be a dense vector subspace of H. We consider B a measurable space with a topological Borel field $\mathscr{B}(B)$. We identify H^* with H, and so that B^* is included in H densely and continuously.

Now let μ be a K quasi-invariant probability measure on B, i.e. for any $k \in K$, $\mu(\cdot+k)$ and $\mu(\cdot)$ are absolutely continuous relative to each other. For any $k \in K$, $k \neq 0$, we define a measure σ_k on B by

$$\sigma_k(E)=\int_R ds \int_B \chi_E(z+sk)\mu(dz) \quad \text{for each } E \in \mathscr{B}(B).$$

σ_k is then a Radon measure on B.

It is easy to see the following.

PROPOSITION 1.1. *For each $k\in K$, $k\neq 0$,*

(1) *σ_k and μ are absolutely continuous relative to each other, and*

(2) $$\int_B F(z)\mu(dz)=\int_{B\times R} F(z+sk)\frac{d\mu}{d\sigma_k}(z+sk)\mu(dz)\otimes ds$$

for any non-negative measurable function F defined on B.

We introduce two notions on the regularity of measurable functions on B.

DEFINITION 1.1. We say that a measurable function u defined on B is *stochastic H Gateaux differentiable with respect to μ* (abbreviated by *SGD*), if there exists a measurable map $Du: B\rightarrow H$ such that for any $k\in K$, the convergence

$$\frac{1}{t}[u(z+tk)-u(z)-t(Du(z),k)_H]\longrightarrow 0,\quad t\longrightarrow 0,$$

takes place in probability with respect to μ. We call Du a *stochastic H Gateaux derivative* (abbreviated by *sgd*).

Obviously $Du(z)$ is uniquely determined for μ-a.e.$z\in B$. Furthermore $u(z)=v(z)$ for μ-a.e.$z\in B$ implies $Du(z)=Dv(z)$ μ-a.e.z in view of the K quasi-invariance of μ.

DEFINITION 1.2. We say that a measurable function u defined on B is *ray absolutely continuous with respect to μ* (abbreviated by *RAC*), if for any $k\in K$, there exists a measurable function $\tilde{u}_k$ defined on B such that

(1) $\tilde{u}_k(z)=u(z)$ for μ-a.e.$z\in B$, and

(2) $\tilde{u}_k(z+tk)$ is absolutely continuous in t for each $z\in B$.

The following notion was first introduced by S. Albeverio and R. Høegh-Krohn [1].

DEFINITION 1.3. We say that μ is *strictly positive*, if for any $k\in K$, $k\neq 0$, there exists a measurable function f_k defined on B such that

(1) $\mu(dz)=f_k(z)\sigma_k(dz)$, and

(2) ess.inf$\{f_k(z+tk);\ -T<t<T\}>0$ for any $z\in B$ and $T>0$.

The following lemma plays a key role in the next section.

LEMMA 1.1. *We assume that μ is strictly positive. Let u_n's $(n=1,2,\cdots)$ be measurable functions defined on B with SGD and RAC properties, u be a measurable function defined on B and F be a measurable map from B into H. Suppose that the following conditions are satisfied:*

(1) $u_n(z)\to u(z)$, $n\to\infty$, *in probability with respect to* μ.

(2) *For any* $k\in K$,

$$\int_B |(F(z), k)_H-(Du_n(z), k)_H|\mu(dz)\longrightarrow 0, \quad n\longrightarrow\infty.$$

Then u *is SGD and RAC, and* $Du(z)=F(z)$ *for* μ-a.e.$z\in B$. *Here* Du_n*'s and* Du *are sgd's of* u_n*'s and* u *respectively.*

PROOF. Fix an arbitrary $k\in K$ and take a function $\tilde{u}_{n,k}$ for u_n as in Definition 1.2. Since $\tilde{u}_{n,k}(z+sk)$ is absolutely continuous in s for each $z\in B$,

$$\frac{1}{t}[\tilde{u}_{n,k}(z+sk+tk)-\tilde{u}_{n,k}(z+sk)]\longrightarrow\frac{d}{ds}\tilde{u}_{n,k}(z+sk), \quad t\longrightarrow 0, \tag{1.1}$$

for a.e.s. (1.1) then holds for a.e.$(z, s)\in B\times\boldsymbol{R}$ with respect to $\mu(dz)\otimes ds$.

On the other hand

$$\frac{1}{t}\{u_{n,k}(z+tk)-u_{n,k}(z)\}\longrightarrow(Du_n(z), k)_H, \quad t\longrightarrow 0, \tag{1.2}$$

in probability with respect to μ. On account of Proposition 1.1 we can conclude from (1.1) and (1.2) that

$$\frac{d}{ds}u_{n,k}(z+sk)=(Du_n(z+sk), k)_H \quad \text{for a.e.}(z, s)\in B\times\boldsymbol{R} \tag{1.3}$$

with respect to $\mu(dz)\otimes ds$. Combining this with our assumption (2) and Proposition 1.1, we get

$$\int_B\mu(dz)\int_R\left|(F(z+sk), k)_H-\frac{d}{ds}\tilde{u}_{n,k}(z+sk)\right|f_k(z+sk)ds\longrightarrow 0 \tag{1.4}$$

as $n\to\infty$, where f_k is the function in Definition 1.3.

Notice that $\tilde{u}_{n,k}(z)\to u(z)$ as $n\to\infty$ in probability with respect to μ. So taking a subsequence if necessary, we may assume that there exists $\Omega_1\in\mathscr{B}(B)$ such that

(1.5) $\mu(\Omega_1)=1$,

(1.6) for any $z\in\Omega_1$,

$$\int_R\left|(F(z+sk), k)_H-\frac{d}{ds}\tilde{u}_{n,k}(z+sk)\right|f_k(z+sk)ds\longrightarrow 0 \text{ as } n\longrightarrow\infty, \text{ and}$$

(1.7) $\tilde{u}_{n,k}(z)\to u(z)$, $n\to\infty$, for each $z\in\Omega_1$.

It follows from (1.6) and the strict positivity of μ that for any real numbers $a<b$ and $z\in\Omega_1$,

$$\int_a^b \left| (F(z+sk),\, k)_H - \frac{d}{ds}\tilde{u}_{n,k}(z+sk) \right| ds \longrightarrow 0, \quad n \longrightarrow \infty.$$

(1.7) then implies that

$$\tilde{u}_{n,k}(z+tk) \longrightarrow u(z) + \int_0^t (F(z+sk),\, k)_H ds, \quad n \longrightarrow \infty, \tag{1.8}$$

for any $z \in \Omega_1$ and $t \in \boldsymbol{R}$.

Since μ is a Radon measure on B, there exists a σ-compact subset Ω_2 of B such that $\Omega_2 \subset \Omega_1$ and $\mu(\Omega_2)=1$. Let $\Omega_0 = \Omega_2 + \boldsymbol{R}k = \{z+tk;\, z \in \Omega_2,\, t \in \boldsymbol{R}\}$, then Ω_0 is σ-compact and $\mu(\Omega_0)=1$. Moreover (1.8) shows that $\{\tilde{u}_{n,k}(z)\}$ is convergent as $n \to \infty$ for each $z \in \Omega_0$. Let us define

$$\tilde{u}_k(z) = \begin{cases} \lim_{n\to\infty} \tilde{u}_{n,k}(z) & \text{if } z \in \Omega_0 \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that $\tilde{u}_k$ is a measurable function on B, $\tilde{u}_k(z)=u(z)$ for μ-a.e. $z \in B$ and $\tilde{u}_k(z+tk)$ is absolutely continuous in t for each $z \in B$. Thus u is *RAC*.

(1.8) also shows that

$$\frac{1}{t}\{\tilde{u}_k(z+sk+tk) - \tilde{u}_k(z+sk)\} \longrightarrow (F(z+sk),\, k)_H, \quad t \longrightarrow 0,$$

for a.e. $(z, s) \in B \times \boldsymbol{R}$ with respect to $\mu(dz) \otimes ds$. By virtue of Proposition 1.1, we obtain

$$\frac{1}{t}\{\tilde{u}_k(z+tk) - \tilde{u}_k(z)\} \longrightarrow (F(z),\, k)_H, \quad t \longrightarrow 0, \text{ for } \mu\text{-a.e. } z \in B,$$

which implies that u is *SGD* and $Du(z)=F(z)$ for μ-a.e. $z \in B$.

This completes the proof.

LEMMA 1.2. *Let u be a measurable function defined on B with SGD and RAC properties and g be a continuous differentiable function defined on $\boldsymbol{R}$. Then $v = g \circ u$ is SGD and RAC, and*

$$Dv(z) = g'(u(z))Du(z) \quad \text{for } \mu\text{-a.e. } z \in B.$$

Here g' is the derivative of g.

PROOF. For each $k \in K$, $k \neq 0$, let $\tilde{u}_k$ be as in Definition 1.2 for u. Then similarly to (1.3), we get

$$\frac{d}{ds}\tilde{u}_k(z+sk) = (Du(z+sk),\, k)_H \quad \text{for a.e. } (z, s) \in B \times \boldsymbol{R}$$

with respect to $\mu(dz) \otimes ds$. Let $\tilde{v}_k(z) = g(\tilde{u}_k(z))$, $z \in B$, then

$$\frac{d}{ds}\tilde{v}_k(z+sk)=(g'(\tilde{u}_k(z+sk))Du(z+sk), k)_H \quad \text{for a.e.}(z, s) \in B\times \boldsymbol{R}$$

with respect to $\mu(dz)\otimes ds$.

Now Lemma 1.2 follows easily.

The following lemma is also useful.

LEMMA 1.3. *Let u be a measurable function defined on B. Assume that there exists a positive constant C such that $|u(z+h)-u(z)|\leqq C\|h\|_H$ for any $z\in B$ and $h\in H$. Then,*

(1) *u is SGD and RAC, and*

(2) *there exists a measurable map Du: $B\to H$ and $\Omega_0\in\mathscr{B}(B)$ such that*

  (i) $\mu(\Omega_0)=1$,

  (ii) $\|Du(z)\|_H\leqq C$ *for any* $z\in B$, *and*

  (iii) $\lim_{t\to 0}\frac{1}{t}(u(z+th)-u(z))=(Du(z), h)_H$ *for any* $z\in\Omega_0$ *and* $h\in H$.

PROOF. It is obvious that u is RAC. So it suffices to prove (2). First, notice that we may choose a countable subset V of K such that V is Q-module and dense in H, where Q is a set of all rational numbers. For each $v\in V$, let us define

$$D_v=\left\{z\in B; \lim_{\substack{r\to 0\\ r\in Q}}\frac{1}{r}[u(z+rv)-u(z)] \text{ exists}\right\},$$

then D_v is a measurable set of B.

Since $u(z+sv)$ is absolutely continuous in s for each $z\in B$, $\lim_{\substack{r\to 0\\ r\in Q}}(1/r)[u(z+sv+rv)-u(z+sv)]$ exists for a.e.$(z, s)\in B\times\boldsymbol{R}$ with respect to $\mu(dz)\otimes ds$. Therefore by virtue of Proposition 1.1, we see that $\mu(D_v)=1$ for each $v\in V$. Moreover the continuity of $u(z+tv)$ in t implies that $\lim_{t\to 0}(1/t)[u(z+tv)-u(z)]$ exists for any $z\in D_v$. Let $G(z; v)=\lim_{t\to 0}(1/t)[u(z+tv)-u(z)]$ for each $v\in V$ and $z\in D_v$, then $G(\cdot; v)$ is a measurable function defined in D_v.

We claim the following (1.9) and (1.10):

(1.9) $G(z; rv)=rG(z; v)$ for each $z\in D_v=D_{rv}$ and $r\in Q$.

(1.10) Let $D_{v,w}=\{z\in D_v\cap D_w\cap D_{v+w}; G(z; v)+G(z; w)=G(z; v+w)\}$ for each $v, w\in V$, then $\mu(D_{v,w})=1$.

(1.9) is obvious, and so is (1.10) whenever v and w are linearly dependent over $\boldsymbol{R}$. Let us show (1.10) when v and w are linearly independent over $\boldsymbol{R}$. Define a measure σ on B by

$$\sigma(E)=\int_{R^2}dxdy\int_B \chi_E(z+xv+yw)\mu(dz) \quad \text{for each } E\in\mathscr{B}(B),$$

then, as we readily see, μ and σ are mutually absolutely continuous. Since $u(z+xv+yw)$ is Lipschitz continuous in (x, y) for each $z\in B$, $u(z+xv+yw)$ is totally differentiable in (x, y) for a.e.$(x, y)\in \boldsymbol{R}^2$ with respect to $dxdy$ by virtue of a theorem in H. Radmacher [7]. This implies $z+xv+yw\in D_{v,w}$ for a.e.$(z, (x, y))\in B\times\boldsymbol{R}^2$ with respect to $\mu(dz)\otimes dxdy$. Hence we get $\sigma(B\backslash D_{v,w})=0$, which proves (1.10).

Now we can prove the assertion (2) of our lemma. Let $\Omega_0=\cap\{D_{v,w}; v, w\in V\}$, then $\mu(\Omega_0)=1$. For each $z\in\Omega_0$, $G(z; \cdot)$ is a Q-linear function defined in V by virtue of (1.9) and (1.10) and furthermore

$$|G(z; v)|=\lim_{t\to 0}\frac{1}{t}|u(z+tv)-u(z)|\leqq C\|v\|_H \quad \text{for each } v\in V. \tag{1.11}$$

So we may extend $G(z; \cdot)$ to a bounded linear functional on H. Thus there exists for each $z\in\Omega_0$ an element $Du(z)$ of H such that $(Du(z), v)_H=G(z; v)$ for each $v\in V$ and $\|Du(z)\|_H\leqq C$.

Let us put $Du(z)=0$ for $z\notin\Omega_0$. Since V is dense in H and $G(\cdot; v)$ is a measurable function on Ω_0 for each $v\in V$, $Du: B\to H$ is measurable. Moreover, for each $z\in\Omega_0$,

$$\begin{aligned}
&\left|\frac{1}{t}(u(z+th)-u(z))-(Du(z), h)_H\right| \\
&\leqq\left|\frac{1}{t}(u(z+th)-u(z+tv))\right|+|(Du(z), h-v)_H| \\
&\quad+\left|\frac{1}{t}(u(z+tv)-u(z))-(Du(z), v)_H\right| \\
&\leqq 2C\|h-v\|_H+\left|\frac{1}{t}(u(z+tv)-u(z))-G(z; v)\right|
\end{aligned}$$

for any $h\in H$ and $v\in V$, which completes the proof.

2. Dirichlet forms on $L^2(B; d\mu)$

Let A be a strongly measurable map from B into $\mathscr{L}^\infty(H, H)$ such that $A(z): H\to H$ is non-negative definite symmetric operator for each $z\in B$.

DEFINITION 2.1. We define a subset $\mathscr{D}(\tilde{\mathscr{E}})$ of $L^2(B; d\mu)$ by

$$\mathscr{D}(\tilde{\mathscr{E}})=\Big\{u\in L^2(B;d\mu);\ u \text{ is } SGD \text{ and } RAC, \text{ and } \int_B (A(z)Du(z), Du(z))_H\mu(dz)<\infty\Big\}$$

and a symmetric bilinear form $\tilde{\mathscr{E}}$ on $\mathscr{D}(\tilde{\mathscr{E}})\times\mathscr{D}(\tilde{\mathscr{E}})$ by

$$\tilde{\mathscr{E}}(u, v)=\int_B (A(z)Du(z), Dv(z))_H\mu(dz) \quad \text{for any } u, v\in\mathscr{D}(\tilde{\mathscr{E}}).$$

We also define a symmetric bilinear form $\tilde{\mathscr{E}}_1$ by

$$\tilde{\mathscr{E}}_1(u, v)=(u, v)_{L^2(B;d\mu)}+\tilde{\mathscr{E}}(u, v), \quad u, v\in\mathscr{D}(\tilde{\mathscr{E}}).$$

By virtue of Lemma 1.2, it is easy to see that $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ is Markovian, i.e. for each $\varepsilon>0$, there exists a real function $\phi_\varepsilon(t)$, $t\in\boldsymbol{R}$, such that $\phi_\varepsilon(t)=t$ for $0<t<1$, $-\varepsilon\leqq\phi_\varepsilon(t)\leqq 1+\varepsilon$ for any $t\in\boldsymbol{R}$, $0\leqq\phi_\varepsilon(s)-\phi_\varepsilon(t)\leqq s-t$ whenever $t<s$, and for any $u\in\mathscr{D}(\tilde{\mathscr{E}})$, we have that $\phi_\varepsilon(u)\in\mathscr{D}(\tilde{\mathscr{E}})$ and $\tilde{\mathscr{E}}(\phi_\varepsilon(u), \phi_\varepsilon(u))\leqq\tilde{\mathscr{E}}(u, u)$.

We shall frequently impose the following conditions (A-1) and (A-2) on A: $B\to\mathscr{L}^\infty(H, H)$.

(A-1) There exists a positive constant c_0 such that $A(z)-c_0I_H$ is a positive definite symmetric operator for any $z\in B$.

(A-2) $\int_B \|A(z)\|_{\mathscr{L}^\infty(H,H)}\mu(dz)<\infty$.

The condition (A-1) is an infinite dimensional analogue to the uniform ellipticity.

THEOREM 1. *Suppose that μ is strictly positive and* (A-1) *is satisfied. Then* $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ is a Dirichlet form on $L^2(B;d\mu)$, i.e. $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$ is a closed Markovian symmetric bilinear form.

PROOF. It suffices to prove that $\mathscr{D}(\tilde{\mathscr{E}})$ is a Hilbert space with inner product $\tilde{\mathscr{E}}_1$. Suppose that $u_n\in\mathscr{D}(\tilde{\mathscr{E}})$, $n=1, 2, \cdots$ and $\tilde{\mathscr{E}}_1(u_n-u_m, u_n-u_m)\to 0$ as $n, m\to\infty$. Then we have

$$\int_B |u_n(z)-u_m(z)|^2\mu(dz)\longrightarrow 0 \quad \text{and}$$

$$c_0\int_B \|Du_n(z)-Du_m(z)\|_H^2\mu(dz)\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.$$

Since $L^2(B;d\mu)$ and $L^2(B\to H;d\mu)$ are Hilbert spaces, there exist a measurable function u on B and a measurable map F: $B\to H$ such that

$$\int_B |u_n(z)-u(z)|^2\mu(dz)\longrightarrow 0 \quad \text{and}$$

$$\int_B \|Du_n(z)-F(z)\|_H^2 \mu(dz) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

By virtue of Lemma 1, u is *SGD* and *RAC*, and $Du(z)=F(z)$ for μ-a.e.$z \in B$. Taking a subsequence if necessary, we may assume that $u_n(z) \to u(z)$ and $\|Du_n(z)-Du(z)\|_H \to 0$, $n \to \infty$, for μ-a.e.$z \in B$. Then by Fatou's lemma,

$$\int_B (A(z)Du(z), Du(z))_H \mu(dz) \leqq \varliminf_{n \to \infty} \int_B (A(z)Du_n(z), Du_n(z))_H \mu(dz) < \infty,$$

which implies $u \in \mathscr{D}(\tilde{\mathscr{E}})$. Similarly

$$\tilde{\mathscr{E}}_1(u-u_n, u-u_n) \leqq \varliminf_{m \to \infty} \tilde{\mathscr{E}}_1(u_m-u_n, u_m-u_n) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

which completes the proof.

The next proposition is an immediate consequence of Lemma 1.3.

PROPOSITION 2.1. *Assume that u is a bounded measurable function on B and there exists a positive constant C such that $|u(z+h)-u(z)| \leqq C\|h\|_H$ for any $z \in B$ and $h \in H$. Then under the condition* (A-2), *u belongs to $\mathscr{D}(\tilde{\mathscr{E}})$.*

We say that a function u defined on B *belongs to $H-C^1$ class* if (1) $u: B \to \boldsymbol{R}$ is continuous, and

(2) there exists a continuous map $Du: B \to H$ such that

$$|u(z+h)-u(z)-(Du(z), h)_H| = o(\|h\|_H) \quad \text{as } \|h\|_H \longrightarrow 0 \quad \text{for each } z \in B.$$

Let $\mathscr{F}_0 = \{u; u$ is a bounded function on B belonging to $H-C^1$ class and $\|Du(\cdot)\|_H$ is bounded on $B\}$.

It follows from Proposition 2.1 that $\mathscr{F}_0 \subset \mathscr{D}(\tilde{\mathscr{E}})$ under the condition (A-2), and it is easy to see that $\phi(u(\cdot)) \in \mathscr{F}_0$ for any $u \in \mathscr{F}_0$ and any bounded smooth function ϕ defined on $\boldsymbol{R}$ with a bounded derivative. Thus we see that $(\tilde{\mathscr{E}}|_{\mathscr{F}_0 \times \mathscr{F}_0}, \mathscr{F}_0)$ is a Markovian symmetric bilinear form on $L^2(B; d\mu)$ under the condition (A-2). Let $\mathscr{F}$ be the closure of $\mathscr{F}_0$ in $\mathscr{D}(\tilde{\mathscr{E}})$ with the norm $\tilde{\mathscr{E}}_1$, and $\mathscr{E}$ be the restriction of $\tilde{\mathscr{E}}$ to $\mathscr{F} \times \mathscr{F}$. By virtue of M. Fukushima [3] Theorem 2.1.1, we get the following.

COROLLARY TO THEOREM 1. *Assume that μ is strictly positive and that* (A-1) *and* (A-2) *are satisfied. Then $(\mathscr{E}, \mathscr{F})$ is a Dirichlet form on $L^2(B; d\mu)$.*

3. A diffusion process on B associated with the Dirichlet form $(\mathscr{E}, \mathscr{F})$

It is well known that there exists a non-positive definite self-adjoint operator

A on $L^2(B; d\mu)$ such that
(1) $\mathscr{F}$ is the domain of $\sqrt{-A}$, and
(2) $\mathscr{E}(u, v)=(\sqrt{-A}u, \sqrt{-A}v)_{L^2(B,d\mu)}$, $u, v \in \mathscr{F}$.
Let $T_t=e^{tA}$, $t\geqq 0$, then $\{T_t\}_{t\geqq 0}$ is a symmetric contraction semi-group on $L^2(B, d\mu)$.

We say that a diffusion process on B, $\{B, x_t, P_z\}$ *is associated with* $(\mathscr{E}, \mathscr{F})$, if for any bounded measurable function f on B and $t\geqq 0$,

$$E_z[f(x_t)]=T_t f(z) \quad \text{for } \mu\text{-a.e.} z \in B.$$

In order to construct an associated diffusion process, we need some additional conditions. Let $\mathscr{P}=\{P: H\to H;\ P$ is an orthogonal projection with finite dimensional range included in $B^*\}$. Then for any $P\in\mathscr{P}$, $Ph=\sum_{n=1}^{m} {}_{B^*}\langle e_n, h\rangle_B e_n$ for each $h\in H$, where $\{e_1, \cdots, e_m\}$ is an orthonormal base of Image P. So $P: H\to$Image P is extensible to a bounded linear operator from B into Image P, which is denoted by $\tilde{P}$.

Our additional conditions are as follows:
(C-1) The inclusion map from H to B is compact.
(C-2) There exists a sequence $\{P_n\}_{n=1}^{\infty}\subset\mathscr{P}$ such that

(1) Image $P_1\subset$Image $P_2\subset\cdots\subset$Image $P_n\subset\cdots$ and P_n converges to I_H strongly as $n\to\infty$,

(2) $\|z-\tilde{P}_n z\|_B\to 0$, $n\to\infty$, in probability with respect to μ, and

(3) there exists a positive constant c such that

$$c\|z\|_B\leqq\sup\{{}_{B^*}\langle v, z\rangle_B;\ v\in V,\ \|v\|_{B^*}=1\}$$

for any $z\in B$, where $V=\cup\{\text{Image } P_n;\ n=1, 2, \cdots\}$.

THEOREM 2. *Suppose that μ is strictly positive and that conditions* (A-1), (A-2), (C-1) *and* (C-2) *are satisfied. Then there exists a diffusion process on B associated with* $(\mathscr{E}, \mathscr{F})$.

REMARK 3.1. The strictly positivity of μ and the condition (A-1) in the statement of Theorem 2 are only necessary to assure the closedness of the form $\mathscr{E}$. Theorem 2 includes the cases treated by S. Albeverio and R. Høegh-Krohn [2].

To prove Theorem 2, we prepare a series of notations and propositions.

Taking a subsequence if necessary, we may assume by the assumptions (C-1) and (C-2) that

$$\|I_H-\tilde{P}_n\|_{\mathscr{L}^\infty(H,B)}\leqq 2^{-n} \quad\text{and}\quad \mu(\{z\in B;\ \|z-\tilde{P}_n z\|_B\geqq 2^{-n}\})\leqq 2^{-n}.$$

We define $q(z)=\left[\sum_{n=0}^{\infty} 2^n\|\tilde{P}_{n+1}z-\tilde{P}_n z\|_B^2\right]^{1/2}$ for each $z\in B$ with the convention that

$\widetilde{P}_0=0$, and $B_0=\{z\in B;\ q(z)<\infty\}$. Then $\mu(B_0)=1$ by Borel Cantelli's lemma. We regard B_0 as a normed space with norm $q(\cdot)$.

PROPOSITION 3.1. *$q(h)\leqq\sqrt{2}\|h\|_H$ for any $h\in H$.*

PROOF. This follows from

$$\|\widetilde{P}_{n+1}h-\widetilde{P}_nh\|_B\leqq 2^{-n}\|P_{n+1}h\|_H\leqq 2^{-n}\|h\|_H.$$

PROPOSITION 3.2. *$\|z\|_B\leqq\sqrt{2}c^{-1}q(z)$ for any $z\in B$. Furthermore, the inclusion map from B_0 to B is compact.*

PROOF. By the assumption (C-2) (3), we get

$$\begin{aligned} c\|z-\widetilde{P}_nz\|_B &\leqq \varlimsup_{m\to\infty}\|\widetilde{P}_mz-\widetilde{P}_nz\|_B \\ &\leqq \varlimsup_{m\to\infty}\sum_{k=n}^{m}\|\widetilde{P}_{k+1}z-\widetilde{P}_kz\|_B \\ &\leqq \left[\sum_{k=n}^{\infty}2^{-k}\right]^{1/2}q(z). \end{aligned}$$

Since $\widetilde{P}_n$'s are compact operators, our proposition has been proved.

PROPOSITION 3.3. *Let E_n's $(n=1,2,\cdots)$ be Banach spaces and define a Banach space E with a norm $\|\cdot\|_E$ by*

$$E=\left\{\{z_k\}_{k=1}^{\infty}\in\prod_{k=1}^{\infty}E_k;\ \sum_{k=1}^{\infty}\|z_k\|_{E_k}^2<\infty\right\} \quad \textit{and}$$

$$\|z\|_E=\left[\sum_{k=1}^{\infty}\|z_k\|_{E_k}^2\right]^{1/2} \quad \textit{for each } z=\{z_k\}_{k=1}^{\infty}\in E.$$

Then

$$E^*=\left\{\{w_k\}_{k=1}^{\infty}\in\prod_{k=1}^{\infty}E_k^*;\ \sum_{k=1}^{\infty}\|w_k\|_{E_k^*}^2<\infty\right\} \quad \textit{and}$$

$$\|w\|_{E^*}=\left[\sum_{k=1}^{\infty}\|w_k\|_{E_k^*}^2\right]^{1/2} \quad \textit{for each } w=\{w_k\}_{k=1}^{\infty}\in E^*.$$

In particularly, if E_n's are reflexive, then E is reflexive.

The proof is easy by virtue of Schwarz's inequality.

PROPOSITION 3.4. *B_0 is a reflexive Banach space.*

PROOF. Since any finite dimensional space is reflexive, the reflexivity follows from Proposition 3.3, provided that B_0 is a Banach space. So it suffices to prove the completeness of B_0.

Suppose that $\{z_n\}_{n=1}^{\infty}\subset B_0$ and $q(z_n-z_m)\to 0$ as $n,m\to\infty$. Then by Proposition

2.2, $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in B. So there exists $z\in B$ such that $z_m\to z$ in B as $m\to\infty$. Thus by Fatou's lemma, we obtain

$$q(z-z_m)^2=\sum_{k=0}^{\infty} 2^k \varliminf_{m\to\infty} \|\widetilde{P}_{k+1}(z_m-z_n)-\widetilde{P}_k(z_m-z_n)\|_B^2$$
$$\leqq \varliminf_{m\to\infty} q(z_m-z_n)^2.$$

This completes the proof.

Let us introduce the space $\mathscr{G}_0$ by

$\mathscr{G}_0=\{u\in\mathscr{F}_0$; there exists an integer n and a smooth function f defined on Image P_n with compact support such that $u(z)=f(\widetilde{P}_n z)$ for any $z\in B\}$.

Obviously $\mathscr{G}_0\subset\mathscr{F}_0$. Let $\mathscr{G}$ be the closure of $\mathscr{G}_0$ in $\mathscr{D}(\widetilde{\mathscr{E}})$ with norm $\widetilde{\mathscr{E}}_1$.

PROPOSITION 3.5. *Fix an integer n. Let g be a bounded, Lipschitz continuous function defined on* Image P_n *and let $u(z)=g(\widetilde{P}_n z)$ for each $z\in B$. Then u belongs to $\mathscr{G}$.*

PROOF. By Proposition 2.1, we see that u belongs to $\mathscr{D}(\widetilde{\mathscr{E}})$. Let $\{e_1,\cdots,e_m\}$ be an orthonormal base of Image P_n. It is well known that there exists a sequence $\{g_r\}_{r=1}^{\infty}$ of smooth functions defined on Image P_n with compact support such that

(3.1) $g_r(y)\to g(y)$, $r\to\infty$, uniformly on each compact subset of Image P_n,

(3.2) $$\frac{\partial}{\partial x_j} g_r\Big(\sum_{i=1}^m x_i e_i\Big)\longrightarrow \frac{\partial}{\partial x_j} g\Big(\sum_{i=1}^m x_i e_i\Big) \quad \text{as } r\longrightarrow\infty$$

for a.e. $x=(x_1,\cdots,x_m)\in \boldsymbol{R}^m$ and $j=1,\cdots,m$, and

(3.3) there exists a positive constant C such that

$$|g_r(y)|\leqq C,\quad \Big|\frac{\partial}{\partial x_j} g_r\Big(\sum_{i=1}^m x_i e_i\Big)\Big|\leqq C \quad \text{for any } y\in \text{Image } P_n,$$
$$x\in\boldsymbol{R}^m,\ j=1,\cdots,m \quad\text{and}\quad r=1,2,\cdots.$$

Let $u_r(z)=g_r(\widetilde{P}_n z)$, $z\in B$, then $u_r\in\mathscr{G}_0$. Suppose that $\{k_1,\cdots,k_m\}$ is a subset of K such that $\{P_n k_1,\cdots,P_n k_m\}$ is linearly independent. Then for each $z\in B$,

$$\frac{\partial}{\partial x_j} u_r\Big(z+\sum_{i=1}^m x_i k_i\Big)\longrightarrow \frac{\partial}{\partial x_j} u\Big(z+\sum_{i=1}^m x_i k_i\Big) \quad \text{as } r\longrightarrow\infty$$

for a.e. $x\in\boldsymbol{R}^m$. By the similar argument as in the proof of Lemma 1.3, it is easy to see that

$$(k_j, Du_r(z))_H\longrightarrow (k_j, Du(z))_H \quad \text{as } r\longrightarrow\infty \text{ for } \mu\text{-a.e.} z\in B.$$

Notice that $Du_r(z)=\sum_{j=1}^m (e_j, Du_r(z))_H e_j$ and

$$Du(z)=\sum_{j=1}^{m}(e_j, Du(z))_H e_j \quad \text{for } \mu\text{-a.e.} z\in B.$$

Since K is dense in H, we obtain $\|Du_r(z)\|_H \leqq \sqrt{m}C$ and $\|Du(z)\|_H \leqq \sqrt{m}C$ for μ-a.e.$z\in B$. Furthermore, we get

$$\|Du_r(z)-\sum_{j=1}^{m}(k_j, Du_r(z))_H e_j\|_H \leqq \sqrt{m}C\sum_{j=1}^{m}\|e_j-k_j\|_H \quad \text{and}$$

$$\|Du(z)-\sum_{j=1}^{m}(k_j, Du(z))_H e_j\|_H \leqq \sqrt{m}C\sum_{j=1}^{m}\|e_j-k_j\|_H,$$

which implies $\|Du(z)-Du_r(z)\|_H \to 0$ as $r\to\infty$ for μ-a.e.$z\in B$. Now it is easy to see that $\tilde{\mathscr{E}}_1(u-u_r, u-u_r)\to 0$, $r\to\infty$, which completes the proof.

PROPOSITION 3.6. $\mathscr{F}=\mathscr{G}$.

PROOF. For any $u\in\mathscr{F}_0$, put $u_n(z)=u(\tilde{P}_n z)$, $z\in B$. Then $u_n\in\mathscr{G}$ by Proposition 3.5. Since $Du_n(z)=P_n Du(\tilde{P}_n z)$, it follows from the assumption (C-2) that $\tilde{\mathscr{E}}_1(u-u_n, u-u_n)\to 0$ as $n\to\infty$, completing the proof.

$\mathscr{G}_0$ is a separable subspace of $C(B)$, a space of all bounded continuous functions on B with a supremum norm. Let S be the closure of $\mathscr{G}_0$ in $C(B)$. Then by the Gelfand representation theorem, there exists a locally compact separable metric space M such that B is densely continuously embedded in M and the restriction to B give rise to an isomorphism from $C_\infty(M)$ onto S, where $C_\infty(M)$ is a set of all real continuous functions on M vanishing at infinity. (See T. Gamelin [4] Chapter 1 Section 8 for an example.)

Let $\tilde{\mu}$ denote the probability measure on M induced by μ through the embedding from B to M. We may identify $L^2(M; d\tilde{\mu})$ with $L^2(B; d\mu)$ and accordingly, regard $(\mathscr{E}, \mathscr{F})$ as a Dirichlet form on $L^2(M; d\tilde{\mu})$. Then in view of Proposition 3.6 and the definition of M, $\mathscr{G}_0$ is a core of $\mathscr{E}$, and consequently $(\mathscr{E}, \mathscr{F})$ becomes a regular Dirichlet form on $L^2(M; d\tilde{\mu})$. Furthermore it is easy to see that $(\mathscr{E}, \mathscr{F})$ has the local property, i.e. if $u, v\in\mathscr{F}$ and $\text{support}[u]\cap\text{support}[v]=\emptyset$, then $\mathscr{E}(u, v)=0$.

By virtue of M. Fukushima [3] Chapter 6, there exists a diffusion process on M, $\{M, x_t, P_x\}$ associated with $(\mathscr{E}, \mathscr{F})$. Note that the continuity of the sample path of x_t is assured only relative to the topology of M, not necessarily to that of B.

The proof of Theorem 2 shall be carried out by the following two propositions. We define the capacity on M by

$$\widetilde{\text{Cap}}(G)=\inf\{\tilde{\mathscr{E}}_1(u, u);\ u\in\mathscr{F} \ \text{ and } \ u(x)\geqq 1 \text{ for } \tilde{\mu}\text{-a.e.}x\in G\}$$

for each open subset G of M, and by

$$\widetilde{\mathrm{Cap}}(E)=\inf\{\widetilde{\mathrm{Cap}}(G);\ E\subset G \text{ and } G \text{ is open in } M\}$$

for any subset E of M. Then $\mathrm{Cap}(\cdot)$ is a Choquet capacity on M.

PROPOSITION 3.7. $\widetilde{\mathrm{Cap}}(M\setminus B_0)=0$.

PROOF. Let $F_r=\{z\in B_0;\ q(z)\leqq r\}$ for $r>0$, then, by Proposition 3.4, F_r is compact in the weak topology of B_0. On account of Proposition 3.2, F_r is compact in the topology of B, so F_r is compact in that of M, which implies that F_r is closed in M.

Take a smooth incresing function ϕ on $\boldsymbol{R}$ such that $\phi(t)=0$, $t\leqq r^2$, $\phi(t)=1$, $t\geqq(r+1)^2$, and $|\phi'(t)|\leqq 2/(r+1)$, $t\in\boldsymbol{R}$. Let $q_m(z)=\left[\sum_{k=0}^{m}2^k\|P_{k+1}z-P_kz\|_B^2\right]^{1/2}$, $z\in B$, then q_m is a semi-norm on B_0 and $q_m(z)\leqq q(z)$ for any $z\in B$. Now let $u_m(z)=\phi(q_m(z)^2)$ and $u(z)=\phi(q(z)^2)$, $z\in B$, with the convention that $\phi(\infty)=1$.

By Proposition 3.5, we see $u_m\in\mathscr{F}$ for $m=1,2,\cdots$. We can also see from Proposition 2.1 and Proposition 3.1 that $u\in\mathscr{D}(\tilde{\mathscr{E}})$. Lemma 1.2 implies that

$$\begin{aligned}&\|Du(z)-Du_m(z)\|_H\\ &=\|\phi'(q(z)^2)D[q(z)^2]-\phi'(q_m(z)^2)D[q_m(z)^2]\|_H\\ &\leqq|\phi'(q(z)^2)-\phi'(q_m(z)^2)|\ \|D[q_m(z)^2]\|_H\\ &\quad+|\phi'(q(z)^2)|\ \|D[q(z)^2-q_m(z)^2]\|_H\end{aligned}$$

for μ-a.e.$z\in B$. By the similar argument as in the proof of Proposition 3.1, we get

$$\begin{aligned}&\|D[q_m(z)^2]\|_H\leqq 2\sqrt{2}\,q_m(z) \quad\text{and}\\ &\|D[q(z)^2-q_m(z)^2]\|_H\leqq 2\sqrt{2}\,2^{-m}q(z) \quad\text{for } \mu\text{-a.e.}z\in B.\end{aligned}$$

Since $\mu(B_0)=1$ and $q_m(z)\leqq q(z)\leqq r+1$ for any $z\in F_{r+1}$, we get

$$\begin{aligned}&\|Du(z)-Du_m(z)\|_H\\ &\leqq 2\sqrt{2}\,q_m(z)|\phi'(q(z)^2)-\phi'(q_m(z)^2)|+2\sqrt{2}\,2^{-m}q(z)|\phi'(q(z)^2)|\\ &\leqq 2\sqrt{2}\,(r+1)|\phi'(q(z)^2)-\phi'(q_m(z)^2)|+4\sqrt{2}\,2^{-m}(r+1)\\ &\longrightarrow 0,\ m\longrightarrow\infty \quad\text{for } \mu\text{-a.e.}z\in B.\end{aligned}$$

Therefore we get $\tilde{\mathscr{E}}_1(u-u_m, u-u_m)\to 0$, $m\to\infty$, which implies $u\in\mathscr{F}$.

Obviously $u(z)=0$ and $Du(z)=0$ for μ-a.e.$z\in F_r$ and $u(z)=1$ for $\tilde{\mu}$-a.e.$z\in M\setminus F_{r+1}$. By Lemma 1.3 and Proposition 3.1, we get $|u(z)|\leqq 1$ and $\|Du(z)\|_H\leqq 4\sqrt{2}$ for μ-a.e.$z\in B$. Hence

$$\begin{aligned}\widetilde{\mathrm{Cap}}(M\backslash B_0)&\leqq\widetilde{\mathrm{Cap}}(M\backslash F_{r+1})\\&\leqq\tilde{\mathscr{E}}_1(u,u)\\&\leqq\int_{B_0\backslash F_r}(1+(4\sqrt{2})^2\|A(z)\|_{\mathscr{L}^\infty(H,H)})\mu(dz)\\&\longrightarrow 0 \text{ as } r\longrightarrow\infty.\end{aligned}$$

The proof of Proposition 3.7 is completed.

We define for each $h\in H$ a continuous function f_h on B by $f_h(z)=\|z-h\|_B$.

PROPOSITION 3.8. *f_h is quasi continuous for each $h\in H$.*

PROOF. According to the proof of the preceding proposition, F_r is compact in B and $\widetilde{\mathrm{Cap}}(M\backslash F_r)\to 0$ as $r\to\infty$. F_r is also compact in M, which means that the topology of F_r induced by B is identical with that induced by M. Thus $f_h|_{F_r}$ is continuous on the induced topology by M. This completes the proof.

Finally we are in a position to prove Theorem 2. Choose a countable dense subset $\{h_n\}_{n=1}^\infty$ of H and denote f_{h_n} by f_n. By virtue of Proposition 3.7, Proposition 3.8 and M. Fukushima [3] Chapter 4, there exists a measurable subset X of B such that

(3.6) X is an invariant set of the diffusion process $\{M, x_t, P_x\}$,

and

(3.7) $P_x[\{\omega; f_n(x_t(\omega))$ is continuous in t for any $n\}]=1$ for any $x\in X$.

However, for any ω in the braces of the left hand side of (3.7), we get

$$\begin{aligned}&\overline{\lim_{s\to t}}\|x_t(\omega)-x_s(\omega)\|_B\\&\leqq\|x_t(\omega)-h_n\|_B+\overline{\lim_{s\to t}}\|x_s(\omega)-h_n\|_B\\&\leqq 2\|x_t(\omega)-h_n\|_B.\end{aligned}$$

Since $\{h_n\}_{n=1}$ is dense in B, we have

$$\overline{\lim_{s\to t}}\|x_t(\omega)-x_s(\omega)\|_B=0 \quad \text{for each } t\geqq 0.$$

Thus

(3.8) $P_x[\{\omega; x_\cdot(\omega)\colon [0,\infty)\to B$ is continuous$\}]=1$

for any $x\in X$. Theorem 2 is proven.

It is useful to define a capacity on B without refering to the enlarged space M. We let

$$\mathrm{Cap}(G)=\inf\{\tilde{\mathscr{E}}_1(u, u);\ u\in\mathscr{F}\ \text{ and }\ u(z)\geqq 1 \text{ for } \mu\text{-a.e.} z\in G\}$$

for any open set G in B and

$$\mathrm{Cap}(E)=\inf\{\mathrm{Cap}(G);\ E\subset G,\ G \text{ is open in } B\}$$

for any subset of B.

PROPOSITION 3.9. *For any subset E of B,*

$$\mathrm{Cap}(E)=\widetilde{\mathrm{Cap}}(E)\quad (=\widetilde{\mathrm{Cap}}(E\cup(M\backslash B))).$$

This follows immediately from the following lemma and M. Fukushima [3] Chapter 3.

LEMMA 3.1. *Let G be a measurable finely open set and u be a bounded measurable function belonging to $\mathscr{F}$ such that $u(x)\geqq 0$ for μ-a.e.$x\in G$. Then for the quasi continuous version $\tilde{u}$ of u, $\tilde{u}(x)\geqq 0$ for q.e.$x\in G$.*

The proof of Lemma 3.1 is informed by M. Fukushima in private communication. We use the notation in M. Fukushima [3].

PROOF. Let $f(x)=\chi_G(x)\tilde{u}(x)$. Then $p_t f(x)=E_x[f(x_t)]$ is non-negative for a.e.x. Since $p_t f$ is quasi-continuous by virtue of M. Fukushima [3] Theorem 4.3.3, we get $p_t f(x)\geqq 0$ for q.e.x. Furthermore $P_x[\lim_{t\to 0} f(x_t(\omega))=\tilde{u}(x)]=1$ for q.e.$x\in G$, because G is finely open. So we get $\lim_{t\to 0} p_t f(x)=\tilde{u}(x)\geqq 0$ for q.e.$x\in G$, which completes the proof.

4. Example

Suppose that μ is a Gaussian probability measure on B such that

$$\int_B \exp(\sqrt{-1}\,{}_{B^*}\langle u, z\rangle_B)\mu(dz)=\exp\left(-\frac{1}{2}\|u\|_H^2\right)$$

for any $u\in B^*\subset H$. So (μ, H, B) is the so-called abstract Wiener space. We know that μ is then H quasi-invariant and strictly positive, and that the conditions (C-1) and (C-2) are fulfilled. (For an example, see H. Kuo [5]).

Suppose further that A is a strongly measurable map from B to $\mathscr{L}^\infty(H, H)$ such that there exist positive constants c_0 and C_0 such that $C_0 I_H-A(z)$ and $A(z)-C_0 I_H$ are positive definite symmetric operators for any $z\in B$. Then the conditions (A-1) and (A-2) are true. By virtue of I. Shigekawa [8] Section 4, it is easy to see that $\mathscr{D}(\tilde{\mathscr{E}})=\mathscr{F}$. Therefore our Theorem 2 provides us with a diffusion process on B, $\{B, x_t, P_z\}$ associated with $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$.

In the special case that $A(z)=(1/2)I_H$ for each $z \in B$, the associated diffusion is the so-called Ornstein-Uhlenbeck process, which is a solution of the stochastic differential equation

$$\begin{cases} dx_t = -\dfrac{1}{2} x_t dt + dw_t \\ x_0 = z \in B \end{cases}$$

where w_t is a standard Wiener process on B associated with (μ, H, B).

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Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan