On the singular Cauchy problem for operators with variable involutive characteristics

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The purpose of this paper is to give an explicit representation of a solution to the Cauchy problem with meromorphic initial data, which implies the results of Y. Hamada [6] and L. Lamport [16], and generalizes those of Y. Hamada and G. Nakamura [8].

Let $a(t, x; \partial_t, \partial_x)$ be a linear differential operator of order $\tilde{m}$ with holomorphic coefficients. We consider the non-characteristic Cauchy problem

$$\text{(CP)} \qquad \begin{cases} a(t, x; \partial_t, \partial_x)u(t, x)=0, \\ \partial_t^d u(0, x)=w_d(x), \quad 0\leqq d\leqq\tilde{m}-1, \end{cases}$$

where at least one of the w_d's has poles along $x_1=0$.

Y. Hamada [5] showed that the singularities of the solution of (CP) propagate along the characteristic surfaces issuing from $x_1=0$ for operators with simple characteristics. This result was extended to operators with characteristics of constant multiplicities ([6], [16], C. Wagschal [27], Y. Hamada, J. Leray and C. Wagschal [7] and others in bibliography of them). Further, S. Ōuchi [21], [22] has investigated the asymptotic behavior of the solution near the characteristic surfaces.

The situation, however, becomes much complicated when the multiplicities of the characteristics are not constant. Y. Hamada and G. Nakamura [8], applying the method of D. Ludwig and B. Granoff [17], dealt with the case when the characteristic surfaces intersect with each other involutively and the maximum multiplicity of the characteristics is at most double.

In this paper we assume the involutiveness, but the size of the multiplicity is arbitrary. Though our method is closely related to that of [8], we use the theory of pseudo-differential operators which act on the functions of the form $\sum_{j=-\infty}^{\infty} f_j(\phi)u_j$. We may derive the necessary results on pseudo-differential operators from the bulk of heavy algebraic analysis of S.K.K. [24], we prefered to give an independent and elementary definition of pseueo-differential operators to suit our problem.

We also use multi-phase functions which correspond to $\#$-products of phase functions defined by H. Kumano-go, K. Toniguchi and Y. Tozaki [15]. In the real hyperbolic case, Y. Morimoto [18] and K. Taniguchi [25] have obtained a similar expression as ours (cf. also H. Kumano-go [13], H. Kumano-go and K. Taniguchi [14], J. C. Nosmas [20]). They use the expression given in [13] and do not solve the transport equations. On the contrary we solve the transport equations, which turn out to be systems of equations of the Goursat type.

We also note that T. Ishii [10] has obtained a representation of the solution of (CP) which corresponds to that of [13], only assuming that the characteristic roots are holomorphic. In the involutive case, however, his results are too general to give explicit informations. Our representation (see Theorem 1.1) enables one to construct a local elementary solution and to decide the propagation of the singular spectra for hyperbolic operators with analytic coefficients and with variable involutive characteristics (cf. T. Kawai and G. Nakamura [11]). This will be published in a separated paper.

§1. Assumptions and the main results

Let

$$a(t, x; \partial_t, \partial_x)=\partial_t^{\tilde{m}}+\sum_{\substack{0\leq\alpha_0+|\alpha|\leq\tilde{m}\\ 0\leq\alpha_0\leq\tilde{m}-1}} a_{\alpha_0\alpha}(t, x)\partial_x^{\alpha}\partial_t^{\alpha_0} \tag{1.1}$$

be a linear differential operator of order $\tilde{m}$ with holomorphic coefficients in a neighborhood (nhbd) of $(t, x)=(t, x_1 \cdots x_n)=(0, 0)\in \boldsymbol{C}^{n+1}$. Here we employ the notation

$$\partial_x^{\alpha}=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}, \qquad \partial_t=\partial/\partial t.$$

We consider the non-characteristic homogeneous Cauchy problem

$$\text{(CP)} \qquad \begin{cases} a(t, x; \partial_t, \partial_x)u(t, x)=0, \\ \partial_t^d u(0, x)=w_d(x), \quad 0\leq d\leq \tilde{m}-1, \end{cases}$$

with meromorphic initial data, that is, the w_d's have poles along $x_1=0$.

We require the principal symbol p of the operator a

$$p(t, x; \tau, \xi)=\tau^{\tilde{m}}+\sum_{\alpha_0+|\alpha|=\tilde{m}} a_{\alpha_0\alpha}(t, x)\xi^{\alpha}\tau^{\alpha_0}$$

to be holomorphically factorizable and to have variable involutive characteristics. More precisely, we require that the principal symbol p is factored in the form

$$p(t, x; \tau, \xi)=\prod_{\sigma=1}^{\kappa}\prod_{i=1}^{m_\sigma}(\tau-\lambda_i^{\sigma}(t, x; \xi)), \tag{1.2}$$

and that the characteristic roots λ_i^{σ} satisfy the following conditions:

(C-1) λ_i^σ's are holomorphic in a nhbd of $(t, x; \xi)=(0, 0; 1, 0 \cdots 0)$ and homogeneous in ξ of degree one,

(C-2) if $\sigma\neq\sigma'$, $\lambda_i^\sigma(t, x; \xi)$ and $\lambda_j^{\sigma'}(t, x; \xi)$ do not coincide at $(t, x; \xi)=(0, 0; 1, 0 \cdots 0)$,

(C-3) for each σ, $1\leqq\sigma\leqq\kappa$, there exists a set of functions $c_{ij}^\sigma(t, x; \xi)$, $1\leqq i, j\leqq m_\sigma$, which are holomorphic in a nhbd of $(t, x; \xi)=(0, 0; 1, 0 \cdots 0)$ and homogeneous in ξ of degree zero such that

$$\{\tau-\lambda_i^\sigma, \tau-\lambda_j^\sigma\}=c_{ij}^\sigma(\lambda_j^\sigma-\lambda_i^\sigma), \quad 1\leqq i, j\leqq m_\sigma. \tag{1.3}$$

Here $\{\cdot, \cdot\}$ stands for the Poisson bracket.

REMARK. Y. Hamada and G. Nakamura [8] require that the Poisson brackets vanish identically, which is stronger than our requirement, (C-3). For example, the symbols

$$\tau^2-(x^j+x^k)\xi\tau+x^{j+k}\xi^2=(\tau-x^j\xi)(\tau-x^k\xi), \quad k\neq j,$$

and

$$\tau^2-2x_1\xi_1\tau-x_1^2(\xi_2^2+\cdots+\xi_n^2)=(\tau-x_1\xi_1-x_1|\xi|)(\tau-x_1\xi_1+x_1|\xi|),$$

do not satisfy their requirement, but do ours (cf. Remark 2.5).

We set for a positive integer m

$$J(m)=\bigcup_{l=1}^{m}\{(i)_l=(i_1 \cdots i_l)\in N^l: 1\leqq i_1<\cdots<i_l\leqq m\}. \tag{1.4}$$

For each σ $(1\leqq\sigma\leqq\kappa)$, we define a set of functions $\varphi_\sigma^{(i)_l}$, $(i)_l\in J(m_\sigma)$, called multi-phase functions, as the solutions of the following equations:

$$\begin{cases}\dfrac{\partial\varphi_\sigma^{(i)_l}}{\partial t}-\lambda_{i_l}^\sigma(t, x; d_x\varphi_\sigma^{(i)_l})=0,\\ \varphi_\sigma^{(i)_l}((t)_{i_l}, x)|_{t=t_{i_{l-1}}}=\varphi_\sigma^{i_1\cdots i_{l-1}}((t)_{i_{l-1}}, x)|_{t=t_{i_{l-1}}},\end{cases} \tag{1.5}$$

for $2\leqq l\leqq m_\sigma$, and

$$\begin{cases}\dfrac{\partial\varphi_\sigma^{i_1}}{\partial t}-\lambda_{i_1}^\sigma(t, x; d_x\varphi_\sigma^{i_1})=0,\\ \varphi_\sigma^{i_1}(t, x)|_{t=0}=x_1,\end{cases} \tag{1.6}$$

where $(t)_{i_k}=(t_{i_1} \cdots t_{i_{k-1}}, t)$ and $d_x\varphi=(\partial\varphi/\partial x_1 \cdots \partial\varphi/\partial x_n)$. It is clear from (C-1) that the multi-phase functions $\varphi_\sigma^{(i)_l}$ are holomorphic at the origins of their arguments. Now we state our main results as follows.

THEOREM 1.1. *There exist open neighborhoods T and X of the origin of $\boldsymbol{C}$ and $\boldsymbol{C}^n$, respectively, such that for an arbitrary point x^0 in X with $x_1^0\neq0$,* *the unique holomorphic solution* $u(t, x)$ *of* (CP) *near* $(t, x)=(0, x^0)$ is represented *as follows*:

$$(1.7)\qquad u(t, x)=\sum_{\sigma=1}^{\kappa}\left\{E_\sigma^1(t, x)+\sum_{l=1}^{m_\sigma-1}\int_0^t dt_l\int_0^{t_l} dt_{l-1}\cdots\int_0^{t_2} dt_1 E_\sigma^{l+1}(t_1\cdots t_l; t, x)\right\} +H(t, x),$$

with

$$\begin{cases} E_\sigma^i=\left\{\dfrac{e_\sigma^i}{(\varphi_\sigma^{12\cdots i})^{p_\sigma^i}}+g_\sigma^i\log\varphi_\sigma^{12\cdots i}\right\}(t_1\cdots t_{i-1}; t, x), \quad 1\leqq i\leqq m_\sigma-1 \quad or \quad i=m_\sigma=1, \\ E_\sigma^{m_\sigma}=\left\{\displaystyle\sum_{j=1}^{\infty}\dfrac{e_{\sigma,j}}{(\varphi_\sigma^{12\cdots m_\sigma})^j}+g_\sigma^{m_\sigma}\log\varphi_\sigma^{12\cdots m_\sigma}\right\}(t_1\cdots t_{m_\sigma-1}; t, x), \quad m_\sigma\geqq 2. \end{cases}$$

Here p_σ^i*'s are non-negative integers,* $H\in\mathcal{O}(T\times X)$*,* $e_\sigma^i, g_\sigma^i, \varphi_\sigma^{12\cdots i}\in\mathcal{O}(T^i\times X)$ *and* $e_{\sigma,j}\in\mathcal{O}(T^{m_\sigma}\times X)$ $(T^k=T\times\cdots\times T\subset \boldsymbol{C}^k$ *and* $\mathcal{O}(U)$ *is the set of all holomorphic functions in* $U)$ *are independent of* x^0*, and the infinite series* $\sum_{j=1}^{\infty}e_{\sigma,j}/(\varphi_\sigma^{12\cdots m_\sigma})^j$ *converges normally in* $T^{m_\sigma}\times X\backslash\{\varphi_\sigma^{12\cdots m_\sigma}=0\}$ *to define a holomorphic function there. Moreover, we have the following estimate for each* σ *such that* $m_\sigma\geqq 2$:

$$(1.8)\qquad \sum_{j=1}^{\infty}\left|\frac{e_{\sigma,j}}{(\varphi_\sigma^{12\cdots m_\sigma})^j}\right|\leqq B\exp C|\varphi_\sigma^{12\cdots m_\sigma}|^{-(1/(m_\sigma/(m_\sigma-1)))},$$

with some positive constants B *and* C *independent of any point in* $T^{m_\sigma}\times X\backslash\{\varphi_\sigma^{12\cdots m_\sigma}=0\}$.

REMARK. An infinite series $\sum_{k=0}^{\infty}u_k$ is said to be normally convergent in an open set U if $\sum_{k=0}^{\infty}\sup_K|u_k|<\infty$ for any compact subset K in U.

REMARK. When the characteristic roots are of constant multiplicities, i.e., $\lambda_1^\sigma(t, x;\xi)=\cdots=\lambda_{m_\sigma}^\sigma(t, x;\xi)$, $1\leqq\sigma\leqq\kappa$, we can determine the locus of the singularities of the solution $u(t, x)$ of (CP) immediately by the expression (1.7). In fact, since the multi-phase functions $\varphi_\sigma^{12\cdots i}(t_1\cdots t_{i-1}; t, x)$ turn out to be $\varphi_\sigma^1(t, x)$, i.e., to be independent of integral arguments, $t_1, t_2, \cdots$, the integrations are easily performed to define a holomorphic function in $T\times X\backslash\bigcup_{\sigma=1}^{\kappa}\{\varphi_\sigma^1=0\}$. Namely, the singularities propagate along the κ characteristic surfaces $\{\varphi_\sigma^1=0\}$, which is the result of [6] and [16].

We carry out the iterated integrations in (1.7) as follows. Since $\varphi_\sigma^{12\cdots i}(0; 0, x^0)=x_1^0\neq 0$, we can choose a sufficiently small positive constant r common to all $\varphi_\sigma^{12\cdots i}$ so that $\varphi_\sigma^{12\cdots i}$ do not vanish on the polydisc $D^{n+i}((0; 0, x^0); r)\subset\boldsymbol{C}^{n+i}$ (of radius r with center $(0; 0, x^0)$). Let us fix branches of $\log\varphi_\sigma^{12\cdots m_\sigma}$, $1\leqq\sigma\leqq\kappa$ so that

$\log \varphi_\sigma^{12\cdots m_\sigma}|_{t_1=\cdots=t=0}$ coincide on $D^n(x^0; r)$ (the fact that $\varphi_\sigma^{12\cdots m_\sigma}|_{t_1=\cdots=t=0}=x_1$ enables us to choose such a system of branches). Using these branches, we define branches of $\log \varphi_\sigma^{12\cdots i}$ on $D^{n+i}((0; 0, x^0); r)$ by the relations:

$$\log \varphi_\sigma^{12\cdots i}=\log \varphi_\sigma^{12\cdots m_\sigma}|_{t_i=\cdots=t_{m_\sigma-1}=t}, \quad 1\leqq i\leqq m_\sigma-1. \tag{1.9}$$

Since E_σ^i are holomorphic and single valued in $D^{n+i}((0; 0, x^0); r)$, when we consider the iterated integrations in $D^{n+i}((0; 0, x^0); r)$, no ambiguity happens and the right side of (1.7) actually defines a holomorphic function $u(t, x)$ on $D^{n+1}((0, x^0); r)$.

We can show that this germ $u(t, x)$ at $(0, x^0)$ can be holomorphically continued along any path in $T\times X\backslash(\Sigma_0\cup\Sigma_\infty)$ with initial point $(0, x^0)$. Namely the singularities of $u(t, x)$ propagate only along $\Sigma_0\cup\Sigma_\infty$. Now we give Σ_0 and Σ_∞ explicitly. Put

$$\Sigma_\sigma^{(i)_l}=\left\{(t, x)\in T\times X: \exists(t)_{i_{l-1}} \text{ s.t. } \varphi_\sigma^{(i)_l}((t)_{i_l}, x)=\frac{\partial\varphi_\sigma^{(i)_l}}{\partial t_{i_1}}((t)_{i_l}, x)=\cdots =\frac{\partial\varphi_\sigma^{(i)_l}}{\partial t_{i_{l-1}}}((t)_{i_l}, x)=0\right\}, \quad (i)_l\in J(m_\sigma), \tag{1.9}$$

where $(t)_{i_{l-1}}=(t_{i_1}\cdots t_{i_{l-1}})$ $((t)_{i_0}=\emptyset)$ and $(t)_{i_l}=(t_{i_1}\cdots t_{i_{l-1}}, t)$ $((t)_{i_1}=t)$. Then Σ_0 is the union of all the closures of $\Sigma_\sigma^{(i)_l}$:

$$\Sigma_0=\bigcup_{\sigma=1}^{\kappa}\ \bigcup_{(i)_l\in J(m_\sigma)} \bar{\Sigma}_\sigma^{(i)_l}. \tag{1.10}$$

See Section 2, especially Theorem 2.2 and Lemma 2.3, for the geometrical meaning of $\Sigma_\sigma^{(i)_l}$.

Next we define Σ_∞. We may assume that $T\subset C$ in Theorem 1.1 is an open disc of radius $R>0$. We define positive valued functions by

$$\rho^{l+1}(t_1\cdots t_l)=|t_1|^2/(R^2-|t_1|^2)+\cdots+|t_l|^2/(R^2-|t_l|^2), \quad (t_1\cdots t_l)\in T^l, \quad l\geqq 1.$$

For a point $(t, x)\in T\times X\backslash\Sigma_0$, we put

$$S_\sigma^{(i)_l}(t, x)=\{(t)_{i_{l-1}}\in T^{l-1}: \varphi_\sigma^{(i)_l}((t)_{i_l}, x)=0\},$$

and

$$D_\sigma^{(i)_l}(t, x)=\{\text{critical values of } \rho^l|_{S_\sigma^{(i)_l}(t,x)}\}, \quad l\geqq 2.$$

Define a function $s\rho$ on $T\times X\backslash\Sigma_0$ by

$$s\rho(t, x)=\left\{\sup r: r\in \bigcup_{\sigma=1}^{\kappa}\ \bigcup_{(i)_l\in J(m_\sigma)} D_\sigma^{(i)_l}(t, x)\right\} \quad (\leqq\infty).$$

Then Σ_∞ is given by

(1.11) $$\Sigma_\infty=\left\{(t,x)\in T\times X\backslash\Sigma_0: \lim_{\delta\to 0}\sup_{(\tilde t,\tilde x)\in D^{n+1}((t,x),\delta)\backslash\{(t,x)\}} s\rho(\tilde t,\tilde x)=\infty\right\}.$$

Σ_∞ is a closed subset in $T\times X\backslash\Sigma_0$.

Example 1.1. $p=\partial_t^2+x_2\partial_1\partial_t$, $\lambda_1=-x_2\xi_1$, $\lambda_2=0$, $\varphi^{12}=x_1-x_2t_1$. $\Sigma^1=\{x_1-x_2t=0\}$, $\Sigma^2=\{x_1=0\}$, $\Sigma^{12}=\{x_1=x_2=0\}$. $\Sigma_0=\Sigma^1\cup\Sigma^2$, $\Sigma_\infty=\{x_2=0\}$.

Example 1.2. $p=\partial_t^2+2x_2\partial_1\partial_t+x_3\partial_2\partial_t$, $\lambda_1=-2x_2\xi_1-x_3\xi_2$, $\lambda_2=0$, $\varphi^{12}=x_1-2x_2t_1+x_3t_1^2$. $\Sigma^1=\{x_1-2x_2t+x_3t^2=0\}$, $\Sigma^2=\{x_1=0\}$, $\Sigma^{12}=\{x_1=x_2=x_3=0\}\cup\{x_3\neq 0, x_1x_3-x_2^2=0\}$ (Σ^{12} is not closed). $\Sigma_0=\Sigma^1\cup\Sigma^2\cup\bar\Sigma^{12}$, $\Sigma_\infty=\{x_3=0\}$.

The above examples are given in [8, Examples 5.5, 5.7] as ones when the origin is an exceptional point for the operator p.

Example 1.3. $p=\partial_t(\partial_t+2x_2\partial_1+\partial_2)(\partial_t+2x_3\partial_1+\partial_3)$, $\lambda_1=-2x_3\xi_1-\xi_3$, $\lambda_2=-2x_2\xi_1-\xi_2$, $\lambda_3=0$, $\varphi^{123}=x_1-2x_2t_2+t_2^2+2(x_2-x_3-t_2)t_1+2t_1^2$. $\Sigma^1=\{x_1-2x_3t+t^2=0\}$, $\Sigma^2=\{x_1-2x_2t+t^2=0\}$, $\Sigma^3=\{x_1=0\}$, $\Sigma^{12}=\{t^2-2(x_2+x_3)t-(x_2-x_3)^2+2x_1=0\}$, $\Sigma^{23}=\{x_1-x_2^2=0\}$, $\Sigma^{13}=\{x_1-x_3^2=0\}$, $\Sigma^{123}=\{x_1-x_2^2-x_3^2=0\}$, $\Sigma_\infty=\emptyset$.

On the singularities of the solution (1.7) we shall describe the details in a separated paper.

§2. Multi-phase functions

Let t be a point in $\boldsymbol{C}$ and (x,ξ) in $T^*\boldsymbol{C}^n$ $(=\boldsymbol{C}^{2n})$. Let $H(t,x;\xi)$ be a holomorphic function in a nhbd of $(t,x;\xi)=(0,x^0;\xi^0)$ where $\xi^0\neq 0$. Let $x=x(s,t,y,\eta)$, $\xi=\xi(s,t,y,\eta)$ be the solutions to the Hamilton-Jacobi system:

(2.1) $$\begin{cases}\dfrac{dx}{dt}=\dfrac{\partial H}{\partial\xi}(t,x;\xi),\\[2mm] \dfrac{d\xi}{dt}=-\dfrac{\partial H}{\partial x}(t,x;\xi),\end{cases}\qquad \begin{cases}x|_{t=s}=y,\\ \xi|_{t=s}=\eta.\end{cases}$$

Then the mapping $\chi_{s,t}\colon(y,\eta)\to(x,\xi)(s,t,y,\eta)$ is a symplectic transformation with parameters s and t. The following theorem is well known (see, for example, V. I. Arnold [1]).

THEOREM 2.1. *Let $\Phi=\Phi(s,t,x,\eta)$ be the solution of the Eikonal equation*:

(2.2) $$\begin{cases}\dfrac{\partial\Phi}{\partial t}+H(t,x;d_x\Phi)=0,\\[2mm] \Phi|_{t=s}=x\cdot\eta.\end{cases}$$

Then $\Phi_{s,t}=\Phi(s, t, \cdots)$ *is a generating function of the symplectic transformation* $\chi_{s,t}$. *Namely,* $\chi_{s,t}$ *is defined implicitly by the relation*

$$y=\partial\Phi_{s,t}/\partial\eta(x, \eta) \quad \text{and} \quad \xi=\partial\Phi_{s,t}/\partial x(x, \eta). \tag{2.3}$$

Given m holomorphic functions $\lambda_i=\lambda_i(t, x; \xi)$, $1\leqq i\leqq m$, on an open nhbd of $(t, x; \xi)=(0, 0; \xi^0)$ $(\xi^0\neq 0)$, we define multi-phase functions $\varphi^{(i)_l}=\varphi^{i_1\cdots i_l}$, $1\leqq i_1, \cdots, i_l\leqq m$, as the solutions of the following equations:

$$\begin{cases} \dfrac{\partial\varphi^{(i)_l}}{\partial t}-\lambda_{i_l}(t, x; d_x\varphi^{(i)_l})=0, \\ \varphi^{(i)_l}((t)_{i_l}, x, \eta)|_{t=t_{i_{l-1}}}=\varphi^{i_1\cdots i_{l-1}}((t)_{i_{l-1}}, x, \eta)|_{t=t_{i_{l-1}}}, \end{cases} \tag{2.4}$$

for $l\geqq 2$, and

$$\begin{cases} \dfrac{\partial\varphi^{i_1}}{\partial t}-\lambda_{i_1}(t, x; d_x\varphi^{i_1})=0, \\ \varphi^{i_1}(t, x, \eta)|_{t=0}=x\cdot\eta. \end{cases} \tag{2.5}$$

Obviously we have for $k=1, 2, \cdots, l-1$

$$\varphi^{(i)_l}((t)_{i_l}, x, \eta)|_{t_{i_k}=t_{i_{k-1}}}=\varphi^{i_1\cdots\hat{i}_k\cdots i_l}(t_{i_1}\cdots\hat{t}_{i_k}\cdots; t, x, \eta), \tag{2.6}$$

where the sign $\hat{}$ means that the index or the variable under $\hat{}$ should be deleted.

Next consider the Hamilton-Jacobi systems associated with (2.4) and (2.5):

$$\begin{cases} \dfrac{dX^{(i)_l}}{dt}=-\dfrac{\partial\lambda_{i_l}}{\partial\xi}(t, X^{(i)_l}, \Xi^{(i)_l}), \\ \dfrac{d\Xi^{(i)_l}}{dt}=\dfrac{\partial\lambda_{i_l}}{\partial x}(t, X^{(i)_l}, \Xi^{(i)_l}), \end{cases} \quad \begin{cases} X^{(i)_l}((t)_{i_l}, y, \eta)|_{t=t_{i_{l-1}}}=X^{(i)_{l-1}}|_{t=t_{i_{l-1}}}, \\ \Xi^{(i)_l}((t)_{i_l}, y, \eta)|_{t=t_{i_{l-1}}}=\Xi^{(i)_{l-1}}|_{t=t_{i_{l-1}}}, \end{cases} \tag{2.7}$$

for $l\geqq 2$, and

$$\begin{cases} \dfrac{dX^{i_1}}{dt}=-\dfrac{\partial\lambda_{i_1}}{\partial\xi}(t, X^{i_1}, \Xi^{i_1}), \\ \dfrac{d\Xi^{i_1}}{dt}=\dfrac{\partial\lambda_{i_1}}{\partial x}(t, X^{i_1}, \Xi^{i_1}), \end{cases} \quad \begin{cases} X^{i_1}(t, y, \eta)|_{t=0}=y, \\ \Xi^{i_1}(t, y, \eta)|_{t=0}=\eta. \end{cases} \tag{2.8}$$

Then for each $(i)_l$ the mapping

$$\chi^{(i)_l}: (y, \eta) \longrightarrow (x, \xi)=(X^{(i)_l}, \Xi^{(i)_l})((t)_{i_l}, y, \eta) \tag{2.9}$$

is a symplectic transformation with parameters $(t)_{i_l}$. By applying Theorem 2.1 inductively on l, we obtain

THEOREM 2.2. *The multi-phase function* $\varphi^{(i)_l}((t)_{i_l}, x, \eta)$ *is a generating function of the symplectic transformation* $\chi^{(i)_l}$ *defined by* (2.9), *namely*

(2.10) $$y=\left(\frac{\partial\varphi^{(i)_l}}{\partial\eta}\right)((t)_{i_l}, X^{(i)_l}, \eta), \qquad \Xi^{(i)_l}=\left(\frac{\partial\varphi^{(i)_l}}{\partial x}\right)((t)_{i_l}, X^{(i)_l}, \eta),$$

where $X^{(i)_l}=X^{(i)_l}((t)_{i_l}, y, \eta)$ *and* $\Xi^{(i)_l}=\Xi^{(i)_l}((t)_{i_l}, y, \eta)$, *or equivalently*

(2.11) $$\begin{cases} x=X^{(i)_l}\left((t)_{i_l}, \dfrac{\partial\varphi^{(i)_l}}{\partial\eta}((t)_{i_l}, x, \eta), \eta\right), \\ \dfrac{\partial\varphi^{(i)_l}}{\partial x}((t)_{i_l}, x, \eta)=\Xi^{(i)_l}\left((t)_{i_l}, \dfrac{\partial\varphi^{(i)_l}}{\partial\eta}((t)_{i_l}, x, \eta), \eta\right). \end{cases}$$

Since the assertion is easily proved by induction on l, we omit the proof.

LEMMA 2.3. *Let* $\varphi^{(i)_l}$ *be a multi-phase function and* k *an integer such that* $1\leqq k\leqq l-1$. *Then we have*

(2.12) $$\frac{\partial\varphi^{(i)_l}}{\partial t_{i_k}}((t)_{i_l}, x, \eta)=(\lambda_{i_k}-\lambda_{i_{k+1}})(t_{i_k}, X^{(i)_k}, \Xi^{(i)_k}),$$

where

$$(X^{(i)_k}, \Xi^{(i)_k})=(X^{(i)_k}, \Xi^{(i)_k})\left(t_{i_1}\cdots t_{i_{k-1}}; t_{i_k}, \frac{\partial\varphi^{(i)_l}}{\partial\eta}((t)_{i_l}, x, \eta), \eta\right).$$

Proof is omitted. See, for example, [16, Theorem 2.3].

From now on, we require that the λ_i's are involutive, that is, there are a set of functions $c_{ij}=c_{ij}(t, x; \xi)$ which are holomorphic in a nhbd of $(t, x; \xi)=(0, 0; \xi^0)$ such that for any pair of integers i and j, $1\leqq i, j\leqq m$, we have

(2.13) $$\{\tau-\lambda_i, \tau-\lambda_j\}=c_{ij}(\lambda_j-\lambda_i).$$

Obviously we may assume that $c_{ij}=c_{ji}$ and $c_{ii}=0$.

REMARK. When λ_i are all homogeneous in ξ of degree one, then c_{ij} are homogeneous in ξ of degree zero and the multi-phase functions $\varphi^{(i)_l}$ homogeneous in η of degree one.

For each c_{ij}, consider the following first order linear differential equation:

(2.14) $$\begin{cases} \{\tau-\lambda_j, a_{ij}\}+c_{ij}a_{ij}=0, \\ a_{ij}(s, t, x; \xi)|_{t=s}=1. \end{cases}$$

Since the hyperplane $t=s$ is non-characteristic, we may assume that the solution a_{ij} of (2.14) is holomorphic in a sufficiently small nhbd of $(s, t, x; \xi)=(0, 0, 0; \xi^0)$ and never vanishes there. Using these a_{ij}, we define as et of functions $a^{(i)_l}_{\mu,\nu}=a^{(i)_l}_{\mu,\nu}((t)_{i_l}, x; \eta)$, $1\leqq\mu\leqq m$, $0\leqq\nu\leqq l-1$, as follows:

(2.15) $$a^{(i)_l}_{\mu,\nu}((t)_{i_l}, x; \eta)=\prod_{k=\nu}^{l-1} a_{\mu, i_{k+1}}(t_{i_k}, t_{i_{k+1}}, X^{(i)_{k+1}}, \Xi^{(i)_{k+1}}),$$

where $(X^{(i)_{k+1}}, \Xi^{(i)_{k+1}})=(X^{(i)_{k+1}}, \Xi^{(i)_{k+1}})(t_{i_1}\cdots t_{i_k}, t_{i_{k+1}}, (\partial\varphi^{(i)_l}/\partial\eta)((t)_{i_l}, x, \eta), \eta)$, $t_{i_0}=0$ and $t_{i_l}=t$. The following theorem will play an important role to construct a formal solution.

THEOREM 2.4. *Suppose that the λ_i's $(1\leqq i\leqq m)$ are involutive in the above sense. Let $a^{(i)_l}_{\mu,\nu}$ be the functions defined by the formula* (2.15). *Then for the multi-phase functions $\varphi^{(i)_l}$, the following equalities hold:*

$$\frac{\partial\varphi^{(i)_l}}{\partial t}-\lambda_{i_k}(t, x; d_x\varphi^{(i)_l})=\begin{cases}-\sum_{\nu=k}^{l-1} a^{(i)_l}_{i_k,\nu}\dfrac{\partial\varphi^{(i)_l}}{\partial t_{i_\nu}}, & 1\leqq k\leqq l-1,\\ 0, & k=l.\end{cases}\tag{2.16}$$

Further, the functions $a^{(i)_l}_{\mu,\nu}$ possess the following properties:

$$a^{(i)_l}_{\mu,\nu}|_{t_{i_1}=\cdots=t=0}=1,\tag{2.17}$$

$$a^{(i)_l}_{\mu,\nu}|_{t_{i_k}=t_{i_{k-1}}}=\begin{cases}a^{(j)_{l-1}}_{\mu,\nu-1}, & 1\leqq k\leqq\nu,\\ a^{(j)_{l-1}}_{\mu,\nu}, & \nu+1\leqq k\leqq l,\end{cases}\tag{2.18}$$

where $(j)_{l-1}=(j_1\cdots j_{l-1})=(i_1\cdots\hat{i}_k\cdots i_l)$. Especially we have

$$(a^{(i)_l}_{\mu,\nu+1}-a^{(i)_l}_{\mu,\nu})|_{t_{i_{\nu+1}}=t_{i_\nu}}=0.\tag{2.19}$$

REMARK 2.5. In the case when the Poisson brackets $\{\tau-\lambda_i, \tau-\lambda_j\}$, $1\leqq i, j\leqq m$, all vanish identically, the functions $a^{(i)_l}_{\mu,\nu}$ are all equal to 1.

Let us put $b_{ij}=1/a_{ij}$. Then the functions b_{ij} satisfy the following first order linear differential equations (cf. (2.14)):

$$\begin{cases}\{\tau-\lambda_j, b_{ij}\}-c_{ij}b_{ij}=0,\\ b_{ij}(s, t, x; \xi)|_{t=s}=1.\end{cases}\tag{2.20}$$

LEMMA 2.6. *The following equality holds:*

$$\begin{aligned}&(\lambda_i-\lambda_{i_{\nu+1}})(t_{i_\nu}, X^{(i)_\nu}, \Xi^{(i)_\nu})\\ &\quad=b_{i,i_{\nu+1}}(t_{i_\nu}, t_{i_{\nu+1}}, X^{(i)_{\nu+1}}, \Xi^{(i)_{\nu+1}})\cdot(\lambda_i-\lambda_{i_{\nu+1}})(t_{i_{\nu+1}}, X^{(i)_{\nu+1}}, \Xi^{(i)_{\nu+1}}),\end{aligned}\tag{2.21}$$

where $(X^{(i)_\mu}, \Xi^{(i)_\mu})=(X^{(i)_\mu}, \Xi^{(i)_\mu})(t_{i_1}\cdots t_{i_{\mu-1}}, t_{i_\mu}, y, \eta)$ for $\mu=\nu$ or $\nu+1$.

PROOF. It is clear that the equality (2.21) holds when $t_{i_{\nu+1}}=t_{i_\nu}$. Hence it is enough to show that the right side of (2.21) is constant along the bicharacteristic strip of $\tau-\lambda_{i_{\nu+1}}$. This follows form (2.13) and (2.20) by differentiating the right side of (2.21) with respect to $t_{i_{\nu+1}}$. Q.E.D.

PROOF OF THEOREM 2.4. Since (2.17)-(2.19) easily follow from the definition of $a^{(i)_l}_{\mu,\nu}$, we only show the equality (2.16) to hold. By Lemmas 2.3 and 2.6, we get

$$\frac{\partial \varphi^{(i)}{}_l}{\partial t_{i_k}} = b_{i_k, i_{k+1}}(t_{i_k}, \cdots)\cdot(\lambda_{i_k} - \lambda_{i_{k+1}})(t_{i_{k+1}}, X^{(i)}{}_{k+1}, \Xi^{(i)}{}_{k+1}).$$

Hence dividing the both sides by $b_{i_k, i_{k+1}}$ and then applying Lemmas 2.3 and 2.6 again, we obtain

$$a_{i_k, i_{k+1}} \frac{\partial \varphi^{(i)}{}_l}{\partial t_{i_k}} + \frac{\partial \varphi^{(i)}{}_l}{\partial t_{i_{k+1}}} = b_{i_k, i_{k+2}}(t_{i_{k+1}}, \cdots)(\lambda_{i_k} - \lambda_{i_{k+2}})(t_{i_{k+2}}, X^{(i)}{}_{k+2}, \Xi^{(i)}{}_{k+2}).$$

Thus after a finite number of applications of this argument, we finally get

$$\sum_{\nu=k}^{l-1} a^{(i)l}_{i_k, \nu} \frac{\partial \varphi^{(i)}{}_l}{\partial t_{i_\nu}} = (\lambda_{i_k} - \lambda_{i_l})(t, X^{(i)}{}_l, \Xi^{(i)}{}_l),$$

which shows, in view of (2.4) and (2.11), the equality (2.21) to hold. Q.E.D.

§ 3. Pseudo-differential operators

Let $y=(z, x)=(z_1 \cdots z_l, x_1 \cdots x_n) \in \boldsymbol{C}^N$ $(N=l+n)$ and let $\mathcal{O}(Y)$ denote the set of all (single valued) holomorphic functions in Y, where Y is an open set in $\boldsymbol{C}^n$. We denote by $\boldsymbol{\mathcal{O}}(Y)$ the Cartesian product of a countable number of copies of $\mathcal{O}(Y)$:

$$\boldsymbol{\mathcal{O}}(Y) = \prod_{j=-\infty}^{\infty} \mathcal{O}(Y)_j, \quad \mathcal{O}(Y)_j = \mathcal{O}(Y). \tag{3.1}$$

We shall write $\boldsymbol{u}, \boldsymbol{v}, \cdots$, for members of $\boldsymbol{\mathcal{O}}(Y)$. The jth component of $\boldsymbol{u}$, resp. $\boldsymbol{v}, \cdots$, is u_j, resp. $v_j, \cdots$, i.e., $\boldsymbol{u}=(u_j)_{j\in \boldsymbol{Z}}$, $\boldsymbol{v}=(v_j)_{j\in \boldsymbol{Z}}, \cdots$. The Cartesian product $\boldsymbol{\mathcal{O}}(Y)$ has a canonical structure of $\mathcal{O}(Y)$-module with componentwise sums and multiplications by holomorphic functions in Y, that is, $\boldsymbol{u}+\boldsymbol{v}=(u_j+v_j)_{j\in \boldsymbol{Z}}$, $a\boldsymbol{u}=(au_j)_{j\in \boldsymbol{Z}}$ for $a \in \mathcal{O}(Y)$.

Throughout in this section we fix a holomorphic function φ in Y, which is called a phase function for convenience. Now we define (formal) derivatives of $\boldsymbol{u} \in \boldsymbol{\mathcal{O}}(Y)$ by the following formulas:

$$\frac{\partial}{\partial y_i}\boldsymbol{u} = \frac{\partial}{\partial y_i}(u_j)_{j\in \boldsymbol{Z}} = \left(\frac{\partial \varphi}{\partial y_i} u_{j+1} + \frac{\partial u_j}{\partial y_i}\right)_{j\in \boldsymbol{Z}}, \quad 1 \leqq i \leqq N. \tag{3.2}$$

It is clear that $(\partial/\partial y_i)\boldsymbol{u} \in \boldsymbol{\mathcal{O}}(Y)$ and $(\partial/\partial y_i)(\partial/\partial y_k)\boldsymbol{u} = (\partial/\partial y_k)(\partial/\partial y_i)\boldsymbol{u}$. Hence for each polynomial $a(y;\eta)$ in η with holomorphic coefficients in Y, we can define in the usual way a linear differential operator $a(y;\partial_y)$ on $\boldsymbol{\mathcal{O}}(Y)$.

PROPOSITION 3.1. *Let* $a(y;\eta) = \sum_{|\alpha|\leqq m} a_\alpha(y)\eta^\alpha$ *be a polynomial in* η *with* $a_\alpha \in$

$\mathcal{O}(Y)$. *For* $\boldsymbol{u}=(u_j)_{j\in \boldsymbol{Z}}\in\mathcal{O}(Y)$ *we set* $\boldsymbol{v}=a(y;\partial_y)\boldsymbol{u}=\sum_{|\alpha|\leq m}a_\alpha(y)\partial_y^\alpha\boldsymbol{u}$, *then the* jth *component of* $\boldsymbol{v}$ *is given by the formula*

$$v_j(y)=\sum_{k,\alpha,\nu}\frac{1}{\alpha!\,\nu!}(\partial_\eta^\alpha a_k)(y;d_y\varphi)\{\partial_{\tilde{y}}^\alpha(\tilde{\varphi}(y;\tilde{y})^\nu u_{j+k-|\alpha|+\nu})|_{\tilde{y}=y}\},\tag{3.3}$$

where $a_k(y;\eta)$ *are homogeneous parts of* $a(y;\eta)$ *of degree* k *and* $\tilde{\varphi}(y;\tilde{y})=\varphi(y)-\varphi(\tilde{y})-\sum_{i=1}^N(\partial\varphi/\partial y_i)(\tilde{y})\cdot(y_i-\tilde{y}_i)$, *which vanishes quadratically on* $\tilde{y}=y$.

REMARK 3.2. The sum in (3.3) is finite. Indeed, the terms in the right side of (3.3) vanish except for the indices k, α, ν such that $0\leqq k\leqq m$, $0\leqq|\alpha|\leqq k$, $0\leqq 2\nu\leqq|\alpha|$.

PROOF. It suffices to show (3.3) when $a(y;\eta)$ is a monomial, say η^β. Once we conceive (3.3) to hold, the proof is easily completed by induction on $|\beta|$. Q.E.D.

A sequence $\boldsymbol{u}_k$, $k=1,2,\cdots$, in $\mathcal{O}(Y)$ is said to be convergent to $\boldsymbol{u}\in\mathcal{O}(Y)$, denoted by $\boldsymbol{u}_k\to\boldsymbol{u}$ in $\mathcal{O}(Y)$ or $\lim\boldsymbol{u}_k=\boldsymbol{u}$ in $\mathcal{O}(Y)$, if for each $j\in\boldsymbol{Z}$ the jth component of $\boldsymbol{u}_k$ converges to that of $\boldsymbol{u}$ uniformly on any compact subset in Y.

COROLLARY 3.3. *If* $\boldsymbol{u}_k\to\boldsymbol{u}$ *in* $\mathcal{O}(Y)$, *then* $a(y,\partial_y)\boldsymbol{u}_k\to a(y,\partial_y)\boldsymbol{u}$ *in* $\mathcal{O}(Y)$ *for any differential operator* $a(y,\partial_y)$ *with holomorphic coefficients in* Y.

Since the sum in (3.3) is finite, the assertion is clear.

We introduce $\mathcal{O}(Y)$-submodule of $\mathcal{O}(Y)$, in which we shall develop calculas of pseudo-differential operators. First in order to facilitate the argument below, we shall prepare some notions. For $\boldsymbol{u}\in\mathcal{O}(Y)$ and $K\Subset Y$ (i.e., $\bar{K}$ is a compact subset in Y), we define a formal Laurent series $N(\boldsymbol{u},K;t)$ in $t\in\boldsymbol{C}$ by

$$N(\boldsymbol{u},K;t)=\sum_{j=-\infty}^{\infty}(|u_j|_K/\gamma(j))t^j,\tag{3.4}$$

where $|u_j|_K=\sup_{y\in K}|u_j(y)|$ and $\gamma(j)$ is

$$\gamma(j)=\begin{cases}j!, & j=0,1,2,\cdots,\\ 1/(-j)!, & j=-1,-2,-3,\cdots.\end{cases}\tag{3.5}$$

Let $g=\sum_{j=-\infty}^{\infty}g_jt^j$ and $G=\sum_{j=-\infty}^{\infty}G_jt^j$ be two formal Laurent series where g_j and G_j are complex numbers. Analogous to the definition of majorants for power series, we call G a majorant of g, denoted by $g\ll G$, if $|g_j|\leqq G_j$ for all $j\in\boldsymbol{Z}$. We define sums and products of two formal Laurent series in the usual way, though the products do not always exist.

LEMMA 3.4. (i) *Suppose that* $g \ll G$ *and that* G *is convergent in an annulus* $\{t \in \boldsymbol{C}: R' < |t| < R\}$. *Then* g *is also convergent in the same annulus.*

(ii) *Suppose that* $g_i \ll G_i$, $i=1, 2$. *Then we have*

$$g_1 + g_2 \ll G_1 + G_2, \tag{3.6}$$

$$g_1 \cdot g_2 \ll G_1 \cdot G_2, \quad \text{if} \quad G_1 \cdot G_2 \quad \text{exists}. \tag{3.7}$$

The proof is trivial.

REMARK. $g \ll G$ does not implies $dg/dt \ll dG/dt$, though this is true for power series. See a trivial example $d/dt(1/t) \not\ll d/dt(2/t)$.

Formal Laurent series $N(\cdot, K; t)$ have properties similar to those of the usual norm:

$$N(\boldsymbol{u}+\boldsymbol{v}, K; t) \ll N(\boldsymbol{u}, K; t) + N(\boldsymbol{v}, K; t), \quad \boldsymbol{u}, \boldsymbol{v} \in \mathcal{O}(Y), \tag{3.8}$$

$$N(a\boldsymbol{u}, K; t) = |a|_K N(\boldsymbol{u}, K; t), \quad a \in \mathcal{O}(Y), \tag{3.9}$$

$$N(\boldsymbol{u}, K; t) = 0 \longleftrightarrow \boldsymbol{u} = 0, \quad \text{i.e.,} \quad u_j = 0 \quad \text{for all} \quad j \in \boldsymbol{Z}, \tag{3.10}$$

(if the interior of K is not void and Y is connected).

We call $N(\boldsymbol{u}, K; t)$ the formal (Laurent series) norm of $\boldsymbol{u}$ on K.

We define shift operators $T_k: \mathcal{O}(Y) \to \mathcal{O}(Y)$ ($k \in \boldsymbol{Z}$) by

$$T_k \boldsymbol{u} = \boldsymbol{v} \quad \text{where} \quad v_j = u_{j-k}, \quad j \in \boldsymbol{Z}. \tag{3.11}$$

LEMMA 3.5. *The following majorant inequalities hold:*

$$N(T_k \boldsymbol{u}, K; t) \ll (2^{|k|}/\gamma(k)) t^k N_2(\boldsymbol{u}, K; t), \quad k \in \boldsymbol{Z}, \tag{3.12}$$

where $N_c(\boldsymbol{u}, K; t) = \sum_{j=-\infty}^{\infty} (|u_j|_K/\gamma(j)) c^{|j|} t^j$ *with* $c > 0$, *and*

$$N((\partial/\partial y)^\alpha u_j)_{j \in Z}, K; t) \ll (\alpha!/\varepsilon^{|\alpha|}) N(\boldsymbol{u}, K_\varepsilon; t), \tag{3.13}$$

where $\varepsilon > 0$ *is so small that the set* K_ε *of all points at distance less than* ε *from* K *is a relatively compact subset in* Y.

PROOF. By the definitions (3.4) and (3.11) we have, by replacing the index j by $j+k$

$$N(T_k \boldsymbol{u}, K; t) = \sum_{j=-\infty}^{\infty} (|u_j|_K/\gamma(j+k)) t^{j+k}.$$

(3.12) now follows from the easy inequalities

$$2^{-(|j|+|k|)} \gamma(j) \gamma(k) \leqq \gamma(j+k) \leqq 2^{|j|+|k|} \gamma(j) \gamma(k), \quad j, k \in \boldsymbol{Z}. \tag{3.14}$$

The inequality (3.13) is a direct consequence of Cauchy's inequality. Q.E.D.

DEFINITION 3.6. $\mathscr{F}(Y)$ is the set of all $\boldsymbol{u}\in\mathscr{O}(Y)$ which satisfy

(F-1) For any $K\Subset Y$ and $\varepsilon>0$, there exist positive constants $B=B(K)$, $h=h(K)$, and $C=C(K,\varepsilon)$ independent of j such that

(3.15) $$|u_j|_K\leqq Bh^j j!,\quad j=0,1,2,\cdots,\quad\text{and}\quad |u_{-j}|_K\leqq C\varepsilon^j/j!,\quad j=1,2,3,\cdots.$$

The requirement (F-1) is equivalent to

(F-2) For any $K\Subset Y$, the formal norm $N(\boldsymbol{u},K;t)$ is convergent in a punctured nhbd $\{t\in\boldsymbol{C}: 0<|t|<R\}$ with some positive constant $R=R(K)$, namely, $N(\boldsymbol{u},K;t)$ is holomorphic there.

For convenience we shall denote by $\mathscr{M}_0$ the set of all (single valued) holomorphic functions in some punctured nhbds $\{t\in\boldsymbol{C}: 0<|t|<R\}$ (R may vary depending on each function).

In view of Lemmas 3.4, 3.5, (3.8), (3.9) and (F-2), $\mathscr{F}(Y)$ is an $\mathscr{O}(Y)$-submodule of $\mathscr{O}(Y)$ and stable under operations of differential operators. We call $\mathscr{F}(Y)$ a formal (coefficient) space and a member of $\mathscr{F}(Y)$ a formal (coefficient) vector. We shall write, if necessary, $\mathscr{F}(\varphi;Y)$ instead of $\mathscr{F}(Y)$ to clarify that the definition (3.2) of the derivatives depends on φ.

DEFINITION 3.7 (Weierstrass M test). An infinite series $\sum_{k=0}^{\infty}\boldsymbol{u}_k$ is said to be normally convergent if for all $K\Subset Y$ there exists $N(K;t)\in\mathscr{M}_0$ such that

(3.16) $$\sum_{k=0}^{\infty}N(\boldsymbol{u}_k,K;t)\ll N(K;t).$$

Inequality (3.16) implies that $\sum_{k=0}^{\infty}|u_{k,j}|_K\leqq\gamma(j)$ times the coefficient of t^j of $N(K;t)$ for all $j\in\boldsymbol{Z}$. In other words, $\sum_{k=0}^{\infty}u_{k,j}$ converges normally in Y in the usual sense. Set $u_j=\sum_{k=0}^{\infty}u_{k,j}$ and $\boldsymbol{u}=(u_j)_{j\in\boldsymbol{Z}}$. Then obviously $\boldsymbol{u}\in\mathscr{F}(Y)$ and $\boldsymbol{u}=\sum_{k=0}^{\infty}\boldsymbol{u}_k$ in $\mathscr{O}(Y)$. Note also that if $\sum_{k=0}^{\infty}\boldsymbol{u}_k$ converges normally, the sum is independent of the order of summations.

Now we clarify the meaning of the formal space $\mathscr{F}(Y)$. Let f_j, $j\in\boldsymbol{Z}$, be a wave form, that is, a sequence of functions of one variable satisfying the relations

(3.17) $$\frac{d}{dt}f_j(t)=f_{j-1}(t),\quad j\in\boldsymbol{Z}.$$

Suppose that for a member $\boldsymbol{u}=(u_j)_{j\in Z}$ of $\mathcal{O}(Y)$ the infinite series $\sum_{j=-\infty}^{\infty} f_j(\varphi)u_j$ converges absolutely and uniformly in a nhbd of $y^0\in Y$. Then the sum (we shall write it $F(\boldsymbol{u})=F_\varphi(\boldsymbol{u})$) is holomorphic at y^0 and we can differentiate termwise and can change the order of summations, yielding

$$\frac{\partial}{\partial y_i}F(\boldsymbol{u})=\frac{\partial}{\partial y_i}\Big(\sum_{j=-\infty}^{\infty} f_j(\varphi)u_j\Big)=F\Big(\frac{\partial}{\partial y_i}\boldsymbol{u}\Big), \tag{3.18}$$

(see (3.2)). Hence for any differential operator $a(y;\partial_y)$ we have

$$a(y;\partial_y)F(\boldsymbol{u})=F(a(y;\partial_y)\boldsymbol{u}). \tag{3.19}$$

The following wave form will be used to construct a formal solution in Section 5:

$$\begin{cases} f_j(t)=\dfrac{t^j}{j!}(\log t-\psi(j+1)), & j=0,1,2,\cdots, \\ f_{-j}(t)=(-1)^{j+1}(j-1)!/t^j, & j=1,2,3,\cdots, \end{cases} \tag{3.20}$$

where ψ is the digamma function, thus $\psi(j+1)=\sum_{k=1}^{j} 1/k-\gamma$, with Euler's constant γ. One of the benefits of defining $\mathscr{F}(Y)$ consists in the next lemma.

LEMMA 3.8. *Let f_j be as in* (3.20). *Suppose that $\boldsymbol{u}=(u_j)_{j\in Z}$ belongs to $\mathscr{F}(Y)$ and that the phase function φ vanishes at $y^0\in Y$. Then the infinite series $F(\boldsymbol{u})=\sum_{j=-\infty}^{\infty} f_j(\varphi)u_j$ converges normally in a* nhbd *of y^0 except $\{\varphi(y)=0\}$.*

PROOF. We have for any $K\Subset Y$

$$\sum_{j=-\infty}^{\infty} |f_j(\varphi)u_j|_K\leqq\sum_{j=1}^{\infty}\frac{|u_{-j}|_K}{\gamma(-j)}\Big|\frac{1}{\varphi}\Big|_K^j+\sum_{j=0}^{\infty}(|\log\varphi|_K+|\psi(j+1)|)\frac{|u_j|_K}{\gamma(j)}|\varphi|_K^j.$$

$\boldsymbol{u}\in\mathscr{F}(Y)$ means that the radius of convergence of $\sum_{j=0}^{\infty}(|u_{-j}|_K/\gamma(-j))t^j$ $\Big($resp. $\sum_{j=0}^{\infty}(|u_j|_K/\gamma(j))t^j\Big)$ is ∞ (resp. $R(K)>0$). Since $\varphi(y^0)=0$ and $|\psi(j+1)|\leqq j+1$, the assertion is now clear. Q.E.D.

Thus, roughly speaking, $\boldsymbol{u}\in\mathscr{F}(Y)$ corresponds to a function of the form $\sum_{j=-\infty}^{\infty} f_j(\varphi)u_j$.

Now according to L. Boutet de Monvel and P. Krée [3], we define (analytic) symbols of pseudo-differential operators. Let Ω^* be an open set in $\boldsymbol{C}_z^l\times(T^*\boldsymbol{C}^n\backslash 0)$.

DEFINITION 3.9. A formal series $P(z,x;\xi)=\sum_{k=-\infty}^{m} p_k(z,x;\xi)$ $(m\in\boldsymbol{Z})$ is said to be a symbol (of a pseudo-differential operator) of order$\leqq m$ in Ω^*, if

(S-1) all $p_k(z,x;\xi)$, $k=m,m-1,m-2,\cdots$, are holomorphic in Ω^* and homogeneous of degree k in ξ,

(S-2) for all $K^*\Subset\Omega^*$ there exist positive constants $C=C(K^*)$ and $B=B(K^*)$ such that

$$|p_{m-k}|_{K^*}\leqq CB^k k!,\quad k=0,1,2,\cdots. \tag{3.21}$$

We denote by $\mathscr{P}(\Omega^*)$ (resp. $\mathscr{P}^m(\Omega^*)$) the set of all symbols (resp. of order not more than m) in Ω^*.

In view of Cauchy's inequality, the requirement (S-2) is equivalent to

(S-3) For all $K^*\Subset\Omega^*$, there exist positive constants $C=C(K^*)$ and $A=A(K^*)$ such that

$$|p_{m-k,(\beta)}^{(\alpha)}|_{K^*}\leqq CA^{k+|\alpha+\beta|}\alpha!\,\beta!\,k!,\quad k=0,1,2,\cdots, \tag{3.22}$$

where $p_{m-k,(\beta)}^{(\alpha)}=(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta p_{m-k}$.

Following [3], we define a power series in $t\in\boldsymbol{C}$ by

$$M^m(P,K^*;t)=\sum_{\alpha,\beta,k}\left(\frac{2(2n)^{-k}k!}{(k+|\alpha|)!\,(k+|\beta|)!}\right)|p_{m-k,(\beta)}^{(\alpha)}|_{K^*}t^{2k+|\alpha+\beta|}. \tag{3.23}$$

Then the requirement (S-3) can be replaced by the following simpler one.

(S-4) $M^m(P,K^*;t)$ is a convergent power series in t for all $K^*\Subset\Omega^*$.

Let $P(z,x;\xi)=\sum_{k=-\infty}^{m}p_k(z,x;\xi)$ and $Q(z,x;\xi)=\sum_{k=-\infty}^{m'}q_k(z,x;\xi)$ be two symbols in $\mathscr{P}(\Omega^*)$. As usual, the symbol $R(z,x;\xi)=\sum_{k=-\infty}^{m+m'}r_k(z,x;\xi)$ defined by

$$r_k(z,x;\xi)=\sum_{i+j-|\alpha|=k}\frac{1}{\alpha!}p_i^{(\alpha)}(z,x;\xi)q_{j(\alpha)}(z,x;\xi), \tag{3.24}$$

is called the composite symbol of P and Q, denoted by $R=P\circ Q$. Since the sum in (3.24) is finite, it is clear that the r_k's are holomorphic in Ω^* and homogeneous of degree k in ξ.

LEMMA 3.10 (L. Boutet de Monvel and P. Krée [2]). *We have*

$$M^{m+m'}(P\circ Q,K^*;t)\ll M^m(P,K^*;t)M^{m'}(Q,K^*;t). \tag{3.25}$$

Consequently, the composite symbol $P\circ Q$ belongs to $\mathscr{P}^{m+m'}(\Omega^*)$.

A phase function φ on Y (i.e., $\varphi\in\mathscr{O}(Y)$) is said to be admissible (in Y with respect to Ω^*) if the graph $\bigcup_{(z,x)\in Y}(z,x;d_x\varphi(z,x))$ is contained in Ω^*. From now on we fix an admissible phase function φ and do not repeat this requirement explicitly, if there is no fear of confusion.

DEFINITION 3.11 (pseudo-differential operators). Let $P(z, x; \xi)=\sum_{k=-\infty}^{m} p_k(z, x; \xi)$ be a symbol in $\mathscr{P}(\Omega^*)$ and let φ be an admissible phase function. Then we define a linear operator $P(z, x; \partial_x): \mathscr{F}(\varphi; Y) \ni \boldsymbol{u} \mapsto \boldsymbol{v} \in \mathscr{F}(\varphi; Y)$ by the formula

$$v_j(z, x)=\sum \frac{1}{\alpha!\,\nu!} p_k^{(\alpha)}(z, x; d_x\varphi)\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}(z, x; \tilde{x})^{\nu} u_{j+k-|\alpha|+\nu}(z, x)|_{\tilde{x}=x}\}, \quad j \in \boldsymbol{Z}, \tag{3.26}$$

where the summation ranges over all k, α, ν such that $-\infty<k\leqq m$, $0\leqq|\alpha|$, $0\leqq 2\nu\leqq|\alpha|$. We call $P(z, x; \partial_x)$ a pseudo-differential operator (of order $\leqq m$) with the symbol $P(z, x; \xi)$.

REMARK 3.12. Summing up the terms for indices $|\alpha|-\nu=l$, (3.26) is rewritten (formally) as follows:

$$v_j(z, x)=\sum_{k=-\infty}^{m} \sum_{l=0}^{\infty} L_l(p_k; \varphi)(z, x; \partial_x) u_{j+k-l}, \quad j \in \boldsymbol{Z}. \tag{3.27}$$

Here $L_l(p_k; \varphi)$ are linear differential operators of order at most l given by

$$L_l(p_k; \varphi)u=\sum_{l\leq|\alpha|\leq 2l} \frac{1}{\alpha!(|\alpha|-l)!} p_k^{(\alpha)}(z, x; d_x\varphi)\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}(z, x; \tilde{x})^{|\alpha|-l} u)|_{\tilde{x}=x}\}. \tag{3.28}$$

For example, the first two operators are well known:

$$L_0(p; \varphi)p(z, x; d_x\varphi(z, x)), \tag{3.29}$$

$$L_1(p; \varphi)=\sum_{i=1}^{n} \frac{\partial p}{\partial \xi_i}(z, x; d_x\varphi)\frac{\partial}{\partial x_i}+\frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 p}{\partial\xi_i\partial\xi_j}(z, x; d_x\varphi)\frac{\partial^2\varphi}{\partial x_i\partial x_j}. \tag{3.30}$$

Definition (3.26) agrees with that of differential operators (see Proposition 3.1). Of course, it is necessary to show that the infinite series in (3.26) converges in a suitable sense to define an element of $\mathscr{F}(Y)$. To do this, we set $\boldsymbol{v}_k^{\alpha,\nu}=(v_{k,j}^{\alpha,\nu})_{j\in \boldsymbol{Z}}$ where

$$v_{k,j}^{\alpha,\nu}(z, x)=\frac{1}{\alpha!\,\nu!} p_k^{(\alpha)}(z, x; d_x\varphi)\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}(z, x; \tilde{x})^{\nu} u_{j+k-|\alpha|+\nu}(z, x))|_{\tilde{x}=x}\}, \tag{3.31}$$

Then we can write (formally)

$$P(z, x; \partial_x)\boldsymbol{u}=\sum_{k,\alpha,\nu} \boldsymbol{v}_k^{\alpha,\nu}. \tag{3.32}$$

The next proposition shows that the pseudo-differential operators are well-defined as operators on $\mathscr{F}(\varphi; Y)$.

PROPOSITION 3.13. *Let $P(z, x; \xi)$ and φ be as in Definition 3.11. Then the infinite series in (3.32) is normally convergent (in the sense of Definition 3.7). Explicitly, there exist positive constants $C=C(m)$ and $c=c(n, \varepsilon, |\varphi|_{K_\varepsilon})$ such that*

$$(3.33)\qquad N(\boldsymbol{Pu}, K; t) \ll \sum_{k,\alpha,\nu} N(\boldsymbol{v}_k^{\alpha,\nu}, K; t) \ll Ct^{-m} M'(P, K^*; ct) N_2(\boldsymbol{u}, K_\varepsilon; t),$$

where $K \Subset Y$, $K^* = \bigcup_{(z,x)\in K} (z, x; d_x\varphi(z, x)) \Subset \Omega^*$, ε is so small that $K_\varepsilon \Subset Y$ and $M'(P, K^*; ct)$ is a power series in t defined by

$$(3.34)\qquad M'(P, K^*; ct) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{|\alpha|=l}^{2l} (|p_{m-k}^{(\alpha)}|_{K^*}/k!\,\alpha!)(ct)^{k+l}.$$

REMARK. From (3.22), $M'(P, K^*; ct)$ is holomorphic in a nhbd of $t=0$ for every $c>0$, $K^* \Subset \Omega^*$ and $P(z, x; \xi) \in \mathscr{P}(\Omega^*)$.

LEMMA 3.14. *For any multi-index α and non-negative integer ν, we have the following estimates:*

$$(3.35)\qquad \left|\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}(z, x; \tilde{x})^{\nu} u)|_{\tilde{x}=x}\right|_K \leq \frac{\alpha!}{\varepsilon^{|\alpha|}} \binom{\alpha+\boldsymbol{\nu}}{\alpha} |\varphi|_{K_\varepsilon}^{\nu} |u|_{K_\varepsilon} \leq \alpha! \left(\frac{2}{\varepsilon}\right)^{|\alpha|} (2^n |\varphi|_{K_\varepsilon})^{\nu} |u|_{K_\varepsilon},$$

where $u \in \mathscr{O}(Y)$, $K \Subset Y$ and $\boldsymbol{\nu} = (\nu, \cdots, \nu)$.

PROOF. Since $\binom{i}{j} \leq 2^i$ $(i \geq j)$, the second inequality of (3.35) is trivial. We show the first inequality. By Leibniz's rule we have

$$\partial_x^{\alpha}(\tilde{\varphi}(z, x; \tilde{x})^{\nu} u) = \sum_{\alpha^1+\cdots+\alpha^{\nu}+\beta=\alpha} \frac{\alpha!}{\alpha^1! \cdots \alpha^{\nu}!\,\beta!} (\partial_x^{\alpha^1}\tilde{\varphi}) \cdots (\partial_x^{\alpha^{\nu}}\tilde{\varphi})(\partial_x^{\beta} u).$$

Substituting $\partial_x^{\gamma}\tilde{\varphi}|_{\tilde{x}=x}=0$ for $|\gamma|<2$ and $\partial_x^{\gamma}\tilde{\varphi}|_{\tilde{x}=x}=\partial_x^{\gamma}\varphi$ for $|\gamma| \geq 2$ and then applying the Cauchy's inequality to φ and u, we obtain

$$\begin{aligned}\left|\partial_x^{\alpha}(\tilde{\varphi}(z, x; \tilde{x})^{\nu} u)|_{\tilde{x}=x}\right|_K &\leq (\alpha!/\varepsilon^{|\alpha|}) |\varphi|_{K_\varepsilon}^{\nu} |u|_{K_\varepsilon} \Big(\sum_{\alpha^1+\cdots+\alpha^{\nu}+\beta=\alpha,\, |\alpha^i| \geq 2} 1\Big) \\ &\leq (\alpha!/\varepsilon^{|\alpha|}) |\varphi|_{K_\varepsilon}^{\nu} |u| \Big(\sum_{\alpha^1+\cdots+\alpha^{\nu}+\beta=\alpha} 1\Big).\end{aligned}$$

The first inequality now follows from the fact that the number of compositions of a integer r into k parts $\sum_{i=1}^{k} x_i = r$ $(x_i \geq 0)$ is just the binomial coefficient $\binom{k+r-1}{r}$. Q.E.D.

PROOF OF PROPOSITION 3.13. Since $\tilde{\varphi}(z, x; \tilde{x})$ vanishes quadratically on $\tilde{x}=x$, $\boldsymbol{v}_k^{\alpha,\nu}$ are null for $|\alpha|<2\nu$. Hence changing $|\alpha|-\nu$ to l and k to $m-k$, we obtain formally

$$P(z, x; \partial_x)\boldsymbol{u}=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\sum_{|\alpha|=l}^{2l}\boldsymbol{v}_{m-k}^{\alpha,|\alpha|-l}\right).$$

On the other hand, it follows from (3.35) that for $K\Subset Y$

$$N(\boldsymbol{v}_{m-k}^{\alpha,|\alpha|-l}, K; t)\ll\left(\frac{2}{\varepsilon}\right)^{|\alpha|}\frac{(2^n|\varphi|_{K_\varepsilon})^{|\alpha|-l}}{(|\alpha|-l)!}|p_{m-k}^{(\alpha)}|_{K^*}N(T_{k+l-m}\boldsymbol{u}, K_\varepsilon; t).$$

Applying Lemma 3.5 and then putting the terms in order, we finally obtain

$$N(\boldsymbol{v}_{m-k}^{\alpha,|\alpha|-l}, K; t)\ll\left(\frac{4^{|m|}\gamma(m)}{t^m}\right)\left(\frac{|p_{m-k}^{(\alpha)}|_{K^*}}{k!\,|\alpha|!}\right)\left(\frac{4\cdot 2^n|\varphi|_{K_\varepsilon}}{\varepsilon}\right)^{|\alpha|}\left(\frac{4t}{2^n|\varphi|_{K_\varepsilon}}\right)^l(4t)^k N_2(\boldsymbol{u}, K_\varepsilon; t).$$

Put $C=4^{|m|}\gamma(m)$ and $c=\max(4, 4/2^n|\varphi|_{K_\varepsilon}, 64\cdot 2^n|\varphi|_{K_\varepsilon}/\varepsilon^2)$, and we have (3.33). Q.E.D.

The last side of (3.33) is holomorphic in a punctured nhbd of $t=0$. Hence either infinite series in (3.26) or (3.27) is normally convergent in Y and $P\boldsymbol{u}=(v_j)_{j\in \boldsymbol{Z}}$ belongs to $\mathscr{F}(Y)$. Thus $P(z, x; \partial_x)$ is well-defined as an operator on $\mathscr{F}(Y)$.

COROLLARY 3.15. *Let $P(z, x; \xi)\in\mathscr{P}(\Omega^*)$. Suppose that an infinite series $\sum_{i=1}^{\infty}\boldsymbol{u}_i$ ($\boldsymbol{u}_i\in\mathscr{F}(Y)$) is normally convergent. Then the pseudo-differential operator $P(z, x; \partial_x)$ operates on the series termwise:*

$$P(z, x; \partial_x)\left(\sum_{i=1}^{\infty}\boldsymbol{u}_i\right)=\sum_{i=1}^{\infty}P(z, x; \partial_x)\boldsymbol{u}_i, \tag{3.36}$$

where the right side is again normally convergent.

PROOF. We put $\boldsymbol{u}_\infty=\sum_{i=1}^{\infty}\boldsymbol{u}_i$ and $\boldsymbol{v}_{h,k}^{\alpha;\nu}=(v_{h,k,j}^{\alpha;\nu})_{j\in\boldsymbol{Z}}$ ($h=1, 2, \cdots$, or ∞) where $v_{h,k,j}^{\alpha;\nu}$ are the functions obtained by replacing $u_{j+k-|\alpha|+\nu}$ by $u_{h,j+k-|\alpha|+\nu}$ in (3.31). By (3.33) we have for all $K\Subset Y$

$$\begin{aligned}(*)\qquad \sum_i N(P\boldsymbol{u}_i, K; t)&\ll\sum_i\sum_{k,\alpha,\nu}N(\boldsymbol{v}_{i,k}^{\alpha;\nu}, K; t)\\ &\ll Ct^{-m}M'(P, K^*; ct)\sum_i N_2(\boldsymbol{u}_i, K; t).\end{aligned}$$

The last side of (*) belongs to $\mathscr{M}_0$ by the hypothesis, which shows that $\sum_{i=1}^{\infty}P\boldsymbol{u}_i$ is normally convergent. The inequality (*) also implies

$$(**)\qquad \sum_{i,k,\alpha,\nu}|v_{i,k,j}^{\alpha;\nu}|_K<\infty,$$

for all $K\Subset Y$ and $j\in\boldsymbol{Z}$. On the other hand we have for all α, ν, k and j

$$(***)\qquad v_{\infty,k,j}^{\alpha;\nu}=\sum_{i=1}^{\infty}v_{i,k,j}^{\alpha;\nu}\quad\text{(normally convergent in } Y),$$

because of the normal convergence of $u_{\infty,j}=\sum_{i=1}^{\infty} u_{i,j}$ in Y by the hypothesis. It follows from (**) and (***) that

$$\sum_{k,\alpha,\nu} v^{\alpha,\nu}_{\infty,k,j}=\sum_{i=1}^{\infty}\left(\sum_{k,\alpha,\nu} v^{\alpha,\nu}_{i,k,j}\right),$$

for all $j\in \boldsymbol{Z}$. This shows $P\boldsymbol{u}_{\infty}=\sum_{i=1}^{\infty} P\boldsymbol{u}_i$. Q.E.D.

The following theorem is fundamental in the theory of pseudo-differential operators.

THEOREM 3.16. *Let $P(z, x; \xi)$ and $Q(z, x; \xi)$ be two symbols in $\mathscr{P}(\Omega^*)$. Let $R(z, x; \xi)$ denote the composite symbol $P\circ Q$. Then the equality*

$$R(z, x; \partial_x)\boldsymbol{u}=P(z, x; \partial_x)\circ Q(z, x; \partial_x)\boldsymbol{u} \tag{3.37}$$

holds for every $\boldsymbol{u}\in\mathscr{F}(Y)$, i.e., $R(z, x; \partial_x)=P(z, x; \partial_x)\circ Q(z, x; \partial_x)$.

Note that under our definition the product formula (3.37) holds without any modulo terms, which enables us to construct a formal solution exactly. We shall give a sketch of the lengthy proof of this theorem in Appendix A.

REMARK. It is often convenient to write $\boldsymbol{u}=\sum_j f_j(\varphi)\colon u_j$, instead of $\boldsymbol{u}=(u_j)_{j\in \boldsymbol{Z}}$, to mean that u_j is the jth component $\boldsymbol{u}\in\mathscr{F}(\varphi; Y)$. For example, $\boldsymbol{u}=f_{-2}(\varphi)\colon 1$ means that the jth component of $\boldsymbol{u}$ is equal to 0 for $j\neq -2$ and equal to 1 for $j=-2$. Suppose that for $\boldsymbol{u}, \boldsymbol{v}\in\mathscr{F}(\varphi; Y)$ the two functions $F(\boldsymbol{u})=\sum_{j=-\infty}^{\infty} f_j(\varphi)u_j$ and $F(\boldsymbol{v})=\sum_{j=-\infty}^{\infty} f_j(\varphi)v_j$ are equal to each other, then, of course, when $P(z, x; \partial_x)$ is a linear differential operator, the equality

$$F(P(z, x; \partial_x)\boldsymbol{u})=F(P(z, x; \partial_x)\boldsymbol{v}) \quad (=P(z, x; \partial_x)F(\boldsymbol{u}))$$

holds. However, this is not true, in general, when P is a pseudo-differential operator. For instance, let $\varphi=x_1$, $\boldsymbol{u}=f_{-2}(x_1)\colon 1$ and $\boldsymbol{v}=f_{-2}(x_1)\colon (1+x_1)+f_{-1}(x_1)\colon 1$, where f_j are defined by (3.20). Then $F(\boldsymbol{u})=F(\boldsymbol{v})=-(1/x_1^2)$, but we have

$$F((\partial/\partial x_1)^{-1}\boldsymbol{u})=F(f_{-1}(x_1)\colon 1)=1/x_1,$$

and

$$F((\partial/\partial x_1)^{-1}\boldsymbol{v})=F(f_{-1}(x_1)\colon (1+x_1))=(1/x_1)+1.$$

Here the symbol of $(\partial/\partial x_1)^{-1}$ is ξ_1^{-1} and $L_l(\xi_1^{-1}; x_1)=(-1)^l(\partial/\partial x_1)^l$ (see (3.27) and (3.28)). Thus we need to distinguish between the formal vector $\sum_j f_j(\varphi)\colon u_j$ and the function $\sum_j f_j(\varphi)u_j$. We will introduce an equivalence relation which seems intrinsic for the structure of $\mathscr{F}(\varphi; Y)$ under the actions of pseudo-differential operators.

§ 4. Preliminary formulas

We go back to the situation in Section 1. Let $\{\lambda_i(t, x; \xi): 1 \leqq i \leqq m\}$ denote an arbitrary one of the sets of characteristic roots $\{\lambda_i^\sigma: 1 \leqq i \leqq m_\sigma\}$, $\sigma = 1, 2, \cdots, \kappa$, in (1.2). Thus the λ_i's are assumed to satisfy

(C-1)′ $\lambda_i(t, x; \xi) \in \mathscr{O}(\Omega^*)$, $1 \leqq i \leqq m$, and homogeneous in ξ of degree one,

(C-3)′ there is a set of holomorphic functions $c_{ij} = c_{ij}(t, x; \xi) \in \mathscr{O}(\Omega^*)$, $1 \leqq i, j \leqq m$, which are homogeneous in ξ of degree 0 such that

$$\{\tau - \lambda_i, \tau - \lambda_j\} = c_{ij}(\lambda_j - \lambda_i).$$

Here Ω^* is an open nhbd of $(t, x; \xi) = (0, 0; 1, 0 \cdots 0) \in \boldsymbol{C} \times (T^*\boldsymbol{C}^n \backslash 0)$.

The multi-phase functions generated by λ_i's with initial datum x_1 (i.e., $\eta = (1, 0 \cdots 0)$ in (2.4) and (2.5)) are denoted by $\varphi^{(i)_l} = \varphi^{(i)_l}((t)_{i_l}, x)$. Since the number of elements of $J(m)$ is finite, we can choose open nhbds $T_0 \subset \boldsymbol{C}$ and $X_0 \subset \boldsymbol{C}^n$ of the origins of $\boldsymbol{C}$ and $\boldsymbol{C}^n$, respectively, which are common to all $(i)_l \in J(m)$ such that

$$\varphi^{(i)_l}((t)_{i_l}, x) \in \mathscr{O}(T_0^l \times X_0), \tag{4.1}$$

$$(t, x; d_x\varphi^{(i)_l}((t)_{i_l}, x)) \in \Omega^* \quad \text{for all} \quad ((t)_{i_l}, x) \in T_0^l \times X, \tag{4.2}$$

where $T_0^l = T_0 \times \cdots \times T_0 \subset \boldsymbol{C}^l$. Note also that

$$\varphi^{(i)_l}(0, x) = x_1, \quad \text{hence} \quad \varphi^{(i)_l}(0, 0) = 0. \tag{4.3}$$

In order to construct a formal solution we need to use the functions of the form

$$\sum_{(i)_l \in J(m)} \int_0^t dt_{i_{l-1}} \int_0^{t_{i_{l-1}}} dt_{i_{l-2}} \cdots \int_0^{t_{i_2}} dt_{i_1} \sum_{j=-\infty}^{\infty} f_j(\varphi^{(i)_l}) u_j^{(i)_l}, \tag{4.4}$$

where the sequence of functions f_j is the wave form defined by (3.20) (in what follows f_j always denote them). So we first modify the definitions in the preceding section slightly.

Let $T_1 \subset T_0$ (resp. $X_1 \subset X_0$) be an open nhbd of the origin of $\boldsymbol{C}$ (resp. $\boldsymbol{C}^n$) and let $\Omega = T_1 \times X_1$. Then we set

$$\tilde{\mathscr{F}}(\Omega) = \bigoplus_{(i)_l \in J(m)} \mathscr{F}(\varphi^{(i)_l}; T_1^l \times X_1) \quad \text{(direct sum)}. \tag{4.5}$$

We shall write $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}, \cdots$, for members of $\tilde{\mathscr{F}}(\Omega)$ whose $(i)_l$-th component is $\boldsymbol{u}^{(i)_l}$,

$v^{(i)_l}, \cdots$, respectively. For $\tilde{u}=\bigoplus_{(i)_l\in J(m)}(u_j^{(i)_l})_{j\in Z}$ we denote by $F(\tilde{u})$ the function defined by (4.4). Indeed, the infinite series $\sum_{j=-\infty}^{\infty} f_j(\varphi^{(i)_l})u_j^{(i)_l}$ are normally convergent in a nhbd of $((t)_{i_l}, x)=(0,0)$ except $\{\varphi^{(i)_l}=0\}$ (see Lemma 3.8) for every $(i)_l\in J(m)$, hence $F(\tilde{u})$ defines at least a germ of a holomorphic function at $(t, x)=(0, x^0)$ with $|x^0|$ sufficiently small such that $x_1^0\neq 0$. In view of this fact we shall write symbolically

$$\tilde{u}=\bigoplus_{(i)_l\in J(m)}\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}):u_j^{(i)_l}. \tag{4.6}$$

DEFINITION 4.1. (i) For a symbol $P(t, x; \xi)\in\mathscr{P}(\Omega^*)$ we define a pseudo-differential operator $P(t, x; \partial_x)$ on $\tilde{\mathscr{F}}(\Omega)$ by

$$P(t, x; \partial_x)\tilde{u}=\bigoplus_{(i)_l\in J(m)}P(t, x; \partial_x)u^{(i)_l}, \tag{4.7}$$

for $\tilde{u}=\bigoplus_{(i)_l\in J(m)}u^{(i)_l}\in\tilde{\mathscr{F}}(\Omega)$ (cf. Definition 3.11).

(ii) The derivatives of $\tilde{u}\in\tilde{\mathscr{F}}(\Omega)$ with respect to t is defined as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{(i)_l\in J(m)}\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}):u_j^{(i)_l} &=\sum_{(i)_l\in J(m)}\int^t\cdots\int\frac{\partial}{\partial t}\Big(\sum_j f_j(\varphi^{(j)_l}):u_j^{(i)_l}\Big)\\ &\quad+\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}):\Big(\sum_{\substack{(j)_{l+1}\in J(m)\\ (j_1\cdots j_l)=(i_1\cdots i_l)}}u_j^{(j)_{l+1}}|_{t_{j_l}=t}\Big).\end{aligned} \tag{4.8}$$

In particular, we have

$$\frac{\partial}{\partial t}\underbrace{\int^t\cdots\int}_{l-1}\sum_j f_j(\varphi^{(i)_l}):u_j=\underbrace{\int^t\cdots\int}_{l-1}\sum_j f_j(\varphi^{(i)_l}):\Big(\frac{\partial\varphi^{(i)_l}}{\partial t}u_{j+1}+\frac{\partial u_j}{\partial t}\Big)+\begin{cases}\underbrace{\int^t\cdots\int}_{l-2}\sum_j f_j(\varphi^{(i)_{l-1}}):(u_j|_{t_{i_{l-1}}=t}), & l\geqq 2,\\ 0, & l=1,\end{cases} \tag{4.9}$$

and

$$\frac{\partial}{\partial x_i}\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}):u_j=\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}):\Big(\frac{\partial\varphi^{(i)_l}}{\partial x_i}u_{j+1}+\frac{\partial u_j}{\partial x_i}\Big),\quad 1\leqq i\leqq n. \tag{4.10}$$

REMARK. It will be more suggestive to rewrite (4.7) in the form

$$(4.11)\qquad P(t, x; \partial_x) \sum_{(i)_l \in J(m)} \int^t \cdots \int \sum_j f_j(\varphi^{(i)_l}) \colon u_j^{(i)_l} = \sum_{(i)_l \in J(m)} \int^t \cdots \int P(t, x; \partial_x)\Big(\sum_j f_j(\varphi^{(i)_l}) \colon u_j^{(i)_l}\Big),$$

that is, pseudo-differential operators with respect to x commute with the (formal) integral sign $\int^t \cdots \int$.

REMARK. The above definition is consistent with that of usual derivatives, that is, the equalities

$$(4.12)\qquad \frac{\partial}{\partial t} F(\tilde{\boldsymbol{u}}) = F\Big(\frac{\partial}{\partial t}\tilde{\boldsymbol{u}}\Big), \qquad \frac{\partial}{\partial x_i} F(\tilde{\boldsymbol{u}}) = F\Big(\frac{\partial}{\partial x_i}\tilde{\boldsymbol{u}}\Big), \quad 1 \leqq i \leqq n,$$

hold for all $\tilde{\boldsymbol{u}} \in \tilde{\mathscr{F}}(\Omega)$. Therefore, when $P(t, x; \partial_t, \partial_x)$ is a linear differential operator, we have

$$(4.13)\qquad P(t, x; \partial_t, \partial_x) F(\tilde{\boldsymbol{u}}) = F(P(t, x; \partial_t, \partial_x)\tilde{\boldsymbol{u}}).$$

DEFINITION 4.2. We require a symbol $P(t, x; \tau, \xi) = \sum_{k=-\infty}^{\tilde{m}} p_k(t, x; \tau, \xi)$ $(\tilde{m} \in \boldsymbol{Z})$ in $\mathscr{P}(\Omega^* \times \boldsymbol{C}_\tau)$ to be a polynomial in τ, that is, all p_k are polynomials in τ of order at most $m \geqq 0$ (independent of k):

$$(4.14)\qquad p_k(t, x; \tau, \xi) = \sum_{d=0}^{m} p_{k-d}^d(t, x; \xi)\tau^d, \quad -\infty < k \leqq \tilde{m},$$

where p_{k-d}^d are homogeneous in ξ of degree $k-d$. We denote by $\tilde{\mathscr{P}}(\Omega^*)$ the set of all such symbols.

REMARK. We put for d, $0 \leqq d \leqq m$,

$$(4.15)\qquad P^d(t, x; \xi) = \sum_{k=-\infty}^{\tilde{m}-d} p_k^d(t, x; \xi),$$

then P^d are symbols in $\mathscr{P}^{\tilde{m}-d}(\Omega^*)$. Therefore we may regard the symbol $P(t, x; \tau, \xi)$ in $\tilde{\mathscr{P}}(\Omega^*)$ as a polynomial whose coefficients are symbols in $\mathscr{P}(\Omega^*)$:

$$(4.16)\qquad P(t, x; \tau, \xi) = \sum_{d=0}^{m} P^d(t, x; \xi)\tau^d.$$

DEFINITION 4.3. Let $P(t, x; \tau, \xi) = \sum_{d=0}^{m} P^d(t, x; \xi)\tau^d$ be a symbol in $\tilde{\mathscr{P}}(\Omega^*)$. Then we define the pseudo-differential operator $P(t, x; \partial_t, \partial_x)$ on $\tilde{\mathscr{F}}(\Omega)$ by

$$(4.17)\qquad P(t, x; \partial_t, \partial_x) = \sum_{d=0}^{m} P^d(t, x; \partial_x)\partial_t^d.$$

It is clear that for two symbols $P(t, x; \tau, \xi)=\sum_{k=-\infty}^{m} p_k(t, x; \tau, \xi)$ and $Q(t, x; \tau, \xi)=\sum_{k=-\infty}^{\tilde{m}'} q_k(t, x; \tau, \xi)$ in $\tilde{\mathscr{P}}(\Omega^*)$, the composite symbol $R=P\circ Q$:

$$
(4.18)\qquad \begin{cases} R(t, x; \tau, \xi)=\sum_{k=-\infty}^{\tilde{m}+\tilde{m}'} r_k(t, x; \tau, \xi), \\ r_k(t, x; \tau, \xi)=\sum_{i+j-\alpha_0-|\alpha|=k} \dfrac{1}{\tilde{\alpha}!} p_i^{(\tilde{\alpha})}(t, x; \tau, \xi) q_{j(\tilde{\alpha})}(t, x; \tau, \xi), \end{cases}
$$

($\tilde{\alpha}=(\alpha_0, \alpha)$) belongs to $\tilde{\mathscr{P}}(\Omega^*)$. Then the product formula holds without any modulo terms.

THEOREM 4.4. *Let $P(t, x; \tau, \xi)$ and $Q(t, x; \tau, \xi)$ be two symbols in $\tilde{\mathscr{P}}(\Omega^*)$ and $R(t, x; \tau, \xi)$ the composite symbol $P\circ Q$. Then we have*

$$
(4.19)\qquad R(t, x; \partial_t, \partial_x)\tilde{\boldsymbol{u}}=P(t, x; \partial_t, \partial_x)\circ Q(t, x; \partial_t, \partial_x)\tilde{\boldsymbol{u}}
$$

for all $\tilde{\boldsymbol{u}}\in\tilde{\mathscr{F}}(\Omega)$.

PROOF. In view of Theorem 3.16, it suffices to show (4.19) when $P=\partial/\partial t$, $Q=q_k(t, x; \xi)\in\mathscr{P}(\Omega^*)$ and $\tilde{\boldsymbol{u}}=\int^t\cdots\int\sum_j f_j(\varphi^{(i)}{}_l): u_j$. After a simple computation, we get

$$
\begin{aligned}
&\frac{\partial}{\partial t} q_k(t, x; \partial_x)\int^t\cdots\int\sum_j f_j(\varphi): u_j \\
&=\int^t\cdots\int\sum_j f_j(\varphi): \Bigg[\sum_{\alpha,\nu}\frac{1}{\alpha!\,\nu!}\Big\{\underbrace{(\partial_t q_k^{(\alpha)})(\cdot)\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}^{\nu} u_{j+k-|\alpha|+\nu})|_{\tilde{x}=x}\}}_{(\mathrm{I})} \\
&\quad\underbrace{+\sum_i q_k^{(\alpha+e_i)}(\cdot)\frac{\partial^2\varphi}{\partial t\partial x_i}\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}^{\nu} u_{j+k-|\alpha|+\nu})|_{\tilde{x}=x}\}}_{(\mathrm{II})} \\
&\quad\underbrace{+q_k^{(\alpha)}(\cdot)\Big\{\partial_{\tilde{x}}^{\alpha}\Big(\tilde{\varphi}^{\nu}\frac{\partial}{\partial t}u_{j+k-|\alpha|+\nu}\Big)\Big|_{\tilde{x}=x}\Big\}}_{(\mathrm{III})} \\
&\quad\underbrace{+\frac{\partial\varphi}{\partial t}q_k^{(\alpha)}(\cdot)\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}^{\nu} u_{j+1+k-|\alpha|+\nu})|_{\tilde{x}=x}\}}_{(\mathrm{IV})}\Big\} \\
&\quad\underbrace{+\sum_{\alpha,\nu,\nu\geq 1}\frac{1}{\alpha!\,\nu!}q_k^{(\alpha)}(\cdot)\Big\{\partial_{\tilde{x}}^{\alpha}\Big(\nu\tilde{\varphi}^{\nu-1}\frac{\partial\tilde{\varphi}}{\partial t}u_{j+k-|\alpha|+\nu}\Big)\Big|_{\tilde{x}=x}\Big\}}_{(\mathrm{V})}\Bigg] \\
&\quad+\int^t\cdots\int\underbrace{\Big\{q_k(t, x; \partial_x)\Big(\sum_j f_j(\varphi): u_j\Big)\Big\}\Big|_{t_{i_{l-1}}=t}}_{(\mathrm{VI})},
\end{aligned}
$$

where $\varphi=\varphi^{(i)_l}$, $(\cdot)=(t, x; d_x\varphi^{(i)_l})$ and $e_i=(0\cdots 0,\overset{i}{1},0\cdots 0)$. Substituting the expression $\partial\tilde{\varphi}/\partial t=(\partial\varphi/\partial t)(t, x)-(\partial\varphi/\partial t)(t, \tilde{x})-\sum_j (\partial^2\varphi/\partial t\partial x_i)(t, \tilde{x})\cdot(x_i-\tilde{x}_i)$ in (V) and then replacing $\nu-1$ by ν, we get

$$\begin{aligned}(\mathrm{V})=&\sum_{\alpha,\nu}\frac{1}{\alpha!\,\nu!}q_k^{(\alpha)}(\cdot)\left\{\partial_{\tilde{x}}^{\alpha}\left(\tilde{\varphi}^{\nu}\frac{\partial\varphi}{\partial t}u_{j+1+k-|\alpha|+\nu}\right)\Big|_{\tilde{x}=x}\right\}-(\mathrm{IV})\\ &\underbrace{-\sum_{\alpha,\nu}\frac{1}{\alpha!\,\nu!}q_k^{(\alpha)}(\cdot)\sum_i\frac{\partial^2\varphi}{\partial t\partial x_i}\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}^{\nu}(x_i-\tilde{x}_i)u_{j+k-|\alpha|+1+\nu})|_{\tilde{x}=x}\}}_{(\mathrm{VII})}.\end{aligned}$$

Changing the index α to $\alpha+e_i$, we obtain the identity (VII)=(II). On the other hand, no terms in (VI) contains the derivatives with respect to t or $t_{i_{l-1}}$, therefore we may carry out the restriction $t_{i_{l-1}}=t$ before operating $q_k(t, x; \partial_x)$. Summarizing the above results, we finally obtain

$$\begin{aligned}&\frac{\partial}{\partial t}q_k(t, x; \partial_x)\int^t\cdots\int\sum_j f_j(\varphi): u_j\\ =&\int^t\cdots\int\sum_j f_j(\varphi): \sum_{\alpha,\nu}\frac{1}{\alpha!\,\nu!}\Big\{(\partial_t q_k^{(\alpha)})(\cdot)\{\partial_{\tilde{x}}^{\alpha}(\tilde{\varphi}^{\nu}u_{j+k-|\alpha|+\nu})|_{\tilde{x}=x}\}\\ &+q_k^{(\alpha)}(\cdot)\{\partial_{\tilde{x}}^{\alpha}\Big(\tilde{\varphi}^{\nu}\Big(\frac{\partial\varphi}{\partial t}u_{j+1+k-|\alpha|+\nu}+\frac{\partial}{\partial t}u_{j+k-|\alpha|+\nu}\Big)\Big)\Big|_{\tilde{x}=x}\}\Big\}\\ &+\int^t\cdots\int q_k(t, x; \partial_x)\Big(\sum_j f_j(\varphi^{(i)_{l-1}}): (u_j|_{t_{i_{l-1}}=t})\Big),\end{aligned}$$

which shows that $(\partial/\partial t)q_k(t, x; \partial_x)=q_k(t, x; \partial_x)(\partial/\partial t)+(\partial_t q_k)(t, x; \partial_x)$. Q.E.D.

LEMMA 4.5. *There exist a set of linear differential operators* $I_{d-k,\mu}^d(\varphi^{(i)_l})$ $(d=0, 1, 2, \cdots, 0\leqq k\leqq d, 0\leqq\mu\leqq d-k)$ *of order* $\leqq\mu$ *such that*

$$(4.20)\qquad \begin{aligned}&\left(\frac{\partial}{\partial t}\right)^d\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}): u_j\\ &=\sum_{k=0}^{l-1}\int^t\cdots\int\sum_j f_j(\varphi^{(i)_{l-k}}): \left(\sum_{\mu=0}^{d-k}I_{d-k,\mu}^d u_{j+d-k-\mu}\right)\Big|_{t_{i_{l-k}}=\cdots=t_{i_{l-1}}=t},\end{aligned}$$

where $(i)_{l-k}=(i_1\cdots i_{l-k})$ *and* $I_{d-k,\mu}^d=0$ *for* $k>d$. *In particular*, $I_{d-k,0}^d$ *are given by*

$$(4.21)\qquad \begin{aligned}I_{d-k,0}^d=I_{d-k,0}^d(\varphi^{(i)_l})=&\sum_{\alpha_0+\cdots+\alpha_k=d-k}\left(\frac{\partial\varphi^{(i)_l}}{\partial t}\right)^{\alpha_0}\left(\frac{\partial\varphi^{(i)_l}}{\partial t}+\frac{\partial\varphi^{(i)_l}}{\partial t_{i_{l-1}}}\right)^{\alpha_1}\cdots\\ &\times\left(\frac{\partial\varphi^{(i)_l}}{\partial t}+\frac{\partial\varphi^{(i)_l}}{\partial t_{i_{l-1}}}+\cdots+\frac{\partial\varphi^{(i)_l}}{\partial t_{i_{l-k}}}\right)^{\alpha_k},\quad 0\leqq k\leqq d.\end{aligned}$$

Note that $I_{0,0}^d=1$ *for all* $d\geqq 0$.

PROOF. We define a set of linear differential operators j_{d-k}^d, $d=0, 1, 2, \cdots$,

$0 \leqq k \leqq d$, as follows:

$$j_{d-k}^{d}=\sum_{\alpha_0+\cdots+\alpha_k=d-k}\left(\frac{\partial}{\partial t}\right)^{\alpha_0}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t_{i_{l-1}}}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t_{i_{l-1}}}+\cdots+\frac{\partial}{\partial t_{i_{l-k}}}\right)^{\alpha_k}. \tag{4.22}$$

Then the following equalities are verified by induction on $d \in N$:

$$\left(\frac{\partial}{\partial t}\right)^{d}\int^{t}\cdots\int\sum_{j} f_j(\varphi^{(i)_l}): u_j=\sum_{k=0}^{l-1}\int^{t}\cdots\int\left\{j_{d-k}^{d}\left(\sum_{j} f_j(\varphi^{(i)_l}): u_j\right)\right\}\Bigg|_{t_{i_{l-k}}=\cdots=t}. \tag{4.23}$$

Hence (4.20) and (4.21) follow from Proposition 3.1. Q.E.D.

We now introduce an equivalence relation in $\tilde{\mathscr{F}}(\Omega)$. Two elements $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$ of $\tilde{\mathscr{F}}(\Omega)$ are said to be equivalent to each other, denoted by $\tilde{\boldsymbol{u}}\equiv\tilde{\boldsymbol{v}}$, if for any pseudo-differential operator P with a symbol in $\tilde{\mathscr{P}}(\Omega^*)$, $F(P\tilde{\boldsymbol{u}})$ and $F(P\tilde{\boldsymbol{v}})$ define the same germ of a holomorphic function. Obviously this equivalence relation possesses the following properties; let $\tilde{\boldsymbol{u}}_i\equiv\tilde{\boldsymbol{v}}_i$, $i=1, 2$, then $P\tilde{\boldsymbol{u}}_1+Q\tilde{\boldsymbol{u}}_2=P\tilde{\boldsymbol{v}}_1+Q\tilde{\boldsymbol{v}}_2$ where P and Q are arbitrary pseudo-differential operators with symbols in $\tilde{\mathscr{P}}(\Omega^*)$. As noted in the last Remark of the preceding section, the equality $F(\tilde{\boldsymbol{u}})=F(\tilde{\boldsymbol{v}})$ does not necessary imply the equivalence between $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$, we have however useful equivalent classes in $\tilde{\mathscr{F}}(\Omega)$.

PROPOSITION 4.6 (integration by parts). *Let l and k be integers such that $l\geqq 2$, $1\leqq k\leqq l-1$. Then we have*

$$\begin{aligned}&\int^{t}\cdots\int\sum_{j} f_j(\varphi^{(i)_l}): \left(\frac{\partial\varphi^{(i)_l}}{\partial t_{i_k}}u_{j+1}+\frac{\partial u_j}{\partial t_{i_k}}\right)\\ &\equiv\int^{t}\cdots\int\sum_{j} f_j(\varphi^{i_1\cdots\hat{i}_{k+1}\cdots i_l}): (u_j|_{t_{i_{k+1}}=t_{i_k}})\\ &\quad-\int^{t}\cdots\int\sum_{j} f_j(\varphi^{i_1\cdots\hat{i}_k\cdots i_l}): (u_j|_{t_{i_k}=t_{i_{k-1}}}),\end{aligned} \tag{4.24}$$

with the convention that $t_{i_l}=t$, $t_{i_0}=0$.

In order to prove the assertion, it is enough to show that for the operator $P(t, x; \partial_x)\partial_t^d$ where $P(t, x; \xi)$ is an arbitrary symbol in $\mathscr{P}(\Omega^*)$ and d an arbitrary non-negative integer, the following equality holds:

$$\begin{aligned}&F\left(P(t, x; \partial_x)\partial_t^d\int^{t}\cdots\int\sum_{j} f_j(\varphi^{(i)_l}): \left(\frac{\partial\varphi^{(i)_l}}{\partial t_{i_k}}u_{j+1}+\frac{\partial u_j}{\partial t_{i_k}}\right)\right)\\ &=F\Big\{P(t, x; \partial_x)\partial_t^d\int^{t}\cdots\int\sum_{j} f_j(\varphi^{i_1\cdots\hat{i}_{k+1}\cdots i_l}): (u_j|_{t_{i_{k+1}}=t_{i_k}})\\ &\quad-P(t, x; \partial_x)\partial_t^d\int^{t}\cdots\int\sum_{j} f_j(\varphi^{i_1\cdots\hat{i}_k\cdots i_l}): (u_j|_{t_{i_k}=t_{i_{k-1}}})\Big\}.\end{aligned}$$

This is easily verified by double induction on the indices $i=l-1-k$ and d. In the way of the induction we only use the usual integration by parts and the fact that the operators $P(t, x; \partial_x)$ and $\partial/\partial t_{i_k}$ are commutative with each other. So we leave the proof to the reader.

By (C-1)′ the pseudo-differential operators $\lambda_i(t, x; \partial_x)$ $(1\leqq i\leqq m)$ are well-defined. We may also assume that the functions $a_{\mu,\nu}^{(i)_l}=a_{\mu,\nu}^{(i)_l}((t)_{i_l}, x)$ $(\eta=(1, 0 \cdots 0))$ in Theorem 2.4 are holomorphic in $T_0^l\times X_0$ for all $(i)_l\in J(m)$, $1\leqq\mu\leqq m$ and $0\leqq\nu\leqq l-1$.

LEMMA 4.7. *Let* $\lambda_i(t, x; \partial_x)$ *and* $a_{\mu,\nu}^{(i)_l}$ *be as above and let* $h_i(t, x; \tau, \xi)=\tau-\lambda_i(t, x; \xi)$. *Then the following equality is true for each* k, $1\leqq k\leqq l-1$:

$$(4.25)\qquad \left(\frac{\partial}{\partial t}-\lambda_{i_k}(t, x; \partial_x)\right)\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l})\colon u_j$$
$$\equiv\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l})\colon\sum_{\mu=1}^{\infty}L_\mu^0(h_{i_k};\varphi^{(i)_l})u_{j+1-\mu}$$
$$+\int^t\cdots\int\sum_j f_j(\varphi^{(j)_{l-1}})\colon a_{i_k,k-1}^{(j)_{l-1}}(u_j|_{t_{i_k}=t_{i_{k-1}}}),$$

where $(j)_{l-1}=(i_1\cdots\hat{i}_k\cdots i_l)$ *and* $L_\mu^0(h_{i_k};\varphi^{(i)_l})$ *are linear differential operators of orders* $\leqq\mu$ *defined by*

$$(4.26)\qquad L_\mu^0(h_{i_k};\varphi^{(i)_l})=\begin{cases}\sum_{\nu=k}^{l-1}\frac{\partial}{\partial t_{i_\nu}}\circ a_{i_k,\nu}^{(i)_l}+\frac{\partial}{\partial t}-L_1(\lambda_{i_k};\varphi^{(i)_l}), & \mu=1,\\ -L_\mu(\lambda_{i_k};\varphi^{(i)_l}), & \mu\geqq 2.\end{cases}$$

See (3.28) *for the definition of* $L_\mu(\lambda_{i_k};\varphi^{(i)_l})$.

PROOF. It follows from (3.27) that

$$(*)\qquad \left(\frac{\partial}{\partial t}-\lambda_{i_k}(t, x; \partial_x)\right)\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l})\colon u_j$$
$$=\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l})\colon\left\{\left(\frac{\partial\varphi^{(i)_l}}{\partial t}-\lambda_{i_k}(t, x; d_x\varphi^{(i)_l})\right)u_{j+1}\right.$$
$$\left.-\sum_{\mu=1}^{\infty}L_\mu(\lambda_{i_k};\varphi^{(i)_l})u_{j+1-\mu}+\frac{\partial u_j}{\partial t}\right\}+\int^t\cdots\int\sum_j f_j(\varphi^{(i)_{l-1}})\colon(u_j|_{t_{i_{l-1}}=t}).$$

Applying Theorem 2.4 and Proposition 4.6, we obtain that the right side of (*) is equivalent to

$$\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l})\colon\sum_{\mu=1}^{\infty}L_\mu^0(h_{i_k};\varphi^{(i)_l})u_{j+1-\mu}$$
$$+\int^t\cdots\int\sum_j f_j(\varphi^{i_1\cdots\hat{i}_k\cdots i_l})\colon((a_{i_k,k}^{(i)_l}\cdot u_j)|_{t_{i_k}=t_{i_{k-1}}})$$
$$+\sum_{\nu=k}^{l-1}\int^t\cdots\int\sum_j f_j(\varphi^{i_1\cdots\hat{i}_{\nu+1}\cdots i_l})\colon((a_{i_k,\nu+1}^{(i)_l}-a_{i_k,\nu}^{(i)_l})u_j|_{t_{i_{\nu+1}}=t_{i_\nu}}).$$

It follows from (2.18) and (2.19) that the third term vanishes and that $a^{(i)l}_{i_k,k}|_{t_{i_k}=t_{i_{k-1}}}$ is equal to $a^{(j)l-1}_{i_k,k-1}$, which completes the proof.

§ 5. Construction of the formal solution

In this section we construct a formal solution of (CP). By the principle of superposition, it suffices to consider the case:

$$\text{(5.1)}\qquad \begin{cases} a(t, x; \partial_t, \partial_x)u(t, x)=0, \\ \partial_t^d u(0, x)=f_{j_0-d}(x_1)w_d(x), \quad 0\leqq d\leqq \tilde{m}-1, \end{cases}$$

where j_0 is an integer and w_d's are holomorphic in X_0 (we use the domains X_0, T_0, Ω^*, $\cdots$, introduced in Section 4). We may assume that $a(t, x; \tau, \xi)\in\tilde{\mathscr{P}}(\Omega^*)$. We look for the solution $u(t, x)$ to (5.1) in the following form:

$$\text{(5.2)}\qquad u(t, x)=\sum_{k=0}^{\infty} u_k(t, x), \qquad u_k(t, x)=\sum_{\sigma=1}^{\kappa} u_k^\sigma(t, x),$$

$$\text{(5.3)}\qquad u_k^\sigma(t, x)=F(\tilde{\boldsymbol{u}}_k^\sigma)=\sum_{j=-\infty}^{\infty} f_j(\varphi_\sigma^1)u_{k,j}^{\sigma,1}+\int_0^t dt_1 \sum_{j=-\infty}^{\infty} f_j(\varphi_\sigma^{12})u_{k,j}^{\sigma,2}+\cdots$$
$$+\int_0^t dt_{m_\sigma-1}\int_0^{t_{m_\sigma-1}} dt_{m_\sigma-2}\cdots\int_0^{t_2} dt_1 \sum_{j=-\infty}^{\infty} f_j(\varphi_\sigma^{12\cdots m_\sigma})u_{k,j}^{\sigma,m_\sigma},$$

where

$$\text{(5.4)}\qquad \tilde{\boldsymbol{u}}_k^\sigma=\sum_j f_j(\varphi_\sigma^1)\colon u_{k,j}^{\sigma,1}+\int^t\cdots\int\sum_j f_j(\varphi_\sigma^{12})\colon u_{k,j}^{\sigma,2}+\cdots+\int^t\cdots\int\sum_j f_j(\varphi_\sigma^{12\cdots m_\sigma})\colon u_{k,j}^{\sigma,m_\sigma}$$

are formal vectors. We also put

$$\text{(5.5)}\qquad u^\sigma(t, x)=\sum_{k=0}^{\infty} u_k^\sigma(t, x), \qquad \tilde{\boldsymbol{u}}^\sigma=\sum_{k=0}^{\infty}\tilde{\boldsymbol{u}}_k^\sigma.$$

Then, if we have

$$\text{(5.6)}\qquad a(t, x; \partial_t, \partial_x)u^\sigma(t, x)=0, \quad 1\leqq\sigma\leqq\kappa,$$

and

$$\text{(5.7)}\qquad \partial_t^d u_k(0, x)=\begin{cases} f_{j_0-d}(x_1)w_d(x), & k=0, \\ 0, & k\geqq 1, \end{cases} \quad 0\leqq d\leqq\tilde{m}-1,$$

obviously $u(t, x)$ is formally the solution of (5.1). First, making use of the calculus of the pseudo-differential operators, we solve the equation (5.6) for each σ separately. To do this, since we have

$$\text{(5.8)}\qquad a(t, x; \partial_t, \partial_x)u^\sigma(t, x)=F(a(t, x; \partial_t, \partial_x)\tilde{\boldsymbol{u}}^\sigma),$$

it suffices to construct $\tilde{\boldsymbol{u}}^{\sigma}=\sum_{k=0}^{\infty}\tilde{\boldsymbol{u}}_k^{\sigma}$ to satisfy the equation

$$a(t, x; \partial_t, \partial_x)\tilde{\boldsymbol{u}}^{\sigma}=\sum_{k=0}^{\infty} a(t, x; \partial_t, \partial_x)\tilde{\boldsymbol{u}}_k^{\sigma} \equiv 0. \tag{5.9}$$

Of course, it is necessary to show that (i) we can choose open nhbds $T_1\subset T_0$ and $X_1\subset X_0$ (of the origin) such that with $\Omega=T_1\times X_1$ $\tilde{\boldsymbol{u}}_k^{\sigma}\in\tilde{\mathscr{F}}(\Omega)$ for all $k=0, 1, 2, \cdots$, and (ii) the infinite series $\sum_{k=0}^{\infty}\tilde{\boldsymbol{u}}_k^{\sigma}$ converges normally in $\tilde{\mathscr{F}}(\Omega)$, which will be proved in Section 6.

Let us put for each i, $1\leqq i\leqq m_{\sigma}$,

$$q^i(t, x; \tau, \xi)=p(t, x; \tau, \xi)\Big/\prod_{j=1}^{i}(\tau-\lambda_j^{\sigma}(t, x; \xi)), \tag{5.10}$$

and

$$H_i(t, x; \partial_t, \partial_x)=(\partial_t-\lambda_1^{\sigma}(t, x; \partial_x))(\partial_t-\lambda_2^{\sigma}(t, x; \partial_x)) \cdots (\partial_t-\lambda_i^{\sigma}(t, x; \partial_x)). \tag{5.11}$$

Then Theorem 4.4 enables us to split the operator a in the following form:

$$a(t, x; \partial_t, \partial_x)=q^i(t, x; \partial_t, \partial_x)\circ H_i(t, x; \partial_t, \partial_x)-B^i(t, x; \partial_t, \partial_x), \tag{5.12}$$

where

$$\begin{cases} q^i(t, x; \tau, \xi)\in\tilde{\mathscr{P}}^{\tilde{m}-i}(\Omega^*), \\ H_i(t, x; \tau, \xi)\in\tilde{\mathscr{P}}^{i}(\Omega^*), \\ B^i(t, x; \tau, \xi)\in\tilde{\mathscr{P}}^{\tilde{m}-1}(\Omega^*), \end{cases} \tag{5.13}$$

From now on, we delete the super- or subscript σ when there is no fear of confusion. Using (5.12), we obtain

$$\begin{aligned} a(t, x; \partial_t, \partial_x)\sum_{k=0}^{\infty}\tilde{\boldsymbol{u}}_k=&\sum_{k=0}^{\infty}\sum_{i=1}^{m-1} q^i\circ H_i\Big(\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots i})\colon u_{k,j}^i\Big) \\ &+\sum_{k=0}^{\infty}\Big\{q^m\circ H_m\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots m})\colon u_{k,j}^m \\ &-\sum_{i=1}^{m} B^i\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots i})\colon u_{k-1,j}^i\Big\}, \end{aligned} \tag{5.14}$$

with the convention that $u_{k,j}^i=0$ for $k<0$. Hence (5.9) reduces to

$$H_i\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots i})\colon u_{k,j}^i\equiv 0, \quad 1\leqq i\leqq m-1, \quad k\geqq 0, \tag{5.15}$$

and

$$(5.16)\qquad q^m\circ H_m\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots m})\colon u_{k,j}^m\equiv\sum_{i=1}^m B^i\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots i})\colon u_{k-1,j}^i,\quad k\geqq 0.$$

LEMMA 5.1. *For* $1\leqq i\leqq m$ *we have*

$$(5.17)\qquad H_i\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots i})\colon u_j$$

$$\equiv\sum_{(i)_l\in J(i)}\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l})\colon\{L^{(i)_l}(\gamma^{(i)_l}u_j)+\mathscr{A}'^{(i)_l}(l+1,\infty;u_{j-1})\}.$$

Here $L^{(i)_l}=L^{(i)_l}((t)_{i_l},x;\partial_{t_{i_1}},\cdots,\partial_{t_{i_l}},\partial_x)$ *are linear differential operators of order* l *defined by*

$$(5.18)\qquad L^{(i)_l}=\Bigl(\prod_{\mu=1}^{i_1-1}a_{\mu,0}^{(i)_l}\Bigr)L_1^0(h_{i_1};\varphi^{(i)_l})\Bigl(\prod_{\mu=i_1+1}^{i_2-1}a_{\mu,1}^{(i)_l}\Bigr)L_1^0(h_{i_2};\varphi^{(i)_l})\cdots$$

$$\times L_1^0(h_{i_{l-1}};\varphi^{(i)_l})\Bigl(\prod_{\mu=i_{l-1}+1}^{i_l-1}a_{\mu,l-1}^{(i)_l}\Bigr)L_1(h_{i_l};\varphi^{(i)_l}),$$

$$(5.19)\qquad \mathscr{A}'^{(i)_l}(l+1,\infty;u_{j-1})=\sum_{\substack{\nu_1+\cdots+\nu_l\geqq l+1\\ \nu_i\geqq 1}}\Bigl(\prod_{\mu=1}^{i_1-1}a_{\mu,0}^{(i)_l}\Bigr)L_{\nu_1}^0(h_{i_1};\varphi^{(i)_l})\Bigl(\prod_{\mu=i_1+1}^{i_2-1}a_{\mu,1}^{(i)_l}\Bigr)\cdots$$

$$\times\Bigl(\prod_{\mu=i_{l-1}+1}^{i_l-1}a_{\mu,l-1}^{(i)_l}\Bigr)L_{\nu_l}(h_{i_l};\varphi^{(i)_l})(\gamma^{(i)_l}u_{j+l-(\nu_1+\cdots+\nu_l)}),$$

(*see* (4.26) *for the definition of* $L_\nu^0(h_{i_k};\varphi^{(i)_l})$), *and* $\gamma^{(i)_l}$ *are the operators of restriction defined by*

$$(5.20)\qquad \gamma^{(i)_l}u=u|_{t_1=\cdots=t_{i_1-1}=0,\,t_{i_1+1}=\cdots=t_{i_2-1}=t_{i_1},\cdots,t_{i_{l-1}+1}=\cdots=t_{i_l-1}=t_{i_{l-1}},\,t_{i_l}=\cdots=t}$$

$$=u(0\cdots\overset{i_1}{0},\,t_{i_1}\cdots\overset{i_2}{t_{i_1}},\,t_{i_2}\cdots t_{i_2},\,\cdots,\,\overset{i_{l-1}}{t_{i_{l-1}}}\cdots\overset{i_l}{t_{i_{l-1}}},\,t\cdots t,\,x).$$

PROOF. We show (5.17) to hold, for example, when $i=3$. Applying Lemma 4.7, we have

$$H_3\int^t\int\sum_j f_j(\varphi^{123})\colon u_j$$

$$=H_2\Bigl\{\int^t\int\sum_j f_j(\varphi^{123})\colon\sum_{\nu=1}^\infty L_\nu(h_3;\varphi^{123})u_{j+1-\nu}+\int^t\sum_j f_j(\varphi^{12})\colon(u_j|_{t_2=t})\Bigr\}$$

$$=H_1\Bigl\{\int^t\int\sum_j f_j(\varphi^{123})\colon\sum_{\nu_1,\nu_2=1}^\infty L_{\nu_1}^0(h_2;\varphi^{123})L_{\nu_2}(h_3;\varphi^{123})u_{j+2-(\nu_1+\nu_2)}$$

$$+\int^t\sum_j f_j(\varphi^{13})\colon\sum_{\nu=1}^\infty a_{21}^{13}L_\nu(h_3;\varphi^{13})(u_{j+1-\nu}|_{t_2=t_1})$$

$$+\int^t\sum_j f_j(\varphi^{12})\colon\sum_{\nu=1}^\infty L_\nu^0(h_2;\varphi^{12})(u_{j+1-\nu}|_{t_2=t})+\sum_j f_j(\varphi^1)\colon(u_j|_{t_1=t_2=t})\Bigr\}$$

$$=\int^t\int\sum_j f_j(\varphi^{123}):\sum_{\nu_1,\nu_2,\nu_3=1}^{\infty}L^0_{\nu_1}(h_1;\varphi^{123})L^0_{\nu_2}(h_2;\varphi^{123})L_{\nu_3}(h_3;\varphi^{123})u_{j+3-(\nu_1+\nu_2+\nu_3)}$$
$$+\int^t\sum_j f_j(\varphi^{23}):\sum_{\nu_1,\nu_2=1}^{\infty}a^{23}_{10}L^0_{\nu_1}(h_2;\varphi^{23})L^0_{\nu_2}(h_3;\varphi^{23})(u_{j+2-(\nu_1+\nu_2)}|_{t_1=0})$$
$$+\int^t\sum_j f_j(\varphi^{13}):\sum_{\nu_1,\nu_2=1}^{\infty}L^0_{\nu_1}(h_1;\varphi^{13})a^{13}_{21}L_{\nu_2}(h_3;\varphi^{13})(u_{j+2-(\nu_1+\nu_2)}|_{t_2=t_1})$$
$$+\int^t\sum_j f_j(\varphi^{12}):\sum_{\nu_1,\nu_2=1}^{\infty}L^0_{\nu_1}(h_1;\varphi^{12})L_{\nu_2}(h_2;\varphi^{12})(u_{j+2-(\nu_1+\nu_2)}|_{t_2=t})$$
$$+\sum_j f_j(\varphi^1):\sum_{\nu=1}^{\infty}L_\nu(h_1;\varphi^1)(u_{j+1-\nu}|_{t_1=t_2=t})+\sum_j f_j(\varphi^2):\sum_{\nu=1}^{\infty}a^2_{10}L_\nu(h_2;\varphi^2)(u_{j+1-\nu}|_{t_1=0,t_2=t})$$
$$+\sum_j f_j(\varphi^3):\sum_{\nu=1}^{\infty}a^3_{10}a^3_{20}L_\nu(h_3;\varphi^3)(u_{j+1-\nu}|_{t_1=t_2=0}),$$

which shows that (5.17) is valid for $i=3$. In general, (5.17) is proved by induction on i. Q.E.D.

REMARK. We can rewrite the differential operator $L^0_1(h_{i_k};\varphi^{(i)l})$ as follows:

$$L^0_1(h_{i_k};\varphi^{(i)l})=a^{(i)l}_{i_k,k}\left(\frac{\partial}{\partial t_{i_k}}+\frac{\partial}{\partial t_{i_{k+1}}}+\cdots+\frac{\partial}{\partial t_{i_{l-1}}}+\frac{\partial}{\partial t}\right)$$
$$+\sum_{\mu=k+1}^{l}(a^{(i)l}_{i_k,\mu}-a^{(i)l}_{i_k,\mu-1})\left(\frac{\partial}{\partial t_{i_\mu}}+\frac{\partial}{\partial t_{i_{\mu+1}}}+\cdots+\frac{\partial}{\partial t}\right)-L_1(\lambda_{i_k};\varphi^{(i)l})$$
$$+0\text{-th order terms.} \tag{5.21}$$

It follows from (4.15) and (4.16) that the operators B^i in (5.12) are in the following form:

$$\begin{cases} B^i(t,x;\partial_t,\partial_x)=\sum_{d=0}^{\tilde m-1}B^i_d(t,x;\partial_x)\partial^d_t, \\ B^i_d(t,x;\xi)=\sum_{k=-\infty}^{\tilde m-1-d}b^i_{d,k}(t,x;\xi)\in\mathscr{P}^{\tilde m-1-d}(\Omega^*), \end{cases} \tag{5.22}$$

with $b^i_{d,k}(t,x;\xi)$ homogeneous in ξ of degree k.

LEMMA 5.2. *Let B^i be as in* (5.22). *Then we have*

$$\sum_{i=1}^{m}B^i(t,x;\partial_t,\partial_x)\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots i}):u^i_j$$
$$=\sum_{l=1}^{m}\int^t\cdots\int\sum_j f_j(\varphi^{12\cdots l}):\sum_{i=0}^{m-l}\mathscr{B}^{l(l)}_i(0,\infty;u^{l+i}_{j+\tilde m-1-i}). \tag{5.23}$$

Here we have set for a sequence of functions u_j

$$\mathscr{B}^{l(l)}_i(0,\infty;u_j)=\sum_{d=i}^{\tilde m-1}\sum_{k=-\infty}^{\tilde m-1-d}\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{d-i}L_{\mu_1}(b^{l+i}_{d,k};\varphi^{12\cdots l})$$
$$\times\{(I^d_{d-i,\mu_2}u_{j-\tilde m+1+d+k-(\mu_1+\mu_2)})|_{t_l=t_{l+1}=\cdots=t_{l+i-1}=t}\},\quad 1\leq l\leq m. \tag{5.24}$$

See Lemma 4.5 for the definition of $I^d_{d-i,\mu}=I^d_{d-i,\mu}(\varphi^{12\cdots l+i})$.

PROOF. First applying Lemma 4.5 and (3.27), and then replacing the index $i-l$ by i, we obtain the desired result. Q.E.D.

LEMMA 5.3. *Let* $q^m(t, x; \tau, \xi)=\sum_{d=0}^{\tilde{m}-m} q_{\tilde{m}-m-d}(t, x; \xi)^d_\tau$, *where* $q_{\tilde{m}-m-d}$ *are homogeneous in* ξ *of degree* $\tilde{m}-m-d$, *then we have*

$$
(5.25)\qquad q^m(t, x; \partial_t, \partial_x)H_m \int^t\cdots\int \sum_j f_j(\varphi^{12\cdots m}) : u_j
$$

$$
=\sum_{(i)_l\in J(m)}\int^t\cdots\int\sum_j f_j(\varphi^{(i)_l}) : \{q^m(t, x; d_{t,x}\varphi^{(i)_l})L^{(i)_l}(\gamma^{(i)_l}u_{j+\tilde{m}-m})
$$

$$
+\mathscr{C}'^{(i)_l}(l+1, \infty; u_{j+\tilde{m}-m-1})\}.
$$

Here we have set for a sequence of functions u_j

$$
(5.26)\qquad \mathscr{C}'^{(i)_l}(l+1, \infty; u_{j-1})
$$

$$
=\sum_{i=0}^{m-l}\sum_{\substack{(j)_{l+i},\ \text{s.t.}\ (j)_l=(i)_l \\ \mu_1+\mu_2+\nu_1+\cdots+\nu_{l+i}\geqq l+1}}\sum_{d=i}^{\tilde{m}-m}\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{d-i}\sum_{\nu_1\cdots\nu_{l+i}=1}^{\infty} L_{\mu_1}(q_{\tilde{m}-m-d}; \varphi^{(i)_l})
$$

$$
\times\left\{\left\{I^d_{d-i,\mu_2}\left(\prod_{\mu=1}^{j_1-1} a^{(j)_{l+i}}_{\mu,0}\right)L^0_{\nu_1}(h_{j_1}; \varphi^{(j)_{l+i}})\cdots L^0_{\nu_{l+i}}(h_{j_{l+i}}; \varphi^{(j)_{l+i}})\right.\right.
$$

$$
\left.\left.\times(\gamma^{(j)_{l+i}}u_{j+l-(\mu_1+\mu_2+\nu_1+\cdots+\nu_{l+i})})\right\}\Big|_{t_{j_l}=t_{j_{l+1}}=\cdots=t_{j_{l+i-1}}=t}\right\},
$$

with $I^d_{d-i,\mu}=I^d_{d-i,\mu}(\varphi^{(j)_{l+i}})$.

PROOF. A simple application of Lemmas 5.1, 4.5 and (3.27) yields the result. Q.E.D.

Let us put $\tilde{u}^i_{k,j}(\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_i, x)$ as follows:

$$
(5.27)\qquad \tilde{u}^i_{k,j}(t_1, t_2-t_1, t_3-t_2, \cdots, t-t_{i-1}; x)=u^i_{k,j}(t_1, t_2, \cdots, t_{i-1}, t, x).
$$

Then we have

$$
\gamma^{(i)_l}u^i_{k,j}=\tilde{u}^i_{k,j}(0\cdots 0, \overset{i_1}{t_{i_1}}, 0\cdots 0, \overset{i_2}{t_{i_2}-t_{i_1}}, 0\cdots 0, \overset{i_l}{t-t_{i_{l-1}}}, 0\cdots 0, x).
$$

Hence it is convenient to use a new coordinate system $(\tilde{t})_{i_l}=(\tilde{t}_{i_1}, \tilde{t}_{i_2}, \cdots, \tilde{t}_{i_l})$ for each $(i)_l$ instead of $(t)_{i_l}$ defined by

$$
(5.28)\qquad \begin{cases}\tilde{t}_{i_1}=t_{i_1}\\ \tilde{t}_{i_2}=t_{i_2}-t_{i_1}\\ \cdots\\ \tilde{t}_{i_l}=t-t_{i_{l-1}}\end{cases}\qquad \begin{cases}t_{i_1}=\tilde{t}_{i_1}\\ t_{i_2}=\tilde{t}_{i_1}+\tilde{t}_{i_2}\\ \cdots\\ t=\tilde{t}_{i_1}+\tilde{t}_{i_2}+\cdots+\tilde{t}_{i_l}.\end{cases}
$$

We use the symbol "$\tilde{\ }$" to functions or differential operators to mean that they are to be considered as functions or differential operators in transformed variables, for example, $\tilde{a}^{(i)_l}_{\mu,\nu}$, $\tilde{L}^0_\nu(h_{i_k};\varphi^{(i)_l})$, $\tilde{\mathscr{A}}'^{(i)_l}(l+1,\infty;u_{j-1})$, etc. Then, since we have by (5.21)

$$(5.29)\qquad \tilde{L}^0_1(h_{i_k};\varphi^{(i)_l})=\tilde{a}^{(i)_l}_{i_k,k}\frac{\partial}{\partial\tilde{t}_{i_k}}+\sum_{\mu=k+1}^{l}(\tilde{a}^{(i)_l}_{i_k,\mu}-\tilde{a}^{(i)_l}_{i_k,\mu-1})\frac{\partial}{\partial\tilde{t}_{i_\mu}}-\tilde{L}_1(\lambda_{i_k};\varphi^{(i)_l})$$
$$+0\text{-th order terms},$$

and by (2.19) the functions $\tilde{a}^{(i)_l}_{i_k,\mu}-\tilde{a}^{(i)_l}_{i_k,\mu-1}$ vanish on $\tilde{t}_{i_\mu}=0$, the differential operators $\tilde{L}^{(i)_l}$ reduce to the following form:

$$(5.30)\qquad \tilde{L}^{(i)_l}=\tilde{a}^{(i)_l}\left\{\frac{\partial^l}{\partial\tilde{t}_{i_1}\partial\tilde{t}_{i_2}\cdots\partial\tilde{t}_{i_l}}-\tilde{R}^{(i)_l}\left((\tilde{t})_{i_l},x;\frac{\partial}{\partial\tilde{t}_{i_1}},\cdots,\frac{\partial}{\partial\tilde{t}_{i_l}},\frac{\partial}{\partial x}\right)\right\}.$$

Here $\tilde{a}^{(i)_l}=\left(\prod_{\mu=1}^{i_1-1}\tilde{a}^{(i)_l}_{\mu,0}\right)\left(\prod_{\mu=i_1+1}^{i_2-1}\tilde{a}^{(i)_l}_{\mu,1}\right)\cdots\left(\prod_{\mu=i_{l-1}+1}^{i_l-1}\tilde{a}^{(i)_l}_{\mu,l-1}\right)$ is equal to 1 on $\tilde{t}_{i_1}=\tilde{t}_{i_2}=\cdots=\tilde{t}_{i_l}=0$ and the differential operator $\tilde{R}^{(i)_l}=\tilde{P}^{(i)_l}+\tilde{Q}^{(i)_l}$ has the following properties:

(5.31) $\tilde{P}^{(i)_l}$ is a homogeneous differential operator of order l in $\partial/\partial\tilde{t}_{i_1}$, $\partial/\partial\tilde{t}_{i_2}$, $\cdots$, $\partial/\partial\tilde{t}_{i_l}$, and all the coefficients vanish on $\tilde{t}_{i_1}=\cdots=\tilde{t}_{i_l}=0$.

(5.32) $\tilde{Q}^{(i)_l}$ is a differential operator of order $\leqq l$ in $\partial/\partial\tilde{t}_{i_1},\cdots,\partial/\partial\tilde{t}_{i_l},\partial/\partial x$, and of order $\leqq l-1$ in $\partial/\partial\tilde{t}_{i_1},\cdots,\partial/\partial\tilde{t}_{i_l}$.

(5.33) Let $\tilde{R}^{(i)_l}((\tilde{t})_{i_l},x;\tilde{\tau}_{i_1}\cdots\tilde{\tau}_{i_l},\xi)=\sum r_{\alpha,\beta}\tilde{\tau}^{\alpha_1}_{i_1}\tilde{\tau}^{\alpha_2}_{i_2}\cdots\tilde{\tau}^{\alpha_l}_{i_l}\xi^\beta$, then $(\underbrace{1\cdots 1}_{l};0\cdots 0)\in\boldsymbol{R}^{l+n}$ is outside the convex hull of the set $\{(\alpha,\beta)\in\boldsymbol{R}^{l+n}: r_{\alpha,\beta}\neq 0,\ |\alpha|+|\beta|=l\}$.

In view of the above facts, Lemmas 5.1, 5.2 and 5.3, the equations (5.15) and (5.16) finally reduce to the following systems of equations of the Goursat type:

$$(T)_{k,j}\quad\begin{cases}\dfrac{\partial^l}{\partial\tilde{t}_{i_1}\cdots\partial\tilde{t}_{i_l}}(\tilde{\gamma}^{(i)_l}\tilde{u}^{\sigma,i}_{k,j-i+1})=\tilde{R}^{(i)_l}(\tilde{\gamma}^{(i)_l}\tilde{u}^{\sigma,i}_{k,j-i+1})+\tilde{\mathscr{A}}^{(i)_l}(l+1,\infty;\tilde{u}^{\sigma,i}_{k,j-i}),\\ \qquad\qquad (i)_l\in J(i),\quad\text{for}\quad i=1,2,\cdots,m_\sigma-1,\\ \dfrac{\partial^l}{\partial\tilde{t}_{i_1}\cdots\partial\tilde{t}_{i_l}}(\tilde{\gamma}^{(i)_l}\tilde{u}^{\sigma,m_\sigma}_{k,j-m_\sigma+1})=\tilde{R}^{(i)_l}(\tilde{\gamma}^{(i)_l}\tilde{u}^{\sigma,m_\sigma}_{k,j-m_\sigma+1})+\tilde{\mathscr{C}}^{(i)_l}(l+1,\infty;\tilde{u}^{\sigma,m_\sigma}_{k,j-m_\sigma})\\ \qquad\qquad +\displaystyle\sum_{i=0}^{m_\sigma-l}\tilde{\mathscr{B}}^{(i)_l}_i(0,\infty;\tilde{u}^{\sigma,l+i}_{k-1,j-(i+l)+1+(l-1)}),\\ \qquad\qquad (i)_l\in J(m_\sigma),\quad\text{for}\quad 1\leqq\sigma\leqq\kappa,\ k\geqq 0\ \text{and}\ j\in\boldsymbol{Z}.\end{cases}$$

Here we have put

$$(5.34)\quad \begin{cases} \tilde{\mathscr{A}}^{(i)}{}_l(l+1, \infty; \tilde{u}_{j-1}) = -(1/\tilde{a}^{(i)}{}_l)\tilde{\mathscr{A}}'^{(i)}{}_l(l+1, \infty; \tilde{u}_{j-i}), \\ \tilde{\mathscr{B}}_i^{(i)}{}_l(0, \infty; \tilde{u}_{j-i}) = \begin{cases} (1/(\tilde{a}^{(i)}{}_l\tilde{q}^m))\tilde{\mathscr{B}}_i'^{(i)}(0, \infty; \tilde{u}_{j-i}), & (i)_l = (1\,2\cdots l), \\ 0, & (i)_l \neq (1\,2\cdots l), \end{cases} \\ \tilde{\mathscr{C}}^{(i)}{}_l(l+1, \infty; \tilde{u}_{j-m}) = -(1/(\tilde{a}^{(i)}{}_l\tilde{q}^m))\tilde{\mathscr{C}}'^{(i)}{}_l(l+1, \infty; \tilde{u}_{j-m}), \end{cases}$$

and

$$(5.35)\qquad \tilde{\gamma}^{(i)}{}_l\tilde{u} = \tilde{u}(0\cdots 0, \overset{i_1}{\tilde{t}_{i_1}}, 0\cdots 0, \overset{i_2}{\tilde{t}_{i_2}}, 0\cdots 0, \overset{i_l}{\tilde{t}_{i_l}}, 0\cdots 0; x).$$

By (C-2), $\tilde{q}^m = \tilde{q}^m(t, x; d_{t,x}\varphi^{(i)}{}_l) \neq 0$ on $\tilde{t}_{i_1} = \cdots = \tilde{t}_{i_l} = 0$, hence we may assume without loss of generality that $\tilde{a}^{(i)}{}_l\tilde{q}^m$ never vanish on $T_0^l \times X_0$. Note also that for a sequence of functions $\tilde{u}_j$ we have

$$\tilde{\mathscr{A}}^{(i)}{}_l(l+1, \infty; \tilde{u}_j) = \sum_{k=0}^{\infty} p_{l+1+k}\tilde{u}_{j-k}, \qquad \tilde{\mathscr{B}}_i^{(i)}{}_l(0, \infty; u_j) = \sum_{k=0}^{\infty} p_k'\tilde{u}_{j-k},$$

and

$$\tilde{\mathscr{C}}^{(i)}{}_l(l+1, \infty; \tilde{u}_j) = \sum_{k=0}^{\infty} p_{l+1+k}''\tilde{u}_{j-k},$$

with some differential operators p_k, p_k' and p_k'' of orders not more than k. We, also, use the notation $\mathscr{O}(\mu, \nu; \tilde{u}_j)$ in [8] when $\nu \neq \infty$, that is, $\mathscr{O}(\mu, \nu; \tilde{u}_j) = \sum_{k=0}^{\nu-\mu} q_{\mu+k}\tilde{u}_{j-k}$, with some differential operators q_k of orders at most k for two non-negative integers μ, ν such that $\mu \leqq \nu$.

Next we observe the initial conditions (5.7). Since the terms which involve the integrals $\int_0^t \cdots$ vanish on $t=0$, applying Lemma 4.5, we obtain

$$(5.36)\quad \partial_t^d u_k(0, x) = \sum_j f_{j-d}(x_1) \sum_{\sigma=1}^{\kappa} \left\{ \sum_{i=1}^{d+1} g_{d,i}^{\sigma} u_{k,j-i+1}^{\sigma;i} - \sum_{i=1}^{d} \mathscr{O}(1, d-i+1; u_{k,j-i}^{\sigma;i})|_{t_1=\cdots=t=0} \right\},$$

where $u_{k,j-i}^{\sigma;i} = 0$ for $i > m_\sigma$, and

$$(5.37)\quad g_{d,i}^{\sigma} = g_{d,i}^{\sigma}(x) = \begin{cases} I_{d-i+1,0}^d(\varphi_\sigma^{12\cdots i})|_{t_1=\cdots=t=0}, & 0 \leqq d \leqq \tilde{m}-1, \quad 1 \leqq i \leqq \min(d+1, m_\sigma), \\ 0, & \text{otherwise}. \end{cases}$$

From (4.21), $g_{d,i}^{\sigma}$ are given explicitly by

$$(5.38)\qquad g_{d,i}^{\sigma}(x) = \sum_{\alpha_1+\cdots+\alpha_i = d-i+1, \alpha_k \geqq 0} (\lambda_1^{\sigma})^{\alpha_1}(\lambda_2^{\sigma})^{\alpha_2}\cdots(\lambda_i^{\sigma})^{\alpha_i},$$

with $\lambda_k^{\sigma} = \lambda_k^{\sigma}(0, x; 1, 0\cdots 0)$.

We set column vectors $\vec{u}_{k,j}$, $\vec{w}$ and $\vec{\mathscr{O}}(1, \tilde{m}-1; \vec{u}_{k,j-1})$ as follows:

$$(5.39)\quad \begin{cases} \vec{u}_{k,j}={}^t(u^{1,1}_{k,j},\cdots,u^{\sigma,1}_{k,j},u^{\sigma,2}_{k,j-1},\cdots,u^{\sigma,m_\sigma}_{k,j-m_\sigma+1},\cdots,u^{\kappa,m_\kappa}_{k,j-m_\kappa+1}) \\ \vec{w}(x)={}^t(w_1(x),w_2(x),\cdots,w_d(x),\cdots,w_{\tilde{m}-1}(x)) \\ \vec{\mathcal{O}}(1,\tilde{m}-1;\vec{u}_{k,j-1})=\sum_{\sigma=1}^{\kappa}{}^t\Big(0,\mathcal{O}(1,1;u^{\sigma,1}_{k,j-1}),\sum_{i=1}^{2}\mathcal{O}(1,2-i+1;u^{\sigma,2}_{k,j-2}),\cdots, \\ \qquad\sum_{i=1}^{d}\mathcal{O}(1,d-i+1;u^{\sigma,i}_{k,j-i}),\cdots,\sum_{i=1}^{\tilde{m}-1}\mathcal{O}(1,\tilde{m}-i;u^{\sigma,i}_{k,j-i})\Big). \end{cases}$$

Let $G=(G^1,\cdots,G^\sigma,\cdots,G^\kappa)$ denote $\tilde{m}$ by $\tilde{m}$ matrix, where $G^\sigma=G^\sigma(x)$ are $\tilde{m}$ by m_σ matrices defined by

$$(5.40)\qquad G^\sigma(x)=\left(g^\sigma_{d,i}(x):\begin{matrix} d=0,1,\cdots,\tilde{m}-1,\downarrow \\ i=1,2,\cdots,m_\sigma,\rightarrow\end{matrix}\right),\quad 1\leqq\sigma\leqq\kappa.$$

Then, from (5.36) the initial conditions (5.7) reduce to

$$(5.41)\qquad G(x)(\vec{u}_{k,j}|_{t_1=\cdots=t=0})=\delta_{k0}\delta_{jj_0}\vec{w}+\vec{\mathcal{O}}(1,\tilde{m}-1;\vec{u}_{k,j-1})|_{t_1=\cdots=t=0},$$

with the Kronecker's delta δ_{ij}. Note that substituting (5.38) we can work out the determinant of G explicitly, that is,

$$(5.42)\qquad \det G(x)=\prod_{i,j,\sigma>\sigma'}(\lambda^\sigma_i-\lambda^{\sigma'}_j)(0,x;1,0\cdots0).$$

Consequently the determinant of G does not vanish at $x=0$ by (C-2), hence we may assume that the inverse matrix $G^{-1}(x)$ of $G(x)$ exists and is holomorphic on X_0. This allows us to replace (5.41) by

$$(I)_{k,j}\qquad \vec{u}_{k,j}|_{t_1=\cdots=t=0}=G^{-1}(x)\{\delta_{k0}\delta_{jj_0}\vec{w}+\vec{\mathcal{O}}(1,\tilde{m}-1;\vec{u}_{k,j-1})|_{t_1=\cdots=t=0}\}.$$

Thus we have obtained transport equations $(T)_{k,j}$ with initial data $(I)_{k,j}$. The existence and the uniqueness of the solutions to the Goursat problem in $(T)_{k,j}$ are guaranteed by Proposition B.4 in Appendix B. Hence we can determine $u^{\sigma,i}_{k,j}$ recursively by solving $(T)_{k,j}$ with $(I)_{k,j}$. We can easily see that

$$(5.43)\qquad u^{\sigma,i}_{k,j}=0,\quad \text{for}\quad j<j_0-i+1-(m^*-1)k,\quad 1\leqq i\leqq m_\sigma,\quad 1\leqq\sigma\leqq\kappa,$$

with $m^*=\max_{1\leqq\sigma\leqq\kappa} m_\sigma$. Furthermore, if we observe $(T)_{k,j}$ and $(I)_{k,j}$ carefully, we obtain more explicitly

$$(5.44)\qquad \vec{u}_{k,j}|_{t_1=\cdots=t=0}=0,\quad \text{for}\quad j<j_0,$$

and

$$(5.45)\qquad u^{\sigma,i}_{k,j}=0,\quad \begin{matrix} j\leqq j_0-i & \text{for} & i=1,2,\cdots,m_\sigma-1, & \text{or} \\ j<j_0-(m_\sigma-1)(k+1) & \text{for} & i=m_\sigma. & \end{matrix}$$

§ 6. Convergence of the formal solution

In this section we prove the convergence of the formal solution constructed in the preceding section by the method of majorant. Our method is essentially the same as that used in [8]. Since we shall only use the transformed variables, we omit "$\tilde{\ }$".

Since the coefficients of differential operators in $(T)_{k,j}$ or $(I)_{k,j}$ are derivatives of finitely many holomorphic functions, we may assume that they are all holomorphic in a polydisc $D^{l+n}(R'')\subset C^{l+n}$ of radius $R''>0$ and that we can choose open nhbd ω of $\xi=(1, 0\cdots 0)$ such that λ_i, $b^i_{d,k}$, $q_{\tilde{m}-m-d}$, $\cdots$, are holomorphic in $D^{l+n}(R'')$ and $d_x\varphi^{(i)_l}((t)_{i_l}, x)\in\omega$ for all $((t)_{i_l}, x)\in D^{l+n}(R'')$. Note that for any holomorphic function $g=g((t)_{i_l}, x)$ on $D^{l+n}(R'')$ the following majorant inequality holds:

$$g\ll R'|g|_{D^{l+n}(R')}/(R'-z), \tag{6.1}$$

where $0<R'<R''$ and $z=\rho(t_{i_1}+\cdots+t_{i_l})+x_1+\cdots+x_n$ with $\rho\geqq 1$.

Let $\mathscr{L}^0_\nu(h_{i_k};\varphi^{(i)_l})$, $\mathscr{P}^{(i)_l}$, $\mathscr{Q}^{(i)_l}$, $\mathscr{I}^d_{d-k,\mu}$, $\cdots$, be majorant differential operators of $L^0_\nu(h_{i_k};\varphi^{(i)_l})$, $P^{(i)_l}$, $Q^{(i)_l}$, $I^d_{d-k,\mu}$, $\cdots$, respectively, that is, the differential operators obtained by replacing the coefficients of corresponding one by the majorant functions of the form in (6.1) (they depend on the choice of R' $(<R'')$, which will be determined later). We denote by $\mathfrak{A}^{(i)_l}(l+1,\infty;u_j)$, $\mathfrak{B}^{(i)_l}_i(0,\infty;u_j)$, $\mathfrak{C}^{(i)_l}(l+1,\infty;u_j)$ the functions obtained by replacing the differential operators in the definition of $\mathscr{A}^{(i)_l}(l+1,\infty;u_j)$, $\mathscr{B}^{(i)_l}_i(0,\infty;u_j)$, $\mathscr{C}^{(i)_l}(l+1,\infty;u_j)$ by the corresponding majorant differential operators.

Consider the following majorant Goursat problem of $(T)_{k,j}$:

$$(MT)_{k,j}\quad\begin{cases}\dfrac{\partial^l}{\partial t_{i_1}\cdots\partial t_{i_l}}(\gamma^{(i)_l}U^{\sigma,i}_{k,j-i+1})\gg\mathscr{R}^{(i)_l}(\gamma^{(i)_l}U^{\sigma,i}_{k,j-i+1})+\mathfrak{A}^{(i)_l}(l+1,\infty;U^{\sigma,i}_{k,j-i}),\\ \qquad\qquad (i)_l\in J(i)\quad\text{for}\quad i=1,2,\cdots,m_\sigma-1,\\ \dfrac{\partial^l}{\partial t_{i_1}\cdots\partial t_{i_l}}(\gamma^{(i)_l}U^{\sigma,m_\sigma}_{k,j-m_\sigma+1})\gg\mathscr{R}^{(i)_l}(\gamma^{(i)_l}U^{\sigma,m_\sigma}_{k,j-m_\sigma+1})+\mathfrak{C}^{(i)_l}(l+1,\infty;U^{\sigma,m_\sigma}_{k,j-m_\sigma})\\ \qquad\qquad +\sum_{i=0}^{m_\sigma-l}\mathfrak{B}^{(i)_l}_i(0,\infty;U^{\sigma,i+l}_{k-1,j-i}),\quad (i)_l\in J(m_\sigma),\end{cases}$$

with initial conditions

$$(MI)_{k,j}\qquad \vec{U}_{k,j}(0\cdots 0,x)\gg\mathscr{G}(\delta_{k0}\delta_{jj_0}\vec{W}(x)+\vec{\mathfrak{D}}(1,\tilde{m}-1;\vec{U}_{k,j-1})|_{t_1=\cdots=t=0}),$$

where $\mathscr{R}^{(i)_l}=\mathscr{P}^{(i)_l}+\mathscr{Q}^{(i)_l}$, $\vec{w}\ll\vec{W}$, $G^{-1}\ll\mathscr{G}$, and $\vec{U}_{k,j}$ and $\vec{\mathfrak{D}}(1,\tilde{m}-1;\vec{U}_{k,j-1})$ are the vectors obtained by replacing $u^{\sigma,i}_{k,j}$ by $U^{\sigma,i}_{k,j}$ in (5.39) (for two matrices M_1 and M_2, $M_1\ll M_2$ means that each component of M_1 is majorized by the corresponding one of M_2). Then applying Proposition B.4 inductively, we have

PROPOSITION 6.1. *Suppose that a family of functions* $U_{k,j}^{\sigma,i}$ *satisfy* $(MT)_{k,j}$ *and* $(MI)_{k,j}$ *and that* $u_{k,j}^{\sigma,i}$ *are the solutions of* $(T)_{k,j}$ *with the initial conditions* $(I)_{k,j}$. *Then each* $U_{k,j}^{\sigma,i}$ *is a majorant of* $u_{k,j}^{\sigma,i}$.

We use the following auxiliary functions of one variable introduced by C. Wagschal and Y. Hamada:

(6.2) $$\theta^k(r, z)=\sum_{s=0}^{\infty}\frac{s!}{(k+s)!}\frac{z^{k+s}}{r^{s+1}} \quad \text{and} \quad \theta^{-k}(r, z)=\frac{k!}{(r-z)^{k+1}}, \quad k=0, 1, 2, \cdots,$$

(6.3) $$\Theta_j^k(R, r; z)=(d/dz)^j\left[\frac{R}{R-z}\theta^k(r, z)\right], \quad k, j=0, 1, 2, \cdots,$$

with constants $R>r>0$. See, for example, [8] for the properties of θ^k and Θ_j^k. In terms of these Θ_j^k, we set

(6.4) $$U_{k,j}^{\sigma,i}=U_{k,j}^i=\begin{cases} BC^{i+j+(m^*-1)k}\Theta_{j+(m^*-1)k-j_0+i-1}^{m^*k}(R, r; z), & \\ & j\geqq j_0-i+1-(m^*-1)k, \\ 0, & j<j_0-i+1-(m^*-1)k, \end{cases}$$

where $m^*=\max_{1\leqq\sigma\leqq\kappa} m_\sigma$, and $z=\rho(t_1+\cdots+t_l)+x_1+\cdots+x_n$ with $\rho\geqq 1$.

PROPOSITION 6.2. *We can choose constants* B, C, ρ $(C>\rho\geqq 1)$ *and* R' *so that the functions defined by* (6.4) *with* r, R *such that* $0<r<R\leqq R'/2$ *satisfy the majorant problem* $(MT)_{k,j}$ *and* $(MI)_{k,j}$.

LEMMA 6.3. *Let* $p=p((t)_{i_l}, x; \xi)\in\mathscr{O}(D^{l+n}(R'')\times\omega)$ *and let* r, R, R' *be such that* $0<r<R\leqq R'/2<R'\leqq R''/2$. *Then there exist constants* C_0, C_1 *and a compact set* K^* *in* $D^{l+n}(R'')\times\omega$ *which are independent of* μ, k, j, p, r, R, R' *and* ρ *such that*

(6.5) $$\mathscr{L}_\mu(p; \varphi^{(i)_l})\Theta_j^k(R, r; z)\ll C_0|p|_{K^*}C_1^\mu\Theta_{j+\mu}^k(R, r; z),$$

where $z=\rho(t_{i_1}+\cdots+t_{i_l})+x_1+\cdots+x_n$.

PROOF. We shall use the convention that $\varphi=\varphi^{(i)_l}$, $D(\cdot)=D^{l+n}(\cdot)$, $\varepsilon=R''/3$ and $R^*=5R''/6$ $(\geqq R'+\varepsilon)$. By (3.28) we can write $L_\mu(p; \varphi)$ in the following form:

$$L_\mu(p; \varphi)=\sum_{\mu\leqq|\alpha|\leqq 2\mu, 0\leqq\beta\leqq\alpha} l_{\alpha,\beta}((t)_{i_l}, x)\partial_x^{\alpha-\beta},$$

where

$$l_{\alpha,\beta}=\frac{1}{\alpha!\,(|\alpha|-\mu)!}\binom{\alpha}{\beta}p^{(\alpha)}((t)_{i_l}, x; d_x\varphi)\{\partial_{\tilde{x}}^\beta(\tilde{\varphi}^{|\alpha|-\mu})|_{\tilde{x}=x}\}, \quad \mu\leqq|\alpha|\leqq 2\mu, \quad 0\leqq\beta\leqq\alpha.$$

Note that $l_{\alpha,\beta}$ vanishes except for the case $|\alpha|-|\beta|\leqq 2\mu-|\alpha|$ $(\leqq\mu)$. Applying Lemma 3.14, we have the estimate:

$$|l_{\alpha,\beta}|_{D(R')}\leqq\frac{\alpha!}{(\alpha-\beta)!\,(|\alpha|-\mu)!}|p|_{K^*}\varepsilon^{-|\alpha|}\left(\frac{2}{\varepsilon}\right)^{|\beta|}(2^n|\varphi|_{D(R'+\varepsilon)})^{|\alpha|-\mu},$$

with $K^*=\bigcup_{((t)_{i_l},x)\in D(R^*)}((t)_{i_l},x;d_x\varphi)\in D(R'')\times\omega$. Since $|\alpha|-\mu\geqq 0$ and $D(R'+\varepsilon)\subset D(R^*)$, we can replace $|\varphi|_{D(R'+\varepsilon)}$ by $|\varphi|_{D(R^*)}$, which yields that

$$(*)\qquad l_{\alpha,\beta}\ll|p|_{K^*}\frac{\alpha!}{(\alpha-\beta)!\,(|\alpha|-\mu)!}C_2^{|\alpha|+|\beta|+\mu}\frac{R'}{R'-z},$$

with a constant C_2 depending only on R'' and φ (by C_i we shall denote constants similar to C_0, C_1). It is easily verified that

$$(6.6)\qquad \Theta_j^k(R,r;z)\ll\frac{r^h}{h!}\Theta_{j+h}^k(R,r;z),\quad h=0,1,2,\cdots,$$

which implies that

$$\partial_x^{\alpha-\beta}\Theta_j^k(R,r;z)\ll\frac{r^{\mu-|\alpha|+|\beta|}}{(\mu-|\alpha|+|\beta|)!}\Theta_{j+\mu}^k(R,r;z)\ll\frac{R''^{\mu-|\alpha|+|\beta|}}{(\mu-|\alpha|+|\beta|)!}\Theta_{j+\mu}^k(R,r;z),\quad |\alpha|-|\beta|\leqq\mu.$$

Hence, replacing $l_{\alpha,\beta}$ with their majorant functions $(*)$ and using the fact

$$\frac{1}{R'-z}\Theta_j^k(R,r;z)\ll\frac{1}{R'-R}\Theta_j^k(R,r;z)\quad (R'>R),$$

we obtain

$$\begin{aligned}&\mathscr{L}_\mu(p;\varphi)\Theta_j^k(R,r;z)\\ &\ll\frac{R'}{R'-R}|p|_{K^*}(R''C_2)^\mu\sum_{\alpha,\beta}(R''C_2)^{|\beta|}\frac{\alpha!}{(|\alpha|-\mu)!\,(\mu-|\alpha|+|\beta|)!\,(\alpha-\beta)!}\left(\frac{C_2}{R''}\right)^{|\alpha|}\Theta_{j+\mu}^k\\ &\ll\frac{R'}{R'-R}|p|_{K^*}C_3^\mu\sum_{\mu\leqq|\alpha|\leqq 2\mu}\left(\sum_{0\leqq\beta\leqq\alpha}\binom{\alpha}{\beta}C_4^{|\beta|}\right)C_5^{|\beta|}\Theta_{j+\mu}^k\\ &\ll\frac{R'}{R'-R}C_0|p|_{K^*}C_1^\mu\Theta_{j+\mu}^k(R,r;z).\end{aligned}$$

Because of $R'/(R'-R)\leqq 2$ we have proved the assertion.

REMARK. Since the number of $\varphi^{(i)_l}$, $(i)_l\in J(m_\sigma)$ is finite, we may assume that the constants C_0, C_1 in Lemma 6.3 are independent of $\varphi^{(i)_l}$.

COROLLARY 6.4. *Let r, R and R' be as in Lemma 6.3. If the two constants C and ρ such that $C>\rho\geqq 1$ in* (6.4) *are sufficiently large, then we have*

$$(6.7)\qquad \begin{cases} \mathfrak{A}^{(i)_l}(l+1,\infty;\, U^i_{k,j-i}) \ll BC_2\rho^{l-1}C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l}(R,r;z), \\ \mathfrak{B}^{(i)_l}_i(0,\infty;\, U^{i+l}_{k-1,j-i}) \ll BC_2C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l}(R,r;z), \\ \mathfrak{C}^{(i)_l}(l+1,\infty;\, U^{m_\sigma}_{k,j-m_\sigma}) \ll BC_2\dfrac{\rho^2}{C}\rho^{l-1}C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l}(R,r;z), \end{cases}$$

with a suitable constant C_2 independent of $(i)_l$, i, k, j, B, C, ρ, r, R and R'.

PROOF. We show, for example, the second inequality to hold. The other inequalities are verified similarly. Since $\mathscr{B}_i^{(i)_l}=0$ for $(i)_l\neq(1\,2\cdots l)$, we consider the case $(i)_l=(1\,2\cdots l)$. We put $(l)=(i)_l$. By definition (see (5.24), (5.34)), we have

$$\mathfrak{B}^{(l)}_i(0,\infty;\, U^{i+l}_{k-1,j-i})=C_3\frac{R'}{R'-z}\sum_{d=i}^{\tilde m-1}\sum_{h=-\infty}^{\tilde m-d-1}\sum_{\mu_1=0}^{\infty}\sum_{\mu_2=0}^{d-i}\mathscr{L}_{\mu_1}(b^{i+l}_{d,h};\varphi^{12\cdots l})$$
$$\{\mathscr{J}^d_{d-i,\mu_2}U^{i+l}_{k-1,j-\tilde m+1+d+h-i-(\mu_1+\mu_2)}|_{t_{l+1}=\cdots=t_{i+l}=0}\},$$

with $C_3\geqq|1/(a^{(l)}q^m)|_{D^{l+n}(R')}$. Hence, by applying Lemma 6.3 we obtain

$$\mathfrak{B}^{(l)}_i(0,\infty;\, U^{i+l}_{k-1,j-i})\ll BC_4\sum_d\sum_h\sum_{\mu_1}\sum_{\mu_2}\left(\frac{C_1}{C}\right)^{\mu_1}\left(\frac{\rho}{C}\right)^{\mu_2}C^{l-m^*}|b^{i+l}_{d,h}|_{K^*}\left(\frac{1}{C}\right)^{\tilde m-d-1-h}$$
$$\times C^{j+(m^*-1)k+1}\Theta^{m^*(k-1)}_{j-j_0+(m^*-1)(k-1)+l-(\tilde m-d-1-h)-1}.$$

The inequalities, $1\leqq\rho<C$, $l-m^*\leqq0$ and (6.6) imply that

$$\ll BC_5\sum_d\underbrace{\left(\sum_{\mu_1=0}^{\infty}\left(\frac{C_1}{C}\right)^{\mu_1}\right)}_{\text{(I)}}\underbrace{\left\{\sum_{h=-\infty}^{\tilde m-d-1}|b^{i+l}_{d,h}|_{K^*}\left(\frac{r}{C}\right)^{\tilde m-d-1-h}\Big/(\tilde m-d-1-h)!\right\}}_{\text{(II)}}$$
$$\times C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l}(R,r;z).$$

Here we have also used the fact that $\Theta^k_j\ll\Theta^{k+1}_{j+1}$. (I) converges if $C>C_1$. In view of Definition 3.9 of symbols, (II) is convergent if we choose C sufficiently large, which completes the proof of the second inequality. Q.E.D.

PROOF OF PROPOSITION 6.2. We shall show the majorant equations $(MT)_{k,j}$ to hold for $i=m_\sigma$. The validity of the other equations are shown analogously. Obviously, the following inequality holds for a suitable constant C_3 independent of B, C, ρ, r, R and R':

$$(*)\qquad \mathscr{R}^{(i)_l}(\gamma^{(i)_l}U^{m_\sigma}_{k,j-m_\sigma+1})\ll BC_2C_3(\|\mathscr{P}^{(i)_l}\|_{R'}\rho^l+\rho^{l-1})C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l},$$

where $\|\mathscr{P}^{(i)_l}\|_{R'}$ is the maximum of the absolute values of the coefficients of $\mathscr{P}^{(i)_l}$ in $D^{l+n}(R')$. Note that since all the coefficients of $\mathscr{P}^{(i)_l}$ vanish on $t_{i_1}=\cdots=t_{i_l}=0$, the value $\|\mathscr{P}^{(i)_l}\|_{R'}$ can be made arbitrary small by letting R' smaller if necessary. In view of Corollary 6.4 and $(*)$, the right side of $(MT)_{k,j}$ is majorized by

$$BC_2\left\{C_3(\|\mathscr{F}^{(i)}{}_l\|_{R'}\rho^l+\rho^{l-1})+m_\sigma+\frac{\rho^{l+1}}{C}\right\}C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l}.$$

On the other hand, the left side of $(MT)_{k,j}$ is equal to

$$B\rho^l C^{j+(m^*-1)k+1}\Theta^{m^*k}_{j-j_0+(m^*-1)k+l}.$$

Hence, it suffices to take R', ρ and C so that

$$1\geqq C_2\left\{C_3\left(\|\mathscr{F}^{(i)}{}_l\|_{R'}+\frac{1}{\rho}\right)+\frac{m_\sigma}{\rho^l}+\frac{\rho}{C}\right\}. \qquad \text{Q.E.D.}$$

In view of Propositions 6.1, 6.2, (5.45) and the inequality (see [12])

$$\frac{R}{R-z}\theta^k(r,z)\ll\left(\frac{R}{R-r}\right)^{k+1}\theta^k(r,z),\quad k\geqq 0, \tag{6.8}$$

we have

$$u^{\sigma,i}_{k,j}\ll\begin{cases} BC^{i+j+(m^*-1)k}\left(\dfrac{R}{R-r}\right)^{m^*k+1}\theta^{k+j_0-i+1-j}(r,z), \\ \qquad\qquad 1\leqq i\leqq m_\sigma-1 \text{ and } j\geqq j_0-i+1 \text{ or} \\ \qquad\qquad i=m_\sigma \text{ and } j\geqq j_0-m_\sigma+1-(m_\sigma-1)k, \\ 0, \qquad\qquad \text{otherwise.}\end{cases} \tag{6.9}$$

Once (6.9) is proved, the rest of the estimates is all the same as in [8] or [12]. So we only state the results without going into details:

$$\begin{cases} |u^{\sigma,i}_{k,j}|_{G^i}\leqq C_3\left(\dfrac{C_3^k}{k!}\right)\left(\dfrac{C_3}{\delta}\right)^j(j+i-1-j_0)!, \\ \sum\limits_{k=0}^{\infty}|u^{\sigma,i}_{k,j}|_{G^i}\leqq C_3(\exp C_3)\left(\dfrac{C_3}{\delta}\right)^j(j+i-1-j_0)!,\end{cases}\quad j\geqq j_0-i+1, \tag{6.10}$$

$$\begin{cases} |u^{\sigma,i}_{k,j}|_{G^i}=0, \quad 1\leqq i\leqq m_\sigma-1 \text{ or } i=m_\sigma=1, \quad j<j_0-i+1, \\ |u^{\sigma,m_\sigma}_{k,j}|_{G^{m_\sigma}}\leqq\dfrac{1}{\delta}C_4^{k+j+1}\Big/(k+j_0-m_\sigma+1-j)!, \quad j<j_0-m_\sigma+1,\end{cases} \tag{6.11}$$

$$\sum_{k=0}^{\infty}|u^{\sigma,m_\sigma}_{k,j}|_{G^{m_\sigma}}\leqq C_5^{j+1}\Big/\Gamma\left(\frac{m_\sigma}{m_\sigma-1}(j_0-m_\sigma+1-j)+1\right),\quad j<j_0-m_\sigma+1. \tag{6.12}$$

Here $G^i=G^i(\delta)$ are the domains defined by

$$G^i=G^i(\delta)=\{(t_1\cdots t_i;x)\in\boldsymbol{C}^{i+n}:\rho(|t_1|+\cdots+|t_i|)+|x_1|+\cdots+|x_n|<r-\delta\},$$

with a sufficiently small positive constant δ. It should be noted that the number $m_\sigma/m_\sigma-1$ in (6.12) comes from the fact that $u^{\sigma,m_\sigma}_{k,j}$ vanish for $j<j_0-(m_\sigma-1)k-m_\sigma+1$.

The inequalities (6.1), (6.11) and (6.12) show that the formal vectors $\tilde{\boldsymbol{u}}_k^g$, $k=0, 1, 2, \cdots$, (see (5.5)) belong to $\tilde{\mathscr{F}}(T_1\times X_1)$ with a sufficiently small nhbd $T_1\times X_1$ of the origin of $\boldsymbol{C}^{n+1}$ and that $\sum_{k=0}^{\infty}\tilde{\boldsymbol{u}}_k^g$ is normally convergent. This guarantees the convergence of the formal solution constructed in Section 5 and justifies the calculus of pseudo-differential operators.

The inequality (1.8) easily follows from (6.12) and the inequality

$$\sum_{j=0}^{\infty}\frac{\tau^j}{\Gamma(sj+1)}\leqq\frac{1}{s}\exp\tau^{1/s}+O\left(\frac{1}{\tau}\right).$$

Thus we have completed the proof of Theorem 1.1.

Appendix A

We sketch the proof of Theorem 3.16. Same notations as in Section 3 are used. Let $y^0\in Y$ and $\xi^0=d_x\varphi(y^0)$. All the components of both side of (3.37) are well-defined as holomorphic functions in Y. Therefore, by the unique continuation theorem, it is enough to show that the equality (3.37) holds in a nhbd of y^0 (i.e., the jth components of both side of (3.37) coincide there). By an affine coordinate change and by shrinking Ω^* and Y if necessary, we may assume that $y^0=(0, 0)\in\boldsymbol{C}^N$, $\xi^0=(1, 0\cdots 0)$ and that Ω^* (resp. Y) is an open nhbd of $(0, 0; 1, 0\cdots 0)\in\boldsymbol{C}^{N+n}$ (resp. $(0, 0)\in\boldsymbol{C}^N$) such that $(y, d_x\varphi(y))\in\Omega^*$ for all $y\in Y$ and $\Omega^*\cap\{\xi_1=0\}=\emptyset$. By the last assumption made above, ξ_1^{-k} are symbols in $\mathscr{P}(\Omega^*)$ for all positive integers k. Let $\partial_{x_1}^{-k}$ denote the pseudo-differential operator with symbol ξ_1^{-k}. Then a direct application of Definition 3.11 implies that

$$\text{(A.1)}\qquad \partial_{x_1}^{-1}\circ\partial_{y_i}=\partial_{y_i}\circ\partial_{x_1}^{-1},\quad 1\leqq i\leqq N,$$

in particular,

$$\text{(A.2)}\qquad \partial_{x_1}^{-1}\circ\partial_{x_1}=\partial_{x_1}\circ\partial_{x_1}^{-1}=\text{identify}.$$

It should be noted that if $\partial_{x_1}^{-1}$ is simply considered as an integral operator $\int_*^{x_1}dx_1$ with some fixed reference point $*$, then $\partial_{x_1}\circ\partial_{x_1}^{-1}=$identity is true, but $\partial_{x_1}^{-1}\circ\partial_{x_1}\neq$ identity in general (cf. [2]). Incidentally, by virtue of (A.2), (3.37) holds without any modulo terms. The first crucial point is to show that

$$\text{(A.3)}\qquad \partial_{x_1}^{-k}=\overbrace{\partial_{x_1}^{-1}\cdots\partial_{x_1}^{-1}}^{k}.$$

This is done by induction on k, using another expression of $\partial_{x_1}^{-1}$: let $\partial_{x_1}^{-1}\boldsymbol{u}=\boldsymbol{v}$. Then the jth component of $\boldsymbol{v}$ is given by

$$v_j=\sum_{k=1}^{\infty}\left\{-\left(\frac{\partial\varphi}{\partial x_1}\right)^{-1}\frac{\partial}{\partial x_1}\right\}^{k-1}\left(\frac{\partial\varphi}{\partial x_1}\right)^{-1}u_{j-k}, \tag{A.4}$$

where the right side converges normally in a nhbd of the origin of $\boldsymbol{C}^N$. The expression (A.4) is proved by the equality

$$\begin{aligned}&\left\{-\left(\frac{\partial\varphi}{\partial x_1}\right)^{-1}\frac{\partial}{\partial x_1}\right\}^{k-1}\left(\frac{\partial\varphi}{\partial x_1}\right)^{-1}u \\ &=\sum_{\nu=0}^{k-1}\frac{1}{(k+\nu-1)!\,\nu!}\,\partial_{\xi_1}^{k+\nu-1}(\xi_1^{-1})\Big|_{\xi_1=\partial\varphi/\partial x_1}\{\partial_{x_1}^{k+\nu-1}(\tilde{\varphi}(z, x; \tilde{x})^{\nu}u(z, x)|_{\tilde{x}=x}\}.\end{aligned} \tag{A.5}$$

In this appendix, we allow the first components of multi-indices to be negative, while we always require the others to be non-negative. Let ∂_x^α denote the pseudo-differential operator with symbol $\xi^\alpha=\xi_1^{\alpha_1}\xi'^{\alpha'}$ ($\alpha=(\alpha_1, \alpha')$, $\alpha_1\in\boldsymbol{Z}$, $\alpha'\geqq 0$, $\xi=(\xi_1, \xi')$). Then it follows from (A.1)–(A.3) that

$$\partial_x^\alpha\circ\partial_x^\beta=\partial_x^{\alpha+\beta}, \tag{A.6}$$

for all multi-indices α, β satisfying the above requirement.

Let $P(z, x; \xi)=\sum_{k=-\infty}^{m}p_k(z, x; \xi)\in\mathscr{P}(\Omega^*)$. By the homogeneity of p_k, we can expand p_k as follows:

$$\begin{cases}p_k(z, x; \xi)=\sum_{\alpha'\geqq 0, \alpha_1=k-|\alpha'|}p_\alpha(z, x)\xi^\alpha, \\ P(z, x; \xi)=\sum_{\alpha'\geqq 0}p_\alpha(z, x)\xi^\alpha,\end{cases} \tag{A.7}$$

$$p_k(z, x; 1, \xi')=\sum_{\alpha'\geqq 0}p_{k-|\alpha'|, \alpha'}(z, x)\xi'^{\alpha'}. \tag{A.8}$$

Then the same expansion as (A.7) is valid for the pseudo-differential operator $P(z, x; \partial_x)$, that is,

$$P(z, x; \partial_x)=\sum_{k=-\infty}^{m}p_k(z, x; \partial_x)=\sum_{\alpha'\geqq 0}p_\alpha(z, x)\partial_x^\alpha. \tag{A.9}$$

Consequently, it is enough to show that (3.37) is valid when $P=\xi_1^{-1}$ and $Q=q(z, x)$, which is easily verified.

Appendix B

We consider the Goursat problem by the method of majorant. Let $y=(y_1, \cdots, y_N)\in\boldsymbol{C}^N$ and $\alpha=(\alpha_1, \cdots, \alpha_N)\in\boldsymbol{N}^N$. When $g=g(y; \partial_y)=\sum_\alpha g_\alpha(y)\partial_y^\alpha$ is a linear differential operator, we denote by $\{g\}$ the set of multi-indices $\{\alpha\in\boldsymbol{N}^N: g_\alpha\neq 0\}$. We say that a linear differential operator $G=\sum_{\alpha} G_\alpha(y)\partial_y^\alpha$ a majorant (differential operator) of g if $\{G\}=\{g\}$ and $g_\alpha \ll G_\alpha$ for all $\alpha\in\{g\}$. A multi-index β is said to dominate a set of indices S $(\subset N^N)$ if β is outside the convex hull of S (cf. [23]).

Let us consider the Goursat problem

$$\text{(gp)}\qquad \begin{cases} \partial_y^\beta v(y)=\sum_{\alpha} g_\alpha(y)\partial_y^\alpha v(y)+h(y), \\ \partial_{y_i}^d v(y)|_{y_i=0}=\partial_{y_i}^d w(y)|_{y_i=0}, \quad 0\leqq d\leqq \beta_i-1, \quad 1\leqq i\leqq N, \end{cases}$$

and the majorant problem

$$\text{(GP)}\qquad \begin{cases} \partial_y^\beta V(y)\gg\sum_{\alpha} G_\alpha(y)\partial_y^\alpha V(y)+H(y), \\ \partial_{y_i}^d V(y)|_{y_i=0}\gg\partial_{y_i}^d w(y)|_{y_i=0}, \quad 0\leqq d\leqq \beta_i-1, \quad 1\leqq i\leqq N, \end{cases}$$

where $g_\alpha(y)$, $h(y)$, $w(y)$, $G_\alpha(y)$ and $H(y)$ are formal power series in y and G_α, H are majorants of g_α, h, respectively.

PROPOSITION B.1. *Suppose that the order of g is not more than $|\beta|$ and that β dominates $\{g\}$. Then the Goursat problem* (gp) *has a unique formal power series solution v. Further, if a formal power series V satisfy the majorant Goursat problem* (GP), *then V is a majorant of v.*

PROOF. Let $v(y)=\sum_{\gamma} v_\gamma y^\gamma$, $g_\alpha(y)=\sum_{\delta} g_\delta^\alpha y^\delta$, $h(y)=\sum_{\gamma} h_\gamma y^\gamma$ and $w(y)=\sum_{\gamma} w_\gamma y^\gamma$. Substituting the above expansions in (gp) and then equating the coefficients of $y^{\gamma-\beta}$ of both side, we get for $\gamma\geqq\beta$

$$v_\gamma=\frac{(\gamma-\beta)!}{\gamma!}\sum_{\alpha\in\{g\}}\ \sum_{0\leqq\delta\leqq\gamma-\beta,\ \gamma'=\gamma+(\alpha-\beta)-\delta}\frac{\gamma'!}{(\gamma'-\alpha)!}g_\delta^\alpha v_{\gamma'}+h_{\gamma-\beta}.$$

Hence

$$\text{(B.1)}\qquad v_\gamma=P_\gamma(g_\delta^\alpha, v_{\gamma'})+h_{\gamma-\beta},$$

where P_γ $(\gamma\geqq\beta)$ are polynomials in $(g_\delta^\alpha, v_{\gamma'})$, $\alpha\in\{g\}$, $0\leqq\delta\leqq\gamma-\beta$, $\gamma'=\gamma+(\alpha-\beta)-\delta$, with non-negative coefficients. Put $B(\gamma)=\{\gamma'=\gamma+(\alpha-\beta)-\delta\in N^N\colon \alpha\in\{g\},\ 0\leqq\delta\leqq\gamma-\beta\}$. Then

(B.2) $\quad v_\gamma$ are determined depending only on $v_{\gamma'}$ such that $\gamma'\in B(\gamma)$.

The boundary condition of (gp) implies that

$$\text{(B.3)}\qquad v_\gamma=w_\gamma \quad \text{for all} \quad \gamma\in C=\{\gamma\in N^N\colon \gamma\ngeqq\beta\}.$$

To prove that v_γ are uniquely determined by (B.1) for $\gamma\in D=\{\gamma\in N^N\colon\gamma\geqq\beta\}$, it is enough, in view of (B.2) and (B.3), to show that $D=\bigcup_{j=1}^{\infty} D_j$ is decomposed into a countable number of disjoint sets D_j so that $\gamma\in D_k$ implies $B(\gamma)\subset C\cup\left(\bigcup_{j=1}^{k-1} D_j\right)$. By convexity, we can choose a vector $\sigma=(\sigma_1,\cdots,\sigma_N)$ with non-negative components σ_i such that $\sigma\cdot\alpha<\sigma\cdot\beta$ for all $\alpha\in\{g\}$ $\left(\sigma\cdot\alpha=\sum_{i=1}^{N}\sigma_i\alpha_i\right)$. We may assume that the elements of the image of D by σ, $\sigma(D)$, are numbered as follows:

$$\sigma(D)=\{s_j\in \boldsymbol{R}: j=1,2,3,\cdots, s_1<s_2<s_3<\cdots\}.$$

Put $D_j=\{\gamma\in D: \sigma\cdot\gamma=s_j\}$. Then $D=\bigcup_{j=1}^{\infty} D_j$ gives the required decomposition.

Since P_γ are polynomials with non-negative coefficients, the second assertion is trivial. Q.E.D.

LEMMA B.2. *Same hypotheses as in Proposition* B.1. *When G_α, H and w are holomorphic at the origin of $\boldsymbol{C}^N$, there exists a holomorphic function V satisfying the majorant Goursat problem* (GP).

PROOF. As in the preceding proof, let $\sigma=(\sigma_1,\cdots,\sigma_N)$ be a vector with non-negative components such that $\sigma\cdot\alpha<\sigma\cdot\beta$ for all $\alpha\in\{G\}$ $(=\{g\})$. Then it is easily seen that we can choose constants M, s, sufficiently large and $0<r<R$, sufficiently small so that the function

$$V(y)=M\Theta_0^0(R,r;z),\quad z=\sum_{j=1}^{N} y_i \exp(s\sigma_i)$$

satisfies (GP). Q.E.D.

As an immediate consequence of Proposition B.1 and Lemma B.2, we have

COROLLARY B.3 (cf. [4], [9], [23]). *Let g_α, h and w be holomorphic at the origin. Suppose that the order of g is not more than $|\beta|$ and that β dominates $\{g\}$. Then there is a unique holomorphic solution of* (gp).

Next we consider a general version of the Goursat problem. Let $u_0=u_0(x)$ and $g^{(i)_l}(t_{i_1},\cdots,t_{i_l},x)$ $((i)_l\in J(m))$ be holomorphic in a nhbd of the origin of $\boldsymbol{C}^n$ and $\boldsymbol{C}^{l+n}$, respectively. Let $R^{(i)_l}$ be a linear differential operators with respect to $(t_{i_1},\cdots,t_{i_l},x)$ of order l with holomorphic coefficients. Let $\mathscr{R}^{(i)_l}$ and $G^{(i)_l}$ be majorants of $R^{(i)_l}$ and $g^{(i)_l}$, respectively.

We consider a system of linear differential equations:

$$\text{(B.4)}\qquad \begin{cases}\dfrac{\partial^l}{\partial t_{i_1}\cdots\partial t_{i_l}}(\gamma^{(i)_l}u)=R^{(i)_l}(\gamma^{(i)_l}u)+g^{(i)_l}, & (i)_l\in J(m),\\ u(0\cdots 0,x)=u_0(x), & \end{cases}$$

and a majorant problem

$$\text{(B.5)}\qquad \begin{cases} \dfrac{\partial^l}{\partial t_{i_1}\cdots\partial t_{i_l}}(\gamma^{(i)_l}U)\gg\mathscr{R}^{(i)_l}(\gamma^{(i)_l}U)+G^{(i)_l}, & (i)_l\in J(m),\\ U(0\cdots 0,x)\gg u_0(x), \end{cases}$$

where $\gamma^{(i)_l}v=v(0\cdots 0,t_{i_1},0\cdots 0,t_{i_2},0\cdots\cdots 0,t_{i_l},0\cdots 0,x)$.

PROPOSITION B.4. *Suppose that* $\beta_l=(1\cdots 1;0\cdots 0)\in N^{l+n}$ *dominates* $\{R^{(i)_l}\}$ *for every* $(i)_l\in J(m)$. *Then there exists a unique holomorphic solution* u *of* (B.4) *in a neighborhood of the origin of* C^{m+n}. *Further, if* U *satisfies* (B.5), *then* U *is a majorant of* u.

PROOF. Let $w^{(i)_l}=w^{(i)_l}(t_{i_1},\cdots,t_{i_l},x)$ $((i)_l\in J(m))$ be holomorphic functions. Since $R^{(i)_l}$ satisfies the hypotheses in Corollary B.3, the Goursat problem

$$(G)_{(i)_l}\qquad \begin{cases} \dfrac{\partial^l}{\partial t_{i_1}\cdots\partial t_{i_l}}u^{(i)_l}(t_{i_1},\cdots,t_{i_l},x)=R^{(i)_l}u^{(i)_l}+g^{(i)_l},\\ u^{(i)_l}|_{t_{i_r}=0}=w^{(i)_l}|_{t_{i_r}=0}, \quad 1\leqq r\leqq l, \end{cases}$$

has a unique holomorphic solution $u^{(i)_l}$. If $w^{(i)_l}$ satisfy the following inductive relations:

$$\text{(B.6)}\qquad \begin{cases} w^1|_{t_{i_1}=0}=\cdots=w^m|_{t_{i_m}=0}=u_0(x),\\ \cdots\\ w^{(i)_l}|_{t_{i_r}=0}=u^{i_1\cdots\hat{i}_r\cdots i_l}, \quad 1\leqq r\leqq l,\\ \cdots\\ w^{12\cdots m}|_{t_i=0}=u^{1\cdots\hat{i}\cdots m}, \quad 1\leqq i\leqq m, \end{cases}$$

where $u^{(i)_l}$ are the unique solutions of $(G)_{(i)_l}$ $((i)_l\in J(m))$, then we can easily see that $u=u^{12\cdots m}$ (i.e., the unique solution of $(G)_{12\cdots m}$) satisfies (B.4). Conversely, let u be a solution of (B.4) and set $u^{(i)_l}=\gamma^{(i)_l}u$. Then $u^{(i)_l}$ is the solution of $(G)_{(i)_l}$. Let us define $w^{(i)_l}$ inductively as follows:

$$\text{(B.7)}\qquad w^{(i)_l}=(-1)^{l+1}u_0+\sum_{k=1}^{l-1}(-1)^{l-k+1}\left\{\sum_{1\leq r_1<\cdots<r_k\leq l}u^{i_{r_1}\cdots i_{r_k}}\right\}, \quad (i)_l\in J(m).$$

Then simple computations show that $w^{(i)_l}$ satisfy (B.6). Hence (B.4) has a unique holomorphic solution. Applying Proposition B.1 to (B.4) and (B.5) inductively, we obtain the second part of the assertions.

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