Growth order of microdifferential operators of infinite order

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ABSTRACT

For each microdifferential operator of infinite order its growth order $0 \leqq \rho \leqq 1$ is defined. Structure of single microdifferential equations with constant multiplicity is studied in the framework of microdifferential operators with restricted growth order.

Introduction.

Let F be a microdifferential operator of infinite order (cf. [3]). Then the total symbol of F is a holomorphic function $F(x, \xi)$ which satisfies the following estimate: for each $\varepsilon>0$ there is a constant $C>0$ such that $|F(x, \xi)| \leqq C \exp(\varepsilon|\xi|)$ (see [1]). Let ρ be a real number such that $0 \leqq \rho < 1$. The operator F is said to be of growth order at most (ρ) if there exist positive constants h, C such that $|F(x, \xi)| \leqq C \exp(h|\xi|^{\rho})$. The sheaf of germs of microdifferential operators of growth order at most (ρ) is denoted by $\mathscr{E}^{\infty}_{(\rho)}$. Then $\mathscr{E}^{\infty}_{(\rho)}$ is a subring of the ring $\mathscr{E}^{\infty}$ of microdifferential operators of infinite order.

In [8], structure of microdifferential equations with constant multiple characteristics is studied as $\mathscr{E}^{\infty}$-modules. We restrict our discussion to the framework of $\mathscr{E}^{\infty}_{(\rho)}$-modules. It corresponds to consider in the category of ultradistributions (see [7]). We prove the following theorem: Let P be a microdifferential operator with principal symbol ξ_1^m defined in a neighborhood of $(\mathring{x}, \mathring{\xi}) = (0; 0, \cdots, 0, 1)$. Suppose that the irregularity of P at $(\mathring{x}, \mathring{\xi})$ is equal to σ (see [2] for the definition of irregularity). Then the microdifferential equation $Pu=0$ is equivalent to the equation $D_1^m v=0$ $(D_1=\partial/\partial x_1)$ as a left $\mathscr{E}^{\infty}_{(\rho)}$-module for $\rho \geqq 1-1/\sigma$.

Moreover we prove the following (see [3], Chap. 3): Let P be an ordinary microdifferential operator with principal symbol x^m. Suppose that the irregularity of P at $x=0$ is equal to σ. Then the microdifferential equation $Pu=0$ is equivalent to the equation $x^m v=0$ as a left $\mathscr{E}^{R}_{(\rho)}$-module for $\rho \geqq 1-1/\sigma$. Here $\mathscr{E}^{R}_{(\rho)}$ is the ring of holomorphic microlocal operators of growth order at most (ρ) (cf. [1]).

1. Growth order.

Let X be an open set in C^n and $\mathring{x}^*$ be a point in the cotangent vector bundle $T^*X \simeq X \times C^n$ of X. We denote by $\mathscr{E}^R$ the sheaf on T^*X of rings of holomorphic microlocal operators. Let us recall the definition (cf. [1], [3], [6]). A holomorphic microlocal operator F in $\mathscr{E}^R_{\mathring{x}^*}$ (= the stalk of $\mathscr{E}^R$) is an equivalence class of holomorphic functions

$$F(x, \xi) \tag{1.1}$$

defined in a conic neighborhood Γ of $\mathring{x}^*$ in T^*X which satisfies the following: for each $\varepsilon > 0$ and each compactly generated cone $\Gamma' \Subset \Gamma$ there exists a positive constant C such that

$$|F(x, \xi)| \leqq C \exp(\varepsilon |\xi|) \quad \text{for } (x, \xi) \in \Gamma' \ (|\xi| \gg 1). \tag{1.2}$$

The holomorphic function $F(x, \xi)$ is called the symbol of F and then F is written as $F = F(x, D_x)$, where $D_x = (D_1, \cdots, D_n)$, $D_j = \partial / \partial x_j$.

DEFINITION 1.1 ([1] Def. 2.3.2). Let ρ be a real number such that $0 \leqq \rho < 1$ (resp. $0 < \rho \leqq 1$). The holomorphic microlocal operator F is called of growth order at most (ρ) (resp. $\{\rho\}$) if there is a conic neighborhood $\Gamma_1 \subset \Gamma$ of $\mathring{x}^*$ so that for each compactly generated cone $\Gamma'' \subset \Gamma_1$ there exist positive constants h, C (resp. for each $h > 0$ and each compactly generated cone $\Gamma'' \subset \Gamma_1$ there exists a positive constant C) such that

$$|F(x, \xi)| \leqq C \exp(h |\xi|^\rho) \quad \text{for } (x, \xi) \in \Gamma''. \tag{1.3}$$

We denote by $\mathscr{E}^R_{(\rho), \mathring{x}^*}$ (resp. $\mathscr{E}^R_{\{\rho\}, \mathring{x}^*}$) the set of all operators in $\mathscr{E}^R_{\mathring{x}^*}$ of growth order at most (ρ) (resp. $\{\rho\}$) and by $\mathscr{E}^R_{(\rho)}$ (resp. $\mathscr{E}^R_{\{\rho\}}$) the subsheaf of $\mathscr{E}^R$ of germs of holomorphic microlocal operators of growth order at most (ρ) (resp. $\{\rho\}$). $\mathscr{E}^R_{(\rho)}$ (resp. $\mathscr{E}^R_{\{\rho\}}$) is a subring of $\mathscr{E}^R$.

Let us denote by $\mathscr{E}^\infty$ and $\mathscr{E}$ the sheaves on T^*X of rings of microdifferential operators of infinite order and of finite order respectively. There are canonical injections

$$\mathscr{E} \hookrightarrow \mathscr{E}^\infty \hookrightarrow \mathscr{E}^R. \tag{1.4}$$

DEFINITION 1.2. A microdifferential operator P of infinite order defined in a neighborhood of $\mathring{x}^*$ is called of growth order at most (ρ) (resp. $\{\rho\}$) if it is of growth order at most (ρ) (resp. $\{\rho\}$) as a holomorphic microlocal operator. We denote as follows.

$$\mathscr{E}^{\infty}_{(\rho)}=\mathscr{E}^{\infty}\cap\mathscr{E}^{R}_{(\rho)}, \tag{1.5}$$

$$\mathscr{E}^{\infty}_{\{\rho\}}=\mathscr{E}^{\infty}\cap\mathscr{E}^{R}_{\{\rho\}}. \tag{1.6}$$

$\mathscr{E}^{\infty}_{(\rho)}$ (resp. $\mathscr{E}^{\infty}_{\{\rho\}}$) is a subring of $\mathscr{E}^{\infty}$. It is proved that $\mathscr{E}^{\infty}_{(\rho)}$ (resp. $\mathscr{E}^{\infty}_{\{\rho\}}$) is invariant under coordinate transformations and quantized contact transformations.

Suppose that a microdifferential operator P is represented as the infinite sum of homogeneous parts:

$$P=\sum_{j\in \boldsymbol{Z}} P_j(x, D_x) \tag{1.7}$$

where $P_j(x,\xi)$ is a holomorphic function defined in a neighborhood Γ of $\mathring{x}^*$, homogeneous of degree j with respect to ξ and $\{P_j(x,\xi)\}$ satisfies the following conditions:

(1.7.1) For every $\varepsilon>0$ and every compactly generated cone $\Gamma'\subset\Gamma$, there is a constant $C>0$ such that

$$|P_j(x,\xi)|\leqq C\frac{\varepsilon^j}{j!}|\xi|^j \quad \text{for } j\geqq 0,\ (x,\xi)\in\Gamma'.$$

(1.7.2) For every compactly generated cone $\Gamma'\subset\Gamma$, there is a constant $R>0$ such that

$$|P_j(x,\xi)|\leqq(-j)!\,R^{-j}|\xi|^j \quad \text{for } j<0,\ (x,\xi)\in\Gamma'.$$

(Recall that if $P_j(x,\xi)\equiv 0$ for every sufficiently large $j>0$, P is said to be of finite order. On the other hand, P is called of infinite order if for any $N>0$ there is $j\geqq N$ such that $P_j(x,\xi)\not\equiv 0$.) Then P is of growth order at most (ρ) (resp. $\{\rho\}$) if and only if

(1.8) for every compactly generated cone $\Gamma'\subset\Gamma$, there are constants $h>0$, $C>0$ (resp. for every $h>0$ and every compactly generated cone $\Gamma'\subset\Gamma$, there is a constant $C>0$) such that

$$|P_j(x,\xi)|\leqq C\frac{h^j}{(j!)^{1/\rho}}|\xi|^j \quad \text{for } j>0,\ (x,\xi)\in\Gamma'.$$

DEFINITION 1.3. Let $\{F_j(x,\xi)\}_{j\geqq 0}$ be a sequence of holomorphic functions defined in a conic neighborhood Γ of $\mathring{x}^*$ such that for each compactly generated cone $\Gamma'\subset\Gamma$ there exist positive constants C, A, h (resp. for each compactly generated cone $\Gamma'\subset\Gamma$ there exists a constant $A>0$ so that for every $h>0$ there is a constant $C>0$) such that

$$|F_j(x,\xi)|\leqq CA^j j!\,|\xi|^{-j}\exp(h|\xi|^{\rho}) \quad \text{for } j>0,\ (x,\xi)\in\Gamma'. \tag{1.9}$$

Then the formal sum $\sum_{j\geqq 0} F_j(x,\xi)$ is called a formal symbol of growth order at most (ρ) (resp. $\{\rho\}$).

THEOREM 1.4 ([1] Th. 2.3.4). *Let* $\sum_{j\geqq 0} F_j(x,\xi)$ *be a formal symbol of growth order* (ρ) *(resp.* $\{\rho\}$*). Then the sum* $\sum_{j\geqq 0} F(x, D_x)$ *converges as a holomorphic microlocal operator and defines a holomorphic microlocal operator of growth order at most* (ρ) *(resp.* $\{\rho\}$*). Moreover if each* $F_j(x,\xi)$ *is a symbol of some microdifferential operator, then the sum is a microdifferential operator of growth order at most* (ρ) *(resp.* $\{\rho\}$*).*

REMARK 1.5. The sheaves $\mathscr{E}^\infty_{(\rho)}$ and $\mathscr{E}^\infty_{\{\rho\}}$ are connected with the theory of ultradistributions (cf. [7]). Set

$$\mathscr{D}^\infty_{(\rho)} = \mathscr{E}^\infty_{(\rho)}|_{T^*_X X}, \tag{1.10}$$

$$\mathscr{D}^\infty_{\{\rho\}} = \mathscr{E}^\infty_{\{\rho\}}|_{T^*_X X}. \tag{1.11}$$

Then $\mathscr{D}^\infty_{(\rho)}$ (resp. $\mathscr{D}^\infty_{\{\rho\}}$) is called the sheaf of ultradifferential operators of growth order at most (ρ) (resp. $\{\rho\}$). Suppose that X is a complexification of a real analytic manifold M. Then the sheaf on M of ultradifferentiable function of class (s) (resp. $\{s\}$) is a left $\mathscr{D}^\infty_{(\rho)}$ (resp. $\mathscr{E}^\infty_{\{\rho\}}$)-module if $s\rho\leqq 1$. Hence the sheaf on M of ultradistributions of class (s) (resp. $\{s\}$) is a left $\mathscr{D}^\infty_{(\rho)}$ (resp. $\mathscr{D}^\infty_{\{\rho\}}$)-module if $s\rho\leqq 1$. See Appendix.

Example 1.6. Growth order of the (ultra)differential operator

$$P_1(D_t) = \cosh\sqrt{D_t} = \sum_{j=0}^{\infty} \frac{1}{(2j)!} D_t^j \tag{1.12}$$

is equal to (1/2). Operate P_1 on the delta function:

$$\varphi(t) = P_1(D_t)\delta(t). \tag{1.13}$$

Then φ is an ultradistribution of class (2).

Example 1.7. Growth order of the (ultra)differential operator

$$P_2(D_t) = \sum_{j=2}^{\infty} \frac{1}{j!(\log j)^j} D_t^j \tag{1.14}$$

is equal to $\{1\}$. Hence the hyperfunction

$$\psi(t) = P_2(D_t)\delta(t) \tag{1.15}$$

does not belong to any class of ultradistributions near the origin.

2. Equivalence of microdifferential operators with constant multiplicity.

Let P be a microdifferential operator of finite order defined in a neighborhood of $\mathring{x}^*=(\mathring{x}, \mathring{\xi})=(0; 0, \cdots, 0, 1)$. Assume that the principal symbol of P is ξ_1^m ($m \in \boldsymbol{N}$). Let $\mathscr{M}$ be the microdifferential equation $Pu=0$, i.e. the left $\mathscr{E}$-module $\mathscr{M}=\mathscr{E}/\mathscr{E}P$.

Set $\mathscr{N}=\mathscr{E}/\mathscr{E}D_1$, that is, $\mathscr{N}$ is the microdifferential equation $D_1v=0$. Then the following theorem is well known ([8] Chap. II, Th. 5.2.1).

THEOREM 2.1. *The equation $\mathscr{M}$ is isomorphic to the direct sum of m copies of $\mathscr{N}$ as a left $\mathscr{E}^\infty$-module in a neighborhood of $\mathring{x}^*$:*

$$\mathscr{E}^\infty \otimes \mathscr{M} \simeq \mathscr{E}^\infty \otimes \Big(\bigoplus_1^m \mathscr{N}\Big). \tag{2.1}$$

The preceding theorem is sufficiently powerful in the framework of hyperfunctions. In the category of ultradistributions, however, slightly more delicate statement is needed. Moreover, it is natural to ask what happens if $\mathscr{E}^\infty$ is replaced by $\mathscr{E}^\infty_{(\rho)}$ in (2.1). For example, (2.1) does not hold if $\mathscr{E}^\infty$ is replaced by $\mathscr{E}$ in general. (Cf. [5])

THEOREM 2.2. *Let σ be the irregularity of P at $\mathring{x}^*$. Then $\mathscr{M}$ is isomorphic to the direct sum of m copies of $\mathscr{N}$ as a left $\mathscr{E}^\infty_{(\rho)}$-module in a neighborhood of $\mathring{x}^*$ if $\rho \geqq 1-1/\sigma$.*

$$\mathscr{E}^\infty_{(\rho)} \otimes \mathscr{M} \simeq \mathscr{E}^\infty_{(\rho)} \otimes \Big(\bigoplus_1^m \mathscr{N}\Big). \tag{2.2}$$

PROOF. By the division theorem of Weierstrass type for microdifferential operators, P can be written in the form

$$P(x, D)=Q(x, D)(D_1^m-P_{m-1}(x, D')D_1^{m-1}-\cdots-P_1(x, D')D_1-P_0(x, D')) \tag{2.3}$$

where Q is an invertible microdifferential operator and P_j is a microdifferential operator of order at most $m-j-1$, which does not contain $D_1(=\partial/\partial x_1)$; D' denotes $(D_2, \cdots, D_n)$. Since Q is invertible, we may assume that

$$P(x, D)=D_1^m-P_{m-1}(x, D')D_1^{m-1}-\cdots-P_1(x, D')D_1-P_0(x, D') \tag{2.4}$$

from the beginning. By the definition of the irregularity σ,

$$\operatorname{Order}(P_j(x, D')D_1^j) \leqq m+\frac{1}{\sigma}(j-m) \tag{2.5}$$

holds for $j=0, 1, \cdots, m-1$.

Now the equation $Pu=0$ is equivalent to the equation

$$D_1U=A(x, D')U \tag{2.6}$$

where $U={}^t(u_1, \cdots, u_m)$ is an unknown vector and

$$A(x, D')=\begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ P_0(x, D') & P_1(x, D') & \cdots\cdots & P_{m-2}(x, D') & P_{m-1}(x, D') \end{pmatrix}. \tag{2.7}$$

To prove the theorem, it is sufficient to find an invertible matrix R of micro-differential operators of growth order at most $(1-1/\sigma)$ satisfying

$$(D_1-A(x, D'))R=RD_1. \tag{2.8}$$

Let $A(x, \xi')$ be the (total) symbol of $A(x, D')$:

$$A(x, \xi')=\begin{pmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ P_0(x, \xi') & P_1(x, \xi') & \cdots\cdots & P_{m-2}(x, \xi') & P_{m-1}(x, \xi') \end{pmatrix} \tag{2.9}$$

where ξ' denotes $(\xi_2, \cdots, \xi_n)$ and $P_j(x, \xi')$ is the symbol of $P_j(x, D')$. Remark that $P_j(x, \xi')$ is a holomorphic function defined in a conic neighborhood Γ of $\mathring{x}^*$ which satisfies the following estimate (see (2.5)): there is a constant $h>0$ such that

$$|P_j(x, \xi')|\leqq h|\xi|^{\rho(m-j)}, \quad (x, \xi)\in\Gamma \tag{2.10}$$

where $\rho=1-1/\sigma$.

Let us define a sequence of matrices of holomorphic functions $\{R_l(x, \xi')\}_{l\geqq 0}$ by solving the following differential equations successively: for $l=0$

$$\begin{cases} \dfrac{\partial}{\partial x_1}R_0(x, \xi')=A(x, \xi')R_0(x, \xi') \\ R_0(x, \xi')|_{x_1=0}=I. \end{cases} \tag{2.11}$$

Here I is the identity matrix of degree m, and for $l\geqq 1$

$$\begin{cases} \dfrac{\partial}{\partial x_1}R_l(x, \xi')=A(x, \xi')R_l(x, \xi')+\displaystyle\sum_{\substack{|\alpha|+k=l \\ k<l}}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}A(x, \xi')\cdot\partial_x^{\alpha}R_k(x, \xi') \\ R_l(x, \xi')|_{x_1=0}=0. \end{cases} \tag{2.12}$$

If each $R_l(x, \xi')$ is a symbol of some microdifferential operator and the formal sum $\sum_{l\geqq 0} R_l(x, \xi')$ is a formal symbol of growth order at most (ρ), then it follows from the composition rule of microdifferential operators in terms of symbols that the operator

$$R=\sum_{l\geqq 0} R_l(x, D')$$

is a microdifferential operator which satisfies (2.8) (see [1]). We have to prove that $\sum_{l\geqq 0} R_l(x, \xi')$ is a formal symbol of growth order at most (ρ) $(\rho=1-1/\sigma)$. Take the matrix

$$\Lambda(\xi)=\begin{pmatrix} \xi_n^{(m-1)\rho} & & & & \\ & \xi_n^{(m-2)\rho} & & & \\ & & \ddots & & \\ & & & \xi_n^{\rho} & \\ & & & & 1 \end{pmatrix} \tag{2.13}$$

which is invertible near $\mathring{x}^*$. Set

$$S_l(x, \xi')=\Lambda(\xi)\cdot R_l(x, \xi')\cdot \Lambda(\xi)^{-1}, \quad l=0, 1, 2, \cdots. \tag{2.14}$$

To show that $\sum R_l(x, \xi')$ is a formal symbol of growth order at most (ρ), it is sufficient to prove that $\sum S_l(x, \xi')$ is a formal symbol of growth order at most (ρ). By (2.14), $\{S_l(x, \xi')\}$ satisfies the following equations: for $l=0$

$$\begin{cases} \dfrac{\partial}{\partial x_1} S_0(x, \xi')=\Lambda(\xi)\cdot A(x, \xi')\cdot \Lambda(\xi)^{-1} S_0(x, \xi') \\ S_0(x, \xi')|_{x_1=0}=I, \end{cases} \tag{2.15}$$

and for $l\geqq 1$

$$\begin{cases} \dfrac{\partial}{\partial x_1} S_l(x, \xi')=\Lambda(\xi)\cdot A(x, \xi')\cdot \Lambda(\xi)^{-1}\cdot S_l(x, \xi') \\ \qquad\qquad + \sum\limits_{\substack{|\alpha|+k=l \\ k<l}} \dfrac{1}{\alpha!}\Lambda(\xi)\partial_\xi^\alpha A(x, \xi')\cdot \Lambda(\xi)^{-1}\cdot \partial_x^\alpha S_k(x, \xi') \\ S_l(x, \xi')|_{x_1=0}=0. \end{cases} \tag{2.16}$$

Observe that

$$\Lambda(\xi)A(x, \xi')\Lambda(\xi)^{-1}=\begin{pmatrix} 0 & \xi_n^{\rho} & & & \\ & 0 & \xi_n^{\rho} & & \\ & & \ddots & \ddots & \\ & & & \ddots & \xi_n^{\rho} \\ P_0(x, \xi')\cdot \xi_n^{-(m-1)\rho} & P_1(x, \xi')\cdot \xi_n^{-(m-2)\rho} & \cdots & \cdots & P_{m-1}(x, \xi') \end{pmatrix}.$$

Hence it follows from (2.10) and Cauchy's integral formula that for each compactly generated cone $\Gamma'\subset\Gamma$,

$$|\Lambda(\xi)\cdot\partial_\xi^\alpha A(x,\xi')\cdot\Lambda(\xi)^{-1}|\leqq\alpha!\,h|\xi|^{\rho-|\alpha|}\varepsilon^{-|\alpha|} \tag{2.17}$$

holds for $(x,\xi)\in\Gamma'$ and for every multi-index α. Here ε is the distance from Γ' to $\partial\Gamma$ on $|\xi|=1$.

Each $S_l(x,\xi')$ satisfies the following estimate: There is a positive constant $M>2$ such that

$$|S_l(x,\xi')|\leqq l!\,\varepsilon^{-Ml}|\xi|^{-l}\sum_{\nu=0}^{l}\frac{(h|x_1|\cdot|\xi|^\rho)^\nu}{\nu!}\exp(h|x_1|\cdot|\xi|^\rho) \tag{2.18}$$

for $(x,\xi)\in\Gamma'$.

Let us prove the estimate by induction on l. First, remark that $S_0(x,\xi')$ satisfies the following integral equation.

$$S_0(x,\xi')=I+\int_0^{x_1}\Lambda(\xi)\cdot A(t,x',\xi')\cdot\Lambda(\xi)^{-1}S_0(t,x',\xi')dt \tag{2.19}$$

where x' denotes $(x_2,\cdots,x_n)$. In view of (2.17) it follows from Gronwall's inequality that

$$|S_0(x,\xi')|\leqq\exp(h|x_1|\cdot|\xi|^\rho)\quad\text{for }(x,\xi)\in\Gamma'. \tag{2.20}$$

Next, suppose that the following inequality holds for each $k<l$:

$$|S_k(x,\xi')|\leqq k!\,\varepsilon^{-Mk}|\xi|^{-k}\sum_{\nu=0}^{k}\frac{(h|x_1|\cdot|\xi|^\rho)^\nu}{\nu!}\exp(h|x_1|\cdot|\xi|^\rho) \tag{2.21}$$

for $(x,\xi)\in\Gamma'$. Then there is a constant N such that $M-1>N>1$ for which

$$|\partial_{x'}^\alpha S_k(x,\xi')|\leqq(|\alpha|+k)!\,\varepsilon^{-Mk-N|\alpha|}|\xi|^{-k}\sum_{\nu=0}^{k}\frac{(h|x_1|\cdot|\xi|^\rho)^\nu}{\nu!}\exp(h|x_1|\cdot|\xi|^\rho) \tag{2.22}$$

for $(x,\xi)\in\Gamma'$ and for each α.

The preceding inequality is proved by induction on $|\alpha|$ as follows. Use the assumption of induction for Γ'' such that $\Gamma''\supset\Gamma'$ and that distance from Γ' to $\partial\Gamma''$ on $|\xi|=1$ is equal to $\varepsilon/(|\alpha|+k)$. Then by the aid of Cauchy's integral formula, we have

$$\begin{aligned}|\partial_{x'}^\alpha S_k(x,\xi')|\leqq&(\varepsilon/(|\alpha|+k))^{-1}(|\alpha|-1+k)!\,((1-1/(|\alpha|+k))\varepsilon)^{-Mk-N|\alpha|+N}|\xi|^{-k}\\&\times\sum_{\nu=0}^{k}\frac{(h|x_1|\cdot|\xi|^\rho)^\nu}{\nu!}\exp(h|x_1|\cdot|\xi|^\rho)\end{aligned}$$

$$\leqq(|\alpha|+k)!\,\varepsilon^{-Mk-N|\alpha|}e^{M}\varepsilon^{N-1}|\xi|^{-k}\sum_{\nu=0}^{k}\frac{(h|x_1|\cdot|\xi|^{\rho})^{\nu}}{\nu!}\exp(h|x_1|\cdot|\xi|^{\rho})$$

for $(x,\xi)\in\Gamma'$. Here $e=2.718\cdots$. Take N as $M-1>N>1$. Since we may assume that ε is sufficiently small, we have $e^{M}\varepsilon^{N-1}\leqq 1$. Hence we have (2.22). Now, $S_l(x,\xi')$ satisfies the following integral equation.

$$S_l(x,\xi')=\int_0^{x_1}\left\{\sum_{\substack{|\alpha|+k=l\\ k<l}}\frac{1}{\alpha!}\Lambda\cdot\partial_{\xi'}^{\alpha}A(t,x',\xi')\cdot\Lambda^{-1}\cdot\partial_{x'}^{\alpha}S_k(t,x',\xi')\right.$$
$$\left.+\Lambda\cdot A(t,x',\xi')\cdot\Lambda^{-1}S_l(t,x',\xi')\right\}dt. \tag{2.23}$$

Hence it follows from (2.17), (2.22), and the Gronwall's inequality that

$$|S_l(x,\xi')|\leqq\sum_{\substack{|\alpha|+k=l\\ k<l}}\frac{1}{\alpha!}\alpha!\,h|\xi|^{\rho-|\alpha|}\varepsilon^{-|\alpha|}l!\,\varepsilon^{-Mk-N|\alpha|}|\xi|^{-k}$$
$$\times\sum_{\nu=0}^{k}\frac{(h|x_1|\cdot|\xi|^{\rho})^{\nu}}{(\nu+1)!}|x_1|\exp(h|x_1|\cdot|\xi|^{\rho})$$
$$\leqq l!\,\varepsilon^{-Ml}|\xi|^{-l}\sum_{\nu=0}^{l}\left(\sum_{j=1}^{l-\nu+1}\varepsilon^{(M-N-1)j}\cdot 2^{n-1+j}\right)\frac{(h|x_1|\cdot|\xi|^{\rho})^{\nu}}{\nu!}\exp(h|x_1|\cdot|\xi|^{\rho}).$$

We may assume that ε is sufficiently small. Therefore we can take $M>N+1$ independently of l for which

$$\sum_{j=1}^{l-\nu-1}\varepsilon^{(M-N-1)j}\cdot 2^{n-1+j}\leqq 1$$

holds. Hence we have estimate (2.18).

Then (2.18) implies

$$|S_l(x,\xi')|\leqq(l+1)!\,\varepsilon^{-Ml}|\xi|^{-l}\exp(2h|x_1|\cdot|\xi|^{\rho}) \tag{2.24}$$

since $t^{\nu}\exp(-t)\leqq\nu!$ for $t>0$. Thus the formal sum $\sum S_l(x,\xi')$ is a formal symbol of growth order at most (ρ). Then the formal sum $\sum R_l(x,\xi')$ is also a formal symbol of growth order at most (ρ). On the other hand, each symbol $R_l(x,\xi')$ defines a microdifferential operator by the definition. Hence we obtain a microdifferential operator

$$R=\sum_{l=0}^{\infty}R_l(x,D')$$

of growth order at most (ρ) defined in a neighborhood of $\mathring{x}^*$, which satisfies

$$(D_1-A(x,D'))R=RD_1. \tag{2.25}$$

Next we have to prove that R is invertible. In the same way as to construct

R, we can find a microdifferential operator T of growth order at most (ρ) satisfying

$$D_1 T = T(D_1 - A(x, D')). \tag{2.26}$$

That is, T is represented by a formal symbol $\sum_{j=0}^{\infty} T_j(x, \xi')$ of growth order at most (ρ) which is defined by solving the following differential equations successively: for $j=0$

$$\begin{cases} \dfrac{\partial}{\partial x_1} T_0(x, \xi') = -T_0(x, \xi') A(x, \xi') \\ T_0(x, \xi')|_{x_1=0} = I, \end{cases} \tag{2.27}$$

and for $j \geqq 1$

$$\begin{cases} \dfrac{\partial}{\partial x_1} T_j(x, \xi') = -T_j(x, \xi') A(x, \xi') - \sum_{\substack{|\alpha|+k=j \\ k<j}} \dfrac{1}{\alpha!} \partial_{\xi'}^{\alpha} T_k(x, \xi') \cdot \partial_{x'}^{\alpha} A(x, \xi') \\ T_j(x, \xi')|_{x_1=0} = 0. \end{cases} \tag{2.28}$$

Consider two operators $F=TR$ and $G=RT$. By the composition rule in terms of formal symbols ([1] Th. 2.2.2), those operators are represented as follows.

$$\begin{cases} F = \sum_{k=0}^{\infty} F_k(x, D') \\ F_k(x, \xi') = \sum_{|\alpha|+j+l=k} \dfrac{1}{\alpha!} \partial_{\xi'}^{\alpha} T_j(x, \xi') \cdot \partial_{x'}^{\alpha} R_l(x, \xi'), \end{cases} \tag{2.29}$$

$$\begin{cases} G = \sum_{k=0}^{\infty} G_k(x, D') \\ G_k(x, \xi') = \sum_{|\alpha|+j+l=k} \dfrac{1}{\alpha!} \partial_{\xi'}^{\alpha} R_l(x, \xi') \cdot \partial_{x'}^{\alpha} T_j(x, \xi'). \end{cases} \tag{2.30}$$

Hence we have

$$F|_{x_1=0} = \mathrm{Id} \quad \text{and} \quad G|_{x_1=0} = \mathrm{Id}.$$

On the other hand, it follows from (2.25) and (2.26) that

$$[D_1, F] = 0 \quad \text{and} \quad [D_1 - A(x, D'), G] = 0.$$

Therefore we have $F=G=\mathrm{Id}$ (see [8], the proof of Th. 5.2.1). Thus R is invertible. This completes the proof of the theorem.

3. Equivalence of degenerate ordinary microdifferential operators.

In this section we assume that $n=1$. Let P be a microdifferential operator defined in a neighborhood of $(\mathring{x}, \mathring{\xi}) = (0, 1) \in T^*\boldsymbol{C}$. Assume that the principal symbol of P is x^m ($m \in N$). Let $\mathscr{M}$ be the microdifferential equation $Pu=0$ and $\mathscr{N}$ be the equation $xv=0$, that is,

$$\mathscr{M}=\mathscr{E}/\mathscr{E}P \quad \text{and} \quad \mathscr{N}=\mathscr{E}/\mathscr{E}x.$$

Then we have the following theorem (cf. [3]).

THEOREM 3.1. *Let σ be the irregularity of P at $(\mathring{x}, \mathring{\xi})$. Then $\mathscr{M}$ is isomorphic to the direct sum of m copies of $\mathscr{N}$ as a left $\mathscr{E}^{\mathbf{R}}_{(\rho)}$-module for $\rho \geqq 1-1/\sigma$:*

$$\mathscr{E}^{\mathbf{R}}_{(\rho)} \otimes \mathscr{M} \simeq \mathscr{E}^{\mathbf{R}}_{(\rho)} \otimes \left(\bigoplus_{1}^{m} \mathscr{N} \right). \tag{3.1}$$

PROOF. By the division theorem of Weierstrass type, we may assume that P is of the form

$$P=x^m-P_{m-1}(D)x^{m-1}-\cdots-P_1(D)x-P_0(D) \tag{3.2}$$

where D denotes d/dx and $P_j(D)$ is a microdifferential operator of order $\leqq -1$ with constant coefficients $(0 \leqq j \leqq m-1)$. The order of $P_j(D)$ is estimated by

$$\mathrm{Order}(P_j(D)) \leqq -\frac{1}{\sigma}(m-j), \quad j=0, 1, \cdots, m-1. \tag{3.3}$$

The equation $Pu=0$ is equivalent to the equation

$$xU=A(D)U \tag{3.4}$$

where $U={}^t(u_1, \cdots, u_m)$ is an unknown vector and

$$A(D)=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ P_0(D) & P_1(D) & \cdots\cdots & P_{m-2}(D) & P_{m-1}(D) \end{pmatrix} \tag{3.5}$$

To prove the theorem, it is sufficient to construct an invertible matrix of holomorphic microlocal operators of growth order at most (ρ) $(\rho=1-1/\sigma)$ such that

$$(x-A(D))R=Rx. \tag{3.6}$$

Let us find R. Take the symbol $A(\xi)$ of $A(D)$:

$$\tag{3.7} A(\xi)=\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 0 \\ P_0(\xi) & P_1(\xi) & \cdots\cdots & P_{m-2}(\xi) & P_{m-1}(\xi) \end{pmatrix}$$

here $P_j(\xi)$ is the symbol of $P_j(D)$. Each $P_j(\xi)$ is holomorphic in some conic neighborhood of $(\mathring{x}, \mathring{\xi})$. In view of (3.3), there are a conic neighborhood Γ of $(\mathring{x}, \mathring{\xi})$ and a positive constant h such that

$$\tag{3.8} |P_j(\xi)| \leqq h|\xi|^{-(m-j)/\sigma} \quad \text{for } (x, \xi) \in \Gamma.$$

Let $R(\xi)$ be a fundamental solution of the following matrix ordinary differential equation.

$$\tag{3.9} -\frac{d}{d\xi} R(\xi) = A(\xi) R(\xi).$$

We define a matrix $\Lambda(\xi)$ by

$$\tag{3.10} \Lambda(\xi)=\begin{pmatrix} 1 & & & & \\ & \xi^p & & & \\ & & \ddots & & \\ & & & \xi^{(m-2)p} & \\ & & & & \xi^{(m-1)p} \end{pmatrix}$$

where $p=1/\sigma$. Set $S(\xi)=\Lambda(\xi)\cdot R(\xi)$. Then $S(\xi)$ satisfies the equation

$$\tag{3.11} -\frac{d}{d\xi} S(\xi) = \left(\Lambda(\xi)\cdot A(\xi)\cdot \Lambda(\xi)^{-1} - \frac{d}{d\xi}\Lambda(\xi)\cdot \Lambda(\xi)^{-1}\right) S(\xi).$$

In view of (3.8) and (3.10) we have

$$|\Lambda(\xi)\cdot A(\xi)\cdot \Lambda(\xi)^{-1}| \leqq h|\xi|^{-p}$$

and

$$\left|\frac{d}{d\xi}\Lambda(\xi)\cdot \Lambda(\xi)^{-1}\right| \leqq h'|\xi|^{-1}$$

for $(x, \xi) \in \Gamma$, where h' is a positive constant. Hence there is a positive constant C_0 such that

$$\tag{3.12} \left|\Lambda(\xi)\cdot A(\xi)\cdot \Lambda(\xi)^{-1} - \frac{d}{d\xi}\Lambda(\xi)\cdot \Lambda(\xi)^{-1}\right| \leqq C_0|\xi|^{-p}$$

for $(x, \xi) \in \Gamma$ ($|\xi| \gg 1$). Then it follows that there exist positive constants C_1, H_1 such that

$$|S(\xi)| \leqq C_1 \exp(H_1 \cdot |\xi|^{1-p})$$

and that

$$|S(\xi)^{-1}| \leqq C_1 \exp(H_1 \cdot |\xi|^{1-p}).$$

Therefore $R(\xi)$ satisfies the following estimates:

$$|R(\xi)| \leqq C_2 \exp(H_2 \cdot |\xi|^{1-1/\sigma}),$$

$$|R(\xi)^{-1}| \leqq C_2 \exp(H_2 \cdot |\xi|^{1-1/\sigma})$$

for some positive constants C_2, H_2. Clearly, $R(\xi)$ and $R(\xi)^{-1}$ are holomorphic in a conic neighborhood of $\mathring{\xi}=1$. Hence $R(\xi)$ defines an invertible matrix R of holomorphic microlocal operators of growth order at most $(1-1/\sigma)$. Since the symbol of the operator $[x, R]$ is equal to $-(d/d\xi)R(\xi)$, R satisfies

$$(x-A(D))R = Rx.$$

This finishes the proof of the theorem.

Example 3.2. Consider the equation $\mathscr{M}: ((xD)^2-D)u=0$. $\mathscr{M}$ is equivalent to

$$\left(xD-\begin{pmatrix} 0 & 1 \\ D & 0 \end{pmatrix}\right)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=0.$$

Set

$$R(D)=\begin{pmatrix} I_0(2\sqrt{D}) & -I_0(2\sqrt{D})\log D+\sum_{k=1}^{\infty}\frac{\varphi_k}{k!^2}D^k \\ -\sqrt{D}\,I_1(2\sqrt{D}) & 1+\sqrt{D}\,I_1(2\sqrt{D})\log D-\sum_{k=0}^{\infty}\frac{\psi_k}{k!\,(k+1)!}D^{k+1} \end{pmatrix}$$

where

$$I_\nu(z)=(z/2)^\nu\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{k!\,(k+\nu)!}, \quad \varphi_k=2\sum_{j=1}^{k}\frac{1}{j}, \quad \psi_k=\varphi_k+\frac{1}{k+1}.$$

Then $R(D)^{-1}$ is equal to

$$\begin{pmatrix} 1+\sqrt{D}\,I_1(2\sqrt{D})\log D-\sum_{k=0}^{\infty}\frac{\psi_k}{k!\,(k+1)!}D^{k+1} & -I_0(2\sqrt{D})\log D+\sum_{k=1}^{\infty}\frac{\varphi_k}{k!^2}D^k \\ \sqrt{D}\,I_1(2\sqrt{D}) & I_0(2\sqrt{D}) \end{pmatrix}$$

and

$$\left(xD-\begin{pmatrix}0 & 1\\ D & 0\end{pmatrix}\right)R(D)=R(D)xD.$$

Hence $\mathcal{M}$ is equivalent to the equations $xDv_1=xDv_2=0$. The irregularity of $\mathcal{M}$ is equal to 2 and $R(D)$ is of growth order at most $(1-1/2)=(1/2)$.

Appendix

Let M be an open set in $\boldsymbol{R}^n$ and $X\subset \boldsymbol{C}^n=\boldsymbol{R}^n\times\sqrt{-1}\boldsymbol{R}^n$ be a complexification of M. We denote by $\mathscr{C}=\mathscr{C}_M$ the sheaf on $T_M^*X\simeq\sqrt{-1}T^*M$ of microfunctions (cf. [8]). Take a point $\mathring{x}^*=(\mathring{x}, \sqrt{-1}\mathring{\eta})$ in $\sqrt{-1}T^*M$ where $(x, \sqrt{-1}\eta)$ is the coordinates in $\sqrt{-1}T^*M\simeq M\times\sqrt{-1}\boldsymbol{R}^n$. Let u be a (germ of) microfunction in $\mathscr{C}_{\mathring{x}^*}$. We can assume that u is represented in a neighborhood of $\mathring{x}^*$ as the spectrum of the boundary value of a holomorphic function φ defined in $U+\sqrt{-1}W$. Here $U\subset M$ is a neighborhood of $\mathring{x}\in M$ and W is a small wedge of the form

$$W=\{w\in\boldsymbol{R}^n;\ \langle w, \eta\rangle>0 \text{ for each } \eta \text{ such that } |\eta-\mathring{\eta}|\ll 1,\ |w|\ll 1\}.$$

The defining function φ is uniquely determined by u modulo holomorphic functions defined in some complex neighborhood of U.

DEFINITION A.1. Let ρ be a real number such that $0<\rho<1$. The microfunction u is said to be of growth order at most (ρ) (resp. $\{\rho\}$) if the following condition (A.1) (resp. (A.1)′) for φ is satisfied.

(A.1) For each compact set $L\Subset U$ and compact wedge $W'\Subset W$ there are positive constants C, h such that

$$\sup_{x\in L}|\varphi(x+\sqrt{-1}y)|\leqq C\cdot\exp(h|y|^{-\rho/(1-\rho)})$$

for $y\in W'$.

(A.1)′ For each compact set $L\Subset U$, compact wedge $W'\Subset W$ and positive number h there is a positive constant C such that

$$\sup_{x\in L}|\varphi(x+\sqrt{-1}y)|\leqq C\cdot\exp(h|y|^{-\rho/(1-\rho)})$$

for $y\in W'$.

The microfunction u is called of growth order (0) if there is a real number m for which for each compact set $L\Subset U$ and compact wedge $W'\Subset W$ there exists a positive constant C such that

$$\sup_{x\in L}|\varphi(x+\sqrt{-1}y)|\leqq C|y|^{-m}$$

for $y\in W'$.

Otherwise, the microfunction u is said to be of growth order $\{1\}$.

Suppose that $0\leqq\rho<1$ (resp. $0<\rho\leqq1$). We denote by $\mathscr{C}_{(\rho),x^*}$ (resp. $\mathscr{C}_{\{\rho\},x^*}$) the set of all microfunctions in $\mathscr{C}_{x^*}$ ($x^*\in\sqrt{-1}T^*M$) of growth order at most (ρ) (resp. $\{\rho\}$). The sheaf on $\sqrt{-1}T^*M$ of germs of microfunctions in $\mathscr{C}_{(\rho),x^*}$ (resp. $\mathscr{C}_{\{\rho\},x^*}$) is denoted by $\mathscr{C}_{(\rho)}$ (resp. $\mathscr{C}_{\{\rho\}}$).

Let $\mathscr{E}^R_{(\rho)}$ (resp. $\mathscr{E}^R_{\{\rho\}}$) be the sheaf on T^*X of holomorphic microlocal operators of growth order at most (ρ) (resp. $\{\rho\}$). The restriction of $\mathscr{E}^R_{(\rho)}$ (resp. $\mathscr{E}^R_{\{\rho\}}$) to $T^*_MX\simeq\sqrt{-1}T^*M$ is also denoted by $\mathscr{E}^R_{(\rho)}$ (resp. $\mathscr{E}^R_{\{\rho\}}$). Then we have the following.

PROPOSITION A.2. *The sheaf* $\mathscr{C}_{(\rho)}$ (resp. $\mathscr{C}_{\{\rho\}}$) *is a left* $\mathscr{E}^R_{(\rho)}$ *(resp.* $\mathscr{E}^R_{\{\rho\}}$*)-module.*

PROOF. If $\rho=1$, the proposition is clear. Suppose that $0<\rho<1$. Take a point $\mathring{x}^*$ in $\sqrt{-1}T^*M$. We can assume that $\mathring{x}^*=(\mathring{x};\sqrt{-1}(1,0,\cdots,0))=(\mathring{x};\sqrt{-1}\mathring{\eta})$. Let F be an operator in $\mathscr{E}^R_{(\rho),\mathring{x}^*}$ (resp. $\mathscr{E}^R_{\{\rho\},\mathring{x}^*}$) and let $L(x,x-x')dx'$ be the defining function of F. The holomorphic function $L(x,x-x')$ is given by

$$L(x,x-x')=\int_{\substack{|\eta-\mathring{\eta}|\ll1\\|\eta|=1}}f(x,\sqrt{-1}\eta,\langle x-x',\sqrt{-1}\eta\rangle)\omega(\sqrt{-1}\eta)\tag{A.2}$$

which is defined on $\Omega=\{(x,x')\in X\times X;|x-\mathring{x}|\ll1,|x-x'|\ll1,\operatorname{Im}(x_1'-x_1)<\varepsilon|\operatorname{Re}(x_1'-x_1)|,$ $|x_1-x_1'|<\varepsilon|x_j-x_j'|,\ j=2,\cdots,n\}$ for some $\varepsilon>0$. Here $\omega(\xi)=\sum_{j=1}^n(-1)^{j-1}\xi_jd\xi_1\wedge\cdots\wedge\widehat{d\xi_j}\wedge\cdots\wedge d\xi_n$ and $f(x,\xi,p)\omega(\xi)$ is the normalized Radon transformation of F (cf. [1], [6]). That is,

$$f(x,\xi,p)=(2\pi\sqrt{-1})^{-n}\int_R^{+\infty}F(x,\tau\xi)e^{p\tau}\tau^{n-1}d\tau$$

where $F(x,\xi)$ is the symbol of F and $R>0$ is a constant. Since the growth order of F is at most (ρ) (resp. $\{\rho\}$), there exist positive constants C_1, h_1 (resp. there is a positive constant C_1 for each $h_1>0$) such that

$$|f(x,\sqrt{-1}\eta,p)|\leqq C_1\cdot\exp(h_1|p|^{-\rho/(1-\rho)})\tag{A.3}$$

for $|x-\mathring{x}|\ll1$, $|\eta-\mathring{\eta}|\ll1$ (see [1] Lemma 2.3.1). Hence for each compact set $\Omega'\Subset\Omega$ there exist constants $C_2>0$, $h_2>0$ (resp. for each compact set $\Omega'\Subset\Omega$ and $h_2>0$ there exists a constant $C_2>0$) such that

$$|L(x, r(x-x'))| \leqq C_2 \exp(h_2 r^{-\rho/(1-\rho)}) \tag{A.4}$$

for $(x, x') \in \Omega'$, $0 < r \leqq 1$.

Let u be a microfunction in $\mathscr{C}_{(\rho),\mathring{x}^*}$ (resp. $\mathscr{C}_{\{\rho\},\mathring{x}^*}$). The defining function φ of u satisfies the estimate of the type (A.1) (resp. (A.1)'). Let $\alpha_\pm$ be points in $\boldsymbol{C}$ sufficiently near the origin such that $\operatorname{Im}\alpha_\pm > \pm\varepsilon \operatorname{Re}\alpha_\pm$. Take the paths $\{\gamma_j\}$ as follows: γ_1 is a path starting from α_-, ending at α_+ around x_1 counterclockwise, and γ_j ($j \geqq 2$) is a cycle rounding x_j counterclockwise (with radius$>\varepsilon^{-1}|x_1 - x_1'|$). Set

$$\psi(x) = \int_{\gamma_1} \cdots \int_{\gamma_n} L(x, x-x')\varphi(x')dx'. \tag{A.5}$$

Then the microfunction Fu is the spectrum of the boundary value of ψ (cf. [4]). It follows from (A.1) (resp. (A.1)') and (A.4) that ψ satisfies an estimate of (A.1) (resp. (A.1)') type. Hence the microfunction Fu is of growth order at most (ρ) (resp. $\{\rho\}$).

Similar argument as above proves the case $\rho = 0$.

Remark A.3. If φ satisfies estimate (A.1) (resp. (A.1)'), then the limit

$$\lim_{\substack{y \to 0 \\ y \in W'}} \varphi(x + \sqrt{-1}y)$$

exists in the space of ultradistributions of class $(1/\rho)$ (resp. $\{1/\rho\}$) and vice versa. Hence $\mathscr{E}^{\boldsymbol{R}}_{(\rho)}$ (resp. $\mathscr{E}^{\boldsymbol{R}}_{\{\rho\}}$) acts on the space of ultradistributions of class (s) (resp. $\{s\}$) for $s\rho \leqq 1$ microlocally.

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