The quasi-classical approximation to Dirac equation, I

By Kenji YAJIMA*)

ABSTRACT

The mathematical foundation to the quasi-classical approximation to the Dirac equation is discussed. After constructing the propagator $U^{\hbar}(t, s)$ and the quasi-classical propagator for the Dirac equation in an external electro-magnetic field, the usual quasi-classical approximation to the wave function $U^{\hbar}(t, s)\ (\exp(iS(x)/\hbar)f)$ is obtained with error estimates in the Hilbert $L^2(\boldsymbol{R}^3, \boldsymbol{C}^4)$.

§1. Introduction.

The present paper is the first of a set of two papers aiming at laying the mathematical foundation to the quasi-classical approximation to the Dirac equation. We study here the problem in the finite time and in the companion we shall deal with the associated scattering operator.

The state of a Dirac particle (mass $m>0$ and change $e<0$) is described by an element of the Hilbert space $\mathscr{H} \equiv L^2(\boldsymbol{R}^3, \boldsymbol{C}^4)$ of $\boldsymbol{C}^4$-valued square integrable functions over $\boldsymbol{R}^3$ and its dynamics in an external electro-magnetic field described by a four vector $(\phi(t, \vec{x}),\ \vec{A}(t, \vec{x}))=(A^{\mu}(t, \vec{x}))_{\mu=0,1,2,3}$ is governed by the Dirac equation

$$
\begin{aligned}
i\hbar\frac{\partial u}{\partial t} &= \sum_{j=1}^{3} c\alpha^j\left(-i\hbar\frac{\partial}{\partial x^j}-\frac{e}{c}A^j(t, \vec{x})\right)u+mc^2\beta u+e\phi(t, \vec{x})u \\
&\equiv H^{\hbar}(t)u.
\end{aligned} \tag{1.1}
$$

Here $\hbar=h/2\pi$, h is Planck's constant, $c>0$ is the velocity of the light, and α^j $(j=1, 2, 3)$ and β are Dirac's 4×4-matrices:

$$
\alpha^j=\begin{bmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{bmatrix}, \quad \beta=\begin{bmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{bmatrix},
$$

where σ^{μ} $(\mu=0, 1, 2, 3)$ are Pauli's spin matrices:

$$
\sigma^0=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2=\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad \sigma^3=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

*) Current address: Department of Mathematics, Princeton University, Fine Hall, Princeton, N.J. 08544, U.S.A.

We suppose throughout this paper that the potential $(A^\mu(t,\vec{x}))$ satisfies the following condition.

ASSUMPTION (A). (1) For any $\mu=0,1,2,3$, $A^\mu(t,\vec{x})$ is a real-valued C^∞-function of $(t,\vec{x})\in[T_1,T_2]\times\boldsymbol{R}^3$. (2) For any multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and any integer $k\geqq 0$ such that $|\alpha|+k\geqq 1$ there exists a constant $C_{\alpha k}>0$ such that

$$|(\partial/\partial t)^k(\partial/\partial\vec{x})^\alpha A^\mu(t,\vec{x})|\leqq C_{\alpha k},\quad (t,\vec{x})\in[T_1,T_2]\times\boldsymbol{R}^3. \tag{1.2}$$

Under this condition we shall first show that the equation (1.1) generates a unitary propagator. $\mathscr{S}\equiv\mathscr{S}(\boldsymbol{R}^3;\boldsymbol{C}^4)$ is the space of rapidly decreasing functions, $\mathscr{S}'\equiv\mathscr{S}'(\boldsymbol{R}^3,\boldsymbol{C}^4)$ is its dual space [15].

THEOREM 1. *Let $(A^\mu(t,\vec{x}))$ satisfy the Assumption* (A). *Then there exists a family of operators $\{U^\kappa(t,s)\colon T_1\leqq s,t\leqq T_2\}$ satisfying the following properties*:

(1) *For each t and s, $U^\kappa(t,s)$ is a unitary operator on $\mathscr{H}$.*

(2) *For any t,s and r, $U^\kappa(t,s)U^\kappa(s,r)=U^\kappa(t,r)$ and $U^\kappa(t,t)=I=$the identity operator on $\mathscr{H}$.*

(3) *If $f\in\mathscr{S}$ then $U^\kappa(t,s)f$ is an $\mathscr{S}$-valued C^∞-function of (t,s) and satisfies*

$$i\hbar\frac{\partial}{\partial t}U^\kappa(t,s)f=H^\kappa(t)U^\kappa(t,s)f; \tag{1.3}$$

$$i\hbar\frac{\partial}{\partial s}U^\kappa(t,s)f=-U^\kappa(t,s)H^\kappa(s)f. \tag{1.4}$$

We call $U^\kappa(t,s)$ the *propagator* associated with the equation (1.1). The main purpose of this paper is to study the asymptotic behavior as $\hbar\downarrow 0$, or *quasi-classical limit*, of $U^\kappa(t,s)(\exp(iS(\vec{x})/\hbar)f)$ in $\mathscr{H}$ for suitable f and $S(\vec{x})$. To state our main results, we introduce some terminology. For any $(t,\vec{x},\vec{\xi})$, the matrix

$$D(t,\vec{x},\vec{\xi})=\sum_{j=1}^{3}\alpha^j(\xi^j-eA^j(t,\vec{x}))+m\beta+e\phi(t,\vec{x})$$

is hermitian and has two eigenvalues

$$H^\pm(t,\vec{x},\vec{\xi})=\pm((\vec{\xi}-e\vec{A}(t,\vec{x}))^2+m^2)^{1/2}+e\phi(t,\vec{x})$$

of multiplicity two (here and hereafter we normalize $c=1$).

$$P^\pm(t,\vec{x},\vec{\xi})\equiv\frac{1}{2}\left(1+\frac{D(t,\vec{x},\vec{\xi})-e\phi(t,\vec{x})}{H^\pm(t,\vec{x},\vec{\xi})-e\phi(t,\vec{x})}\right)$$

is the orthogonal projection to the eigenspace $V^\pm(t,\vec{x},\vec{\xi})$ of $D(t,\vec{x},\vec{\xi})$ corresponding to the eigenvalue $H^\pm(t,\vec{x},\vec{\xi})$. The function $H^+(t,\vec{x},\vec{\xi})$ is the Hamiltonian for a relativistic classical particle of mass m and charge e in the external electromagnetic field $(A^{\mu}(t,\vec{x}))$ and the other $H^{-}(t,\vec{x},\vec{\xi})$ can be regarded as the one for its anti-particle since $H^{-}(t,\vec{x},\vec{\xi})$ can be obtained from $H^{+}(t,\vec{x},\vec{\xi})$ by charge conjugation $e\to -e$ and the time reflection $t\to -t$. We write as $(\vec{x}^{\pm}(t,s,\vec{y},\vec{\eta}),\ \vec{\xi}^{\pm}(t,s,\vec{y},\vec{\eta}))$ the trajectory of the classical particle, i.e. the solution of the Hamilton equation

$$(1.5)\qquad \begin{cases} \dfrac{d\vec{x}^{\pm}}{dt}=\dfrac{\partial H^{\pm}}{\partial \xi}(t,\vec{x}^{\pm},\vec{\xi}^{\pm}),\\[2mm] \dfrac{d\vec{\xi}^{\pm}}{dt}=-\dfrac{\partial H^{\pm}}{\partial x}(t,\vec{x}^{\pm},\vec{\xi}^{\pm}),\end{cases}$$

satisfying the initial condition $\vec{x}^{\pm}(s,s,\vec{y},\vec{\eta})=\vec{y},\ \vec{\xi}^{\pm}(s,s,\vec{y},\vec{\eta})=\vec{\eta}$. For sufficiently small $|t-s|<\delta$, the mapping $(\vec{y},\vec{\eta})\to(\vec{x}^{\pm}(t,s,\vec{y},\vec{\eta}),\vec{\eta})$ is a global diffeomorphism on $\boldsymbol{R}^6$. We write the inverse of this map as $(\vec{y}^{\pm}(t,s,\vec{x},\vec{\eta}),\vec{\eta})$. Being given $(t,\vec{x},\vec{\xi})$, we write as $\xi_0^{\pm}=H^{\pm}(t,\vec{x},\vec{\xi})$ and $p_{\mu}^{\pm}(t,s,\vec{y},\vec{\eta})=\xi_{\mu}^{\pm}(t,s,\vec{y},\vec{\eta})-eA_{\mu}(t,\vec{x}^{\pm}(t,s,\vec{y},\vec{\eta}))$ $(\mu=0,1,2,3)$. $\gamma^0=\beta,\ \gamma^j=\beta\alpha^j$ $(j=1,2,3)$; $\sigma^{\mu\nu}=(i/2)\,(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})$ is the spinor tensor. We often consider $(t,\vec{x})$ as a four vector and write as $x=(x_0,\vec{x})$, $x_0=x^0=t$. Latin letters j,k etc. run over $1,2,3$ and Greek letters μ,ν, etc. run over $0,1,2,3$. To distinguish three vector from four vector, we write it as $\vec{x},\vec{\xi}$, etc. However if there is no fear of confusion, we write three vectors as, x, ξ, etc. for typographical reasons. $F^{\mu\nu}(t,\vec{x})=(\partial A^{\nu}/\partial x_{\mu})-(\partial A^{\mu}/\partial x_{\nu})$ is the field strength tensor. $L^{\pm}(t,\vec{x},\vec{\xi})=(\partial H^{\pm}/\partial\vec{\xi})(t,\vec{x},\vec{\xi})\cdot\vec{\xi}-H^{\pm}(t,\vec{x},\vec{\xi})$ is the Lagrangian for the Hamiltonian $H^{\pm}(t,\vec{x},\vec{\xi})$.

$$(1.6)\qquad \widetilde{S}^{\pm}(t,s,\vec{y},\vec{\eta})=\int_s^t L^{\pm}(u,\vec{x}^{\pm}(u,s,\vec{y},\vec{\eta}),\vec{\xi}^{\pm}(u,s,\vec{y},\vec{\eta}))du$$

is the action integral along the trajectory. We write for $|t-s|<\delta$

$$(1.7)\qquad S^{\pm}(t,s,\vec{x},\vec{\eta})=\widetilde{S}^{\pm}(t,s,\vec{y}^{\pm}(t,s,\vec{x},\vec{\eta}),\vec{\eta})+\vec{y}^{\pm}(t,s,\vec{x},\vec{\eta})\cdot\vec{\eta}.$$

For $j=0,1,2,\cdots$, $E_j^{\pm}(t,s,\vec{x},\vec{\eta})$ is the solution of the transport equation

$$(1.8)_j\qquad \begin{aligned}&[2\{(-(\partial S^{\pm}/\partial t)(t,s,\vec{x},\vec{\eta})-e\phi(t,\vec{x}))(\partial/\partial t)\\ &+\sum_{j=1}^{3}((\partial S^{\pm}/\partial x^j)(t,s,\vec{x},\vec{\eta})-eA^j(t,\vec{x}))(\partial/\partial x^j)\}\\ &-\Box S^{\pm}(t,s,\vec{x},\vec{\eta})-\sum(ie/2)\sigma^{\mu\nu}F_{\mu\nu}(t,\vec{x})]E_j^{\pm}(t,s,\vec{x},\vec{\eta})\\ &+i\Box E_{j-1}^{\pm}(t,s,\vec{x},\vec{\eta})=0\end{aligned}$$

with the initial condition

$$(1.9)_0\qquad E_0^{\pm}(s,s,\vec{x},\vec{\eta})=P^{\pm}(s,\vec{x},\vec{\eta}),$$

$$(1.9)_j \qquad E_j^{\mp}(s, s, \vec{x}, \vec{\eta}) = -(2p_0^{\pm}(s, s, \vec{x}, \vec{\eta}))^{-1}$$
$$(i(\partial/\partial t) - \sum \alpha^j(-i\partial/\partial x_j))(E_{j-1}^{+}(t, s, \vec{x}, \vec{\eta}) + E_{j-1}^{-}(t, s, \vec{x}, \vec{\eta}))|_{t=s}, \quad j \geqq 1,$$

where $E_{-1}^{\pm}(t, s, \vec{x}, \vec{\eta}) \equiv 0$. Using the functions $S^{\pm}(t, s, \vec{x}, \vec{\eta})$ and $E_j^{\mp}(t, s, \vec{x}, \vec{\eta})$, we define the operator $G_{\pm,N}^{\hbar}(t, s)$ and $G_N^{\hbar}(t, s)$ on $\mathscr{H}$ as follows:

$$(1.10) \qquad G_{\pm,N}^{\hbar}(t, s)f(\vec{x}) = (2\pi\hbar)^{-3/2}\int e^{-iS^{\pm}(t,s,\vec{x},\vec{\eta})/\hbar}\left\{\sum_{j=0}^{N} \hbar^j E_j^{\pm}(t, s, \vec{x}, \vec{\eta})\right\} \hat{f}^{\hbar}(\vec{\eta})d\vec{\eta},$$

and

$$(1.11) \qquad G_N^{\hbar}(t, s)f = G_{+,N}^{\hbar}(t, s)f + G_{-,N}^{\hbar}(t, s)f,$$

where $\hat{f}^{\hbar}(\eta)$ is the Fourier transform of f:

$$\hat{f}^{\hbar}(\xi) \equiv (\mathscr{F}^{\hbar}f)(\xi) \equiv (2\pi\hbar)^{-3/2}\int e^{-ix\cdot\xi/\hbar}f(x)dx.$$

For a subset $K \subset \{1, 2, 3\}$. x_K is the K-component of x, e.g. if $K=\{1, 2\}$, $x_K = (x_1, x_2)$, etc. $\mathscr{F}_K^{\hbar}$ is the partial Fourier transform:

$$(\mathscr{F}_K^{\hbar}f)(x_{K^c}, \xi_K) = (2\pi\hbar)^{-|K|/2}\int \exp(-ix_K\cdot\xi_K/\hbar)f(x)dx_K.$$

Our results are summarized in the following two theorems.

THEOREM 2. *Let Assumption* (A) *be satisfied and* $\delta>0$ *be sufficiently small. Then for* $|t-s|<\delta$, $G_{\pm,N}^{\hbar}(t, s)$ *and* $G_N^{\hbar}(t, s)$ *are bounded operators on* $\mathscr{H}$ *and there exists a constant* $C_N>0$ *such that*

$$(1.12) \qquad \|U^{\hbar}(t, s) - G_N^{\hbar}(t, s)\| \leqq C_N \min(|t-s|\hbar^N, \hbar^{N+1}).$$

Moreover there exists a constant $C>0$ *such that*

$$(1.13) \qquad \|G_{+,N}^{\hbar}(t, s)G_{-,N}^{\hbar}(s, r)\| \leqq C\hbar;$$

$$(1.13)' \qquad \|G_{-,N}^{\hbar}(t, s)G_{+,N}^{\hbar}(s, r)\| \leqq C\hbar.$$

REMARK 3. In view of the property (1.12), we call $G_N^{\hbar}(t, s)$ the *quasi-classical propagator* and $G_+^{\hbar}(t, s)$ (and $G_-^{\hbar}(t, s)$) the positive (and negative) part of $G_N^{\hbar}(t, s)$.

For $f \in \mathscr{H}$, we write as $f^{\pm}(t, s, \vec{y}, \vec{\eta})$ the solution of the ordinary differential equation

$$(1.14) \qquad \hat{M}^{\pm}(t, s, \vec{y}, \vec{\eta})f^{\pm}(t, s, \vec{y}, \vec{\eta}) \equiv p_0^{\pm}(t, s, \vec{y}, \vec{\eta})(df^{\pm}/dt)(t, s, \vec{y}, \vec{\eta})$$
$$-\sum_{\mu,\nu} (ie/4)\sigma^{\mu\nu}F_{\mu\nu}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}))f^{\pm}(t, s, \vec{y}, \vec{\eta}) = 0.$$

with the initial condition $f^{\pm}(s, s, \vec{y}, \vec{\eta})=f(\vec{y})$.

THEOREM 4. *Let Assumption* (A) *be satisfies, $T_1 \leqq t$, $s \leqq T_2$ and $S(\vec{x}) \in C^{\infty}(\boldsymbol{R}^3)$ be a real valued function. Then there exists a locally finite covering $\{\Omega_\lambda\}_\lambda$ of $\boldsymbol{R}^3$ satisfying the following properties*:

1) *For each λ, there exists a subset K_λ of $\{1, 2, 3\}$ such that the mapping $(x^{\pm}_{K^c_\lambda}(t, s, \vec{y}, \partial S/\partial\vec{y}), \xi^{\pm}_{K_\lambda}(t, s, \vec{y}, \partial S/\partial\vec{y}))$ is a diffeomorphism on the set $\bar{\Omega}_\lambda$.*

2) *If $f \in C_0^{\infty}(\Omega_\lambda)$ and $P^{\pm}(s, \vec{x}, \partial S/\partial\vec{x})f(\vec{x})=f(\vec{x})$, then the following estimate holds*:

$$
\begin{aligned}
(1.15)\qquad &\|\mathscr{F}^{\hbar}_{K_\lambda} U^{\hbar}(t, s)(\exp(iS(\vec{x})/\hbar)f)(x_{K^c_\lambda}, \xi_{K_\lambda}) \\
&\quad -\exp(i(S^{\pm}(t, s, \vec{y}, \partial S/\partial\vec{y})+S(\vec{y})-x^{\pm}_{K_\lambda}(t, s, \vec{y}, \partial S/\partial\vec{y})\cdot\xi_{K_\lambda})/\hbar) \\
&\quad \times\exp(-(i\pi/2)\operatorname{Ind}\gamma^{\pm}(t, s, \vec{y})+i|K_\lambda|\pi/4-i\operatorname{Inert}\partial x^{\pm}_{K_\lambda}/\partial\xi_{K_\lambda}(\vec{y})) \\
&\quad \times|\det(\partial(x^{\pm}_{K^c}(t, s, \vec{y}, \partial S/\partial\vec{y}), \xi^{\pm}_{K_\lambda}(t, s, \vec{y}, \partial S/\partial\vec{y}))/\partial\vec{y})|^{-1/2} \\
&\quad \times(p_0^{\pm}(t, s, \vec{y}, \partial S/\partial\vec{y})/p_0^{\pm}(s, s, \vec{y}, \partial S/\partial\vec{y}))^{-1/2} \\
&\quad \times f^{\pm}(t, s, \vec{y}, \partial S/\partial\vec{y})|_{(x_{K^c}, \xi_K)=(x^{\pm}_{K^c}(t, s, \vec{y}, \partial S/\partial\vec{y}), \xi^{\pm}_K(t, s, \vec{y}, \partial S/\partial\vec{y}))}\| \\
&\leqq C\hbar\|f\|_2,
\end{aligned}
$$

where $\operatorname{Ind}\gamma^{\pm}(t, s, \vec{y})$ *is the Keller-Maslov-index of* $\{(u, -H^{\pm}(u, \vec{x}^{\pm}(u, s, \vec{y}, \partial S/\partial\vec{y}(\vec{y})), \vec{\xi}^{\pm}(u, s, \vec{y}, \partial S/\partial\vec{y})), \vec{x}^{\pm}(u, s, \vec{y}, \partial S/\partial\vec{y}))\vec{\xi}^{\pm}(u, s, \vec{y}, \partial S/\partial\vec{y}(\vec{y})))$; Inert A *for symmetric matrix A is the inertia of the matrix; $|K|$ is the cardinal number of the set K; $\|f\|_2$ is the Sobolev norm of order* 2.

REMARK 5. We will see in Section 2 that $f^{\pm}(t, s, \vec{y}, \vec{\eta}) \in V^{\pm}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta}))$ and $|f^{\pm}(t, s, \vec{y}, \vec{\eta})|^2=p_0^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(s, s, \vec{y}, \vec{\eta})$. Thus if we interpret that the wave function $\exp(iS(\vec{x})/\hbar)f(x)$, $f(\vec{x}) \in V^{+}(s, \vec{x}, \partial S/\partial\vec{x})$ (or $V^{-}(s, \vec{x}, \partial S/\partial\vec{x})$) represents as $\hbar \downarrow 0$ at time s the ensemble of independent classical particles on the Lagrangian manifold $\{(\vec{x}, \partial S/\partial\vec{x}) : \vec{x} \in R^3\}$ with the density $|f(x)|^2dx$, then Theorem 2 and Theorem 4 may be interpreted as follows: As $\hbar \downarrow 0$ the Dirac equation represents the motions of two independent particles, the classical electron and its anti-particle, simultaneously. They have an internal degree of freedom (degree two) whose dynamics is governed by the equation (1.14) which is related to that for the magnetic dipole moment (see Proposition 2.16). The dynamics of two particles are independent and the negative and the positive energy parts $f^{\pm}(t, s, \vec{y}, \vec{\eta})$ of the wave function does not mix up. Thus for sufficiently small $\hbar>0$, we may neglect modulo the error of order $\hbar$, the problem about the negative energy.

The quasi-classical limit problem for the Dirac equation has a long history and there are many references, among which we mention the papers of Pauli [11], Rubinow-Keller [13], Maslov [8] and Plebanski-Stachel [12]. After Rubinow-Keller

[13], formal asymptotic series for the quasi-classical solution has been known for short time and for any finite time this has been extended by Maslov [8]. Unfortunately, however, the estimates like (1.12) or (1.15) have not been proved and hence it has been obscure in what sense the quasi-classical solution obtained by these authors is the approximate solution to the Dirac equation.

The plan of this paper is as follows. Section 2 is preliminary and we shall study the properties of the classical trajectories, the action integrals and the solution of the transport equation (1.8) and (1.14). Using these materials, we shall construct the propagator $U^{\kappa}(t, s)$ in Section 3. On the way of the construction, we shall construct the quasi-classical propagator $G_N^{\kappa}(t, s)$. We shall prove Theorem 2 in Section 4. Applying the stationary phase method to the result of Theorem 2, we shall prove Theorem 4 in Section 5. Section 6 is an appendix and two theorems on oscillatory integral operators are reproduced from Asada-Fujiwara [3] and Yajima [15].

In addition to the terminology introduced above, we shall use the following notation and the conventions. $(g^{\mu\nu})$ is the Minkowski metrix tensor: $g^{00}=1$, $g^{jj}=-1$ $(j=1, 2, 3)$, $g^{\mu\nu}=0$ if $\mu\neq\nu$. $(g_{\mu\nu})=(g^{\mu\nu})^{-1}$. Einstein's summation convention is used: every time the same index appears twice, the summation is taken over the index. For four vector $p=(p^{\mu})$, $p_{\mu}=g_{\mu\nu}p^{\nu}$. Feynman's slash notation is adopted: For four vector $p=(p^{\mu})$, $\not{p}=\gamma^{\mu}p_{\mu}$. $\partial=(\partial^{\mu})=(\partial/\partial x_{\mu})$. Using the slash notation, we sometime write the Dirac equation as

$$(i\hbar\not{\partial}-e\not{A}(x)-m)u(x)=0.$$

$\Box=\partial\cdot\partial=\partial^{\mu}\partial_{\mu}$.

For multi-index $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, $(\partial/\partial\vec{x})^{\alpha}=(\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}(\partial/\partial x_3)^{\alpha_3}$, $\vec{x}^{\alpha}=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}$, and $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. For a function $f(\vec{x})$, $\partial f/\partial\vec{x}$ is a 3-vector $(\partial f/\partial x^j)$. For a vector valued function $f(x)=(f_j(x))$, $\partial f/\partial x$ is the matrix $(\partial f_j/\partial x^k)_{j,k}$. Thus for a function $f(x, y)$, $\partial^2 f/\partial x\partial y=(\partial^2 f/\partial x^j\partial y^k)_{j,k}$ etc. $\mathrm{Hess}_x f=(\partial^2 f/\partial x^j\partial x^k)_{j,k}$ is the Hessian of the function $f(x, y)$ with respect to the variable x.

$\mathscr{H}=L^2(\boldsymbol{R}^3, \boldsymbol{C}^4)$ is the Hilbert space of all $\boldsymbol{C}^4$-valued square integrable functions over $\boldsymbol{R}^3$. The inner product and the norm of $\mathscr{H}$ is denoted as $(\ ,\ )$ and $\|\ \ \|$. For an integer $m\geqq 0$, $H^m=H^m(\boldsymbol{R}^3, \boldsymbol{C}^4)$ is the Sobolev space of order m:

$$H^m=\{f\in\mathscr{H}: (\sum_{|\alpha|\leqq m}\|(\partial/\partial x)^{\alpha}f\|^2)^{1/2}=\|f\|_m<\infty\}.$$

For a subset $\Omega\subset\boldsymbol{R}^n$

$$B^m(\Omega)=\{f\in C^m(\Omega): \sup_{x\in\Omega}\sum_{|\alpha|\leqq m}|(\partial/\partial x)^{\alpha}f(x)|=\|f\|_{B^m}<\infty\}.$$

$B(\Omega)=\bigcap_{m\geqq 0} B^m(\Omega)$ is the Fréchet space with the seminorms $\|\ \ \|_{B^m}$.

§ 2. Preliminaries.

In the following sections we shall construct the propagator and the quasi-propagator associated with the Dirac equation (1.1). For this purpose we need several properties of the classical orbits, the action integrals and the solutions of the associated transport equation. We collect these properties here.

2.1. Classical orbits and the action integrals.

We write as $(\vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta}))$ the solution of the Hamilton equation (1.5) with the initial condition $(\vec{x}^{\pm}(s, s, \vec{y}, \vec{\eta}), \vec{\xi}^{\pm}(s, s, \vec{y}, \vec{\eta}))=(\vec{y}, \vec{\eta})$. Since by Assumption (A) the functions $\partial H^{\pm}(t, \vec{x}, \vec{\xi})/\partial\vec{x}$ and $\partial H^{\pm}(t, \vec{x}, \vec{\xi})/\partial\vec{\xi}$ are smooth and belong to the class $B(\boldsymbol{R}^6)$ as functions of $(\vec{x}, \vec{\xi})$, an elementary calculation shows the following proposition. (See Coddington-Levinson [5], Chapt. 1.)

PROPOSITION 2.1. (1) *$(\vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta}))$ is a C^∞-function of $(t, s, \vec{y}, \vec{\eta})$ and for any multi-index α and β, there exists a constant $C_{\alpha\beta}>0$ such that*

$$|(\partial/\partial\vec{y})^\alpha(\partial/\partial\vec{\eta})^\beta(\vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta})-\vec{y})|\leqq C_{\alpha\beta}|t-s|,$$
$$|(\partial/\partial\vec{y})^\alpha(\partial/\partial\vec{\eta})^\beta(\vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta})-\vec{\eta})|\leqq C_{\alpha\beta}|t-s|.$$

By Proposition 2.1 and the global implicit function theorem (see Schwartz [14]), it follows that there exists a constant $\delta>0$ such that if $|t-s|\leqq\delta$, the mappings $(\vec{y}, \vec{\eta})\to(\vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})$ and $(\vec{y}, \vec{\eta})\to(\vec{y}, \vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta}))$ are global diffeomorphisms on $\boldsymbol{R}^6$ and their inverse functions $(\vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta})$ and $(\vec{y}, \vec{\eta}^{\pm}(t, s, \vec{y}, \vec{\xi}))$ satisfy the following properties.

PROPOSITION 2.2. *For any multi-index α and β, there exists a constant $C_{\alpha\beta}>0$ such that*

$$|(\partial/\partial\vec{x})^\alpha(\partial/\partial\vec{\eta})^\beta(\vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta})-\vec{x})|\leqq C_{\alpha\beta}|t-s|,$$
$$|(\partial/\partial\vec{y})^\alpha(\partial/\partial\vec{\xi})^\beta(\vec{\eta}^{\pm}(t, s, \vec{y}, \vec{\xi})-\vec{\xi})|\leqq C_{\alpha\beta}|t-s|.$$

Using the action integral along the trajectory $(\vec{x}^{\pm}(t, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta}), \vec{\xi}^{\pm}(t, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta}))$, we define for $|t-s|\leqq\delta$ as

$$S^{\pm}(t, s, \vec{x}, \vec{\eta})\equiv\vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta})\cdot\vec{\eta} + \int_s^t L^{\pm}(u, \vec{x}^{\pm}(u, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta}), \vec{\xi}^{\pm}(u, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta}))du, \tag{2.1}$$

where

$$L^{\pm}(t, \vec{x}, \vec{\xi})=(\partial H^{\pm}/\partial\vec{\xi})(t, \vec{x}, \vec{\xi})\cdot\vec{\xi}-H^{\pm}(t, \vec{x}, \vec{\xi})$$

is the Lagrangian of the system. The function $S^{\pm}(t, s, \vec{x}, \vec{\eta})$ satisfies the following properties.

PROPOSITION 2.3. (1) *$S^{\pm}(t, s, \vec{x}, \vec{\eta})$ is a C^{∞}-function of all variables $(t, s, \vec{x}, \vec{\eta})$.*

(2) $(\partial S^{\pm}/\partial x_j)(t, s, \vec{x}, \vec{\eta})=\xi_j^{\pm}(t, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta})$, $(j=1, 2, 3)$.

(3) $(\partial S^{\pm}/\partial \eta_j)(t, s, \vec{x}, \vec{\eta})=y_j^{\pm}(t, s, \vec{x}, \vec{\eta})$, $(j=1, 2, 3)$.

(4) *$S^{\pm}(t, s, \vec{x}, \vec{\eta})$, satisfies the Hamilton Jacobi equation for the relativistic classical particle:*

$$(\partial S^{\pm}/\partial t)(t, s, \vec{x}, \vec{\eta})+H^{\pm}(t, \vec{x}, (\partial S^{\pm}/\partial \vec{x})(t, s, \vec{x}, \vec{\eta}))=0. \tag{2.2}$$

$$(\partial S^{\pm}/\partial s)(t, s, \vec{x}, \vec{\eta})-H^{\pm}(s, (\partial S^{\pm}/\partial \vec{\eta})(t, s, \vec{x}, \vec{\eta}), \vec{\eta})=0 \tag{2.2$'$}$$

and the initial condition

$$S^{\pm}(s, s, \vec{x}, \vec{\eta})=\vec{x}\cdot\vec{\eta}. \tag{2.3}$$

(5) *For any $j, k=1, 2, 3$, there exists a constant C_{jk} such that*

$$|(\partial^2 S^{\pm}/\partial x_j\partial\eta_k)(t, s, \vec{x}, \vec{\eta})-\delta_{jk}|\leqq C_{jk}|t-s|, \tag{2.4}$$

where δ_{jk} is Kronecker's delta.

(6) *For integers $n, l\geqq 0$ and multi-indices α and β such that $|\alpha|+|\beta|+n+l\geqq 2$, there exists a constant $C_{nl\alpha\beta}>0$ such that*

$$|(\partial/\partial S)^l(\partial/\partial t)^n(\partial/\partial \vec{x})^{\alpha}(\partial/\partial\vec{\eta})^{\beta}S^{\pm}(t, s, \vec{x}, \vec{\eta})|\leqq C_{nl\alpha\beta}, \tag{2.5}$$

PROOF. Statements (1), (2), (3) and (4) are easy to prove and are well-known (cf. Arnold [2], Chapt. 9). The estimate (2.4) is clear from (2) or (3) and Proposition 2.2. (2.5) is also clear from (2), (3), (2.2) and Proposition 2.2. Q.E.D.

2.2. Transport equation.

Along with the trajectory, we consider the transport equation (1.8). Write as

$$k_{\mu}^{\pm}(t, s, \vec{x}, \vec{\eta})\equiv p_{\mu}^{\pm}(t, s, \vec{y}^{\pm}(t, s, x, \vec{\eta}), \vec{\eta}), \tag{2.6}$$

$$\begin{aligned} L^{\pm}(t, \vec{x}; s, \vec{\eta})\equiv\, & 2k_{\mu}^{\pm}(t, s, \vec{x}, \vec{\eta})(\partial/\partial x_{\mu}) \\ & -(\Box S^{\pm})(t, s, \vec{x}, \vec{\eta})-(ie/2)\sigma^{\mu\nu}F_{\mu\nu}(t, \vec{x}). \end{aligned} \tag{2.7}$$

By Proposition 2.3, (2), the equation (1.8) can be written as

(2.8) $$L^{\pm}(t, \vec{x}; s, \vec{\eta})E_j^{\pm}(t, s, \vec{x}, \vec{\eta})+i\partial E_{j-1}^{\pm}(t, s, \vec{x}, \vec{\eta})=0.$$

We first study the initial value problem

(2.8)′ $$L^{\pm}(t, \vec{x}; s, \vec{\eta})u^{\pm}(t, s, \vec{x}, \vec{\eta})=f(t, s, \vec{x}, \vec{\eta}),$$

(2.9) $$u^{\pm}(s, s, \vec{x}, \vec{\eta})=a(s, \vec{x}, \vec{\eta}).$$

Notice that by Proposition 2.3, (2)

$$\begin{aligned}(dx_{\mu}^{\pm}/dt)(t, s, \vec{y}, \vec{\eta})&=(\partial H^{\pm}/\partial\xi_{\mu})(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta}))\\ &=p_{\mu}^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(t, s, \vec{y}, \vec{\eta}).\end{aligned}$$

It follows that via the substitution $v^{\pm}(t, s, \vec{y}, \vec{\eta})=u^{\pm}(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})$ the equation (2.8)′ and (2.9) is equivalent to the initial value problem for the ordinary differential equation:

(2.10) $$\begin{aligned}&\hat{L}^{\pm}(t, s, \vec{y}, \vec{\eta})v^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &\equiv 2p_0^{\pm}(t, s, \vec{y}, \vec{\eta})(d/dt)v^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &\quad-(\Box S^{\pm})(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})v^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &\quad-(ie/2)\sigma^{\mu\nu}F_{\mu\nu}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}))v^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &=f(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta}),\end{aligned}$$

(2.11) $$v^{\pm}(s, s, \vec{x}, \vec{\eta})=a(s, \vec{x}, \vec{\eta}).$$

Applying the standard theorems on the uniqueness, the dependence on the parameters and initial conditions of the solution of the ordinary differential equations, we obtain the following proposition (cf. Coddington-Levinson [5]; Chapt. 1).

PROPOSITION 2.4. *Let* $a(s, \vec{x}, \vec{\eta})\in B((T_1, T_2)\times \boldsymbol{R}^6)$ *and* $f(t, s, \vec{x}, \vec{\eta})\in B((T_1, T_2)^2\times \boldsymbol{R}^6)$. *Then the initial value problem* (2.8)′ *and* (2.9) *has a unique solution* $u^{\pm}(t, s, \vec{x}, \vec{\eta})\in B(\{|t-s|<\delta\}\times \boldsymbol{R}^6)$.

PROOF. Since $p_0^{\pm}(t, s, y, \eta)^{-1}$, $(\Box S^{\pm})(t, s, x^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta})$ and $F_{\mu\nu}(t, \vec{x}(t, s, \vec{y}, \vec{\eta}))$ belong to $B(\{|t-s|<\delta\}\times \boldsymbol{R}^3)$ by Proposition 2.1 and 2.3, the initial value problem (2.10) and (2.11) has a unique solution $v^{\pm}(t, s, \vec{y}, \vec{\eta})\in B(\{|t-s|<\delta\}\times \boldsymbol{R}^6)$. Hence, by Proposition 2.2, $u^{\pm}(t, s, \vec{x}, \vec{\eta})=v^{\pm}(t, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta})$ is the unique solution of (2.8)′ and (2.9) and it satisfies the property of the proposition. Q.E.D.

LEMMA 2.5. *For* $j=0, 1, \cdots$, *the equation* $(1.8)_j$ *with the initial condition* $(1.9)_j$ *has a unique solution* $E_j^{\pm}(t, s, \vec{x}, \vec{\eta})$ *for* $|t-s|<\delta$ *and* $E_j^{\pm}(t, s, \vec{x}, \vec{\eta})$ *is bounded with its all derivatives.*

PROOF. By Assumption (A),

(2.12) $$P^{\pm}(s, \vec{x}, \vec{\eta})=1/2+D(s, \vec{x}, \vec{\eta})/p_0^{\pm}(s, s, \vec{x}, \vec{\eta})$$

is smooth and bounded with its all derivatives. Hence Proposition 2.4 implies Lemma 2.5. Q.E.D.

PROPOSITION 2.6. *If $f(t, s, \vec{x}, \vec{\eta})$ and $u^{\pm}(t, s, \vec{x}, \vec{\eta})$ satisfy the equations*

(2.13) $$(k^{\pm}(t, s, \vec{x}, \vec{\eta})+m)f(t, s, \vec{x}, \vec{\eta})=0,$$

(2.14) $$L^{\pm}(t, \vec{x}; s, \vec{\eta})u^{\pm}(t, s, \vec{x}, \vec{\eta})+\partial f(t, s, \vec{x}, \vec{\eta})=0,$$

(2.15) $$(k^{\pm}(s, s, \vec{x}, \vec{\eta})-m)u^{\pm}(s, s, \vec{x}, \vec{\eta})+f(s, s, \vec{x}, \vec{\eta})=0,$$

then $u^{\pm}(t, s, \vec{x}, \vec{\eta})$ satisfies the following equations:

(2.16) $$(k^{\pm}(t, s, \vec{x}, \vec{\eta})-m)u^{\pm}(t, s, \vec{x}, \vec{\eta})+f(t, s, \vec{x}, \vec{\eta})=0,$$

(2.17) $$(k^{\pm}(t, s, \vec{x}, \vec{\eta})+m)\partial u^{\pm}(t, s, \vec{x}, \vec{\eta})=0.$$

PROOF. A little calculation shows that

$$L^{\pm}(t, \vec{x}; s, \vec{\eta})=\partial\cdot k^{\pm}(t, s, \vec{x}, \vec{\eta})+k^{\pm}(t, s, \vec{x}, \vec{\eta})\cdot\partial$$

and Proposition 2.3, (4) shows that

$$k^{\pm}(t, s, \vec{x}, \vec{\eta})^2=m^2.$$

It follows that $(k^{\pm}-m)\cdot L^{\pm}=L^{\pm}\cdot(k^{\pm}-m)$, and by (2.13) and (2.14) we have

$$\begin{aligned}L^{\pm}\{(k^{\pm}-m)u^{\pm}+f\}&=(k^{\pm}-m)L^{\pm}u^{\pm}+L^{\pm}f\\&=-(k^{\pm}-m)\partial f+(\partial\cdot k^{\pm}+k^{\pm}\cdot\partial)f\\&=\partial(k^{\pm}+m)f=0.\end{aligned}$$

By the uniqueness of the solution of the initial value problem (2.8)′ and (2.9) (Proposition 2.4), we obtain the equation (2.16). Since (2.14) is equivalent to

$$\partial\{(k^{\pm}-m)u^{\pm}+f\}+(k^{\pm}+m)\partial u^{\pm}=0,$$

(2.16) implies (2.17). Q.E.D.

LEMMA 2.7. *For any $j=0, 1, \cdots$, the function $E_j^{\pm}(t, s, \vec{x}, \vec{\eta})$ satisfies the following equations*:

$(2.18)_j$ $$(k^{\pm}(t, s, \vec{x}, \vec{\eta})-m)E_j^{\pm}(t, s, \vec{x}, \vec{\eta})+i\partial E_{j-1}^{\pm}(t, s, \vec{x}, \vec{\eta})=0.$$

$(2.19)_j$ $$(k^{\pm}(t, s, \vec{x}, \vec{\eta})+m)\partial E_j^{\pm}(t, s, \vec{x}, \vec{\eta})=0.$$

(2.20) $$E_j^{+}(s, s, \vec{x}, \vec{\eta})+E_j^{-}(s, s, \vec{x}, \vec{\eta})=0, \quad j\geqq 1.$$

(2.21) $$E_0^{+}(s, s, \vec{x}, \vec{\eta})+E_0^{-}(s, s, \vec{x}, \vec{\eta})=I.$$

PROOF. The equations (2.20) and (2.21) are obvious. By Proposition 2.6 (with $f=0$) and the definition (2.12) of $P^{\pm}(s, \vec{x}, \vec{\eta})$, $(2.18)_0$ and $(2.19)_0$ are obvious. Suppose that the equations $(2.18)_j$ and $(2.19)_j$ hold for $j \leqq n$. Set $u^{\pm}(t, s, \vec{x}, \vec{\eta})=E^{\pm}_{n+1}(t, s, \vec{x}, \vec{\eta})$ and $f^{\pm}(t, s, \vec{x}, \vec{\eta})=i\partial E^{\pm}_n(t, s, \vec{x}, \vec{\eta})$ in Proposition 2.6. $(2.19)_n$ implies (2.13) and $(2.8)_{n+1}$ implies (2.14). Note that $k^{\pm}_j(s, s, \vec{x}, \vec{\eta})=\eta_j-eA_j(s, \vec{x})$ for $j=1, 2, 3$, $k^{+}_0(s, s, \vec{x}, \vec{\eta})=-k^{-}_0(s, s, \vec{x}, \vec{\eta})$ and that the equation $(k\!\!\!/^{\pm}(s, s, \vec{x}, \vec{\eta})+m)f^{\pm}(s, s, \vec{x}, \vec{\eta})=0$ is equivalent to $(\gamma^0 k^{\pm}_0+\gamma_j k^{\pm}_j+m)\gamma^0 f^{\pm}(s, s, \vec{x}, \vec{\eta})=0$. Hence

$$
\begin{aligned}
&(k\!\!\!/^{\pm}(s, s, \vec{x}, \vec{\eta})-m)u^{\pm}(s, s, \vec{x}, \vec{\eta})\\
&=-(2k^{\pm}_0(s, s, \vec{x}, \vec{\eta}))^{-1}(\gamma^0 k^{\pm}_0-\gamma_j k^{\pm}_j-m)\gamma^0\{f^{+}(s, s, \vec{x}, \vec{\eta})+f^{-}(s, s, \vec{x}, \vec{\eta})\}\\
&=-(2k^{\pm}_0)^{-1}\{(k^{\pm}_0+k^{+}_0)f^{+}+(k^{\pm}_0+k^{-}_0)f^{-}\}\\
&=-f^{\pm}(s, s, \vec{x}, \vec{\eta}).
\end{aligned}
$$

By Proposition 2.10, it follows that $(2.18)_{n+1}$ and $(2.19)_{n+1}$ hold and the lemma is proved. Q.E.D.

LEMMA 2.8. $E^{\pm}_0(t, s, \vec{x}, \vec{\eta})=P^{\pm}(t, \vec{x}, \vec{\xi}^{\pm}(t, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta}))E^{\pm}_0(t, s, \vec{x}, \vec{\eta})$
$=E^{\pm}_0(t, s, \vec{x}, \vec{\eta})P^{\pm}(s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta})$.

PROOF. The first equation is clear by $(2.18)_0$. Since both $E^{\pm}_0(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta})$ and $E^{\pm}_0(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})$ satisfy the same equation $L^{\pm}(t, s, \vec{y}, \vec{\eta})u=0$ and the same initial condition, the uniqueness of the Cauchy problem implies $E^{\pm}_0(t, s, x^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta})=E^{\pm}_0(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})$. By setting $\vec{y}=\vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta})$, we obtain the second equation. Q.E.D.

So far the equation (2.8) and (2.10) are considered only locally in time $|t-s|<\delta$, since the function $S^{\pm}(t, s, x, \eta)$ is defined only for that region. However by factoring out of the solution $u^{\pm}(t, s, x, \eta)$ the part corresponding to the summand $\Box S^{\pm}(t, s, x, \eta)u^{\pm}(t, s, x, \eta)$ in the equations (2.8) and (2.10), we can find the equation which plays as a substitute of (2.8) and (2.10) and can be considered globally in $[T_1, T_2]\times[T_1, T_2]$.

PROPOSITION 2.9.

$$
\begin{aligned}
\hat{J}^{\pm}(t, s, \vec{y}, \vec{\eta}) &\equiv \exp\left(\int_s^t \frac{(\Box S^{\pm})(u, s, \vec{x}^{\pm}(u, s, \vec{y}, \vec{\eta}), \vec{\eta})}{2p^{\pm}_0(u, s, \vec{x}^{\pm}(u, s, \vec{y}, \vec{\eta}), \vec{\eta})}du\right)\\
&=\left\{\frac{p^{\pm}_0(t, s, \vec{y}, \vec{\eta})}{p^{\pm}_0(s, s, \vec{y}, \vec{\eta})}\left|\det\left(\frac{\partial x^{\pm}(t, s, \vec{y}, \vec{\eta})}{\partial y}\right)\right|\right\}^{-1/2}.
\end{aligned}
$$

PROOF. We omit the variables of $p^{\pm}_\mu$ in this proof. By definition

$$2p_0^{\pm}\cdot(d\hat{J}^{\pm}/dt)(t, s, \vec{y}, \vec{\eta})=(\square S^{\pm})(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})\hat{J}^{\pm}(t, s, \vec{y}, \vec{\eta}).$$

Using the Hamilton equation (1.5), we have

$$\begin{aligned}&(d/dt)(p_0^{\pm}(t, s, \vec{y}, \vec{\eta})dx^{\pm}(t, s, \vec{y}, \vec{\eta}))\\ &=\{\partial p_0^{\pm}/\partial t+(\partial p_0^{\pm}/\partial x_j)(dx_j^{\pm}/dt)+p_0^{\pm}(\partial/\partial x_j)(p_j^{\pm}/p_0^{\pm})\}dx^{\pm}\\ &=\{\partial p_\mu^{\pm}/\partial x_\mu+(dx_\mu^{\pm}/dt-p_\mu^{\pm}/p_0^{\pm})(\partial p_0^{\pm}/\partial x_\mu)\}dx^{\pm}\\ &=-(\square S^{\pm})(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})dx^{\pm}(t, s, \vec{y}, \vec{\eta}).\end{aligned}$$

Combining these two equations, we have

$$(d/dt)(p_0^{\pm}(t, s, \vec{y}, \vec{\eta})\hat{J}^{\pm}(t, s, \vec{y}, \vec{\eta})^2dx^{\pm}(t, s, \vec{y}, \vec{\eta}))=0.$$

It follows that $p_0^{\pm}(t, s, \vec{y}, \vec{\eta})\hat{J}^{\pm}(t, s, \vec{y}, \vec{\eta})^2d\vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta})=p_0^{\pm}(s, s, \vec{y}, \vec{\eta})dy$. Since $\hat{J}^{\pm}(t, s, \vec{y}, \vec{\eta})>0$, we have the desired result. Q.E.D.

REMARK 2.10. The last expression in Proposition 2.9 can be defined for any $t, s\in[T_1, T_2]$, through it might be infinity or $S^{\pm}(t, s, \vec{x}, \vec{\eta})$ may not be defined throughout the region $(t, s)\in[T_1, T_2]\times[T_1, T_2]$.

COROLLARY 2.11. *Let $J^{\pm}(t, s, \vec{x}, \vec{\eta})=\hat{J}^{\pm}(t, s, \vec{y}^{\pm}(t, s, \vec{x}, \vec{\eta}), \vec{\eta})$. Then the function $u^{\pm}(t, s, \vec{x}, \vec{\eta})$ is the solution of the initial value problem* (2.8)′ *and* (2.9) *if and only if $\hat{u}^{\pm}(t, s, \vec{x}, \vec{\eta})=J^{\pm}(t, s, \vec{x}, \vec{\eta})u^{\pm}(t, s, \vec{x}, \vec{\eta})$ is the solution of initial value problem*

$$\begin{aligned}[2k_\mu^{\pm}(t, s, \vec{x}, \vec{\eta})(\partial/\partial x_\mu)-(ie/2)\sigma^{\mu\nu}F_{\mu\nu}(t, \vec{x})]u^{\pm}(t, s, \vec{x}, \vec{\eta})&\\ =J^{\pm}(t, s, \vec{x}, \vec{\eta})^{-1}f(t, s, \vec{x}, \vec{\eta}),&\end{aligned} \tag{2.22}$$

$$u^{\pm}(s, s, \vec{x}, \vec{\eta})=a(s, \vec{x}, \vec{\eta}). \tag{2.23}$$

Notice that (2.22) does not contain $S^{\pm}(t, s, x, \eta)$ explicitly and by the substitution $w^{\pm}(t, s, \vec{y}, \vec{\eta})=u^{\pm}(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})$ (2.22) and (2.23) are equivalent for $|t-s|<\delta$ to

$$\begin{aligned}M^{\pm}(t, s, \vec{y}, \vec{\eta})w^{\pm}(t, s, \vec{y}, \vec{\eta})&\equiv 2p_0^{\pm}(t, s, \vec{y}, \vec{\eta})(dw^{\pm}/dt)(t, s, \vec{y}, \vec{\eta})\\ &\quad-(ie/2)\sigma^{\mu\nu}F_{\mu\nu}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}))w^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &=\hat{J}^{\pm}(t, s, \vec{y}, \vec{\eta})^{-1}f(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta}),\end{aligned} \tag{2.22$'$}$$

$$w^{\pm}(s, s, \vec{y}, \vec{\eta})=a(s, \vec{y}, \vec{\eta}). \tag{2.23$'$}$$

Clearly the equation (2.22)′ has its meaning on the entire region $(t, s)\in[T_1, T_2]\times[T_1, T_2]$.

PROPOSITION 2.12. *Let $K^{\pm}(t, s, \vec{y}, \vec{\eta})$ and $F^{\pm}(t, s, \vec{y}, \vec{\eta})$ be the solutions of the initial value problems*

$$\hat{M}^{\pm}(t, s, \vec{y}, \vec{\eta})K^{\pm}(t, s, \vec{y}, \vec{\eta})=\hat{M}^{\pm}(t, s, \vec{y}, \vec{\eta})F^{\pm}(t, s, \vec{y}, \vec{\eta})=0, \tag{2.24}$$

(2.25) $$K^{\pm}(s, s, \vec{y}, \vec{\eta})=I,$$

(2.26) $$F^{\pm}(s, s, \vec{y}, \vec{\eta})=P^{\pm}(s, \vec{y}, \vec{\eta}).$$

Then

(2.27) $$\begin{aligned} K^{\pm}(t, s, \vec{y}, \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta}) &= F^{\pm}(t, s, \vec{y}, \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta}) \\ &= P^{\pm}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\xi}^{\pm}(t, s, \vec{y}, \vec{\eta}))F^{\pm}(t, s, \vec{y}, \vec{\eta}) \\ &= F^{\pm}(t, s, \vec{y}, \vec{\eta}). \end{aligned}$$

Thus if $f^{\pm}(s, \vec{y}, \vec{\eta}) \in V^{\pm}(s, \vec{y}, \vec{\eta})$, then $F^{\pm}(t, s, \vec{y}, \vec{\eta})f^{\pm}(s, \vec{y}, \vec{\eta})$ is the unique solution of the initial value problem

$$\hat{M}^{\pm}(t, s, \vec{y}, \vec{\eta})u^{\pm}(t, s, \vec{y}, \vec{\eta})=0, \quad u^{\pm}(s, s, \vec{y}, \vec{\eta})=f^{\pm}(s, \vec{y}, \vec{\eta}).$$

PROOF. Since $F^{\pm}(t, s, \vec{y}, \vec{\eta})=\hat{J}(t, s, \vec{y}, \vec{\eta})^{-1}E^{\pm}(t, s, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\eta})$, the second and the third equations of (2.27) is obvious by Lemma 2.8. To prove the first, we need only to check that $K^{\pm}(t, s, \vec{y}, \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta})$ and $F^{\pm}(t, s, \vec{y}, \vec{\eta})$ satisfy the same equation (2.24) and the initial condition (2.26), which, however, is clear. The last statement of the proposition is clear. Q.E.D.

LEMMA 2.13. *Let $T_1 \leqq s=t_0<t_1<\cdots<t_N=t\leqq T_2$ be such that $0<t_j-t_{j-1}<\delta$; $x_j^{\pm}=x^{\pm}(t_j, s, \vec{y}, \vec{\eta})$, $\vec{\xi}_j^{\pm}=\vec{\xi}^{\pm}(t_j, s, \vec{y}, \vec{\eta})$, $j=0, 1, \cdots, N$. Then for any $f(s) \in \boldsymbol{C}^4$,*

(2.29) $$\begin{aligned} &E^{\pm}(t, t_{N-1}, \vec{x}_N^{\pm}, \vec{\xi}_{N-1}^{\pm})\cdots E_0^{\pm}(t_2, t_1, \vec{x}_2^{\pm}, \xi_1^{\pm})E_0^{\pm}(t_1, \vec{s}, \vec{x}_1^{\pm}, \vec{\eta})f(s) \\ &=\left(\left(\prod_{j=1}^{N} \det \partial\vec{x}^{\pm}(t_j, t_{j-1}, \vec{y}, \vec{\eta})/\partial\vec{y}\,\Big|_{\vec{y}=\vec{x}_{j-1}, \vec{\eta}=\vec{\xi}_{j-1}^{\pm}}\right)\Big|(p_0^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(s, s, \vec{y}, \vec{\eta}))\Big|\right)^{-1/2} \\ &\qquad \times f^{\pm}(t, s, \vec{y}, \vec{\eta}), \end{aligned}$$

where $f^{\pm}(t, s, \vec{y}, \vec{\eta})$ is the solution of

$$\hat{M}^{\pm}(t, s, \vec{y}, \vec{\eta})f^{\pm}(t, s, \vec{y}, \vec{\eta})=0, \quad f^{\pm}(s, s, \vec{y}, \vec{\eta})=P^{\pm}(s, \vec{y}, \vec{\eta})f(s).$$

PROOF. By Proposition 2.9 and Corollary 2.11 the LHS of (2.29) is equal to

$$\left[\prod_{j=1}^{N} J^{\pm}(t_j, t_{j-1}, \vec{x}_j^{\pm}, \vec{\xi}_{j-1}^{\pm})\right]F^{\pm}(t, t_{N-1}, \vec{x}_{N-1}^{\pm}, \vec{\xi}_{N-1}^{\pm})\cdots F^{\pm}(t_1, s, \vec{y}, \vec{\eta})f(s).$$

By Proposition 2.12, and an elementary property of the ordinary differential equation

$$\begin{aligned} &F^{\pm}(t, t_{N-1}, \vec{x}_{N-1}^{\pm}, \vec{\xi}_{N-1}^{\pm})\cdots F^{\pm}(t, s, y, \eta)f(s) \\ &=K^{\pm}(t, t_{N-1}, \vec{x}_{N-1}^{\pm}, \vec{\xi}_{N-1}^{\pm})\cdots K^{\pm}(t_1, s, \vec{y}, \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta})f(s) \\ &=K^{\pm}(t, s, \vec{y}, \vec{\eta})P^{\pm}(s, \vec{y}, \vec{\eta})f(s)=F^{\pm}(t, s, \vec{y}, \vec{\eta})f(s). \end{aligned}$$

Thus Proposition 2.9 implies Lemma 2.13. Q.E.D.

2.3. Miscellaneous properties of transport equation.

Transport equation enjoys some physically interesting properties. We collect them here. We write as $f^{\pm}(t, s, \vec{y}, \vec{\eta})$ the solution of

$$\hat{M}^{\pm}(t, s, \vec{y}, \vec{\eta})f^{\pm}(t, s, \vec{y}, \vec{\eta})=0, \quad f^{\pm}(s, s, \vec{y}, \vec{\eta})=f^{\pm}(s, \vec{y}, \vec{\eta}) \in V^{\pm}(s, \vec{y}, \vec{\eta}).$$

PROPOSITION 2.14. *For* $j=1, 2, 3$

$$(f^{\pm}(t, s, \vec{y}, \vec{\eta}), \alpha_j f^{\pm}(t, s, \vec{y}, \vec{\eta}))_{C^4}=(p_j^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(t, s, \vec{y}, \vec{\eta}))|f^{\pm}(t, s, \vec{y}, \vec{\eta})|^2. \tag{2.30}$$

PROOF. Let $(p^{\mu}) \in \boldsymbol{R}^4$ and $f \in \boldsymbol{C}^4$ satisfy $(\not{p}-m)f=0$. Then since γ^j and $\gamma^j\gamma^k$ $(j=1, 2, 3,\ j\neq k)$ are anti-hermitian, we have $0=\mathrm{Re}(\not{p}f, \gamma^j f)=-p^j(f, f)+p^0(f, \alpha^j f)$. Since $f^{\pm}(t, s, \vec{y}, \vec{\eta})$ satisfies $(\not{p}^{\pm}(t, s, \vec{y}, \vec{\eta})-m)f^{\pm}=0$, we obtain (2.30). Q.E.D.

PROPOSITION 2.15. $|f^{\pm}(t, s, \vec{y}, \vec{\eta})|^2=(p_0^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(s, s, \vec{y}, \vec{\eta}))|f^{\pm}(s, s, \vec{y}, \vec{\eta})|^2$.

PROOF. By an elementary calculation

$$(d/dt)p_0^{\pm}(t, s, \vec{y}, \vec{\eta})=eF_{0j}(t, x^{\pm}(t, s, \vec{y}, \vec{\eta}))p_j^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(t, s, \vec{y}, \vec{\eta}). \tag{2.31}$$

Since $\sigma^{\mu\nu}$ $(\mu, \nu\neq 0)$ are hermite and $\sigma^{0j}=i\alpha^j$, it follows by (2.30) that and (2.31) that

$$\begin{aligned}(d/dt)&(|f^{\pm}(t, s, \vec{y}, \vec{\eta})|^2/p_0^{\pm}(t, s, \vec{y}, \vec{\eta}))\\ &=-p_0^{\pm}(t, s, \vec{y}, \vec{\eta})^{-2}\{|f^{\pm}(t, s, \vec{y}, \vec{\eta})|^2(d/dt)p_0^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &\quad -2\,\mathrm{Re}\,(f^{\pm}(t, s, \vec{y}, \vec{\eta}), (ie/4)\sigma^{\mu\nu}F_{\mu\nu}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}))f^{\pm}(t, s, \vec{y}, \vec{\eta}))\}\\ &=0.\end{aligned} \tag{2.32}$$

Equation (2.32) clearly implies Proposition 2.15. Q.E.D.

We write $\vec{\sigma}=(\sigma^1, \sigma^2, \sigma^3)\equiv(\sigma^{23}, \sigma^{31}, \sigma^{12})$; $\vec{E}(t, x)=(F^{01}, F^{02}, F^{03})$, $\vec{H}(t, s)=(F^{23}, F^{31}, F^{12})$; $\vec{v}^{\pm}(t, s, \vec{y}, \vec{\eta})=p^{\pm}(t, s, \vec{y}, \vec{\eta})/p_0^{\pm}(t, s, \vec{y}, \vec{\eta})$ and $\vec{\sigma}^{\pm}(t, s, \vec{y}, \vec{\eta})=(f^{\pm}(t, s, \vec{y}, \vec{\eta}), \vec{\sigma}f^{\pm}(t, s, \vec{y}, \vec{\eta}))/p_0^{\pm}(t, s, \vec{y}, \vec{\eta})$.

PROPOSITION 2.16.

$$\begin{aligned}p_0^{\pm}(t, s, y, \eta)(d/dt)\vec{\sigma}^{\pm}(t, s, \vec{y}, \vec{\eta})&=e\vec{H}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}))\wedge\vec{\sigma}^{\pm}(t, s, \vec{y}, \vec{\eta})\\ &\quad+\vec{v}^{\pm}(t, s, \vec{y}, \vec{\eta})\wedge(e\vec{E}(t, \vec{x}^{\pm}(t, s, \vec{y}, \vec{\eta}))\wedge\vec{\sigma}^{\pm}(t, s, \vec{y}, \vec{\eta})).\end{aligned} \tag{2.33}$$

PROOF. We omit the variables $(t, s, \vec{y}, \vec{\eta})$ in the following expressions. By taking the real part of $(i\gamma^1\gamma^2\gamma^3 f^{\pm}, (\not{p}^{\pm}-m)f^{\pm})=0$, we obtain

$$(i\gamma^0\gamma^1\gamma^2\gamma^3 f^{\pm}, f^{\pm})=(f^{\pm}, (\vec{\sigma}, \vec{v}^{\pm})f^{\pm}). \tag{2.34}$$

After a little gymnasium with γ-matrices, we see that

$$p_0^\pm(d\vec{\sigma}^\pm/dt)=\mathrm{Re}\,(f^\pm,\ \vec{\sigma}(e\sigma^{\mu\nu}F_{\mu\nu})f^\pm)-(dp_0^\pm/dt)\sigma^\pm$$
$$+e\,\mathrm{Re}\,(f^\pm,\ \vec{\sigma}(\vec{\alpha}\cdot\vec{E}-i\vec{\sigma}\cdot\vec{H})f^\pm)-(dp_0^\pm/dt)\vec{\sigma}^\pm$$
$$+e\vec{H}\wedge\vec{\sigma}^\pm+e\vec{E}(f^\pm,\ i\gamma^0\gamma^1\gamma^2\gamma^3f^\pm)-(dp_0^\pm/dt)\vec{\sigma}.$$

Hence using (2.31) and (2.34), we obtain the desired result:

$$p_0^\pm(d\vec{\sigma}^\pm/dt)=e\vec{H}\wedge\vec{\sigma}^\pm+e\vec{E}(\vec{\sigma}^\pm,\ \vec{v}^\pm)-(e\vec{E},\ \vec{v}^\pm)\vec{\sigma}^\pm$$
$$=e\vec{H}\wedge\vec{\sigma}^\pm+\vec{v}^\pm\wedge(e\vec{E}^\pm\wedge\vec{\sigma}^\pm). \qquad \text{Q.E.D.}$$

(2.33) can be regarded as the equation for a magnetic dipole moment.

The following lemma which is supplementary to Proposition 2.3 will be necessary in Section 5.

LEMMA 2.17. *If $\delta>0$ is sufficiently small, $\partial^2S^+(t,s,\vec{x},\vec{\eta})/\partial\vec{\eta}^2$ (or $\partial^2S^-(t,s,\vec{x},\vec{\eta})/\partial\vec{\eta}^2$) is negative (or positive) definite for $0<t-s<\delta$, $(x,\eta)\in\boldsymbol{R}^6$.*

PROOF. Since the matrices $\partial x^\pm(t,s,y,\eta)/\partial\eta$ and $\partial\xi^\pm(t,s,y,\eta)/\partial\eta$ satisfy the equations

$$\frac{d}{dt}\left(\frac{\partial x^\pm}{\partial\eta}\right)=\frac{\partial^2H^\pm}{\partial\xi\partial x}\,\frac{\partial x^\pm}{\partial\eta}+\frac{\partial^2H^\pm}{\partial\xi\partial\xi}\,\frac{\partial\xi^\pm}{\partial\eta} \tag{2.35}$$

$$\frac{d}{dt}\left(\frac{\partial\xi^\pm}{\partial\eta}\right)=-\frac{\partial^2H^\pm}{\partial x\partial x}\,\frac{\partial x^\pm}{\partial\eta}-\frac{\partial^2H^\pm}{\partial x\partial\xi}\,\frac{\partial\xi^\pm}{\partial\eta} \tag{2.36}$$

and the initial conditions $\partial x^\pm(s,s,y,\eta)/\partial\eta=0$, $\partial\xi^\pm(s,s,y,\eta)/\partial\eta=I$ and the coefficients of the equations (2.35) and (2.36) belong to $B((T_1,T_2)\times\boldsymbol{R}^6)$, it is easy to see that

$$|\partial x^\pm(t,s,y,\eta)/\partial\eta-(t-s)\partial^2H^\pm/\partial\xi^2(s,y,\eta)|=o(|t-s|). \tag{2.37}$$

Explicitly

$$\frac{\partial^2H^\pm}{\partial\xi_j\partial\xi_\nu}(s,y,\eta)=p_0^\pm(s,s,y,\eta)\left\{\delta_{jk}-\frac{(\eta_j-eA_j(s,\vec{y}))(\eta_k-eA_k(s,\vec{y}))}{p_0^\pm(s,s,y,\eta)^2}\right\}$$

and is positive definite for "+"-case and negative definite for "−"-case. By Proposition 2.3 and 2.2,

$$\frac{\partial^2S^\pm}{\partial\vec{\eta}\partial\vec{\eta}}(t,s,\vec{x},\vec{\eta})=\frac{\partial\vec{y}^\pm}{\partial\vec{\eta}}(t,s,x,\vec{\eta})=-(1+o(|t-s|))\frac{\partial x^\pm}{\partial\eta}(t,s,y,\eta)$$
$$=-(t-s)\frac{\partial^2H^\pm}{\partial\xi^2}(s,y,\eta)(1+o(|t-s|))$$

and thus the lemma is proved.

§3. Construction of the propagator.

In this section we construct the propagator $U^{\kappa}(t, s)$ associated with the equation (1.1), prove Theorem 1 and give some additional properties. We follow a general scheme for constructing the propagator associated with evolution equations of hyperbolic type using oscillatory integrals (cf. Kumano-go [7]). Recall that $G^{\kappa}_{\pm,N}(t, s)$ and $G^{\kappa}_{N}(t, s)$ are defined as

$$G^{\kappa}_{\pm,N}(t, s)f(\vec{x})=(2\pi\hbar)^{-3/2}\int\exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/\hbar)\left\{\sum_{j=0}^{N}\hbar^{j}E^{\pm}_{j}(t, s, \vec{x}, \vec{\eta})\right\}\hat{f}^{\kappa}(\vec{\eta})d\vec{\eta} \tag{1.10}$$

and

$$G^{\kappa}_{N}(t, s)f(\vec{x})=G^{\kappa}_{+,N}f(\vec{x})+G^{\kappa}_{-,N}f(\vec{x}),\quad f\in\mathscr{S}. \tag{1.11}$$

We also define as

$$E^{\kappa}_{\pm,j}(t, s)f(\vec{x})=(2\pi\hbar)^{-3/2}\int\exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/h)E^{\pm}_{j}(t, s, \vec{x}, \vec{\eta})\hat{f}^{\kappa}(\vec{\eta})d\vec{\eta}. \tag{3.1}$$

$$G^{\kappa}_{\pm,N}(t, s)=\sum_{j=0}^{N}\hbar^{j}E^{\kappa}_{\pm,j}(t, s). \tag{3.2}$$

We write as $\langle\vec{x}\rangle=(1+|\vec{x}|^2)^{1/2}$ and $\langle\vec{D}\rangle=(1-\Delta)^{1/2}$, where $\Delta=\partial^j\partial^j$ is the Laplacian. For real numbers m and p, $H^m_p=H^m_p(\boldsymbol{R}^3, \boldsymbol{C}^4)$ is the weighted Sobolev space:

$$H^m_p=\{f\in\mathscr{S}'(\boldsymbol{R}^3, \boldsymbol{C}^4)\colon \|\langle\vec{x}\rangle^p\langle\vec{D}\rangle^m f\|=\|f\|_{m,p}<\infty\}.$$

H^m_p is a Hilbert space and the norms $\|\langle x\rangle^p\langle\vec{D}\rangle^m f\|$ and $\|\langle\vec{D}\rangle^m\langle x\rangle^p f\|$ are equivalent; $\bigcap_{m,p\geqq 0}H^m_p=\mathscr{S}$ and the norms $\{\|f\|_{m,p}\colon m, p\geqq 0\}$ furnish $\mathscr{S}$ with the ordinary topology of $\mathscr{S}$.

PROPOSITION 3.1. (1) *For any integers $m, p\geqq 0$, the operator $E^{\kappa}_{\pm,j}(t, s)$ $(j=0, 1, \cdots, N;\ |t-s|<\delta)$ originally defined on $\mathscr{S}$ can be extended to a bounded operator on H^m_p and there exists a constant $C_{mp}>0$ independent of $0<\hbar<1$, $|t-s|<\delta$ and $f\in\mathscr{S}$ such that*

$$\|E^{\kappa}_{\pm,j}(t, s)f\|_{m,p}\leqq C_{mp}\hbar^{-m}\|f\|_{m,p}. \tag{3.3}$$

(2) *Let $m, p, k\geqq 0$ be integers. If $f\in H^{m+k}_{p+k}$, $E^{\kappa}_{\pm,j}(t, s)f$ is an H^m_p-valued C^k-function of (t, s) for $|t-s|<\delta$ and there exists a constant C_{mpk} such that*

$$\sum_{0\leqq l+n\leqq k}\|(\partial/\partial t)^l(\partial/\partial s)^n E^{\kappa}_{\pm,j}(t, s)f\|_{m,p}\leqq C_{mpk}\hbar^{-m-k}\|f\|_{m+k,p+k}. \tag{3.4}$$

PROOF. We prove for $E^{\kappa}_{+,0}(t, s)$ only and omit the subscript or superscript $+$ and 0 in the following expressions. Other cases can be proved similarly. By

(2.4), we may take $\delta>0$ such that

$$\|\partial^2 S(t, s, \vec{x}, \vec{\eta})/\partial x_j \partial\eta_k - \delta_{jk}\| \leqq \frac{1}{2} \quad \text{for } |t-s|<\delta. \tag{3.5}$$

We write $R \equiv (T_2 - T_1)C_{00}$, where C_{00} is the constant appeared in Proposition 2.2. By assumption (A) and Propositions 2.1–2.3, we have the following estimates for $|t-s|<\delta$:

$$|\partial S(t, s, \vec{x}, \vec{\eta})/\partial x| \leqq C\langle\vec{\eta}\rangle; \tag{3.6}$$

$$|\partial S(t, s, \vec{x}, \vec{\eta})/\partial\vec{\eta}| \leqq C\langle\vec{x}\rangle: \tag{3.7}$$

$$|\partial S(t, s, \vec{x}, \vec{\eta})/\partial t| + |\partial S(t, s, \vec{x}, \vec{\eta})/\partial s| \leqq C(\langle\vec{x}\rangle + \langle\vec{\eta}\rangle); \tag{3.8}$$

(2.5) the higher derivatives of $S(t, s, x, \eta)$ are bounded.

Thus by (2.5), Lemma 2.5 and (3.5), applying Theorem A.1 of Appendix, we see that there exist a constant C independent of $|t-s|<\delta$, $0<\hbar<1$ and $f\in\mathscr{S}$ such that

$$\|E^{\kappa}(t, s)f\| \leqq C\|f\|,$$

which is a special case of (3.3). To obtain (3.3) in general, and (3.4) we proceed as follows. Take $\psi(\vec{x}) \in C_0^\infty(\boldsymbol{R}^3, \boldsymbol{R}^1)$ such that $\psi(\vec{x})=1$ for $|\vec{x}| \leqq 2R+1$ and $\psi(\vec{x})=0$ for $|x| \geqq 3R+1$ and split $E^{\kappa}(t, s)f(\vec{x})$ into two parts:

$$\begin{aligned} E^{\kappa}(t, s)f(\vec{x}) &= \psi(\vec{x})E^{\kappa}(t, s)f(\vec{x}) + (1-\psi(\vec{x}))E^{\kappa}(t, s)f(\vec{x}) \\ &\equiv I_1^{\kappa}(t, s, \vec{x}) + I_2^{\kappa}(t, s, \vec{x}). \end{aligned}$$

By Lemma 2.5, (2.5), (3.6) and (3.8), we can write for $|\beta| \leqq m$ and $l+n \leqq k$,

$$\begin{aligned} &\vec{x}^p(\partial/\partial t)^l(\partial/\partial s)^n(\partial/\partial\vec{x})^\beta[\exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar)\cdot\psi(\vec{x})E(t, s, \vec{x}, \vec{\eta})] \\ &= \exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar)\hbar^{-k-m}A^{\kappa}_{p\beta ln;mk}(t, s, \vec{x}, \vec{\eta})\langle\vec{\eta}/\hbar\rangle^{k+m} \end{aligned} \tag{3.9}$$

where $A^{\kappa}_{p\beta ln;mk} \in C^\infty$ and for any multi-indices α and γ

$$|(\partial/\partial x)^\alpha(\partial/\partial\vec{\eta})^\gamma A^{\kappa}_{p\beta ln;mk}(t, s, \vec{x}, \vec{\eta})| \leqq C_{\alpha\gamma} \tag{3.10}$$

for $0<\hbar<1$, $|t-s|<\delta$. Hence by (3.9), (3.10) and Theorem A.1,

$$\sum_{0\leqq l+n\leqq k} \|(\partial/\partial s)^n(\partial/\partial t)^l I_1^{\kappa}(t, s, \vec{x})\|_{m,p} \leqq C_{mpk}\hbar^{-(k+m)}\|f\|_{0,k+m}. \tag{3.11}$$

We treat $I_2^{\kappa}(t, s, x)$ by partial integration. On the support of $1-\psi(\vec{x})$, we set

$$K \equiv -i(\partial S(t, s, \vec{x}, \vec{\eta})/\partial\vec{\eta})^{-2}(\partial S(t, s, \vec{x}, \vec{\eta})/\partial\eta_j)\partial/\partial\eta_j. \tag{3.12}$$

By Proposition 2.2 and 2.3 (2), we have on $\mathrm{supp}(1-\psi(\vec{x}))$,

$$|\partial S(t, s, \vec{x}, \vec{\eta})/\partial \vec{\eta}| \geqq |\vec{x}| - |\vec{y}(t, s, \vec{x}, \vec{\eta}) - \vec{x}| \geqq |\vec{x}| - R \geqq \frac{1}{2}|\vec{x}|, \tag{3.13}$$

and clearly

$$K^{p+k}(\exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar)) = \hbar^{-p-k} \exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar).$$

By (2.5) and (3.13), the formal adjoint ${}^tK^{p+k}$ of K^{p+k} can be written, on supp$(1-\psi)$, as

$${}^tK^{p+k} = \sum_{|\alpha| \leqq p+k} a_\alpha(t, s, \vec{x}, \vec{\eta})(\partial/\partial\vec{\eta})^\alpha.$$

where $a_\alpha \in C^\infty$ and

$$|(\partial/\partial s)^n(\partial/\partial t)^l(\partial/\partial\vec{x})^\beta(\partial/\partial\vec{\eta})^\gamma a_\alpha(t, s, \vec{x}, \vec{\eta})| \leqq C_{\alpha\beta\gamma nl}\langle\vec{x}\rangle^{-p-k}.$$

It follows that for any $0 \leqq l+n < k$. $|\beta| < m$,

$$\begin{aligned} &\hbar^{p+k}\vec{x}^p(\partial/\partial t)^l(\partial/\partial s)^n(\partial/\partial\vec{x})^\beta[\exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar)^tK^{p+k}\{(1-\psi(\vec{x}))E(t, s, \vec{x}, \vec{\eta})\hat{f}^\kappa(\vec{\eta})\} \\ &= \exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar)\hbar^{-m-k} \sum_{|\alpha| \leqq p+k} B^\kappa_{\alpha p\beta ln;mk}(t, s, x, \eta)\langle\vec{\eta}/\hbar\rangle^{m+k}\hbar^{p+k}(\partial/\partial\vec{\eta})^\alpha\hat{f}^\kappa(\vec{\eta}) \end{aligned} \tag{3.14}$$

where $B^\kappa_{\alpha p\beta ln;mk} \in C^\infty$ and for any multi-indices α, δ,

$$|(\partial/\partial\vec{x})^\gamma(\partial/\partial\vec{\eta})^\delta B^\kappa_{\alpha p\beta ln;mk}(t, s, \vec{x}, \vec{\eta})| \leqq C_{\alpha\delta}; \ |t-s| < \delta, \ 0 < \hbar < 1. \tag{3.15}$$

Since, by partial integration

$$I^\kappa_2(t, s, \vec{x}) = (2\pi\hbar)^{-3/2}\hbar^{p+k}\int \exp(iS(t, s, \vec{x}, \vec{\eta})/\hbar)^tK^{p+k}[(1-\psi(\vec{x}))E(t, s, \vec{x}, \vec{\eta})\hat{f}^\kappa(\eta)]d\vec{\eta},$$

(3.14), (3.15) and Theorem A.1 imply

$$\begin{aligned} &\sum_{0 \leqq l+n \leqq k} \|(\partial/\partial t)^l(\partial/\partial s)^n I^\kappa_2(t, s, \vec{x})\|_{m,p} \\ &\leqq C_{mpk}\hbar^{-k-m} \sum_{|\alpha| \leqq p+k} \|\langle\vec{\eta}/\hbar\rangle^{m+k}(\hbar\partial/\partial\vec{\eta})^\alpha\hat{f}^\kappa(\vec{\eta})\| \\ &\leqq C_{mpk}\hbar^{-k-m}\|f\|_{m+k,p+k}. \end{aligned} \tag{3.16}$$

Combining (3.11) with (3.16), we obtain

$$\sum_{0 \leqq l+n \leqq k} \|(\partial/\partial t)^l(\partial/\partial s)^n E^\kappa(t, s)f(\vec{x})\|_{m,p} \leqq C_{mpk}\hbar^{-(m+k)}\|f\|_{m+k,p+k},$$

from which the statements Proposition 3.1 follow easily. Q.E.D.

COROLLARY 3.2. *If $f \in \mathscr{S}$, then $E^\kappa_{\pm,j}(t, s)f$ is an $\mathscr{S}$-valued C^∞-function of (t, s) for $|t-s| < \delta$.*

COROLLARY 3.3. (1) *For any integer $m, p \geqq 0$, $G^\kappa_{\pm,N}(t, s)$ and $G^\kappa_N(t, s)$ are bounded operators on H^m_p for $|t-s| < \delta$ and there exists a constant C_{mp} such that*

$$\|G^{\kappa}_{\pm,N}(t, s)f\|_{m,p} \leqq C_{mp}\hbar^{-m}\|f\|_{m,p}; \ \|G^{\kappa}_{N}(t, s)f\|_{m,p} \leqq C_{mp}\hbar^{-m}\|f\|_{m,p}. \tag{3.17}$$

(2) *Let $m, p, k \geqq 0$ be integers and $f \in H^{m+k}_{p+k}$. Then $G^{\kappa}_{\pm,N}(t, s)f$ is an H^{m}_{p}-valued C^{k}-function of (t, s) for $|t-s|<\delta$ and there exists a constant $C_{mpk}>0$ such that*

$$\begin{cases} \sum_{0\leqq l+n\leqq k} \|(\partial/\partial s)^{n}(\partial/\partial t)^{l}G^{\kappa}_{\pm,N}(t, s)f\|_{m,p} \leqq C_{mpk}\hbar^{-(m+k)}\|f\|_{m,p}. \\ \sum_{0\leqq l+n\leqq k} \|(\partial/\partial s)^{n}(\partial/\partial t)^{l}G^{\kappa}_{N}(t, s)f\|_{m,p} \leqq C^{(1)}_{mpk}\hbar^{-(m+k)}\|f\|_{m,p}. \end{cases} \tag{3.18}$$

(3) $G^{\kappa}_{N}(s, s)=I$.

Notice that the exponents of h in the RHS of (3.17) and (3.18) do not depend on the index p of weight. This reflects the finiteness of propagation speed. We write

$$F^{\kappa}_{\pm,N}(t, s)f(\vec{x})=(2\pi\hbar)^{-3/2}\int \exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/\hbar)\gamma^{0}\partial_{x}E^{\pm}_{N}(t, s, \vec{x}, \vec{\eta})\hat{f}^{\kappa}(\vec{\eta})d\vec{\eta}; \tag{3.19}$$

$$F^{\kappa}_{N}(t, s)f(\vec{x})=F^{\kappa}_{+,N}(t, s)f(\vec{x})+F^{\kappa}_{-,N}(t, s)f(\vec{x}). \tag{3.20}$$

Since $\gamma^{0}\partial_{x}E^{\pm}_{N}(t, s, \vec{x}, \vec{\eta}) \in B((|t-s|<\delta)\times \boldsymbol{R}^{6})$, the proof of Proposition 3.1 shows that if $f \in \mathscr{S}$, $F^{\kappa}_{\pm,N}(t, s)f$ and $F^{\kappa}_{N}(t, s)f$ is an $\mathscr{S}$-valued C^{∞}-function of (t, s) for $|t-s|<\delta$ and

$$\sum_{0\leqq l+n\leqq k} \|(\partial/\partial s)^{n}(\partial/\partial t)^{l}F^{\kappa}_{N}(t, s)f\|_{m,p} \leqq C^{(2)}_{mpk}\hbar^{-m-k}\|f\|_{m+k,p+k}, \tag{3.21}$$

$$\sum_{0\leqq l+n\leqq k} \|(\partial/\partial s)^{n}(\partial/\partial t)^{l}F^{\kappa}_{\pm,N}(t, s)f\|_{m,p} \leqq C_{mpk}\hbar^{-m-k}\|f\|_{m+k,p+k}. \tag{3.22}$$

LEMMA 3.4. *If $f \in \mathscr{S}$, then*

$$(i\hbar\partial/\partial t-H^{\kappa}(t))G^{\kappa}_{\pm,N}(t, s)f=i\hbar^{N+1}F^{\kappa}_{\pm,N}(t, s)f \tag{3.23}$$

$$(i\hbar\partial/\partial t-H^{\kappa}(t))G^{\kappa}_{N}(t, s)f=i\hbar^{N+1}F^{\kappa}_{N}(t, s)f. \tag{3.24}$$

PROOF. By Proposition 2.3 (2), (2.2), (2.6) and Lemma 2.7, we have

$$\begin{aligned} &(i\hbar\partial/t-H^{\kappa}(t))[\exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/\hbar \sum_{j=0}^{N} \hbar^{j}E^{\pm}_{j}(t, s, \vec{x}, \vec{\eta})] \\ &=\gamma^{0}(i\hbar\partial-eA(t, \vec{x})-m)[\exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/\hbar) \sum_{j=0}^{N} \hbar^{j}E^{\pm}_{j}(t, s, x, \vec{\eta})] \\ &=\exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/\hbar)\gamma^{0}[(-\partial S^{\pm}-eA-m)E^{\pm}_{0} \\ &\quad+\sum_{j=1}^{N} \hbar^{j}\{(-\partial S^{\pm}-eA-m)E_{j}+i\partial E^{\pm}_{j-1}\}+i\hbar^{N+1}\partial E^{\pm}_{N}(t, s, \vec{x}, \vec{\eta})] \\ &=i\hbar^{N+1}\exp(iS^{\pm}(t, s, \vec{x}, \vec{\eta})/\hbar)\gamma^{0}\partial E^{\pm}_{N}(t, s, \vec{x}, \vec{\eta}). \end{aligned}$$

Hence the differentiation under the sign of integration shows (3.23). The equation (3.24) is clear from (3.23). Q.E.D.

Suppose temporally that there exists a nice propagator $U^{\kappa}(t, s)$ for the equation (1.1). Then, since $G_N^{\pm}(s, s)=I$, by solving (3.24), we obtain

$$G_N^{\kappa}(t, s)=U^{\kappa}(t, s)+\hbar^N\int_s^t U^{\kappa}(t, \sigma)F_N^{\kappa}(\sigma, s)d\sigma. \tag{3.25}$$

If follows that $U^{\kappa}(t, s)$ can be obtained by solving (3.25) by iteration as

$$\begin{aligned} U^{\kappa}(t, s)=&G_N^{\kappa}(t, s)-\hbar^N\int_s^t G_N^{\kappa}(t, \sigma)F_N^{\kappa}(\sigma, s)d\sigma \\ &+\sum_{j=2}^{\infty}(-\hbar^N)^j\int_s^t d\sigma_{j-1}\int_{\sigma_{j-1}}^t d\sigma_{j-2}\cdots\int_{\sigma_1}^t d\sigma G_N^{\kappa}(t, \sigma)F_N^{\kappa}(\sigma, \sigma_1), \cdots, F_N^{\kappa}(\sigma_{j-1}, s). \end{aligned} \tag{3.26}$$

We trace the argument backward: We define $U^{\kappa}(t, s)$ by the equation (3.26) and check that it is in fact satisfies Theorem 1. We write

$$(G_N^{\kappa}\# F_N^{\kappa})(t, s)\equiv\int_s^t G_N^{\kappa}(t, s)F_N^{\kappa}(\sigma, s)d\sigma \tag{3.27}$$

and for any integers $\gamma\geqq 0$,

$$(G_N^{\kappa}\overset{(\gamma)}{\#}F_N^{\kappa})(t, s)=(\overbrace{(\cdots((G_N^{\kappa}\# F_N^{\kappa})\# F_N^{\kappa})\#\cdots)\# F_N^{\kappa}}^{\gamma\text{-times}})(t, s). \tag{3.28}$$

LEMMA 3.5. *Let $p, m, k\geqq 0$ be integers.*

(1) *If $f\in H_{p+k}^{m+k}$, then $(G_N^{\kappa}\overset{(j)}{\#}F_N^{\kappa})(t, s)f$ is an H_p^m-valued C^k-function of (t, s) for any $j=1, 2, \cdots$.*

(2) *There exist constants C_1 and C_2 independent of $j, f\in\mathscr{S}$, $0<\hbar<1$ and $|t-s|<\delta_1$ such that for any $0\leqq k'\leqq k$*

$$\begin{aligned} &\sup_{n+l=k'}\{\|(\partial/\partial s)^n(\partial/\partial t)^l(G_N^{\kappa}\overset{(j)}{\#}F_N^{\kappa})(t, s)f\|_{m,p}\} \\ &\qquad\leqq C_2\frac{(C_1|t-s|)^{(j-k')_+}}{(j-k')_+!}\hbar^{-(m+k')(j+1)}\|f\|_{m+k', p+k'}. \end{aligned} \tag{3.29}$$

PROOF. The first statement is clear by Corollary 3.3 and (3.22). We prove (3.29) first for $j\leqq k$. A little calculation shows that $(\partial/\partial t)^{k-l}(\partial/\partial s)^l(G_N^{\kappa}\overset{(j)}{\#}F_N^{\kappa})(t, s)$ can be expressed as a sum of at most $2^{k(k+1)/2}$ terms of the form $(\partial^{p_1+q_1}G_N^{\kappa}/\partial t^{p_1}\partial s^{q_1})\circ(\partial^{p_2+q_2}F_N^{\kappa}/\partial t^{p_2}\partial s^{q_2})\circ\cdots\circ(\partial^{p_j+q_j}F_N^{\kappa}/\partial t^{p_j}\partial s^{q_j})$ or $(\partial^{p_1+q_1}F_N^{\kappa}/\partial t^{p_1}\partial s^{q_1})\circ\cdots\circ(\partial^{p_{j-1}+q_{j-1}}F_N^{\kappa}/\partial t^{p_{j-1}}\partial s^{q_{j-1}})$, where $\sum_{\lambda=1}^{j}(p_\lambda+q_\lambda)\leqq k$, $\sum_{\lambda=1}^{j-1}(p_\lambda+q_\lambda)\leqq k-1$ and $\circ$ stands for $\#$-product or operator product. Thus, applying (3.18) and (3.21), we can easily get (3.29). We show (3.29) for general $j\geqq k+1$ by induction. We write $R_j^{\kappa}(t, s)=(G_N^{\kappa}\overset{(j)}{\#}F_N^{\kappa})(t, s)$.

Clearly $R^{\kappa}_{j+1}(t,s)=\int_s^t R^{\kappa}_j(t,\sigma)F^{\kappa}_N(\sigma,s)d\sigma$ and it is easy to see that

$$(\partial/\partial s)^n(\partial/\partial t)^l R^{\kappa}_{j+1}(t,s)=\int_s^t(\partial^l R^{\kappa}_j/\partial t^l)(t,\sigma)(\partial^n F^{\kappa}_N/\partial s^n)(\sigma,s)d\sigma$$
$$+\sum_{0\le p\le l-1}(\partial/\partial t)^{l-p-1}\{(\partial^p R^{\kappa}_j/\partial t^p/\partial t^p)(t,t)(\partial^n F^{\kappa}_N/s^n)(t,s)\}$$
$$-\sum_{0\le q\le n-1}(\partial/\partial s)^{n-q-1}(\partial^l R^{\kappa}_j/\partial s^l)(t,s)(\partial^q F^{\kappa}_N/\partial s^q)(s,s)\}. \tag{3.30}$$

Notice that (3.30) implies

$$(\partial^p R^{\kappa}_j/\partial t^p)(t,t)=0 \quad \text{for } 0\leqq p\leqq j-1,$$

and if $l\leqq j$, the second summand in the RHS of (3.30) does not appear. Hence granting (3.29) up to j, $0\leqq k'\leqq k$, we obtain for $n+l=k'\leqq k$, $j\geqq k+1$,

$$\|(\partial/\partial s)^n(\partial/\partial t)^l R^{\kappa}_{j+1}(t,s)f\|_{m,p}$$
$$\leqq C_2\left\{\int_s^t C^{(2)}_{(m+k'-n)(p+k'-n)n}\hbar^{-(m+k')}\left\{\frac{(C_1|t-\sigma|)^{(j-l)_+}}{(j-l)_+!}\hbar^{-(m+l)(j+1)}\right\}\|f\|_{m+k',p+k'}d\sigma\right.$$
$$\left.+2^n\sum_{0\le q\le n-1}C^{(2)}_{(m+k'-q)(p+k'-q)q}\hbar^{-(m+k')}\left\{\frac{(C_1|t-s|)^{(j-k'+q+1)_+}}{(j-k'+q+1)_+!}\hbar^{-(m+q)(j+1)}\right\}\|f\|_{m+k',p+k'}\right\}.$$

Here we used (3.21) and $C^{(2)}_{mpk}$ is the constant appeared there. Thus if we choose $C_1=\max\{1, \max_{0\le k'\le k}2^k\sum_{0\le q\le k}C^{(2)}_{(m+k'-q)(p+k'-q)q}\}$ and $\delta_1=\min\{\delta, C_1^{-1}\}$, we have for $|t-s|<\delta_1$

$$\|(\partial/\partial s)^n(\partial/\partial t)^l R^{\kappa}_{j+1}(t,s)f\|_{m,p}$$
$$\leqq C_2\hbar^{-(m+k')(j+2)}\left\{2^n\sum_{0\le q\le n}C^{(2)}_{(m+k'-q)(p+k'-q)q}\right\}\frac{C_1^{(k-1')}|t-s|^{(j-k'+1)_+}}{(j-k'+1)_+!}\|f\|_{m+k',p+k'}$$
$$\leqq C_2\frac{(C_1|t-s|)^{(j-k'+1)}}{(j-k'+1)!}\hbar^{-(m+k')(j+2)}\|f\|_{m+k',p+k'},$$

which proves the stated result. Q.E.D.

COROLLARY 3.6. *Let $m, p, k\geqq 0$ be integers and $\delta_1>0$ be as in Lemma 3.5.*
(1) *If $f\in H^{m+k}_{p+k}$, then for $|t-s|<\delta_1$,*

$$U^{\kappa}(t,s)f\equiv\sum_{j=0}^{\infty}(-\hbar^N)^j(G^{\kappa}_N\overset{(j)}{\#}F^{\kappa}_N)(t,s)$$

defines an H^m_p-valued C^k-function. In particular if $f\in\mathscr{S}$, $U^{\kappa}(t,s)f$ is an $\mathscr{S}$-valued C^∞-function of $|t-s|<\delta_1$.

(2) *There exists a constant $C^{(3)}_{pmk}$ such that for $|t-s|<\delta_1$*

$$\sum_{n+l=k}\|(\partial/\partial s)^n(\partial/\partial t)^l U^{\kappa}(t,s)f\|_{m,p}$$
$$\leqq C^{(3)}_{pmk}\hbar^{-(m+k)(k+1)}\exp(C^{(3)}_{pmk}\hbar^{N-m-k}|t-s|)\|f\|_{m,p}. \tag{3.31}$$

In particular

$$\|U^{\kappa}(t, s)f\|_{0,p} \leqq C_1 \exp(C_2|t-s|)\|f\|_{0,p}. \tag{3.31$'$}$$

LEMMA 3.7. *If $f \in \mathscr{S}$, then*

$$(i\hbar\partial/\partial t - H^{\kappa}(t))U^{\kappa}(t, s)f = 0. \tag{3.32}$$

PROOF. Since $G_N^{\kappa}(t, t) = I$, we easily see that

$$\partial/\partial t(G_N^{\kappa} \overset{(j)}{\#} F_N^{\kappa})(t, s) = (F_N^{\kappa} \overset{(j-1)}{\#} F_N^{\kappa})(t, s)f + (\partial G^{\kappa}/\partial t \overset{(j)}{\#} F_N^{\kappa})(t, s)f$$

understanding $(F_N^{\kappa} \overset{(0)}{\#} F_N^{\kappa})(t, s) = F_N^{\kappa}(t, s)$. It follows by (3.24) and (3.30) that

$$\begin{aligned}
&(i\hbar\partial/\partial t - H^{\kappa}(t))U^{\kappa}(t, s)f \\
&= (i\hbar\partial/\partial t - H^{\kappa}(t))G_N^{\kappa}(t, s)f \\
&\quad + \sum_{j=1}^{\infty} (-\hbar^N)^j \{i\hbar(F_N^{\kappa} \overset{(j-1)}{\#} F_N^{\kappa})(t, s)f + (((i\hbar\partial/\partial t - H^{\kappa}(t))G^{\kappa}) \overset{(j)}{\#} F_N^{\kappa})(t, s)f\} \\
&= i\hbar\Big[\hbar^N F_N^{\kappa}(t, s)f + \sum_{j=1}^{\infty} (-\hbar^N)^j \{(F_N^{\kappa} \overset{(j-1)}{\#} F_N^{\kappa})(t, s)f + \hbar^N (F_N^{\kappa} \overset{(j)}{\#} F_N^{\kappa})(t, s)f\}\Big] \\
&= 0.
\end{aligned}$$

Q.E.D.

COROLLARY 3.8. (1) $\|U^{\kappa}(t, s)f\| = \|f\|$, $f \in L^2$.

(2) *If $f \in H_1^1$, then $U^{\kappa}(t, s)f$ is the unique solution of equation* (1.1) *which is H_1^1-valued continuous and L^2-valued C^1-function of t.*

PROOF. Since $H^{\kappa}(t) \in B(H_1^1, \mathscr{H})$ and $H^{\kappa}(t)|_{H_1^1}$ is essentially selfadjoint (cf. Chernoff [4]), we see (3.32) holds for $f \in H_1^1$ and that

$$(\hbar\partial/\partial t)\|U^{\kappa}(t, s)f\|^2 = (-iH(t)U^{\kappa}(t, s)f, U^{\kappa}(t, s)f) + (U^{\kappa}(t, s)f, -iH^{\kappa}(t)U^{\kappa}(t, s)f = 0,$$

which implies $\|U^{\kappa}(t, s)f\| = \|U^{\kappa}(s, s)f\| = \|f\|$. The second statement can be proved similarly. Q.E.D.

LEMMA 3.9. (1) *If $f \in \mathscr{S}$,*

$$i\hbar(\partial/\partial s)U^{\kappa}(t, s)f = -U^{\kappa}(t, s)H^{\kappa}(s)f. \tag{3.33}$$

(2) *If $|t-s|$, $|s-r|$, $|t-r| < \delta_1$ then*

$$U^{\kappa}(t, s)U^{\kappa}(s, r) = U^{\kappa}(t, r) \quad \text{on } H_p^m. \tag{3.34}$$

PROOF. (1) Write $V^{\kappa}(t, s)f = i\hbar(\partial/\partial s)U^{\kappa}(t, s)f$. Differentiating $U^{\kappa}(t, t)f = f$ by t and using (3.32), we have $V^{\kappa}(s, s)f = -H^{\kappa}(s)f$. Differentiating (3.31) by s, we have $(i\hbar\partial/\partial t - H^{\kappa}(t))V^{\kappa}(t, s)f = 0$. Since $V^{\kappa}(t, s)f$ is an $\mathscr{S}$-valued C^{∞}-function by Corollary 3.6 (1), Corollary 3.8 (2) implies $V^{\kappa}(t, s)f = U^{\kappa}(t, s)(-H^{\kappa}(s)f)$ as is desired.

(2) It suffices to prove the equation $U^{\kappa}(t, s)U^{\kappa}(s, r)f=U^{\kappa}(t, r)f$ for $f\in\mathscr{S}$. By (3.32) and (3.33), it is easy to see that $(\partial/\partial s)U^{\kappa}(t, s)U^{\kappa}(s, r)f=0$ in the region being considered. Hence we obtain the desired result. Q.E.D.

COROLLARY 3.10. *$U^{\kappa}(t, s)$ is a unitary operator on $\mathscr{H}$ for $|t-s|<\delta_1$.*

PROOF OF THEOREM 1. For any $T_1<s<t<T_2$, we define $U^{\kappa}(t, s)$ as follows: Take a subdivision $s=t_0<t_1<\cdots<t_{J-1}<t_J=t$ such that $|t_j-t_{j-1}|<\delta_1$ for $j=1,\cdots,J$. Define

$$U^{\kappa}(t, s)=U^{\kappa}(t_J, t_{J-1})\cdots U^{\kappa}(t_1, t_0)$$

where $U^{\kappa}(t_j, t_{j-1})$ in the RHS is defined by (3.30). If $T_1<t<s<T_2$, $U^{\kappa}(t, s)\equiv U^{\kappa}(s, t)^{-1}$. By (3.34), this definition does not depend on the choice of subdivision and by Corollary 3.6 — Corollary 3.10, $U^{\kappa}(t, s)$ satisfies the statements (1)-(3) of Theorem 1. Q.E.D.

COROLLARY 3.11. *The propagator $\{U^{\kappa}(t, s),\ T_1<t, s<T_2\}$ satisfies Corollary 3.8 — Corollary 3.10 without the restriction $|t-s|<\delta_1$.*

§4. Proof of Theorem 2.

We give a proof of Theorem 2 here. Here and hereafter we omit the arrow $\rightarrow$, for x and η are used only for denoting three vectors.

LEMMA 4.1. *For any integer $p\geqq 0$, there exists a constant $C>0$ independent of $0<\hbar<1$, $|t-s|<\delta_1$ and $f\in\mathscr{S}$ such that*

$$\|U^{\kappa}(t, s)f-G_N^{\kappa}(t, s)f\|_{0,p}\leqq C\hbar^N|t-s|\exp(C|t-s|)\|f\|_{0,p}. \tag{4.1}$$

PROOF. Applying (3.29) to (3.30) with $m=k=0$, we readily obtain (4.1). Q.E.D.

COROLLARY 4.2. *For $|t-s|<\delta_1$*

$$\|U^{\kappa}(t, s)f-G_N^{\kappa}(t, s)f\|_{0,p}\leqq C\hbar^{N+1}\|f\|_{0,p}. \tag{4.2}$$

PROOF. $G_N^{\kappa}(t, s)=G_{N+1}^{\kappa}(t, s)-\hbar^{N+1}E_{N+1}^{\kappa}(t, s)$, $E_j^{\kappa}=E_{+,j}^{\kappa}+E_{-,j}^{\kappa}$. Hence by Proposition 3.1 and (4.1), we obtain (4.2). Q.E.D.

REMARK 4.3. By duality argument, (4.1) and (4.2) hold equally for $p\leqq 0$.

Corollary 4.2 provides a proof of the first half of Theorem 2. We prove the second half. Since (1.13)′ is proved similarly, we prove (1.13) only. By Proposition 3.1 it suffices to prove (1.13) for $N=0$ and we omit the index $N=0$ in the following expressions. Applying Theorem A.1, we see by Proposition 2.3, Lemma 2.5 and (3.5) that $G_+^\kappa(t, s)G_-^\kappa(s, r)f(x)$ can be written as an oscillating integral

$$G_+^\kappa(t, s)G_-^\kappa(s, r)f(x)$$
$$=(2\pi\hbar)^{-6}\int_{R^{12}}\exp(i(S^+(t, s, x, \xi)-\xi\cdot z+S^-(s, r, z, \eta)-\eta\cdot y)/\hbar)$$
$$\times E^+(t, s, x, \xi)E^-(s, r, z, \eta)f(y)dyd\eta dzd\xi.$$

Write as $\theta=(\xi, z, \eta)\in \boldsymbol{R}^9$, $\phi(x, \theta, y)=S^+(t, s, x, \xi)-\xi\cdot z+S^-(s, r, z, \eta)-\eta\cdot y$ and $A(x, \theta, y)=E^+(t, s, x, \xi)E^-(s, r, z, \eta)$ and set

$$C_\phi=\{(x, \theta, y)\colon \mathrm{grad}_\theta\,\phi(x, \theta, y)=0\}.$$

Since $\mathrm{grad}_\theta\,\phi(x, \theta, y)=0$ is equivalent to $y=\partial S^-(s, r, z, \eta)/\partial\eta$, $\xi=S^-(s, r, z, \eta)/\partial z$ and $z=\partial S^+(t, s, x, \xi)/\partial\xi$, Proposition 2.3 shows that $(x, \theta, y)\in C_\phi$ if and only if $z=x^-(s, r, y, \eta)$, $\xi=\xi^-(s, r, y, \eta)$ and $z=y^+(t, s, x, \xi)$. Hence, by Lemma 2.8, if $(x, \theta, y)\in C_\phi$, $A(x, \theta, y)=E^+(t, s, x, \xi)P^+(s, y^+(t, s, x, \xi), \xi)P^-(s, z, \xi^+(s, r, y, \eta))E^-(s, r, z, \eta)=0$. Thus by Theorem A.1, we obtain (1.13). Q.E.D.

§5. Quasi-classical approximation in finite time, proof of Theorem 3.

We prove Theorem 3 here. For a real valued smooth function $S(y)$ given at time s, we associate a Lagrangian manifold $\Lambda(s)=\{(y, \partial S(y)/\partial y\colon y\in \boldsymbol{R}^3\}$ generated by S in the phase space $T^*\boldsymbol{R}^3=\boldsymbol{R}^3\times\boldsymbol{R}^3$. After time $t-s$, the canonical transformation $(x^\pm(t, s, y, \eta), \xi^\pm(t, s, y, \eta))$ on the phase space generated by the Hamilton equation (1.5) transforms $\Lambda(s)$ into another Lagrangian manifold $\Lambda^\pm(t)=\{(x^\pm(t, s, y, \partial S(y)/\partial y), \xi^\pm(t, s, y, \partial S(y)/\partial y))\colon y\in \boldsymbol{R}^3\}$. Since $\Lambda^\pm(t)$ is Lagrangian, the existence of covering $\{\Omega_\lambda\}$ of $\boldsymbol{R}^3$ with the property Theorem 3 (1) is well-known (see, Abraham-Marsden [1], § 5.3). We prove the second statement by applying Theorem A.2 or the stationary phase method. Given s, t such that $T_1\leqq s<t\leqq T_2$, we take a subdivision

$$\Delta\colon\ s=t_0<t_1<\cdots<t_N=t \tag{5.1}$$

such that $|t_j-t_{j-1}|<\delta_1$, $j=1, 2, \cdots, N$ and write

$$G_\pm^\kappa(t, s, \Delta)\equiv G_{\pm,0}^\kappa(t_N, t_{N-1})\cdots G_{\pm,0}^\kappa(t_1, t_0). \tag{5.1}$$

The following lemma is obvious from Theorem 1, Theorem 2 and the unitarity of the Fourier transform.

LEMMA 5.1. *Let $G_\pm^\kappa(t, s, \Delta)$ be as above and $K\subset\{1, 2, 3\}$. Then there exists* *a constant* $C>0$ *independent of* $0<\hbar\leqq 1$ *and* $f\in\mathscr{S}$ *such that*

$$\|U^{\hbar}(t,s)f-G_{+}^{\hbar}(t,s,\varDelta)f-G_{-}^{\hbar}(t,s,\varDelta)f\|\leqq C\hbar\|f\|, \tag{5.2}$$

$$\|\mathscr{F}_{K}^{\hbar}U^{\hbar}(t,s)f-\mathscr{F}_{K}^{\hbar}(G_{+}^{\hbar}(t,s,\varDelta)f-\mathscr{F}_{K}^{\hbar}G_{-}^{\hbar}(t,s,\varDelta)f\|\leqq C\hbar\|f\|. \tag{5.3}$$

5.1. Proof for the case $K_{\lambda}=\phi$.

We first prove the case $\operatorname{supp} f\subset\Omega_{\lambda}$ and $K_{\lambda}=\phi$. We write $x=x_N$, $y=x_0$, and $\theta=(\eta_{N-1}, x_{N-1}, \eta_{N-2},\cdots,\eta_1, x_1, \eta_0)\in \boldsymbol{R}^{6N-3}$ and set

$$\phi^{\pm}(x,\theta,y)=\sum_{j=1}^{N}(S^{\pm}(t_j, t_{j-1}, x_j, \eta_{j-1})-x_{j-1}\cdot\eta_{j-1})+S(y); \tag{5.4}$$

$$A^{\pm}(x,\theta,y)=E_0^{\pm}(t_N, t_{N-1}, x_N, \eta_{N-1})\cdots E_0^{\pm}(t_1, t_0, x_1, \eta_0). \tag{5.5}$$

Since, by Proposition 2.3 and Lemma 2.5, $S^{\pm}(t_j, t_{j-1}, x_j, \eta_{j-1})$ and $E^{\pm}(t_j, t_{j-1}, x_j, \eta_{j-1})$ (or $x_{j-1}\eta_{j-1}$ and I=identity matrix) satisfy conditions (C.1)-(C.3) of Appendix, Theorem A.1 (3) implies that the functions $\phi^{\pm}(x,\theta,y)$ and $A^{\pm}(x,\theta,y)$ also satisfy (C.1)-(C.3) with $n=3$, $m=6N-3$ and $G_{\pm}^{\hbar}(t,s,\varDelta)(e^{iS(y)/\hbar}f)(x)$ is written as

$$\begin{aligned} &G_{\pm}^{\hbar}(t,s,\varDelta)(e^{iS(y)/\hbar}f)(x)\\ &=(2\pi\hbar)^{-3N}\int_{R^{6N}}\exp(i\phi^{\pm}(x,\theta,y)/\hbar)A^{\pm}(x,\theta,y)f(y)dyd\theta. \end{aligned} \tag{5.6}$$

The following lemma shows that $\phi^{\pm}(x,\theta,y)$ satisfies the condition (C.4) as well.

LEMMA 5.2. *The equation* $\operatorname{grad}_{(\theta,y)}\phi^{\pm}(x,\theta,y)=0$ *determines a function* $(x,\theta)=(x^{\pm}(y),\theta^{\pm}(y))$ *of* $y\in\boldsymbol{R}^3$ *as follows:*

$$\eta_0=\partial S(y)/\partial y; \tag*{(5.7)$_0$}$$

$$x_j=x_j^{\pm}(y)=x^{\pm}(t_j, s, y, \eta_0),\ \ \eta_{j-1}^{\pm}(y)=\xi^{\pm}(t_{j-1}, s, y, \eta_0),\ \ j=1,2,\cdots,N. \tag*{(5.7)$_j$}$$

$$|\det\partial x^{\pm}(y)/\partial y|\geqq\varepsilon>0 \ \textit{for some}\ \varepsilon>0\ \textit{and all}\ y\in\Omega_{\lambda}. \tag{5.8}$$

PROOF. The equation $\operatorname{grad}_{(\theta,y)}\phi^{\pm}(x,\theta,y)=0$ is written as a set of equations

$$\eta_0=\partial S(y)/\partial y; \tag*{(5.9)$_0$}$$

$$\begin{aligned} &\partial S^{\pm}(t_j, t_{j-1}, x_j, \eta_{j-1})/\partial\eta_{j-1}=x_{j-1}\quad (j=1,\cdots,N),\\ &\partial S^{\pm}(t_j, t_{j-1}, x_j, \eta_{j-1})/\partial x_j=\eta_j\quad (j=1,\cdots,N-1). \end{aligned} \tag*{(5.9)$_j$}$$

By Proposition 2.3, (5.9) implies (5.7). (5.8) is clear by the assumption $K_{\lambda}=\phi$.

Q.E.D.

By Lemma 5.2, we can apply Theorem A.2 to $G^{\hbar}(t,s,\varDelta)(e^{iS(y)/\hbar}f)(x)$. Thus by virtue of Lemma 5.1, the following lemma with Theorem A.2 concludes the proof of (1.15) for the case $K_\lambda=\phi$.

LEMMA 5.3. *Let* $(x^\pm(y), \theta^\pm(y))$ *be as in Lemma* 5.2. *Then the following statements hold.*

(1) *Let* $f^\pm(t, s, y)$ *be the solution of initial value problem*

$$\hat{M}^\pm(t, s, y, \eta_0(y))f^\pm(t, s, y)=0, \quad f^\pm(s, s, y)=f(y).$$

Then

$$A^\pm(x^\pm(y), \theta^\pm(y), y)=\sum_{j=1}^{N}\left|\det(\partial x^\pm(t_j, t_{j-1}, z, \eta)/\partial z)\Big|_{z=x^\pm_{j-1}(y), \eta=\eta^\pm_{j-1}(y)}\right|^{1/2} \times(p_0^\pm(t, s, y, \eta_0(y))/p_0^\pm(s, s, y, \eta_0(y)))^{-1/2}f^\pm(t, s, y). \tag{5.10}$$

(2) *For the phase function,*

$$\phi(x^\pm(y), \theta^\pm(y), y)=S(y)+\int_s^t L^\pm(u, x^\pm(u, s, y, \eta_0(y)), \xi^\pm(u, s, y, \eta_0(y)))du; \tag{5.11}$$

$$\begin{aligned}&|\det(\mathrm{Hess}_{(\theta,y)}\phi^\pm)(x^\pm(y), \theta^\pm(y), y)|\\ &=\left|\det \partial x^\pm(y)/\partial y \prod_{j=1}^{N}\det(\partial x^\pm(t_j, t_{j-1}, z, \eta)/\partial z)^{-1}\Big|_{z=x^\pm_{j-1}(y), \eta=\eta^\pm_{j-1}(y)}\right|;\end{aligned} \tag{5.12}$$

$$\mathrm{Inert}(\mathrm{Hess}_{(\theta,y)}\phi^\pm)(x^\pm(y), \theta^\pm(y), y)=3N+\mathrm{Ind}\,\gamma^\pm(t, s, y, \eta_0(y)). \tag{5.13}$$

PROOF. Statement (1) is a repetition of Lemma 2.13 by (5.5) and (5.7). Since, by definition (2.1) of $S^\pm(t, s, y, \eta)$,

$$\begin{aligned}&S^\pm(t_j, t_{j-1}, x^\pm(t_j, s, y, \eta_0), \xi^\pm(t_{j-1}, s, y, \eta_0))-x^\pm(t_{j-1}, s, y, \eta_0)\cdot\xi^\pm(t_{j-1}, s, y, \eta_0)\\ &=\int_{t_{j-1}}^{t_j} L^\pm(u, x^\pm(u, s, y, \eta_0), \xi^\pm(u, s, y, \eta_0))du,\end{aligned}$$

(5.11) is obvious. We prove (5.12). $\mathrm{Hess}_{(\theta,y)}\phi^\pm(x, \theta, y)$ is written as

(5.14) (See Diagram A on page 187)

where we wrote $S_j^\pm(x_j, \eta_{j-1})=S^\pm(t_j, t_{j-1}, x_j, \eta_{j-1})$, $j=1, \cdots, N$. Differentiating (5.9) by y after plugging (5.7) into (5.9), we have

$$\begin{cases}(\partial^2 S_j^\pm/\partial x_j^2)(x_j^\pm(y), \eta_{j-1}^\pm(y))\partial x_j^\pm(y)/\partial y\\ \qquad\qquad +(\partial^2 S_j^\pm/\partial x_j\partial\eta_{j-1})(x_j^\pm(y))\partial\eta_{j-1}^\pm(y)/\partial y=\partial\eta_j^\pm(y)/\partial y;\\ (\partial^2 S_j^\pm/\partial\eta_{j-1}\partial x_j)(x_j^\pm(y), \eta_{j-1}^\pm(y))\partial x_j^\pm(y)/\partial y\\ \qquad\qquad +(\partial^2 S_j/\partial\eta_{j-1}^2)(x_j^\pm(y), \eta_{j-1}^\pm(y)\partial\eta_{j-1}^\pm(y)/\partial\eta=\partial x_{j-1}^\pm(y)/\partial y;\\ (\partial^2 S/\partial y^2)(y)=\partial\eta_0(y)/\partial y.\end{cases} \tag{5.15}$$

We apply the following determinant preserving operation to (5.14): Add $(2j-1)-$

DIAGRAM A

$$\begin{bmatrix}
\frac{\partial^2 S_N^\pm}{\partial \eta_{N-1}^2}(x, \eta_{N-1}) & -I & 0 & 0 & & & \\
-I & \frac{\partial^2 S_{N-1}}{\partial x_{N-1}^2}(x_{N-1}, \eta_{N-2}) & \frac{\partial^2 S_{n-1}^\pm}{\partial x_{N-1}\partial \eta_{N-2}}(x_{N-1}, \eta_{N-2}) & 0 & & & \\
0 & \frac{\partial^2 S_{N-1}^\pm}{\partial \eta_{N-2}\partial x_{N-1}}(x_{N-1}, \eta_{N-2}) & \frac{\partial^2 S_{N-1}^\pm}{\partial \eta_{N-2}^2}(x_{N-1}, \eta_{N-2}) & -I & & & \\
0 & 0 & -I & * & * & & \\
 & & 0 & * & * & & \\
 & & & * & \frac{\partial^2 S_1^\pm}{\partial x_1^2}(x_1, \eta_0) & \frac{\partial^2 S_1^\pm}{\partial x_1 \partial \eta_0}(x_1, \eta_0) & 0 \\
 & & & & \frac{\partial^2 S_1^\pm}{\partial \eta_0 \partial x_1}(x_1, \eta_0) & \frac{\partial^2 S_1^\pm}{\partial \eta_0^2}(x_1, \eta_0) & -I \\
 & & & & 0 & -I & \frac{\partial^2 S}{\partial y^2}(y)
\end{bmatrix}$$

th $(j=1,\cdots,N)$ and $2j-th$ $(j=1,\cdots,N-1)$ columns to the last column after multiplying $\partial\eta^{\pm}_{N-j}(y)/\partial y$ to $(2j-1)-th$ and $\partial x^{\pm}_{N-j}(y)/\partial y$ to $(2j)-th$. Then by (5.15) it is easy to see that

$$(5.16)\qquad |\det(\mathrm{Hess}\,\phi^{\pm})(x^{\pm}(y),\theta^{\pm}(y),y)| = \left\{\prod_{j=1}^{N}|\det(\partial^2 S_j/\partial\eta_{j-1}\partial x_j)(x^{\pm}_j(y),\eta^{\pm}_{j-1}(\eta)|\right\}|\det\partial x^{\pm}_N(y)/\partial y|.$$

Since

$$(\partial S^{\pm}_j/\partial\eta_{j-1}\partial x_j)(x_j,\eta_{j-1})=\partial y^{\pm}(t_j,t_{j-1},x_j,\eta_{j-1})/\partial x_j = (\partial x^{\pm}(t_j,t_{j-1},y,\eta_{j-1})/\partial y)^{-1}\Big|_{y=y^{\pm}(t_j,t_{j-1},x_j,\eta_{j-1})}$$

by Propositions 2.2 and 2.3, (5.16) shows (5.12). We prove (5.13) next. Recall that by Lemma 2.17

$$(5.17)\qquad \partial^2 S^{+}_j(x_j,\eta_{j-1})/\partial\eta^2_{j-1}<0,\quad \partial^2 S^{-}_j(x_j,\eta_{j-1})/\partial\eta^2_{j-1}>0,\quad j=1,\cdots,N$$

and hence, by implicit function theorem, the mapping $\boldsymbol{R}^3\ni\eta_{j-1}\to\partial S_j(x_j,\eta_{j-1})/\partial\eta_{j-1}=y^{\pm}(t_j,t_{j-1},x_j,\eta_{j-1})$ is a local diffeomorphism. Since $y^{\pm}(t_j,t_{j-1},x^{\pm}_j(y),\eta^{\pm}_{j-1}(y))=x^{\pm}_{j-1}(y)$, it follows that there exists an open neighborhood $U^{\pm}_j\subset\boldsymbol{R}^3$ of $x^{\pm}_j(y)$ $(j=0,\cdots,N)$ such that for any $(x_j,x_{j-1})\in U^{\pm}_j\times U^{\pm}_{j-1}$ there exists unique $\eta^{\pm}_{j-1}=\eta^{\pm}_{j-1}(x_j,x_{j-1})$ near $\eta^{\pm}_{j-1}(y)$ such that $x_{j-1}=y^{\pm}(t_j,t_{j-1},x_j,\eta^{\pm}_{j-1})$ or $x_j=x^{\pm}(t_j,t_{j-1},x_{j-1},\eta^{\pm}_{j-1}(x_j,x_{j-1}))$. Write $\widetilde{S}^{\pm}_j(x_j,x_{j-1})\equiv S^{\pm}_j(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1}))-x_{j-1}\cdot\eta^{\pm}_{j-1}(x_j,x_{j-1})$ and

$$(5.18)\qquad \widetilde{S}^{\pm}(x,x_{N-1},\cdots,x_1,y)\equiv\sum_{j=1}^{N}\widetilde{S}^{\pm}_j(x_j,x_{j-1})+S(y).$$

$\widetilde{S}^{\pm}(x,x_{N-1},\cdots,x_1,y)$ is defined on a neighborhood $U\subset\boldsymbol{R}^{3N+1}$ of $(x^{\pm}_N(y),\cdots,x^{\pm}_1(y),y)$. We write $\tilde{\theta}=(x_{N-1},\cdots,x_1)$. We remark that $\widetilde{S}^{\pm}_j(x_j,x_{j-1})$ is the action integral of the trajectory $\{(x^{\pm}(u,t_{j-1},x_{j-1},\eta^{\pm}_{j-1}(x_j,x_{j-1})),\xi^{\pm}(u,t_{j-1},x_{j-1},\eta^{\pm}_{j-1}(x_j,x_{j-1})):t_{j-1}\leqq u\leqq t_j\}$ and that it satisfies the following properties:

$$(5.19)\qquad \partial\widetilde{S}^{\pm}_j(x_j,x_{j-1})/\partial x_{j-1}=-\eta^{\pm}_j(x_j,x_{j-1});$$

$$(5.20)\qquad \partial^2\widetilde{S}^{\pm}_j(x_j,x_{j-1})/\partial x^2_{j-1}=-(\partial^2 S^{\pm}_j/\partial\eta^2_{j-1})(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1}));$$

$$(5.21)\qquad \partial^2\widetilde{S}^{\pm}_j(x_j,x_{j-1})/\partial x^2_j=(\partial^2 S^{\pm}_j/\partial x^2_j)(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1})) + (\partial^2 S^{\pm}_j/\partial x_j\partial\eta_{j-1})(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1}))\partial\eta^{\pm}_j(x_j,x_{j-1})/\partial x_j;$$

$$(5.22)\qquad \partial^2\widetilde{S}^{\pm}_j(x_j,x_{j-1})/\partial x_j\partial x_{j-1}=(\partial^2 S^{\pm}_j/\partial\eta^2_{j-1})(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1}))^{-1} \times(\partial^2 S^{\pm}_j/\partial x_j\partial\eta_{j-1})(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1})).$$

$$(5.23)\qquad \partial^2 S^{\pm}_j/\partial\eta^2_{j-1}(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1}))\partial\eta^{\pm}_{j-1}(x_j,x_{j-1})/\partial x_j + (\partial^2 S^{\pm}_j/\partial\eta_{j-1}\partial x_j)(x_j,\eta^{\pm}_{j-1}(x_j,x_{j-1}))=0.$$

We write for $1 \leqq j \leqq N$,

$$
\begin{array}{c}
B_j^{\pm} = \left[\begin{array}{c:cc:c}
I \ddots I & 0 & 0 & 0 \\
\hdashline
0 & I & 0 & 0 \\
 & \partial^2 S_j^{\pm}/\partial \eta_{j-1}^2 (x_j^{\pm}, \eta_{j-1}^{\pm})^{-1} & I & \\
\hdashline
0 & 0 & & I
\end{array}\right] \begin{array}{l} \\ \\ \\ \cdots 2(N-j) \\ \\ \end{array} \\
(2N-j)+1
\end{array}
$$

and

$$
\begin{array}{c}
D_j^{\pm} = \left[\begin{array}{c:cc:c}
I \ddots I & 0 & & 0 \\
\hdashline
 & I & {}^t(\partial \eta_{j-1}^{\pm}/\partial x_j (x_j^{\pm}, x_{j-1}^{\pm})) & \\
0 & & & 0 \\
 & 0 & I & \\
\hdashline
0 & 0 & & I \ddots I
\end{array}\right] \begin{array}{l} \\ \\ \cdots 2(N-j)+1. \\ \\ \\ \\ \end{array} \\
2(N-j)+2
\end{array}
$$

Clearly $B_j^{\pm}$ and $D_j^{\pm}$ are non-singular and

(5.24) $\quad \mathrm{Inert}(\mathrm{Hess}_{(\theta,y)}\phi^{\pm})(x^{\pm}(y), \theta^{\pm}(y), y)$
$= \mathrm{Inert}(B_N^{\pm} D_{N-1}^{\pm} \cdots D_1^{\pm} B_1^{\pm} \cdot \mathrm{Hess}_{(\theta,y)}\phi^{\pm} \cdot {}^t B_1^{\pm}\, {}^t D_1^{\pm} \cdots {}^t D_{N-1}^{\pm}\, {}^t B_N^{\pm})(x^{\pm}(y), \theta^{\pm}(y), y).$

By virtue of (5.17), (5.19)-(5.23) and (5.24) it is easy to see that

(5.25)$_+$ $\quad \mathrm{Inert}(\mathrm{Hess}_{(\theta,y)}\phi^{+})(x^{+}(y), \theta^{+}(y), y)$
$= 3N + \mathrm{Inert}(\mathrm{Hess}_{(\tilde{\theta},y)}\tilde{S}^{+})(x_N^{+}(y), \tilde{\theta}^{+}(y), y);$

(5.25)$_-$ $\quad \mathrm{Inert}(\mathrm{Hess}_{(\theta,y)}\phi^{-})(x^{-}(y), \theta^{-}(y), y)$
$= \mathrm{Inert}(\mathrm{Hess}_{(\tilde{\theta},y)}\tilde{S}^{-})(x_N^{-}(y), \tilde{\theta}^{-}(y), y),$

since $\pm \partial^2 H(t, x, \xi)/\partial \xi^2$ is positive definite, a well-known theorem on Morse index (cf. Theorem 15.1 of Milnor [9] and Theorem 17.4 of Morse [10]) shows that

(5.26)$_+$ $\quad \mathrm{Inert}(\mathrm{Hess}_{(\tilde{\theta},y)}\tilde{S}^{+})(x^{+}(y), \tilde{\theta}^{+}(y), y)$
$= \mathrm{Ind}\{(x^{+}(u, s, y, \eta_0(y)), \xi^{+}(u, s, y, \eta_0(y)) : s \leqq u \leqq t\};$

(5.26)$_-$ $\quad 3N - \mathrm{Inert}(\mathrm{Hess}_{(\tilde{\theta},y)}\tilde{S}^{-})(x^{-}(y), \tilde{\theta}-(y), y)$
$= -\mathrm{Ind}\{(x^{-}(u, s, y, \eta_0(y)), \xi^{-}(u, s, y, \eta_0(y)) : s \leqq u \leqq t\}.$

Thus (5.25) and (5.26) imply the desired equation (5.13). Q.E.D.

5.2. Proof for the case $K_\lambda\neq\phi$.

Since other case can be proved similarly, we prove the case $K_\lambda=\{1\}$ only. Write

$$\psi^{\pm}(\eta_{\{1\}}, x, \theta, y)=-x_{\{1\}}\cdot\eta^{\pm}_{\{1\}}+\phi^{\pm}(x, \theta, y).$$

Then the same argument as in Subsection 5.1 shows that the functions $A^{\pm}(x, \theta, y)$ and $\psi^{\pm}(\eta_{\{1\}}, x, \theta, y)$ satisfy conditions (C.1)–(C.3) of Appendix (take $(\eta_{\{1\}}, x_{\{2,3\}})$ and $(x_{\{1\}}, \theta)$ to be x and θ there, respectively), and

$$\begin{aligned}(5.27)\qquad &\mathscr{F}^{\hbar}_{\{1\}}G^{\hbar}(t, s; \Delta)(\exp(iS(y)/\hbar)f)(x_{\{2,3\}}, \eta_{\{1\}})\\ &=(2\pi\hbar)^{-3N-1/2}\int_{R^{6N+1}}\exp(\psi^{\pm}(\eta_{\{1\}}, x, \theta, y)/\hbar)A(x, \theta, y)f(y)dyd\theta dx_{\{1\}}.\end{aligned}$$

LEMMA 5.4. *The equation* $\mathrm{grad}_{(x_{\{1\}},\theta,y)}\psi^{\pm}(\eta_{\{1\}}, x, \theta, y)=0$ *determines a function* $(\eta_{\{1\}}, x, \theta)=(\eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y))$ *as* (5.7) *and*

$$(5.28)\qquad \eta^{\pm}_{\{1\}}(y)=\xi^{\pm}_{\{1\}}(t, s, y, \eta_0(y)).$$

$$(5.29)\qquad |\det \partial(x^{\pm}_{\{2,3\}}(y), \eta^{\pm}_{\{1\}}(y))/\partial y|\geqq\varepsilon>0 \quad \textit{for some } \varepsilon>0$$

and all $y\in\Omega_\lambda$.

PROOF. The equation $\mathrm{grad}_{(x_{\{1\}},\theta,y)}\psi^{\pm}(\eta_{\{1\}}, x, \theta, y)=0$ is written as

$$(5.30)\qquad \mathrm{grad}_{(\theta,y)}\phi^{\pm}(x, \theta, y)=0 \quad\text{and}\quad \eta_{\{1\}}=\partial S^{\pm}_N(t_N, t_{N-1}, x, \eta_{N-1})/\partial x_{\{1\}}.$$

Hence by Lemma 5.2, $x_j=x_j(y)$ and $\eta_{j-1}=\eta^{\pm}_{j-1}(y)$ are determined as (5.7), and by Proposition 2.3, $\eta_{\{1\}}=\xi^{\pm}_{\{1\}}(t_N, s, y, \eta_0(y))$. (5.29) is clear by (5.7), (5.28), and the assumption $K_\lambda=\{1\}$. Q.E.D.

The estimate (5.29) shows that $(x_{\{2,3\}}, \eta_{\{1\}})$ can be taken as a local coordinate of the manifold $\{(x^{\pm}(t, s, y, \eta_0(y)), \xi^{\pm}(t, s, y, \eta_0(y))): y\in\Omega_\lambda\}$ and $(x_{\{1\}}, \eta_{\{2,3\}})$ can be written as a function of $(x_{\{2,3\}}, \eta_{\{1\}})$ as $(x_{\{1\}}, \eta_{\{2,3\}})=(x^{\pm}_{\{1\}}(x_{\{2,3\}}, \eta_{\{1\}}), \eta^{\pm}_{\{2,3\}}, (x_{\{2,3\}}, \eta_{\{1\}}))$. By (5.29), we apply Theorem A.2 to (5.27). Thus by virtue of Lemma 5.1, Lemma 5.4 and the following lemma show that (1.15) holds for the case $K_\lambda=\{1\}$.

LEMMA 5.5. *Let* $(\eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y))$ *be as above. Then the following statements hold:*

(5.31) $\psi^{\pm}(\eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y), y) = -x^{\pm}_{\{1\}}(t, s, y, \eta_0(y)) \cdot \xi^{\pm}_{\{1\}}(t, s, y, \eta_0(y))$
$$+S(y) + \int_s^t L^{\pm}(u, x^{\pm}(u, s, y, \eta_0(y)), \xi^{\pm}_{\{1\}}(u, s, y, \eta_0(y)))du;$$

(5.32) $|\det(\mathrm{Hess}_{(x_{\{1\}}, \theta, y)} \psi^{\pm})(\eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y), y)|$
$$= |\det \partial(x^{\pm}_{\{2,3\}}(y), \eta^{\pm}_{\{1\}}(y))/\partial y \prod_{j=1}^{N} \det(\partial x^{\pm}(t_j, t_{j-1}, z, \eta)/\partial z)^{-1}_{z=x^{\pm}_{j-1}(y), \eta_{j-1}=\eta^{\pm}_{j-1}(y)}|;$$

(5.33) $\mathrm{Inert}(\mathrm{Hess}_{(x_{\{1\}}, \theta, y)} \psi^{\pm})(\eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y), y)$
$$\equiv \mathrm{Ind}\, \gamma^{\pm}(t, s, y, \eta_0(y)) + \mathrm{Inert}(\partial x^{\pm}_{\{1\}}/\partial \eta_{\{1\}})(x^{\pm}_{\{2,3\}}(y), \eta^{\pm}_{\{1\}}(y)) + 3N \pmod 4.$$

PROOF. The equation (5.31) is clear by (5.7), (5.28) and (5.11). The equation (5.32) can be proved in a similar way as was used to prove (5.12) and we omit its proof here. We prove (5.33). Write for subset $K \subset \{1, 2, 3\}$ and $s \leqq u$,

$$\Sigma^{\pm}(u, K) = \{y \in \boldsymbol{R}^3 \colon \det \partial(x^{\pm}_{K^c}(u, s, y, \eta_0(y)), \xi^{\pm}_K(u, s, y, \eta_0(y)))/\partial y \neq 0\}.$$

For each u, $\Sigma^{\pm}(u, K)$ is an open set and $\bigcup_K \Sigma^{\pm}(u, K) = \boldsymbol{R}^3$. For sufficiently small $\varepsilon > 0$, we set

$$\psi^{\pm}(t', \eta_{\{1\}}, x, \theta, y) = -x_{\{1\}} \cdot \eta_{\{1\}} + \sum_{j=1}^{N} (S^{\pm}(t_j, t_{j-1}, x_j, \eta_{j-1}) - x_{j-1} \cdot \eta_{j-1}) + S(y)$$

for $0 \leqq |t - t'| \leqq \varepsilon$, where $t_0, \cdots, t_{N-1}$ are taken as in (5.1) and $t_N = t'$. As in the proof of Lemma 5.4, the equation $\mathrm{grad}_{(x_{\{1\}}, \theta, y)} \psi^{\pm}(t', \eta_{\{1\}}, x, \theta, y) = 0$ determines $(\eta_{\{1\}}, x, \theta)$ as a function (5.7) and (5.28) of $y \in \boldsymbol{R}^3$ with the change t to t'. Since the functions $\psi^{\pm}(t', \eta_{\{1\}}, x, \theta, y)$, (5.7) and (5.28) are smooth, it is clear that for some $\delta_0 > 0$ the inequality

(5.34) $$|\det \partial(x^{\pm}_{\{2,3\}}(t', s, y, \eta_0(y)), \eta^{\pm}_{\{1\}}(t', s, y, \eta_0(y)))/\partial y| \geqq \delta_0$$

holds for $0 \leqq |t - t'| \leqq \varepsilon$, $y \in \Omega_\lambda$. Hence by (5.32), the matrix $(\mathrm{Hess}_{(x_{\{1\}}, \theta, y)} \psi^{\pm})(t', \eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y), y)$ is non-singular for $y \in \Omega_\lambda$, $0 \leqq |t - t'| \leqq \varepsilon$ and its index of inertia is constant there. Fix $y \in \Omega_\lambda$. Since the focal points on the trajectory $\{(x^{\pm}(u, s, y, \eta_0(y)), \xi^{\pm}(u, s, y, \eta_0(y))\}$ are discrete (cf. Morse [10], Theorem 17.4), we may assume that there exists a point $t - \varepsilon < \tau < t$ such that $y \in \Sigma^{\pm}(\tau, \phi)$. Take a neighborhood U_λ of y such that $U_\lambda \subset \Sigma^{\pm}(\tau, \phi) \cap \Omega_\lambda$, and a function $f \in C_0^\infty(U_\lambda)$. Then by the result for the case $K_\lambda = \phi$, $G^{\kappa}_{\pm}(\tau, s, \Delta)(e^{iS(y)/\kappa} f)(x)$ satisfies (1.15). Hence applying Theorem A.2 to the partial transform $\mathscr{F}^{\kappa}_{\{1\}}$ of the second term in the L.H.S. of (1.15), we have

(5.35) $\Big\| \mathscr{F}^{\kappa}_{\{1\}} G^{\kappa}_{\pm}(\tau, s; \Delta)(e^{iS(y)/\kappa} f)(x_{\{2,3\}}, \eta_{\{1\}})$
$$- |\det \partial(x^{\pm}_{\{2,3\}}(\tau, s, y, \eta_0(y)), \xi^{\pm}_{\{1\}}(\tau, s, y, \eta_0(y)))/\partial y|^{-1/2}$$
$$\times (p_0^{\pm}(\tau, s, y, \eta_0(y))/p_0^{\pm}(s, s, y, \eta_0(y)))^{-1/2}$$

$$\times \exp\Big(-\frac{i\pi}{2}\operatorname{Ind}\gamma^{\pm}(\tau, s, y, \eta_0(y))+\frac{i\pi}{4}-\frac{i\pi}{2}\operatorname{Inert}\frac{\partial x^{\pm}_{\{1\}}}{\partial \eta_{\{1\}}}(x^{\pm}_{\{2,3\}}(\tau, s, y, \eta_0(y)),$$
$$\xi^{\pm}_{\{1\}}(\tau, s, y, \eta_0(y)))+i(S^{\pm}(t, y, \eta, \eta_0(y))+S(y)-x^{\pm}_{\{1\}}(t, s, y, \eta_0(y))\cdot\eta_{\{1\}})/\hbar)$$
$$\times P^{\pm}(s, y, \eta_0(y))f(y)\Big|_{(x_{\{2,3\}}, \eta_{\{1\}})=(x^{\pm}_{\{2,3\}}(\tau, s, y, \eta_0(y)), \eta^{\pm}_{\{1\}}(\tau, s, y, \eta_0(y)))}\Big\|$$
$$\leqq C\hbar\|f\|_2.$$

Comparing (5.35) with the formulas (6.2) and (6.3) for the expression in the RHS of (5.27), we immediately see that

$$3N+\frac{1}{2}-\operatorname{Inert}(\operatorname{Hess}_{(x_{\{1\}}, \theta, y)}\psi^{\pm})(\tau, \eta^{\pm}_{\{1\}}(y), x^{\pm}(y), \theta^{\pm}(y), y) \tag{5.36}$$
$$\equiv -\operatorname{Ind}\gamma^{\pm}(\tau, s, y, \eta_0(y))-\operatorname{Inert}(\partial x_{\{1\}}/\partial\eta_{\{1\}})(x^{\pm}_{\{2,3\}}(\tau, s, y, \eta_0(y)),$$
$$\eta^{\pm}_{\{1\}}(\tau, s, y, \eta_0(y)))+\frac{1}{2} \pmod 4.$$

Since LHS of (5.36) is constant for $t-\varepsilon\leqq\tau\leqq t$ and

$$\operatorname{Ind}\gamma^{\pm}(t, s, y, \eta_0(y))-\operatorname{Ind}\gamma^{\pm}(\tau, s, y, \eta_0(y))$$
$$=\operatorname{Inert}(\partial x^{\pm}_{\{1\}}/\partial\eta_{\{1\}})(x^{\pm}_{\{2,3\}}(\tau), \eta^{\pm}_{\{1\}}(\tau))-\operatorname{Inert}(\partial x^{\pm}_{\{1\}}/\partial\eta_{\{1\}})(x^{\pm}_{\{2,3\}}(t), \eta^{\pm}_{\{1\}}(t))$$

by (5.34) (cf. Fujiwara [3], p. 124), we obtain (5.33). Q.E.D.

§6. Appendix.

We collect here two theorems about L^2-boundedness and asymptotic expansion of oscillatory integrals which are proved elsewhere. We are concerned with integral operators of the following form.

$$A(\nu)f(x)=(\nu/2\pi)^{(n+m)/2}\int_{R^n\times R^m}\exp(i\nu\phi(x, \theta, y))a(x, \theta, y)f(y)dyd\theta, \tag{6.1}$$

where the integral is taken in the sense of oscillatory integral (cf. Asada-Fujiwara [3]). We assume the following conditions.

(C.1) $a(x, \theta, y)\in B(\boldsymbol{R}^{n+m+n})$, $(x, \theta, y)\in \boldsymbol{R}^n\times\boldsymbol{R}^m\times\boldsymbol{R}^n$.

(C.2) $\phi(x, \theta, y)\in C^\infty(\boldsymbol{R}^{n+m+n})$ is real valued and there exists a constant $\varepsilon>0$ such that $|\det D(\phi)(x, \theta, y)|\geqq\varepsilon>0$, where

$$D(\phi)=\begin{bmatrix}\partial^2\phi/\partial x\partial y & \partial^2\phi/\partial x\partial\phi\\ \partial^2\phi/\partial\theta\partial y & \partial^2\phi/\partial\theta\partial\theta\end{bmatrix}.$$

(C.3) Any entry of $D(\phi)$ belongs to the class $B(\boldsymbol{R}^{n+m+n})$. We assume $n\geqq1$, but $m\geqq0$ and $m=0$ is *not excluded*.

We write

$$C(\phi)=\{(x, \theta, y)\in \boldsymbol{R}^{n+m+n}: \mathrm{grad}_\theta\, \phi(x, \theta, y)=0\}.$$

THEOREM A.1 (due to Asada-Fujiwara [3]). *Let $\phi(x, \theta, y)$ and $a(x, \theta, y)$ satisfy conditions* (C.1)-(C.3) *and $A(\nu)$ be defined by* (6.1). *Then the following statements hold.*

(1) *There exists a constant $C>0$ independent of $\nu\geqq 1$ and $f\in C_0^\infty(\boldsymbol{R}^n)$ such that*

$$\|A(\nu)f\|\leqq C\|f\|.$$

(2) *Suppose $a(x, \theta, y)=0$ for any $(x, \theta, y)\in C(\phi)$. Then there exists a constant $C>0$ independent of $\nu\geqq 1$ and $f\in C_0^\infty(\boldsymbol{R}^n)$ such that*

$$\|A(\nu)f\|\leqq C\nu^{-1}\|f\|.$$

(3) *Let $b(x', \theta', y')\in B(\boldsymbol{R}^{n+m+n})$ and $\psi(x', \theta', y')\in C^\infty(\boldsymbol{R}^{n+m'+n})$ satisfy conditions* (C.1)-(C.3) *and $B(\nu)f(x')$ be defined by* (6.1) *with obvious alternations. We set as $\tilde\theta=(\theta, z, \theta')\in \boldsymbol{R}^{n+m+n'}$, $c(x, \tilde\theta, y)=a(x, \theta, z)b(z, \theta', y)$ and $\tilde\phi(x, \tilde\theta, y)=\phi(x, \theta, z)+\psi(z, \theta', y)$. Then the functions $c(x, \tilde\phi, y)$ and $\tilde\phi(x, \tilde\theta, y)$ satisfies conditions* (C.1)-(C.3) *with changing m to $n+m+m'$ and $\varepsilon>0$ to some other constant $\varepsilon'>0$. Moreover*

$$A(\nu)B(\nu)f(x)=(\nu/2\pi)^{(2n+m+m')/2}\int_{R^{2n+m+m'}}\exp(i\nu\tilde\phi(x, \tilde\theta, y))c(x, \tilde\theta, y)f(y)dyd\tilde\theta. \tag{6.2}$$

Notice that condition (C.2) implies that the equation $\mathrm{grad}_{(\theta,y)}\phi(x, \theta, y)=0$ uniquely determines a smooth function $(x, \theta)=(x(y), \theta(y))$ globally on $\boldsymbol{R}^n$. Let $U\subset\boldsymbol{R}^n$ be open subset.

(C.4) The mapping $x=x(y)$ is a diffeomorphism on U and there exists a constant $\varepsilon_1>0$ such that

$$|\det \partial x/\partial y|\geqq\varepsilon_1, \quad y\in U.$$

THEOREM A.2 (cf. [16], §5). *Let conditions* (C.1)-(C.4) *be satisfied and let $K\Subset U$. Then there exists a constant $C>0$ independent of $f\in C_0^\infty(K)$ such that*

$$\|A(\nu)f(x)-A^c(\nu)f(x)\|\leqq C^{-1}\|f\|_2,$$

where

$$A^c(\nu)f(x)=\begin{cases}\exp(i(n+m)\pi/4-i\,\mathrm{Inert}(\mathrm{Hess}_{(\theta,y)}\phi)(x(y), \theta(y), y)/2+i\nu\phi(x(y), \theta(y), y)) \\ \quad\times|\det \mathrm{Hess}_{(\theta,y)}\phi(x(y), \theta(y), y)|^{-1/2}a(x(y), \theta(y), y)f(y) \\ \quad\textit{when } x=x(y),\ y\in U; \\ 0 \quad \textit{otherwise.}\end{cases} \tag{6.3}$$

Acknowledgement.

The author is indebted to Professor D. Fujiwara for stimulating discussions. He enjoyed the hospitality of the Institut für Theoretische Physik, ETH-Zürich, Department of Physics, Princeton University and Courant Institute for mathematical sciences at various stages of the work. He acknowledges partial support by Sakko-kai Foundation and the U.S. National Science Foundation under Grant No. MCS-78-01885.

References

[1] Abraham, R. and J. E. Marsden, Foundations of Mechanics, Second ed. Benjamin, Reading, Mass. 1978.

[2] Arnold, V. I., Mathematical Methods of Classical Mechanics, Springer, New York-Heiderberg-Berlin, 1978 (English translation from Russian).

[3] Asada, K. and D. Fujiwara, On some oscillatory integral transformations in $L^2(\boldsymbol{R}^n)$, Japan. J. Math. 4 (1978), 299-361.

[4] Chernoff, P., Essential selfadjointness of powers of generators of hyperbolic equations, J. Funct. Anal. 12 (1973), 401-414.

[5] Coddington, E. A. and N. Levinson, Theory of ordinary differential equations, McGraw-Hill, New York, 1955.

[6] Fujiwara, D., Asymptotic methods in the theory of partial defferential equations, I, II, in Monographs for elementary math., edited by K. Kodaira, Iwanami-shoten, Tokyo, 1977 (in Japanese).

[7] Kumano-go, H., A calculus of Fourier integral operators on $\boldsymbol{R}^n$ and the fundamental solution for an operator of hyperbolic type, Comm. Partial Differential Equations, 1 (1976), 1-44.

[8] Maslov, V. P., Théorie des perturbations et méthodes asymptotiques, Dunod, Paris, 1970 (French translation from Russian).

[9] Milnor, J., Morse theory, Ann. of Math. Studies 51, Princeton Univ. Press, Princeton, 1963.

[10] Morse, M., Analysis in the large, Lecture Notes, Inst. Adv. Study, Princeton, 1937.

[11] Pauli, W., Diracs Wellengleichung der Elektrons und geometrische Optik, Helv. Phys. Acta 5 (1932), 179-199.

[12] Plebanski, J. and J. Stachel, Classical particles with spin I, The WKBJ approximation, J. Math. Phys. 18 (1977), 2368-74.

[13] Rubinow, S. I. and J. B. Keller, Asymptotic solution of the Dirac equation, Phys. Rev. **131** (1963), 2789-2796.

[14] Schwartz, J. T., Non-linear functional analysis, Gordon and Breach, New York, 1969.

[15] Schwartz, L., Théorie des distributions, Dunod, Paris, 1974.

[16] Yajima, K., The quasi-classical limit of quantum scattering theory II, Long-range scattering, to appear in Duke Math. J.

(Received January 6, 1981)

Department of Mathematics
Faculty of Science
University of Tokyo
Hongo, Tokyo
113 Japan