Full-discrete finite element approximation of evolution equation $u_t+A(t)u=0$ of parabolic type

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1. Introduction

In the present paper, we make an error analysis of the full-discrete finite element method for the parabolic equation.

Let $\Omega\subset R^2$ be a bounded domain whose boundary $\partial\Omega$ is smooth, and let $-\mathscr{A}=-\mathscr{A}(t, x, D)$ be an elliptic differential operator of second order with smooth coefficients. We consider the parabolic equation

$$\frac{\partial}{\partial t}u+\mathscr{A}(t, x, D)u=0 \quad (0<t\leqq T,\ x\in\Omega) \tag{1.1}$$

with the boundary condition either

$$u=0 \quad (0<t\leqq T,\ x\in\partial\Omega) \tag{1.2}$$

or

$$\frac{\partial}{\partial\nu_A}u+\sigma u=0 \quad (0<t\leqq T,\ x\in\partial\Omega) \tag{1.2$'$}$$

and with the initial condition

$$u|_{t=0}=a(x) \quad (x\in\Omega). \tag{1.3}$$

We discretize the equation (1.1) with (1.2) (or (1.2′)) and with (1.3) as follows. For the space variable $x=(x_1, x_2)$ we adopt the finite element method, triangulating Ω regularly with the size parameter $h>0$. Adopting "piece-wise linear" trial functions, we firstly obtain the semidiscrete finite element approximation of u, denoted by $u_h=u_h(t)$. In the next place, we adopt the backward difference method with the size parameter $\tau>0$ with respect to the time variable t and obtain the full-discrete finite element approximation of u, denoted by $u_h^\tau=u_h^\tau(t)$ $(t=n\tau)$. For details, see § 2.

Our purpose is to estimate $\|u(t)-u_h^\tau(t)\|_{L^2(\Omega)}$. To this end we estimate $\|u(t)-u_h(t)\|_{L^2(\Omega)}$ and $\|u_h(t)-u_h^\tau(t)\|_{L^2(\Omega)}$, respectively. The estimate

$$\|u(t)-u_h(t)\|_{L^2(\Omega)}\leqq Ch^2/t\|a\|_{L^2(\Omega)} \tag{1.4}$$

has been obtained for the boundary condition (1.2) in Fujita-Suzuki [7]. A weaker estimate

$$\|u(t)-u_h(t)\|_{L^2(\Omega)} \leqq C_\delta (h^2/t)^{1-\delta} \|a\|_{L^2(\Omega)} \quad (0<\delta\leqq 1) \tag{1.4'}$$

was given for the boundary condition (1.2′) in Suzuki [18]. On the other hand the estimate

$$\|u_h(t)-u_h^\tau(t)\|_{L^2(\Omega)} \leqq C_\gamma (\tau/t)^{1-\gamma} \|a\|_{L^2(\Omega)} \quad (0<\gamma\leqq 1) \tag{1.5'}$$

was shown in Suzuki [19]. In the present paper we show that (1.5′) can be improved as

$$\|u_h(t)-u_h^\tau(t)\|_{L^2(\Omega)} \leqq C\tau/t \|a\|_{L^2(\Omega)}. \tag{1.5}$$

Furthermore, we show that (1.4) holds even under the boundary condition (1.2′), which gives our final estimate

$$\|u(t)-u_h^\tau(t)\|_{L^2(\Omega)} \leqq C(h^2/t+\tau/t) \|a\|_{L^2(\Omega)}. \tag{1.6}$$

Fujita-Mizutani [6] showed (1.6) in the case that $A(t)$ is independent of t, where $A(t)$ is the realization in $L^2(\Omega)$ of the operator $A(t, x, D)$ with the boundary condition (1.2) or (1.2′). Baker-Bramble-Thomée [3] also showed (1.6) in that case, assuming that $A(t)\equiv A$ is self-adjoint. Sammon [15] showed (1.6) under the boundary condition (1.2), assuming $A(t)$ to be self-adjoint. For other references, see Suzuki [18, 19, 20].

The present paper is composed of six sections. In § 2, we shall take some preliminaries. Sections 3, 4 and 5 are devoted to the proof of (1.5). The proof of (1.4) for the boundary condition (1.2′) is given in § 6.

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2. Preliminaries

Let Ω be a bounded domain with boundary $\partial\Omega$ of C^2-class, and let $-\mathscr{A}=-\mathscr{A}(t, x, D)$ $(0\leqq t\leqq T,\ x\in\bar{\Omega})$ be an elliptic differential operator of second order:

$$\begin{aligned} -\mathscr{A} &= -\mathscr{A}(t, x, D) \\ &= \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{ij}(t, x) \frac{\partial}{\partial x_j} + \sum_{j=1}^{2} b_j(t, x) \frac{\partial}{\partial x_j} + c(t, x). \end{aligned} \tag{2.1}$$

Uniform ellipticity

$$\mathrm{Re}\sum_{i,j=1}^{2} a_{ij}(t, x)\zeta_i\bar{\zeta}_j \geqq \delta'|\zeta|^2_{C^2} \quad (\zeta=(\zeta_1, \zeta_2)\in C^2) \tag{2.2}$$

is assumed, δ' being a positive constant. As we stated in §1, we consider the parabolic equation

$$\frac{\partial}{\partial t}u+\mathscr{A}(t, x, D)u=0 \quad (0<t\leqq T,\ x\in\Omega) \tag{2.3}$$

with the boundary condition either

$$u=0 \quad (0<t\leqq T,\ x\in\partial\Omega) \tag{2.4}$$

or

$$\begin{aligned}\mathscr{B}u&=\mathscr{B}(t, x, D)u\\ &\equiv\frac{\partial}{\partial\nu_A}u+\sigma u=0 \quad (0<t\leqq T,\ x\in\partial\Omega)\end{aligned} \tag{2.4$'$}$$

and with the initial condition

$$u|_{t=0}=a(x) \quad (x\in\Omega). \tag{2.5}$$

In (2.4′), $\sigma=\sigma(t, x)$ is a given function on $[0, T]\times\partial\Omega$ and $\partial/\partial\nu_A$ means the differentiation along the outer conormal vector ν_A:

$$\frac{\partial}{\partial\nu_A}=\sum_{i,j=1}^{2} n_i a_{ij}(t, x)\frac{\partial}{\partial x_j}, \tag{2.6}$$

where $n=(n_1, n_2)$ is the outer unit normal to $\partial\Omega$. In what follows, the standard norm in $H^j(\Omega)$ $(j=0, 1, 2, \cdots)$, the Sobolev space $W^{2,j}(\Omega)$ of order j, is denoted by $\|\cdot\|_j$ and various generic constants are denoted indifferently by C. If C depends on some parameters, say $\alpha, \beta, \cdots$, we shall denote it by $C_{\alpha,\beta,\ldots}$. However, sometimes we write C in stead of C_T.

Let $V=H_0^1(\Omega)$ or $H^1(\Omega)$ according to the boundary condition (2.4) or (2.4′), where $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, the C^∞-functions with compact support in Ω. We consider the following sesquilinear form $a_t(\ ,\ )$ on $V\times V$:

$$\begin{aligned}a_t(u, v)=&\sum_{i,j=1}^{2}\int_\Omega a_{ij}(t, x)\frac{\partial}{\partial x_j}u\overline{\frac{\partial}{\partial x_i}v}\,dx-\sum_{i,j=1}^{2}\int_\Omega b_j(t, x)\frac{\partial}{\partial x_j}u\cdot\bar{v}\,dx\\ &-\int_\Omega c(t, x)u\cdot\bar{v}\,dx+\int_{\partial\Omega}\sigma(t, \xi)u\cdot\bar{v}\,dS(\xi) \quad (u, v\in V).\end{aligned} \tag{2.7}$$

Then an m-sectorial operator $A(t)$ in $X=L^2(\Omega)$ associated with $a_t(\ ,\ )$ is defined through the identity

(2.8) $$a_t(u, v)=(A(t)u, v) \quad (u \in D(A(t)) \subset V, v \in V),$$

where (,) is the L^2-inner product. See, Kato [12] for instance. Assuming $a_{ij}(t, \cdot) \in C^1(\bar{\Omega})$, $b_j(t, \cdot) \in L^\infty(\Omega)$, $c(t, \cdot) \in L^\infty(\Omega)$, $\sigma(t, \cdot) \in C^0(\partial\Omega)$ $(0 \leqq t \leqq T)$ and $a \in L^2(\Omega)$, we can reduce the equation (2.3) with (2.4) (or (2.4')) and with (2.5) to the evolution equation

(2.9) $$\frac{du}{dt}+A(t)u=0 \quad (0<t \leqq T)$$

in X with

(2.10) $$u(0)=a.$$

In fact we have

(2.11) $$|a_t(u, v)| \leqq C\|u\|_1\|v\|_1 \quad (u, v \in V)$$

and

(2.12') $$\operatorname{Re} a_t(u, u) \geqq \delta\|u\|_1^2-\lambda_1\|u\|_0^2 \quad (u \in V)$$

with constants $C, \delta>0$ and $\lambda_1 \in R$. Also we note that the equality

$$D(A(t))=\{u \in H^2(\Omega) \mid u_{|\partial\Omega}=0\}$$

holds if $V=H_0^1(\Omega)$ and the equality

$$D(A(t))=\{u \in H^2(\Omega) \mid \mathscr{B}(t, x, D)u_{|\partial\Omega}=0\}$$

holds if $V=H^1(\Omega)$. See Agmon [1] and Lions-Magenes [14], for example. See also Tanabe [22]. From the view point of our problem, however, we may consider $A(t)+\lambda_1$ in stead of $A(t)$ by considering $v(t)=e^{-\lambda_1 t}u(t)$ in stead of $u(t)$. Therefore, we shall assume

(2.12) $$\operatorname{Re} a_t(u, u) \geqq \delta\|u\|_1^2 \quad (u \in V)$$

in stead of (2.12') without loss of generality, hereafter.

We now assume that $a_{ij}(t, x)$, $b_j(t, x)$, $c(t, x)$ and $\sigma(t, x)$ are uniformly Hölder continuous in t with the exponent θ. Hence we have

(2.13) $$|a_t(u, v)-a_s(u, v)| \leqq C|t-s|^\theta\|u\|_1\|v\|_1 \quad (t, s \in [0, T];\ u, v \in V).$$

Assuming $\theta>1/2$, the inequalities (2.11), (2.12) and (2.13) assure us of the existence of evolution operators: $X \rightarrow X$ of C^1-class denoted by $\{U(t, s) \mid T \geqq t \geqq s \geqq 0\}$ of which generator is $A(t)$, by virtue of the generation theorem of Y. Fujie and H. Tanabe

(Fujie-Tanabe [4]). Namely, the continuously differentiable solution $u=u(t)$ of the equation

$$\frac{du}{dt}+A(t)u=0 \quad (s<t\leqq T) \tag{2.9'}$$

with

$$u(s)=a \tag{2.10'}$$

is given by

$$u(t)=U(t, s)a. \tag{2.14'}$$

Therefore, the solution $u=u(t)$ of (2.9) with (2.10) is given by

$$u(t)=U(t, 0)a. \tag{2.14}$$

Furthermore, the inequalities

$$\|U(t, s)\|\leqq C \quad (T\geqq t\geqq s\geqq 0) \tag{2.15}$$

and

$$\|A(t)U(t, s)\|\leqq C(t-s)^{-1} \quad (T\geqq t>s\geqq 0) \tag{2.16}$$

hold.

REMARK. The generation theorem of Kato [10] or Sobolevskii [17], which is based on the assumption of the Hölder continuity in t of fractional powers of $A(t)$ in a certain sense, is also applicable. In fact the inequalities (2.11), (2.12) and (2.13) yield

$$\|A(t)^{\rho}A(s)^{-\rho}-1\|\leqq C_{\rho}|t-s|^{\theta} \quad (T\geqq t, s\geqq 0) \tag{2.17}$$

for each ρ in $0<\rho<1/2$, by a theorem due to T. Kato (Kato [11]). The generation theorem of Kato-Tanabe [13] is also applicable under some more assumptions on the smoothness in t of a_{ij}, b_j, c and σ, and is made use of in later sections in deriving the estimate (1.6). See, Proposition 4.1. When we consider the boundary condition (2.4), we can apply the generation theorem of Tanabe [21] or Sobolevskii [16]. In fact we make an integration by parts in (2.7) and obtain

$$\|A(t)A(s)^{-1}-1\|\leqq C|t-s|^{\theta} \quad (T\geqq t, s\geqq 0), \tag{2.18}$$

by virtue of the elliptic estimate of $A(t)$ due to Agmon-Douglis-Nirenberg [2]. In this case, the assumption $1\geqq\theta>1/2$ can be weakened as $1\geqq\theta>0$.

Now we proceed to the discretization of the equation (2.9) with (2.10). When

Ω is a convex polygon, we triangulate it regularly with the size parameter $h>0$ and put

$$V_h = \text{all functions in } V \text{ which are linear in each element.}$$

In our case, however, since $\partial\Omega$ is curved, we must modify V_h in a certain way. See, Zlámal [24], in this connection. Anyway, we can construct a finite dimensional space V_h contained in V, and V_h satisfies

$$\inf_{\chi \in V_h} \|\chi - v\|_1 \leqq Ch\|v\|_2 \quad (v \in V \cap H^2(\Omega)) \tag{2.19}$$

in virtue of the theorem of M. Zlámal (Zlámal [24]). We restrict $a_t(\ ,\ )$ to $V_h \times V_h$ and get an m-sectorial operator $A_h(t)$: $V_h \to V_h$ through the identity

$$a_t(u, v) = (A_h(t)u, v) \quad (u, v \in V_h). \tag{2.20}$$

Let P_h: $X \to V_h$ be the L^2-orthogonal projection. The equation

$$\frac{du_h}{dt} + A_h(t)u_h = 0 \quad (0 \leqq t \leqq T) \tag{2.21}$$

in V_h with

$$u_h(0) = P_h a \tag{2.22}$$

can be regarded as an approximation of (2.9) with (2.10), and the solution $u_h = u_h(t) \in V_h$ is called the "semi-discrete finite element approximation" of u. The generation theorem of Y. Fujie and H. Tanabe is also applicable to $A_h(t)$ and we obtain

$$u_h(t) = U_h(t, 0)P_h a, \tag{2.23}$$

where $\{U_h(t, s) \mid T \geqq t \geqq s \geqq 0\}$ is the family of evolution operators: $V_h \to V_h$, generated by $A_h(t)$. Furthermore, we have

$$\|U_h(t, s)\| \leqq C \tag{2.24}$$

and

$$\|A_h(t)U_h(t, s)\| \leqq C(t-s)^{-1} \tag{2.25}$$

by re-examining their theorem. In the next place, we discretize the equation (2.21) with (2.22) with respect to the time variable t and consider the equation

$$(u_h^\tau(t+\tau) - u_h^\tau(t))/\tau + A_h(t+\tau)u_h^\tau(t+\tau) = 0 \quad (t = n\tau;\ n = 0, 1, \cdots, N) \tag{2.26}$$

with

(2.27) $$u_h^\tau(0)=P_h a,$$

$\tau>0$ being a small parameter with $\tau N=T$. Let $t_n=n\tau$. Then, we have

(2.28) $$u_h^\tau(t)=U_h^\tau(t, 0)P_h a \quad (t=t_n;\ n=0, 1, \cdots, N),$$

where

(2.29) $$U_h^\tau(t_n, t_j)=\begin{cases}(1+\tau A_h(t_n))^{-1}\ (1+\tau A_h(t_{n-1}))^{-1}\cdots(1+\tau A_h(t_{j+1}))^{-1} & (n>j)\\ 1 & (n=j).\end{cases}$$

$u_h^\tau=u_h^\tau(t)\in V_h$ $(t=t_n)$ is called the "backward difference full-discrete finite element approximation" of u.

As is stated in § 1, we shall give the estimate (1.6) in later sections. To this end, we add more assumptions on the smoothness of a_{ij}, b_j, c and σ. Namely, we assume that they are continuously differentiable in t with uniformly Hölder continuous derivatives whose exponent is α $(1\geqq\alpha>0)$. Furthermore, we assume $b_j(t, \cdot)\in C^1(\bar{\Omega})$ in order that the adjoint operator $A(t)^*$ of $A(t)$ is nicely defined. Note that from these assumptions follows $\theta=1$ in (2.13) and (2.17). Then, the inequalities (1.4) and (1.5) are claimed in the form of the following Theorems 1 and 2, and will be proved in § 6 and § 5, respectively.

THEOREM 1. *Under the assumptions stated above, the estimate*

(2.30) $$\|U(t, s)-U_h(t, s)P_h\|\leqq Ch^2/(t-s) \quad (T\geqq t>s\geqq 0)$$

holds.

THEOREM 2. *Under the assumptions stated above, the estimate*

(2.31) $$\|U_h(t, 0)-U_h^\tau(t, 0)\|\leqq C(\tau/t+\tau^\alpha) \quad (t=t_n)$$

holds, $C>0$ being independent of h.

Indeed, the estimate (1.6) follows from (2.30) and (2.31), if we assume that a_{ij}, b_j, c and σ are so smooth that $\alpha=1$ holds.

Before concluding this section, we state the following two lemmas for the proof of Theorem 2.

LEMMA 1. *Under the same assumptions as in Theorems, the inequality*

(2.32) $$\|U_h^\tau(t_n, t_j)A_h(t_{j+1})^\beta\|\leqq C_\beta(t_n-t_j)^{-\beta} \quad (n-j>\beta)$$

holds for each β in $0\leqq\beta<4/3$.

LEMMA 2. *Under the same assumptions as in Theorems, the equality*

(2.33) $$A_h(t)U_h(t,s)-A_h(r)U_h(r,s)=A_h(t)[e^{-(t-s)A_h(t)}-e^{-(r-s)A_h(t)}]+A_h(t)^\beta Z_{h,\beta}(t,r,s)$$

holds with

(2.34) $$\|Z_{h,\beta}(t,r,s)\|\leqq C_\beta(t-r)^\alpha(r-s)^{\beta-1}\quad (T\geqq t>r>s\geqq 0)$$

for each β *in* $0<\beta<1/2$.

Lemmas 1 and 2 are proved in § 3 and § 4, respectively.

3. Proof of Lemma 1

It is easy to see that (2.32) is reduced to

(3.1) $$\|A_h(t_n)^\beta U_h^\tau(t_n,t_j)\|\leqq C_\beta(t_n-t_j)^{-\beta}\quad (0\leqq\beta<4/3),$$

by considering the adjoint operator of $U_h^\tau(t_n,t_j)A_h(t_{j+1})^\beta$. On the other hand, in the same way as in (2.17), the inequality (2.13) yields

(3.2) $$\|A_h(t)^\rho A_h(s)^{-\rho}-1\|\leqq C_\rho|t-s|^\theta\quad (T\geqq t,s\geqq 0)$$

for each ρ in $0<\rho<1/2$ by means of the theorem in Kato [11]. Therefore, the following abstract Lemma 1′ gives our Lemma 1, when applied for $X=V_h$, $A=A_h$, $U^\tau=U_h^\tau$, $\theta=1$ and $m=3$.

LEMMA 1′. *Let X be a Banach space, and let $A(t)$ be a bounded operator in X which satisfies the following* (A0) *and* (A1).

(A0) *The relation*

$$\rho(A(t))\supset G_1\equiv\{z\in C\mid |\arg z|>\omega_1\}\cup\{0\}\quad (0<\omega_1<\pi/2)$$

and the inequality

(3.3) $$\|(\lambda-A(t))^{-1}\|\leqq M/(1+|\lambda|)\quad (\lambda\in G_1)$$

hold where $\rho(A(t))$ is the resolvent set of $A(t)$.

(A1) *The inequality*

(3.4) $$\|A(t)^\rho A(s)^{-\rho}-1\|\leqq L|t-s|^\theta\quad (T\geqq t,\ s\geqq 0)$$

holds for $\rho=1/m$ and $\rho+\theta>1$, where m is a positive integer.

Furthermore, let $\tau>0$ be a small parameter with $\tau N=T$ and put

(3.5) $$U^\tau(t_n,t_j)=\begin{cases}(1+\tau A(t_n))^{-1}\cdots(1+\tau A(t_{j+1}))^{-1} & (n>j)\\ 1 & (n=j),\end{cases}$$

where $t_n=n\tau$. Then the inequality

(3.6) $$\|A(t_n)^\beta U^\tau(t_n,t_j)\|\leqq C_\beta(t_n-t_j)^{-\beta}\quad (n-j>\beta)$$

holds for each β *in* $0 \leqq \beta < \theta + \rho$, C_β *being a positive constant depending only on* ω_1, M, L, ρ, θ, β *and* T.

In this section, we shall prove Lemma 1′, which is a discrete version of a theorem by T. Kato. Indeed, he showed

$$\|A(t)^\beta U(t, s)\| \leqq C_\beta (t-s)^{-\beta} \tag{3.6′}$$

for each β in $0 \leqq \beta < \theta + \rho$ (Kato [10]).

Put

$$D(t, s) = A(t)^\rho A(s)^{-\rho} - 1. \tag{3.7}$$

Then, Sobolevskii's identity

$$A(t) - A(s) = \sum_{p=1}^{m} A(t)^{1-p\rho} D(t, s) A(s)^{p\rho} \tag{3.8}$$

gives

$$\begin{aligned} U^\tau(t_n, t_j) &- (1+\tau A(t_n))^{-(n-j)} \\ &= \sum_{k=j}^{n-1} [(1+\tau A(t_n))^{-(n-k-1)} U^\tau(t_{k+1}, t_j) - (1+\tau A(t_n))^{-(n-k)} U^\tau(t_k, t_j)] \\ &= \sum_{k=j}^{n-1} (1+\tau A(t_n))^{-(n-k)} [(1+\tau A(t_n)) - (1+\tau A(t_{k+1}))] U^\tau(t_{k+1}, t_j) \\ &= \tau \sum_{k=j}^{n-1} (1+\tau A(t_n))^{-(n-k)} [A(t_n) - A(t_{k+1})] U^\tau(t_{k+1}, t_j) \\ &= \sum_{p=1}^{m} \tau \sum_{k=j}^{n-2} (1+\tau A(t_n))^{-(n-k)} A(t_n)^{1-p\rho} D(t_n, t_{k+1}) A(t_{k+1})^{p\rho} U^\tau(t_{k+1}, t_j). \end{aligned} \tag{3.9}$$

Now we introduce a few notations. Let $K_i(t_n, t_j)$ $(i=1, 2)$ be operator-valued functions defined on $D = D^\tau = \{(t_n, t_j) \mid N \geqq n \geqq j \geqq 0\}$. We define another operator-valued function $K = K_1 * K_2 = K_1 \overset{\tau}{*} K_2$ on $D = D^\tau$ by

$$\begin{aligned} (K_1 * K_2)(t_n, t_j) &= (K_1 \overset{\tau}{*} K_2)(t_n, t_j) \\ &= \tau \sum_{k=j}^{n-2} K_1(t_n, t_{k+1}) K_2(t_{k+1}, t_j). \end{aligned} \tag{3.10}$$

We put

$$W^\tau(t_n, t_j) = U^\tau(t_n, t_j) - (1+\tau A(t_n))^{-(n-j)} \tag{3.11}$$

and

$$Y_q^\tau(t_n, t_j) = A(t_n)^{q\rho} W^\tau(t_n, t_j). \tag{3.12}$$

Then, (3.9) gives

$$Y_q^\tau = \sum_{p=1}^{m} H_{q,p}^\tau \overset{\tau}{*} Y_p^\tau + Y_{q,0}^\tau \tag{3.13}$$

where

$$H_{q,p}^\tau(t_n, t_j) = A(t_n)^{1-p\rho+q\rho}(1+\tau A(t_n))^{-(n-j+1)} D(t_n, t_j) \tag{3.14}$$

and

$$Y_{q,0}^\tau = \sum_{p=1}^{m} H_{q,p}^\tau \overset{\tau}{*} Y_{p,-1}^\tau \tag{3.15}$$

with

$$Y_{p,-1}^\tau(t_n, t_j) = A(t_n)^{p\rho}(1+\tau A(t_n))^{-(n-j)}. \tag{3.16}$$

Therefore, Y_q^τ $(1 \leqq q \leqq m)$ satisfies

$$Y_q^\tau = \sum_{i=0}^{\infty} Y_{q,i}^\tau, \tag{3.17}$$

where

$$Y_{q,i+1}^\tau = \sum_{p=1}^{m} H_{q,p}^\tau \overset{\tau}{*} Y_{p,i}^\tau \quad (i=0, 1, \cdots). \tag{3.18}$$

In order to give estimates on $\|Y_{q,i}^\tau\|$, we make the following

DEFINITION. An operator-valued function K on D^τ is said to belong to $Q^\tau(a, M)$ if

$$\|K(t_n, t_j)\| \leqq M(t_n - t_j)^{a-1} \quad (N \geqq n \geqq j \geqq 1) \tag{3.19}$$

holds.

The following elementary proposition is useful throughout the present paper.

PROPOSITION 3.1. *Let $0<a\leqq b$ and $a \leqq 1$. Then the inequality*

$$B^N(a, b) \equiv \frac{1}{N} \sum_{k=1}^{N-1} \left(1-\frac{k}{N}\right)^{a-1} \left(\frac{k}{N}\right)^{b-1} \leqq B(a, b) \tag{3.20}$$

$$\equiv \int_0^1 (1-x)^{a-1} x^{b-1}\, dx$$

holds.

PROOF. If $b \geqq 1 \geqq a$, then $f(x)=(1-x)^{a-1}x^{b-1}$ is monotonously increasing in $[0, 1)$, which implies (3.20). If $a, b \leqq 1$, then $f(x)$ is convex in $(0, 1)$, which implies (3.20). □

From now on, we drop the suffix τ for simplicity.

PROPOSITION 3.2. *Let $K_i \in Q(a_i, M_i)$ $(i=1,2)$ with $0<a_1\leqq 1$ and $0<a_2$. Then we have $K_1 * K_2 \in Q(a_1+a_2, M_1M_2B(a_1, a_2))$.*

PROOF. By Proposition 3.1, we have

$$\begin{aligned}\|(K_1*K_2)(t_n, t_j)\| &\leqq \tau \sum_{k=j}^{n-2} \|K_1(t_n, t_{k+1})\| \cdot \|K_2(t_{k+1}, t_j)\| \\ &\leqq M_1M_2\tau \sum_{k=j}^{n-2} (t_n-t_{k+1})^{a_1-1}(t_{k+1}-t_j)^{a_2-1} \\ &\leqq M_1M_2\tau^{a_1+a_2-1}(n-j)^{a_1+a_2-1}B^{n-j}(a_1, a_2) \\ &\leqq M_1M_2(t_n-t_j)^{a_1+a_2-1}B(a_1, a_2) \quad (\because (3.20)).\end{aligned}$$ □

PROPOSITION 3.3. *The relation*

$$H_{q,p} \in Q(\theta-q\rho+p\rho, C_{q,p}) \tag{3.21}$$

holds if $0\leqq 1-p\rho+q\rho<2$.

PROOF. Fujita-Mizutani [6] showed

$$\|A(t)^\beta(1+\tau A(t))^{-m}\| \leqq C_\beta(m\tau)^{-\beta} \quad (0\leqq t\leqq T) \tag{3.22}$$

for $m>\beta\geqq 0$, which will be proved in another way at the end of this section. Hence we get

$$\begin{aligned}\|H_{q,p}(t_n, t_j)\| &\leqq \|A(t_n)^{1-p\rho+q\rho}(1+\tau A(t_n))^{-(n-j+1)}\| \cdot \|D(t_n, t_j)\| \\ &\leqq C_{q,p}(t_n-t_j)^{-1+p\rho-q\rho+\theta} \quad (\because (3.4)).\end{aligned}$$ □

PROPOSITION 3.4. *The inequality*

$$\|H_{q,m}(t_n, t_{k+1})-H_{q,m}(t_n, t_j)\| \leqq C_q(t_n-t_{k+1})^{-q\rho}(t_{k+1}-t_j)^\theta \quad (N\geqq n>k+1>j\geqq 0) \tag{3.23}$$

holds if $0\leqq q\rho<2$.

PROOF. We have

$$\begin{aligned} &H_{q,m}(t_n, t_{k+1})-H_{q,m}(t_n, t_j) \\ &=A(t_n)^{q\rho}(1+\tau A(t_n))^{-(n-k)}D(t_n, t_{k+1})-A(t_n)^{q\rho}(1+\tau A(t_n))^{-(n-j+1)}D(t_n, t_j) \\ &=A(t_n)^{q\rho}[(1+\tau A(t_n))^{-(n-k)}-(1+\tau A(t_n))^{-(n-j+1)}]D(t_n, t_{k+1}) \\ &\quad +A(t_n)^{q\rho}(1+\tau A(t_n))^{-(n-j+1)}[D(t_n, t_{k+1})-D(t_n, t_j)].\end{aligned} \tag{3.24}$$

The following inequality is proved at the end of this section for $0\leqq\beta<j\leqq n\leqq N$ and $0\leqq\alpha\leqq 1$:

$$\begin{aligned}&\|A(t)^\beta[(1+\tau A(t))^{-n}-(1+\tau A(t))^{-j}]\| \\ &\leqq C_\beta(t_n-t_j)^\alpha t_j^{-\alpha-\beta}=C_\beta\tau^{-\beta}(n-j)^\alpha j^{-\alpha-\beta} \quad (0\leqq t\leqq T).\end{aligned} \tag{3.25}$$

By means of (3.25), we get

$$\begin{aligned}(3.26)\qquad &\|A(t_n)^{q\rho}[(1+\tau A(t_n))^{-(n-k)}-(1+\tau A(t_n))^{-(n-j+1)}]\|\cdot\|D(t_n, t_{k+1})\| \\ &\leqq C_q(\tau(n-j+1)-\tau(n-k))^{\theta}(\tau(n-k-1))^{-q\rho-\theta}(t_n-t_{k+1})^{\theta} \\ &=C_q(t_{k+1}-t_j)^{\theta}(t_n-t_{k+1})^{-q\rho}.\end{aligned}$$

On the other hand, the identity

$$(3.27)\qquad D(t, r)-D(t, s)=A(t)^{\rho}A(s)^{-\rho}D(s, r)$$

yields

$$\begin{aligned}(3.28)\qquad &\|A(t_n)^{q\rho}(1+\tau A(t_n))^{-(n-j+1)}\|\cdot\|D(t_n, t_{k+1})-D(t_n, t_j)\| \\ &\leqq C_q((n-j+1)\tau)^{-q\rho}(t_{k+1}-t_j)^{\theta}\quad(\because\ (3.4),\ (3.22)) \\ &\leqq C_q(t_n-t_{k+1})^{-q\rho}(t_{k+1}-t_j)^{\theta}.\end{aligned}$$

(3.24) combined with (3.26) and (3.28) gives (3.23). □

PROPOSITION 3.5. *We have*

$$(3.29)\qquad Y_{q,0}\in Q(1+\theta-q\rho, C_q)$$

with a constant $C_q>0$ *if* $0\leqq q\rho<\theta+\rho$.

PROOF. By (3.22), $Y_{p,-1}\in Q(1-p\rho, C_p)$ holds. Therefore, by Propositions 3.3 and 3.2, we obtain

$$(3.30)\qquad H_{q,p}*Y_{p,-1}\in Q(1+\theta-q\rho, C_q)$$

for $p=1, \cdots, m-1$, because of $\theta-q\rho+p\rho>0$ $(1\leqq p\leqq m)$ and $1-p\rho>0$ $(1\leqq p\leqq m-1)$.

On the other hand, we have

$$\begin{aligned}(3.31)\qquad &H_{q,m}*Y_{m,-1}(t_n, t_j) \\ &=\tau\sum_{k=j}^{n-2}H_{q,m}(t_n, t_{k+1})A(t_{k+1})(1+\tau A(t_{k+1}))^{-(k+1-j)} \\ &=\tau\sum_{k=j}^{n-2}H_{q,m}(t_n, t_{k+1})[A(t_{k+1})(1+\tau A(t_{k+1}))^{-(k+1-j)}-A(t_j)(1+\tau A(t_j))^{-(k+1-j)}] \\ &\quad+\tau\sum_{k=j}^{n-2}[H_{q,m}(t_n, t_{k+1})-H_{q,m}(t_n,t_j)]A(t_j)(1+\tau A(t_j))^{-(k+1-j)} \\ &\quad+\tau\sum_{k=j}^{n-2}H_{q,m}(t_n, t_j)A(t_j)(1+\tau A(t_j))^{-(k+1-j)}.\end{aligned}$$

The following inequality is proved at the end of this section for $0\leqq\beta\leqq1$ and $n=1, 2, \cdots$:

$$(3.32)\qquad \|A(t)^{\beta}(1+\tau A(t))^{-n}-A(s)^{\beta}(1+\tau A(s))^{-n}\|\leqq C|t-s|^{\theta}(n\tau)^{-\beta}\quad(0\leqq t, s\leqq T).$$

Therefore, the first term of the right hand side of (3.31) is estimated as

$$(3.33)\qquad \tau \sum_{k=j}^{n-2} \|H_{q,m}(t_n, t_{k+1})\| \cdot \|A(t_{k+1})(1+\tau A(t_{k+1}))^{-(k+1-j)} - A(t_j)(1+\tau A(t_j))^{-(k+1-j)}\|$$
$$\leqq C_q\tau \sum_{k=j}^{n-2} (t_n - t_{k+1})^{-q\rho+\theta}(t_{k+1}-t_j)^{\theta}((k+1-j)\tau)^{-1} \quad (\because\ (3.21),\ (3.32))$$
$$\leqq C_q(t_n-t_j)^{-q\rho+2\theta} \quad (\because\ (3.20))$$
$$\leqq C_q(t_n-t_j)^{-q\rho+\theta}.$$

In the next place, we note

$$(3.34)\qquad (t_n-t_j)\cdot\tau \sum_{k=j}^{n-2} [H_{q,m}(t_n, t_{k+1}) - H_{q,m}(t_n, t_j)]A(t_j)(1+\tau A(t_j))^{-(k+1-j)}$$
$$= \tau \sum_{k=j}^{n-2} (t_n-t_{k+1})[H_{q,m}(t_n, t_{k+1}) - H_{q,m}(t_n, t_j)]A(t_j)(1+\tau A(t_j))^{-(k+1-j)}$$
$$+ \tau \sum_{k=j}^{n-2} (t_{k+1}-t_j)[H_{q,m}(t_n, t_{k+1}) - H_{q,m}(t_n, t_j)]A(t_j)(1+\tau A(t_j))^{-(k+1-j)}.$$

By Proposition 3.4 we have

$$(3.35)\qquad \tau \sum_{k=j}^{n-2} (t_n-t_{k+1})\|[H_{q,m}(t_n, t_{k+1}) - H_{q,m}(t_n, t_j)]\| \cdot \|A(t_j)(1+\tau A(t_j))^{-(k+1-j)}\|$$
$$\leqq C_q\tau \sum_{k=j}^{n-2} (t_n-t_{k+1})^{1-q\rho}(t_{k+1}-t_j)^{\theta-1}$$
$$\leqq C_q(t_n-t_j)^{1-q\rho+\theta} \quad (\because\ (3.20)),$$

and by Proposition 3.3 we have

$$(3.36)\qquad \tau \sum_{k=j}^{n-2} (t_{k+1}-t_j)(\|H_{q,m}(t_n, t_{k+1})\| + \|H_{q,m}(t_n, t_j)\|) \cdot \|A(t_j)(1+\tau A(t_j))^{-(k+1-j)}\|$$
$$\leqq C_q\tau \sum_{k=j}^{n-2} \{(t_n-t_{k+1})^{-q\rho+\theta} + (t_n-t_j)^{-q\rho+\theta}\}$$
$$\leqq C_q(t_n-t_j)^{-q\rho+\theta+1} \quad (\because\ (3.20)).$$

Therefore, the second term of the right hand side of (3.31) is estimated as

$$(3.37)\qquad \left\|\tau \sum_{k=j}^{n-2} [H_{q,m}(t_n, t_{k+1}) - H_{q,m}(t_n, t_j)]A(t_j)(1+\tau A(t_j))^{-(k+1-j)}\right\| \leqq C_q(t_n-t_j)^{-q\rho+\theta}.$$

Finally, we have

$$(3.38)\qquad \left\|\tau \sum_{k=j}^{n-2} H_{q,m}(t_n, t_j)A(t_j)(1+\tau A(t_j))^{-(k+1-j)}\right\|$$
$$= \left\|H_{q,m}(t_n, t_j) \sum_{k=j}^{n-2} [(1+\tau A(t_j))^{-(k-j)} - (1+\tau A(t_j))^{-(k+1-j)}]\right\|$$
$$= \|H_{q,m}(t_n, t_j)[1-(1+\tau A(t_j))^{-(n-1-j)}]\|$$
$$\leqq \|H_{q,m}(t_n, t_j)\|(1+\|(1+\tau A(t_j))^{-(n-1-j)}\|)$$
$$\leqq C_q(t_n-t_j)^{-q\rho+\theta} \quad (\because\ (3.21),\ (3.22)).$$

Thus, (3.31) combined with (3.33), (3.37) and (3.38) gives

(3.39) $$H_{q,m} * Y_{m,-1} \in Q(1+\theta-q\rho, C_q).$$

We now recall (3.15) and obtain (3.29) by means of (3.30) and (3.39). □

PROPOSITION 3.6. *The relation*

(3.40) $$Y_q \in Q(\theta-q\rho+1, C) \quad (q=1, \cdots, m)$$

holds.

PROOF. From (3.29), (3.21), (3.18) and Proposition 3.1, the relation

(3.41) $$Y_{q,i} \in Q(1+(i+1)\theta-q\rho, L_i) \quad (q=1, \cdots, m)$$

follows for $i=0, 1, 2, \cdots$ by an induction, where

(3.42) $$L_{i+1}/L_i = CmB(\theta-1+\rho, (i+1)\theta).$$

We obtain (3.40) by (3.17) and (3.41). □

PROOF OF LEMMA 1′. We first show

(3.40′) $$Y_q \in Q(\theta-q\rho+1, C_q)$$

for each q in $0 \leqq q\rho < \theta+\rho$ which is not necessarily an integer. In fact, by (3.40) and (3.21) we have

(3.43) $$H_{q,p} * Y_p \in Q(\theta-q\rho+1, C_q) \quad (p=1, \cdots, m),$$

provided that $\theta-q\rho+p\rho>0$ $(1 \leqq p \leqq m)$. Therefore, (3.43), (3.29) and (3.13) give (3.40′) for each q in $0 \leqq q\rho < \theta+\rho$.

Putting $\beta=q\rho$, we have (3.6) by means of (3.11), (3.12), (3.22) and (3.40′). □

In order to complete the proof, we show (3.22), (3.25) and (3.32) before concluding this section.

PROOF OF (3.22). For simplicity, we put $A=A(t)$. By differentiating $(m-1)$-times in λ both sides of the well-known identity

(3.45) $$(A-\lambda)^{-1} = \int_0^\infty e^{\lambda r} e^{-rA}\, dr \quad (\mathrm{Re}\, \lambda \leqq 0),$$

we obtain

(3.46) $$(1+\tau A)^{-m} = \frac{1}{(m-1)!} \int_0^\infty r^{m-1} e^{-r} e^{-\tau r A}\, dr$$

by putting $\lambda=-1/\tau$ ($\tau>0$). The known estimate

$$\|A^{\beta}e^{-rA}\|\leq C_{\beta}r^{-\beta}, \tag{3.47}$$

which is derived from the assumption (A0), gives

$$\begin{aligned}\|A^{\beta}(1+\tau A)^{-m}\|&\leq \frac{C_{\beta}\tau^{-\beta}}{(m-1)!}\int_0^{\infty} r^{m-1-\beta}e^{-r}\,dr\\ &\leq C_{\beta}(\tau m)^{-\beta}\quad (m>\beta)\end{aligned}$$

by means of Stirling's formula. □

PROOF OF (3.25). We first note that the elementary inequality

$$|(1+\lambda)^{-n}-1|\leq n|\lambda|\quad (\mathrm{Re}\,\lambda\geq 0) \tag{3.48}$$

holds. Actually, we have

$$\begin{aligned}|(1+\lambda)^{-n}-1|&=\left|\int_{\Gamma_1}\frac{\partial}{\partial z}(1+z)^{-n}dz\right|\leq n\int_{\Gamma_1}|(1+z)^{-n-1}|\,|dz|\\ &\leq n\int_{\Gamma_1}|dz|\quad (\text{in fact } \mathrm{Re}(z)\geq 0 \text{ on } \Gamma_1)\\ &=n|\lambda|,\end{aligned}$$

Γ_1 being the segment connecting λ with the origin. Furthermore, the estimate

$$|(1+\tau\lambda)^{-j}|\leq(1+\gamma_0(j\tau|\lambda|)^{\beta_0})^{-1}\quad (|\arg\lambda|=\omega_1) \tag{3.49}$$

is derived by the binomial theorem and is proved in Fujita-Mizutani [6], where β_0 is the smallest integer which is greater than $\beta+1$, and where γ_0 is a positive constant. Let Γ be a positively oriented boundary of $G_1\equiv\{\lambda\in C\mid |\arg\lambda|>\omega_1\}\cup\{0\}$. Putting $A=A(t)$, we get by (3.48) and (3.49)

$$\begin{aligned}&\|A^{\beta}[(1+\tau A)^{-n}-(1+\tau A)^{-j}]\|\\ &=\left\|\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\lambda^{\beta}(1+\tau\lambda)^{-j}[(1+\tau\lambda)^{-(n-j)}-1](\lambda-A)^{-1}\,d\lambda\right\|\\ &\leq C\int_0^{\infty} r^{\beta}(1+\gamma_0(j\tau r)^{\beta_0})^{-1}(n-j)\tau r\frac{dr}{r}\quad (\because\ (3.3),\ (3.48),\ (3.49))\\ &=C_{\beta}(n-j)j^{-\beta-1}\tau^{-\beta}.\end{aligned} \tag{3.50}$$

On the other hand, we have

$$\begin{aligned}\|A^{\beta}[(1+\tau A)^{-n}-(1+\tau A)^{-j}]\|&\leq\|A^{\beta}(1+\tau A)^{-n}\|+\|A^{\beta}(1+\tau A)^{-j}\|\\ &\leq C_{\beta}(j\tau)^{-\beta}\end{aligned} \tag{3.51}$$

by (3.22). In virtue of (3.50) and (3.51), we obtain (3.25). □

PROOF OF (3.32). If $\beta=n=1$, we have

$$
\begin{aligned}
(3.52)\qquad & \|A(t)^{\beta}(1+\tau A(t))^{-n}-A(s)^{\beta}(1+\tau A(s))^{-n}\| \\
&=\|A(t)(1+\tau A(t))^{-1}-A(s)(1+\tau A(s))^{-1}\| \\
&=\tau^{-1}\|(1+\tau A(t))^{-1}-(1+\tau A(s))^{-1}\| \\
&\leqq \sum_{p=1}^{m}\|A(t)^{1-p\rho}(1+\tau A(t))^{-1}\|\cdot\|D(t, s)\|\cdot\|A(s)^{p\rho}(1+\tau A(s))^{-1}\| \quad (\because\ (3.8)) \\
&\leqq C\tau^{-1}|t-s|^{\theta} \quad (\because\ (3.22),\ (3.4)).
\end{aligned}
$$

Otherwise, since the inequality

$$
(3.53)\qquad \|(\lambda-A(t))^{-1}-(\lambda-A(s))^{-1}\|\leqq C|t-s|^{\theta}/|\lambda| \quad (\lambda\in G_1)
$$

can be shown in the same way as in (3.52), we obtain

$$
\begin{aligned}
(3.54)\qquad & \|A(t)^{\beta}e^{-rA(t)}-A(s)^{\beta}e^{-rA(s)}\| \\
&=\left\|\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\lambda^{\beta}e^{-r\lambda}[(\lambda-A(t))^{-1}-(\lambda-A(s))^{-1}]\,d\lambda\right\| \\
&\leqq C\int_0^{\infty}\mu^{\beta}e^{-r\mu\cos\omega_1}\frac{d\mu}{\mu}|t-s|^{\theta} \quad (\because\ (3.53)) \\
&=C_{\beta}|t-s|^{\theta}r^{-\beta}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(3.55)\qquad & \|A(t)^{\beta}(1+\tau A(t))^{-n}-A(s)^{\beta}(1+\tau A(s))^{-n}\| \\
&\leqq\frac{1}{(n-1)!}\int_0^{\infty}r^{n-1}e^{-r}\|A(t)^{\beta}e^{-\tau rA(t)}-A(s)^{\beta}e^{-\tau rA(s)}\|\,dr \quad (\because\ (3.46)) \\
&\leqq\frac{C_{\beta}}{(n-1)!}\int_0^{\infty}r^{n-1}e^{-r}|t-s|^{\theta}(\tau r)^{-\beta}\,dr \quad (\because\ (3.54)) \\
&=C_{\beta}\tau^{-\beta}|t-s|^{\theta}\frac{1}{(n-1)!}\int_0^{\infty}r^{n-1-\beta}e^{-r}\,dr \quad (\text{in fact, } n>\beta) \\
&\leqq C_{\beta}\tau^{-\beta}|t-s|^{\theta}n^{-\beta}
\end{aligned}
$$

by means of Stirling's formula. □

4. Proof of Lemma 2

We define another sesquilinear form $\dot{a}_t(\ ,\ )$ on $V\times V$ by

$$
\begin{aligned}
(4.1)\qquad \dot{a}_t(u, v)=&\sum_{i,j=1}^{2}\int_{\Omega}\frac{\partial}{\partial t}a_{ij}(t, x)\frac{\partial}{\partial x_j}u\cdot\overline{\frac{\partial}{\partial x_i}v}\,dx-\sum_{j=1}^{2}\int_{\Omega}\frac{\partial}{\partial t}b_j(t, x)\frac{\partial}{\partial x_j}u\cdot\bar{v}\,dx \\
&-\int_{\Omega}\frac{\partial}{\partial t}c(t, x)u\cdot\bar{v}\,dx+\int_{\partial\Omega}\frac{\partial}{\partial t}\sigma(t, \xi)u\cdot\bar{v}\,dS(\xi) \quad (u, v\in V).
\end{aligned}
$$

Then, by means of the assumptions stated in § 2 we have

$$|\dot{a}_t(u, v)| \leqq C\|u\|_1\|v\|_1 \quad (u, v \in V), \tag{4.2}$$

$$|\dot{a}_t(u, v) - \dot{a}_s(u, v)| \leqq C|t-s|^\alpha \|u\|_1\|v\|_1 \quad (u, v \in V) \tag{4.3}$$

and

$$\lim_{t \to s} \sup_{u, v \in V, \|u\|_1, \|v\|_1 \leqq 1} \left| \frac{1}{t-s}(a_t - a_s)(u, v) - \dot{a}_s(u, v) \right| = 0. \tag{4.4}$$

Henceforce we write $(a_t - a_s)(u, v)$ in stead of $a_t(u, v) - a_s(u, v)$. We shall also write $(\lambda - a_t)(u, v)$ in stead of $\lambda(u, v) - a_t(u, v)$, hereafter. In the first place, we assert that the hypotheses of the generation theorem in Kato-Tanabe [13] are satisfied for $A_h(t)$ uniformly in h. Namely, we have the following

PROPOSITION 4.1. *The inequalities and the relation* (2.11), (2.12), (2.13) (*with* $\theta = 1$), (4.2), (4.3) *and* (4.4) *make the following four conditions to be satisfied:*

(A0) *The relation*

$$\rho(A_h(t)) \supset G_1 \equiv \{\lambda \in C \mid |\arg \lambda| > \omega_1\} \cup \{0\} \quad (0 < \omega_1 < \pi/2)$$

and the inequality

$$\|(\lambda - A_h(t))^{-1}\| \leqq M/(1 + |\lambda|) \quad (\lambda \in G_1) \tag{4.5}$$

hold where $\rho(A_h(t))$ *is the resolvent set of* $A_h(t)$.

(B1) $A_h(t)^{-1}$ *is continuously differentiable in* t.

(B2) *The inequality*

$$\left\| \frac{d}{dt} A_h(t)^{-1} - \frac{d}{ds} A_h(s)^{-1} \right\| \leqq K|t-s|^\alpha \quad (t, s \in [0, T]) \tag{4.6}$$

holds with a constant $K > 0$.

(B3) *The inequality*

$$\left\| \frac{\partial}{\partial t}(\lambda - A_h(t))^{-1} \right\| \leqq L/|\lambda| \quad (\lambda \in G_1) \tag{4.7}$$

holds with a constant $L > 0$.

PROOF OF (B3). We put

$$w_h(t, \lambda) = (\lambda - A_h(t))^{-1} P_h \phi \quad (\lambda \in G_1,\ \phi \in X)$$

and show that it is continuously differentiable in t and that the inequality

$$\left\| \frac{\partial}{\partial t} w_h(t, \lambda) \right\|_0 \leqq C/|\lambda| \, \|\phi\|_0 \tag{4.7'}$$

holds. To this end, we recall that (2.11) and (2.12) give

$$\delta_1(|\lambda|\,\|\phi\|_0^2+\|\phi\|_1^2)\leqq|(\lambda-a_t)(\phi,\phi)| \quad (\lambda\in G_1,\ \phi\in V) \tag{4.8}$$

with a constant $\delta_1>0$ by Fujita-Mizutani [6]. We note

$$(\lambda-a_t)(w_h(t,\lambda),\chi)=(\phi,\chi) \quad (\chi\in V_h) \tag{4.9}$$

and obtain

$$\begin{aligned}\delta_1(|\lambda|\,\|w_h\|_0^2+\|w_h\|_1^2)&\leqq|(\lambda-a_t)(w_h,w_h)|\\ &=|(\phi,w_h)| \quad (\because\ (4.8))\\ &\leqq\|\phi\|_0\cdot\|w_h\|_0.\end{aligned}$$

Hence we get

$$\|w_h\|_0\leqq\delta_1^{-1}\|\phi\|_0/|\lambda| \tag{4.10}$$

and

$$\begin{aligned}\|w_h\|_1&\leqq\delta_1^{-1/2}\|\phi\|_0^{1/2}\cdot\|w_h\|_0^{1/2}\\ &\leqq\delta_1^{-1}\|\phi\|_0/|\lambda|^{1/2}.\end{aligned} \tag{4.11}$$

Now, we differentiate formally in t the both hand sides of (4.9) and get

$$(\lambda-a_t)(\dot{w}_h(t,\lambda),\chi)=\dot{a}_t(w_h(t,\lambda),\chi) \quad (\chi\in V_h). \tag{4.12}$$

Since the right hand side of (4.12) is a bounded antilinear form of $\chi\in V_h$, we can define $\dot{w}_h(t,\lambda)\in V_h$ by the equality (4.12) by means of Lax-Milgram's theorem. Recall (4.2) and (4.8). Defining $\dot{w}_h$ as above, we show that actually

$$\lim_{t\to s}\left\|\frac{w_h(t,\lambda)-w_h(s,\lambda)}{t-s}-\dot{w}_h(s,\lambda)\right\|_1=0 \tag{4.13}$$

holds. In fact, we put $\chi(t)=(w_h(t,\lambda)-w_h(s,\lambda))/(t-s)-\dot{w}_h(s,\lambda)\in V_h$, and we have

$$\begin{aligned}&(\lambda-a_t)(\chi(t),\chi(t))\\ &=\frac{1}{t-s}(\lambda-a_t)(w_h(t)-w_h(s),\chi)-(\lambda-a_t)(\dot{w}_h(s,\lambda),\chi) \quad (\chi=\chi(t)\in V_h)\\ &=\frac{1}{t-s}(\lambda-a_t)(w_h(t)-w_h(s),\chi)-\dot{a}_s(w_h(s,\lambda),\chi)+(a_t-a_s)(\dot{w}_h(s,\lambda),\chi) \quad (\because\ (4.12))\\ &=\frac{1}{t-s}((\lambda-a_s)-(\lambda-a_t))(w_h(s,\lambda),\chi)-\dot{a}_s(w_h(s,\lambda),\chi)+(a_t-a_s)(\dot{w}_h(s,\lambda),\chi) \quad (\because\ (4.9))\\ &=\left\{\frac{1}{t-s}(a_t-a_s)(w_h(s,\lambda),\chi)-\dot{a}_s(w_h(s,\lambda),\chi)\right\}+(a_t-a_s)(\dot{w}_h(s,\lambda),\chi).\end{aligned}$$

By (4.8), we have

$$\|\chi(t)\|_1^2 \leqq \delta_1^{-1} \left| \frac{1}{t-s}(a_t - a_s)(w_h(s, \lambda), \chi(t)) - \dot{a}_s(w_h(s, \lambda), \chi(t)) \right| + \delta_1^{-1} C|t-s| \cdot \|\dot{w}_h(s, \lambda)\|_1 \|\chi(t)\|_1, \tag{4.14}$$

hence

$$\|\chi(t)\|_1 \leqq C|t-s| \cdot \|\dot{w}_h(s, \lambda)\|_1 + C \sup_{\chi \in V, \|\chi\|_1 \leqq 1} \left| \frac{1}{t-s}(a_t - a_s)(w_h(s, \lambda), \chi) - \dot{a}_s(w_h(s, \lambda), \chi) \right|. \tag{4.15}$$

Therefore, we have by (4.4)

$$\lim_{t \to s} \|\chi(t)\|_1 = 0. \tag{4.16}$$

Now we show (4.7′). By (4.8) we have

$$\begin{aligned} \delta_1(|\lambda| \cdot \|\dot{w}_h(t, \lambda)\|_0^2 + \|\dot{w}_h(t, \lambda)\|_1^2) &\leqq |(\lambda - a_t)(\dot{w}_h(t, \lambda), \dot{w}_h(t, \lambda))| \\ &= |\dot{a}_t(w_h(t, \lambda), \dot{w}_h(t, \lambda))| \\ &\leqq C\|w_h(t, \lambda)\|_1 \|\dot{w}_h(t, \lambda)\|_1. \end{aligned} \tag{4.17}$$

Hence we get

$$\begin{aligned} \|\dot{w}_h(t, \lambda)\|_1 &\leqq C\|w_h(t, \lambda)\|_1 \\ &\leqq C\|\phi\|_0 / |\lambda|^{1/2} \quad (\because\ (4.11)) \end{aligned} \tag{4.18}$$

so that

$$\begin{aligned} \|\dot{w}_h(t, \lambda)\|_0 &\leqq C\|w_h(t, \lambda)\|_1^{1/2} \cdot \|\dot{w}_h(t, \lambda)\|_1^{1/2} / |\lambda|^{1/2} \quad (\because\ (4.17)) \\ &\leqq C\|\phi\|_0 / |\lambda| \quad (\because\ (4.11),\ (4.18)). \end{aligned} \tag{4.19}$$

□

PROOF OF (B1) AND (B2). The proof of (B1) is similar to that of (B3). We put

$$f_h(t) = A_h(t)^{-1} P_h \phi \quad (\phi \in X)$$

and note

$$a_t(f_h(t), \chi) = (\phi, \chi) \quad (\chi \in V_h). \tag{4.20}$$

Defining $\dot{f}_h(t) \in V_h$ by

$$a_t(\dot{f}_h(t), \chi) = -\dot{a}_t(f_h(t), \chi) \quad (\chi \in V_h), \tag{4.21}$$

we can show

$$\lim_{t \to s} \left\| \frac{f_h(t) - f_h(s)}{t-s} - \dot{f}_h(s) \right\|_1 = 0. \tag{4.22}$$

In order to prove (4.6), we show

$$\|\dot{f}_h(t) - \dot{f}_h(s)\|_1 \leqq C|t-s|^\alpha \|\phi\|_{V^*}, \tag{4.6′}$$

$\|\cdot\|_{V^*}$ being the norm in V^*, which is weaker than $\|\cdot\|_0$:

$$\|\phi\|_{V^*}=\sup_{\chi\in V,\|\chi\|_V\leq 1}|(\phi,\chi)|.$$

In fact, we put

$$\chi=\chi(t)=\dot{f}_h(t)-\dot{f}_h(s)\in V_h$$

and get

$$\begin{aligned}\delta\|\chi\|_1^2&\leqq \mathrm{Re}\, a_t(\dot{f}_h(t)-\dot{f}_h(s),\chi) \quad (\because\ (2.12))\\ &=-\mathrm{Re}\,\dot{a}_t(f_h(t),\chi)-\mathrm{Re}\,(a_t-a_s)(\dot{f}_h(s),\chi)+\mathrm{Re}\,\dot{a}_s(f_h(s),\chi)\\ &=\mathrm{Re}\,(\dot{a}_s-\dot{a}_t)(f_h(t),\chi)+\mathrm{Re}\,\dot{a}_s(f_h(s)-f_h(t),\chi)-\mathrm{Re}\,(a_t-a_s)(\dot{f}_h(s),\chi)\\ &\leqq C(|t-s|^\alpha\|f_h(t)\|_1\|\chi\|_1+\|f_h(s)-f_h(t)\|_1\|\chi\|_1+|t-s|\,\|\dot{f}_h(s)\|_1\|\chi\|_1).\end{aligned}$$

Hence we get

$$\begin{aligned}\|\dot{f}_h(t)-\dot{f}_h(s)\|_1&=\|\chi\|_1\\ &\leqq C(|t-s|^\alpha\|f_h(t)\|_1+\|f_h(s)-f_h(t)\|_1+|t-s|\,\|\dot{f}_h(s)\|_1).\end{aligned}\tag{4.23}$$

Now we have

$$\begin{aligned}\delta\|f_h(t)\|_1^2&\leqq \mathrm{Re}\, a_t(f_h(t),f_h(t)) \quad (\because\ (2.12))\\ &=\mathrm{Re}\,(\phi,f_h(t))\leqq C\|\phi\|_{V^*}\|f_h(t)\|_1 \quad (\because\ (4.20)),\end{aligned}$$

hence

$$\|f_h(t)\|_1\leqq C\|\phi\|_{V^*}.\tag{4.24}$$

Similarly, we have

$$\begin{aligned}\delta\|\dot{f}_h(s)\|_1^2&\leqq \mathrm{Re}\, a_s(\dot{f}_h(s),\dot{f}_h(s))\\ &=-\mathrm{Re}\,\dot{a}_s(f_h(s),\dot{f}_h(s)) \quad (\because\ (4.21))\\ &\leqq C\|f_h(s)\|_1\|\dot{f}_h(s)\|_1,\end{aligned}$$

and hence

$$\|\dot{f}_h(s)\|_1\leqq C\|f_h(s)\|_1\leqq C\|\phi\|_{V^*} \quad (\because\ (4.24)).\tag{4.25}$$

Finally, we have

$$\begin{aligned}\delta\|f_h(s)-f_h(t)\|_1^2&\leqq \mathrm{Re}\, a_t(f_h(s)-f_h(t),f_h(s)-f_h(t))\\ &=\mathrm{Re}\,(a_t-a_s)(f_h(s),f_h(s)-f_h(t)) \quad (\because\ (4.20))\\ &\leqq C|t-s|\,\|f_h(s)\|_1\|f_h(s)-f_h(t)\|_1\end{aligned}$$

whence

$$\begin{aligned}\|f_h(s)-f_h(t)\|_1&\leqq C|t-s|\,\|f_h(s)\|_1\\ &\leqq C|t-s|\,\|\phi\|_{V^*} \quad (\because\ (4.24)).\end{aligned}\tag{4.26}$$

Thus, by (4.23), (4.24), (4.25) and (4.26), we have (4.6′). □

In future, we may drop the suffix h for simplicity. In virture of Proposition 4.1, we can apply the generation theorem by T. Kato and H. Tanabe to $A(t)$ $(=A_h(t))$. Hence the evolution operator $U(t, s)$ $(=U_h(t, s))$ is given by

$$U(t, s)=e^{-(t-s)A(t)}+W(t, s) \tag{4.27}$$

with

$$W(t, s)=\int_s^t e^{-(t-z)A(t)}R(z, s)\,dz. \tag{4.28}$$

Here $R(t, s)$ is the solution of the integral equation

$$R(t, s)=R_1(t, s)+\int_s^t R_1(t, z)R(z, s)\,dz, \tag{4.29}$$

where

$$\begin{aligned} R_1(t, s) &= -\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial s}\right)e^{-(t-s)A(t)} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_\Gamma e^{-(t-s)\lambda}\frac{\partial}{\partial t}(\lambda-A(t))^{-1}\,d\lambda, \end{aligned} \tag{4.30}$$

and satisfies

$$\|R(t, s)\|\leqq C \quad (T\geqq t\geqq s\geqq 0) \tag{4.31}$$

and

$$\|R(t, s)-R(r, s)\|\leqq C_{\gamma,\delta}\{(t-r)^\gamma(r-s)^{-\gamma}+(t-r)^\delta(r-s)^{\alpha-\delta-1}\} \quad (T\geqq t>r>s\geqq 0) \tag{4.32}$$

for each γ, δ in $0<\gamma<1$, $0<\delta<\alpha$. See, Kato-Tanabe [13]. Therefore, we have

$$\begin{aligned} &A(t)U(t, s)-A(r)U(r, s) \\ &=A(t)(e^{-(t-s)A(t)}+W(t, s))-A(r)(e^{-(r-s)A(r)}+W(r, s)) \\ &=A(t)[e^{-(t-s)A(t)}-e^{-(r-s)A(t)}]+A(t)^\beta[A(t)^{1-\beta}e^{-(r-s)A(t)}-A(r)^{1-\beta}e^{-(r-s)A(r)}] \\ &\quad +A(t)^\beta(1-A(t)^{-\beta}A(r)^\beta)A(r)^{1-\beta}e^{-(r-s)A(r)} \\ &\quad +A(t)^\beta[A(t)^{1-\beta}W(t, s)-A(r)^{1-\beta}W(r, s)] \\ &\quad +A(t)^\beta(1-A(t)^{-\beta}A(r)^\beta)A(r)^{1-\beta}W(r, s), \end{aligned} \tag{4.33}$$

so that

$$\begin{aligned} Z_\beta(t, r, s)=&\{A(t)^{1-\beta}e^{-(r-s)A(t)}-A(r)^{1-\beta}e^{-(r-s)A(r)}\} \\ &+(1-A(t)^{-\beta}A(r)^\beta)A(r)^{1-\beta}e^{-(r-s)A(r)}+\{A(t)^{1-\beta}W(t, s)-A(r)^{1-\beta}W(r, s)\} \\ &+(1-A(t)^{-\beta}A(r)^\beta)A(r)^{1-\beta}W(r, s) \end{aligned} \tag{4.34}$$

holds.

In the first place, by (3.54) the first term of the right hand side of (4.34) is estimated as

$$\|A(t)^{1-\beta}e^{-(r-s)A(t)}-A(r)^{1-\beta}e^{-(r-s)A(r)}\|\leqq C_\beta(t-r)(r-s)^{\beta-1}. \tag{4.35}$$

In the next place, considering the adjoint operator of $1-A(t)^{-\beta}A(s)^\beta$, we have

$$\|1-A(t)^{-\beta}A(s)^\beta\|\leqq C|t-s| \quad (0<\beta<1/2) \tag{4.36}$$

by (3.2). Hence the second term of the right hand side of (4.34) is estimated as

$$\|1-A(t)^{-\beta}A(r)^\beta\|\cdot\|A(r)^{1-\beta}e^{-(r-s)A(r)}\|\leqq C_\beta(t-r)(r-s)^{\beta-1}, \tag{4.37}$$

by (3.47). On the other hand, we have

$$\begin{aligned}\|A(t)^{1-\beta}W(t,s)\|&=\left\|\int_s^t A(t)^{1-\beta}e^{-(t-z)A(t)}R(z,s)\,dz\right\|\\ &\leqq C_\beta\int_s^t (t-z)^{-1+\beta}\,dz \quad (\because\ (3.47),\ (4.31))\\ &=C_\beta(t-s)^\beta,\end{aligned} \tag{4.38}$$

so that the last term of the right hand side of (4.34) is estimated as

$$\|1-A(t)^{-\beta}A(r)^\beta\|\cdot\|A(r)^{1-\beta}W(r,s)\|\leqq C_\beta(t-r)(r-s)^\beta \quad (0<\beta<1/2). \tag{4.39}$$

In this way, Lemma 2 is reduced to the following

LEMMA 2′. *The estimate*

$$\|A(t)^{1-\beta}W(t,s)-A(r)^{1-\beta}W(r,s)\|\leqq C_\beta(t-r)^\alpha(r-s)^{\beta-1} \quad (0<\beta<1/2) \tag{4.40}$$

holds.

In order to prove Lemma 2′, we prepare a few propositions below.

PROPOSITION 4.2. *The estimates*

$$\left\|A(t)^\beta\frac{\partial}{\partial t}A(t)^{-\beta}\right\|\leqq C_\beta \quad (0<\beta<1/2), \tag{4.41}$$

$$\left\|\frac{\partial}{\partial t}w(t,\lambda)-\frac{\partial}{\partial s}w(s,\lambda)\right\|_1\leqq C|t-s|^\alpha\|\phi\|_{V^*} \tag{4.42}$$

and

$$\left\|A(t)^\beta\frac{\partial}{\partial t}A(t)^{-\beta}-A(s)^\beta\frac{\partial}{\partial s}A(s)^{-\beta}\right\|\leqq C_\beta|t-s|^\alpha \quad (0<\beta<1/2) \tag{4.43}$$

hold, where $w(t,\lambda)=w_h(t,\lambda)=(\lambda-A_h(t))^{-1}P_h\phi$ $(\lambda\in G_1,\ \phi\in X)$.

PROOF OF (4.41). Clearly,

$$\frac{\partial}{\partial t} A(t)^{-\beta} = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} \lambda^{-\beta} \frac{\partial}{\partial t} (\lambda - A(t))^{-1} d\lambda. \tag{4.44}$$

exists. Hence making $\varepsilon \to 0$ in the inequality

$$\| A(t)^{\beta} (A(t+\varepsilon)^{-\beta} - A(t)^{-\beta}) / \varepsilon \| \leqq C_{\beta},$$

which is derived by (3.2), we obtain (4.41). □

PROOF OF (4.42). We put $\chi = \chi(t) = \dot{w}(t, \lambda) - \dot{w}(s, \lambda)$ and obtain

$$\begin{aligned}(\lambda - a_t)(\chi, \chi) &= (\lambda - a_t)(\dot{w}(t, \lambda) - \dot{w}(s, \lambda), \chi) \\ &= \dot{a}_t(w(t, \lambda), \chi) + (a_t - a_s)(\dot{w}(s, \lambda), \chi) - \dot{a}_s(w(s, \lambda), \chi) \quad (\because \ (4.12)) \\ &= (\dot{a}_t - \dot{a}_s)(w(t, \lambda), \chi) + (a_t - a_s)(\dot{w}(s, \lambda), \chi) + \dot{a}_s(w(t, \lambda) - w(s, \lambda), \chi).\end{aligned}$$

By (4.8), we get

$$\delta_1(\|\chi\|_1^2 + |\lambda| \|\chi\|_0^2) \leqq C|t-s|^{\alpha} \|w(t, \lambda)\|_1 \|\chi\|_1 + C|t-s| \|\dot{w}(s, \lambda)\|_1 \|\chi\|_1 + C\|w(t, \lambda) - w(s, \lambda)\|_1 \|\chi\|_1.$$

Hence we obtain

$$\|\chi\|_1 \leqq C|t-s|^{\alpha} \|w(t, \lambda)\|_1 + C|t-s| \|\dot{w}(s, \lambda)\|_1 + C\|w(t, \lambda) - w(s, \lambda)\|_1. \tag{4.45}$$

On the other hand, we have

$$\begin{aligned}\delta_1 \|w(t, \lambda)\|_1^2 &\leqq |(\lambda - a_t)(w(t, \lambda), w(t, \lambda))| \quad (\because \ (4.8)) \\ &= |(\phi, w(t, \lambda))| \leqq \|\phi\|_{V^*} \|w(t, \lambda)\|_1 \quad (\because \ (4.9)), \\ \delta_1 \|\dot{w}(t, \lambda)\|_1^2 &\leqq |(\lambda - a_t)(\dot{w}(t, \lambda), \dot{w}(t, \lambda))| \quad (\because \ (4.8)) \\ &= |\dot{a}_t(w(t, \lambda), \dot{w}(t, \lambda))| \quad (\because \ (4.12)) \\ &\leqq C\|w(t, \lambda)\|_1 \|\dot{w}(t, \lambda)\|_1\end{aligned}$$

and

$$\begin{aligned}\delta_1 \|w(t, \lambda) - w(s, \lambda)\|_1^2 &\leqq |(\lambda - a_t)(w(t, \lambda) - w(s, \lambda), w(t, \lambda) - w(s, \lambda))| \quad (\because \ (4.8)) \\ &= |(a_t - a_s)(w(s, \lambda), w(t, \lambda) - w(s, \lambda))| \quad (\because \ (4.9)) \\ &\leqq C|t-s| \|w(s, \lambda)\|_1 \|w(t, \lambda) - w(s, \lambda)\|_1,\end{aligned}$$

so that the inequalities

$$\|w(t, \lambda)\|_1 \leqq C\|\phi\|_{V^*}, \tag{4.46}$$

$$\|\dot{w}(s, \lambda)\|_1 \leqq C\|\phi\|_{V^*} \tag{4.47}$$

and

$$\|w(t, \lambda) - w(s, \lambda)\|_1 \leqq C|t-s| \|\phi\|_{V^*} \tag{4.48}$$

follow. (4.45), (4.46), (4.47) and (4.48) imply (4.42). □

PROOF OF (4.43). We note the identity

$$A(t)^{\beta}\frac{\partial}{\partial t}A(t)^{-\beta}=-\Big(\frac{\partial}{\partial t}A(t)^{\beta}\Big)A(t)^{-\beta} \tag{4.49}$$

and obtain

$$\begin{aligned} &A(t)^{\beta}\frac{\partial}{\partial t}A(t)^{-\beta}-A(s)^{\beta}\frac{\partial}{\partial s}A(s)^{-\beta} \\ &=-\Big(\frac{\partial}{\partial t}A(t)^{\beta}\Big)A(t)^{-\beta}+\Big(\frac{\partial}{\partial s}A(s)^{\beta}\Big)A(s)^{-\beta} \\ &=-\Big(\frac{\partial}{\partial t}A(t)^{\beta}-\frac{\partial}{\partial s}A(s)^{\beta}\Big)A(t)^{-\beta}+\Big(\frac{\partial}{\partial s}A(s)^{\beta}\Big)A(s)^{-\beta}[1-A(s)^{\beta}A(t)^{-\beta}]. \end{aligned} \tag{4.50}$$

Since (3.2), (4.49) and (4.41) give

$$\Big\|\Big(\frac{\partial}{\partial s}A(s)^{\beta}\Big)A(s)^{-\beta}\Big\|\cdot\|1-A(s)^{\beta}A(t)^{-\beta}\|\leqq C|t-s|, \tag{4.51}$$

(4.43) is reduced to

$$\Big\|\Big(\frac{\partial}{\partial t}A(t)^{\beta}-\frac{\partial}{\partial s}A(s)^{\beta}\Big)\phi\Big\|\leqq C_{\beta}|t-s|^{\alpha}\|A(s)^{\beta}\phi\| \quad (\phi\in V_h;\ 0<\beta<1/2). \tag{4.52}$$

PROOF OF (4.52). Recall

$$\begin{aligned} A(t)^{\beta}\phi&=A(t)^{\beta-1}A(t)\phi \\ &=\frac{\sin\pi\beta}{\pi}\int_0^{\infty}\mu^{\beta-1}(\mu+A(t))^{-1}A(t)\phi\,d\mu \\ &=\frac{\sin\pi\beta}{\pi}\int_0^{\infty}\mu^{\beta-1}(1-\mu(\mu+A(t))^{-1})\phi\,d\mu. \end{aligned} \tag{4.53}$$

See, Tanabe [22], for instance. By this identity, we have

$$\begin{aligned} \Big(\frac{\partial}{\partial t}A(t)^{\beta}\phi,\chi\Big)&=\frac{\partial}{\partial t}(A(t)^{\beta}\phi,\chi) \\ &=-\frac{\sin\pi\beta}{\pi}\int_0^{\infty}\mu^{\beta}\Big(\frac{\partial}{\partial t}(\mu+A(t))^{-1}\phi,\chi\Big)d\mu \quad (\phi,\chi\in V_h), \end{aligned} \tag{4.54}$$

if $0<\beta<1/2$. To see this, we have to show $\mu^{\beta}((\partial/\partial t)(\mu+A(t))^{-1}\phi,\chi)\in L^1(0,\infty)$. But, since this can be done by the same arguments as in the proof of (4.61) below, we omit the proof.

Let $a_t^*(\ ,\ )$ be another sesquilinear form on $V\times V$ such that

$$a_t^*(u,v)=\overline{a_t(v,u)} \quad (u,v\in V). \tag{4.55}$$

We put

$$a_t^0(u, v)=\frac{1}{2}\{a_t(u, v)+a_t^*(u, v)\} \quad (u, v \in V). \tag{4.56}$$

Then the m-sectorial operator $H_h(t)$ associated with $a_t^0(\ ,\ )$ on $V_h\times V_h$ is self-adjoint and satisfies

$$C^{-1}\|v\|_1\leqq\|H_h(t)^{1/2}v\|_0\leqq C\|v\|_1 \quad (v\in V_h). \tag{4.57}$$

See Kato [12], for example. We again drop the suffix h and write the equality

$$\begin{aligned}-\Big(\frac{\partial}{\partial t}(\mu+A(t))^{-1}-\frac{\partial}{\partial s}(\mu+A(s))^{-1}\Big)\\ =H(t)^{1/2}(\mu+H(t))^{-1}B(t, s)H(s)^{1/2}(\mu+H(s))^{-1}\end{aligned} \tag{4.58}$$

and claim

$$\|B(t, s)\|\leqq C|t-s|^{\alpha}, \tag{4.59}$$

which will be proved later. We also note the following known identity

$$\int_0^\infty \mu^{2\beta}\|H^{1/2}(\mu+H)^{-1}\phi\|^2\,d\mu=C_\beta\|H^\beta\phi\|^2 \quad (\phi\in D(H^\beta)) \tag{4.60}$$

for $0\leqq\beta<1/2$, H being a non-negative self-adjoint operator, which will be also proved later for completeness.

By means of these relations, we get

$$\begin{aligned}&\Big(\Big(\frac{\partial}{\partial t}A(t)^\beta-\frac{\partial}{\partial s}A(s)^\beta\Big)\phi, \chi\Big)\\ &=-\frac{\sin\pi\beta}{\pi}\int_0^\infty \mu^\beta\Big(\Big(\frac{\partial}{\partial t}(\mu+A(t))^{-1}-\frac{\partial}{\partial s}(\mu+A(s))^{-1}\Big)\phi, \chi\Big)\,d\mu\\ &=\frac{\sin\pi\beta}{\pi}\int_0^\infty \mu^\beta(B(t, s)H(s)^{1/2}(\mu+H(s))^{-1}\phi, H(t)^{1/2}(\mu+H(t))^{-1}\chi)\,d\mu,\end{aligned}$$

so that we have

$$\begin{aligned}&\Big|\Big(\Big(\frac{\partial}{\partial t}A(t)^\beta-\frac{\partial}{\partial s}A(s)^\beta\Big)\phi, \chi\Big)\Big|\\ &\leqq C\frac{\sin\pi\beta}{\pi}\Big(\int_0^\infty \mu^{2\beta}\|H(s)^{1/2}(\mu+H(s))^{-1}\phi\|^2 d\mu\Big)^{1/2}\\ &\quad\times\Big(\int_0^\infty \|H(t)^{1/2}(\mu+H(t))^{-1}\chi\|^2\,d\mu\Big)^{1/2}|t-s|^\alpha \quad (\because\ (4.59))\\ &=C_\beta\|H(s)^\beta\phi\|\cdot\|\chi\|\,|t-s|^\alpha \quad (\because\ (4.60)).\end{aligned} \tag{4.61}$$

On the other hand, as Kato [11] shows, the inequality

$$C_\beta^{-1}\|A_h(t)^\beta\phi\| \leqq \|H_h(t)^\beta\phi\| \leqq C_\beta\|A_h(t)^\beta\phi\| \quad (\phi\in V_h) \tag{4.62}$$

is true for $0\leqq\beta<1/2$. Therefore, we get (4.52). □

In order to complete our proof, we prove (4.60) and (4.59).

PROOF OF (4.60). Let $H=\int_0^\infty \lambda\, dE(\lambda)$ be the spectral representation of H. Then (4.60) is given as

$$\begin{aligned}\int_0^\infty \mu^{2\beta}\|H^{1/2}(\mu+H)^{-1}\phi\|^2\, d\mu &= \int_0^\infty \mu^{2\beta}\, d\mu \int_0^\infty \lambda(\mu+\lambda)^{-2}\, d\|E(\lambda)\phi\|^2 \\ &= \int_0^\infty \lambda\, d\|E(\lambda)\phi\|^2 \int_0^\infty \mu^{2\beta}(\mu+\lambda)^{-2}\, d\mu \\ &= \int_0^\infty \lambda\, d\|E(\lambda)\phi\|^2\cdot\lambda^{2\beta-1}\int_0^\infty \mu^{2\beta}(\mu+1)^{-2}\, d\mu \\ &= C_\beta\int_0^\infty \lambda^{2\beta}\, d\|E(\lambda)\phi\|^2 = C_\beta\|H^\beta\phi\|^2.\end{aligned}$$

□

PROOF OF (4.58) WITH (4.59). We define an operator $A_h'(t)\colon V_h\to V_h$ by

$$\dot{a}_t(u, v)=(A_h'(t)u, v) \quad (u, v\in V_h). \tag{4.63}$$

Then, we have

$$\frac{\partial}{\partial t}(\mu+A_h(t))^{-1}=-(\mu+A_h(t))^{-1}A_h'(t)(\mu+A_h(t))^{-1}. \tag{4.64}$$

We again drop the suffix h and obtain

$$\begin{aligned}&-\left(\frac{\partial}{\partial t}(\mu+A(t))^{-1}-\frac{\partial}{\partial s}(\mu+A(s))^{-1}\right)\\ &=(\mu+A(t))^{-1}A'(t)(\mu+A(t))^{-1}-(\mu+A(s))^{-1}A'(s)(\mu+A(s))^{-1}\\ &=(\mu+A(t))^{-1}A'(t)[(\mu+A(t))^{-1}-(\mu+A(s))^{-1}]\\ &\quad+(\mu+A(t))^{-1}[A'(t)-A'(s)](\mu+A(s))^{-1}\\ &\quad+[(\mu+A(t))^{-1}-(\mu+A(s))^{-1}]A'(s)(\mu+A(s))^{-1}\\ &=-(\mu+A(t))^{-1}A'(t)(\mu+A(t))^{-1}(A(t)-A(s))(\mu+A(s))^{-1}\\ &\quad+(\mu+A(t))^{-1}(A'(t)-A'(s))(\mu+A(s))^{-1}\\ &\quad-(\mu+A(t))^{-1}(A(t)-A(s))(\mu+A(s))^{-1}A'(s)(\mu+A(s))^{-1}.\end{aligned} \tag{4.65}$$

Therefore, we have

$$B(t, s)=-B_1(t, s)+B_2(t, s)-B_3(t, s) \tag{4.66}$$

with

$$(4.67)\qquad B_1(t,s)=H(t)^{-1/2}(\mu+H(t))(\mu+A(t))^{-1}H(t)^{1/2}\cdot H(t)^{-1/2}A'(t)H(t)^{-1/2}$$
$$\times H(t)^{1/2}(\mu+A(t))^{-1}H(t)^{1/2}\cdot H(t)^{-1/2}(A(t)-A(s))H(s)^{-1/2}$$
$$\times H(s)^{1/2}(\mu+A(s))^{-1}(\mu+H(s))H(s)^{-1/2},$$

$$(4.68)\qquad B_2(t,s)=H(t)^{-1/2}(\mu+H(t))(\mu+A(t))^{-1}H(t)^{1/2}\cdot H(t)^{-1/2}(A'(t)-A'(s))H(s)^{-1/2}$$
$$\times H(s)^{1/2}(\mu+A(s))^{-1}(\mu+H(s))H(s)^{-1/2}$$

and

$$(4.69)\qquad B_3(t,s)=H(t)^{-1/2}(\mu+H(t))(\mu+A(t))^{-1}H(t)^{1/2}\cdot H(t)^{-1/2}(A(t)-A(s))H(s)^{-1/2}$$
$$\times H(s)^{1/2}(\mu+A(s))^{-1}H(s)^{1/2}\cdot H(s)^{-1/2}A'(s)H(s)^{-1/2}$$
$$\times H(s)^{1/2}(\mu+A(s))^{-1}(\mu+H(s))H(s)^{-1/2}.$$

Thus, (4.59) is reduced to

PROPOSITION 4.3. *The following inequalities hold:*

$$(4.70)\qquad \|H_h(t)^{-1/2}(\mu+H_h(t))(\mu+A_h(t))^{-1}H_h(t)^{1/2}\|\leqq C,$$

$$(4.71)\qquad \|H_h(t)^{-1/2}A_h'(t)H_h(t)^{-1/2}\|\leqq C,$$

$$(4.72)\qquad \|H_h(t)^{1/2}(\mu+A_h(t))^{-1}H_h(t)^{1/2}\|\leqq C,$$

$$(4.73)\qquad \|H_h(t)^{-1/2}(A_h(t)-A_h(s))H_h(s)^{-1/2}\|\leqq C|t-s|,$$

$$(4.74)\qquad \|H_h(t)^{1/2}(\mu+A_h(t))^{-1}(\mu+H_h(t))H_h(t)^{-1/2}\|\leqq C,$$

$$(4.75)\qquad \|H_h(t)^{-1/2}(A_h'(t)-A_h'(s))H_h(s)^{-1/2}\|\leqq C|t-s|^{\alpha}.$$

PROOF OF PROPOSITION 4.3. Let $\|\cdot\|_{V_h^*}$ be the norm in V_h defined by

$$\|f\|_{V_h^*}=\sup_{\chi\in V_h,\|\chi\|_{V_h}\leq 1}|(f,\chi)|\quad (f\in V_h).$$

Then we have

$$(4.76)\qquad C^{-1}\leqq\|A_h(t)\|_{V_h\to V_h^*}\leqq C,$$

$$(4.77)\qquad C^{-1}\leqq\|H_h(t)\|_{V_h\to V_h^*}\leqq C$$

and

$$(4.78)\qquad \|A_h'(t)\|_{V_h\to V_h^*}\leqq C.$$

We again drop the suffix h. Since

$$H(t)^{-1/2}(\mu+H(t))(\mu+A(t))^{-1}H(t)^{1/2}$$
$$=H(t)^{-1/2}(1+(H(t)-A(t)))(\mu+A(t))^{-1}H(t)^{1/2}$$
$$=1+H(t)^{1/2}(\mu+A(t))^{-1}H(t)^{1/2}-H(t)^{-1/2}A(t)(\mu+A(t))^{-1}H(t)^{1/2},$$

(4.70) is reduced to (4.72) and

(4.79) $$\|H(t)^{-1/2}A(t)(\mu+A(t))^{-1}H(t)^{1/2}\|\leqq C.$$

(4.79) is proved as follows: We recall the inequality

(4.80) $$\|(\lambda-A(t))^{-1}\|_{V^*\to V^*}\leqq C/(|\lambda|+1) \quad (\lambda\in G_1).$$

See, Tanabe [22], for instance. Taking $\phi\in V_h$, we have

$$\begin{aligned}
&\|H(t)^{-1/2}A(t)(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_0\\
&\leqq C\|A(t)(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_{V^*} \quad (\because (4.57),\ (4.77))\\
&\leqq C(\|H(t)^{1/2}\phi\|_{V^*}+\mu\|(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_{V^*})\\
&\leqq C\|H(t)^{1/2}\phi\|_{V^*} \quad (\because (4.80))\\
&\leqq C\|\phi\|_0 \quad (\because (4.57),\ (4.77)),
\end{aligned}$$

which shows (4.79).

Taking $\phi\in V_h$, we can prove (4.71) as follows:

$$\begin{aligned}
\|H(t)^{-1/2}A'(t)H(t)^{-1/2}\phi\|_0 &\leqq C\|A'(t)H(t)^{-1/2}\phi\|_{V^*} \quad (\because (4.57),\ (4.77))\\
&\leqq C\|H(t)^{-1/2}\phi\|_1 \quad (\because (4.76))\\
&\leqq C\|\phi\|_0 \quad (\because (4.57)).
\end{aligned}$$

Similarly, (4.72) is proved as

$$\begin{aligned}
&\|H(t)^{1/2}(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_0\\
&\leqq C\|(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_1 \quad (\because (4.57))\\
&\leqq C\|A(t)(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_{V^*} \quad (\because (4.76))\\
&\leqq C\|H(t)^{-1/2}A(t)(\mu+A(t))^{-1}H(t)^{1/2}\phi\|_0 \quad (\because (4.57),\ (4.77))\\
&\leqq C\|\phi\|_0 \quad (\because (4.79)).
\end{aligned}$$

Noting the inequality

(4.81) $$\|A_h(t)-A_h(s)\|_{V_h\to V_h^*}\leqq C|t-s|,$$

which is derived from (2.13), we get (4.73) as

$$\begin{aligned}
\|H(t)^{-1/2}(A(t)-A(s))H(s)^{-1/2}\phi\|_0 &\leqq C\|(A(t)-A(s))H(s)^{-1/2}\phi\|_{V^*} \quad (\because (4.57),\ (4.77))\\
&\leqq C|t-s|\,\|H(s)^{-1/2}\phi\|_V \quad (\because (4.81))\\
&\leqq C|t-s|\,\|\phi\|_0 \quad (\because (4.57)).
\end{aligned}$$

(4.74) is obtained by considering the adjoint operator in (4.70). Finally, the proof of (4.75) is similar to that of (4.73). □

Thus, Proposition 4.2 has been obtained. In virtue of it, we have the following

PROPOSITION 4.4. *The estimate*

$$\|A(t)^{-\beta}R_1(t, s)-A(r)^{-\beta}R_1(r, s)\| \leqq C_\beta(t-r)^\alpha(r-s)^{\beta-1} \quad (T\geqq t>r>s\geqq 0) \tag{4.82}$$

holds true if $0<\beta<1/2$.

PROOF. By means of (4.30), we have

$$\begin{aligned}
(4.83)\qquad & A(t)^{-\beta}R_1(t, s) \\
&= -A(t)^{-\beta}\frac{\partial}{\partial t}e^{-(t-s)A(t)}-A(t)^{-\beta}\frac{\partial}{\partial s}e^{-(t-s)A(t)} \\
&= -\left\{\frac{\partial}{\partial t}(A(t)^{-\beta}e^{-(t-s)A(t)})+A(t)^{1-\beta}e^{-(t-s)A(t)}\right\}+\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)e^{-(t-s)A(t)} \\
&= -\frac{\partial}{\partial t}\left(\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}e^{-(t-s)\lambda}(\lambda-A(t))^{-1}\,d\lambda\right) \\
&\quad -\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{1-\beta}e^{-(t-s)\lambda}(\lambda-A(t))^{-1}\,d\lambda+\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)e^{-(t-s)A(t)} \\
&= -\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}e^{-(t-s)\lambda}\frac{\partial}{\partial t}(\lambda-A(t))^{-1}\,d\lambda+\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)e^{-(t-s)A(t)},
\end{aligned}$$

so that we obtain

$$\begin{aligned}
(4.84)\qquad & A(t)^{-\beta}R_1(t, s)-A(r)^{-\beta}R_1(r, s) \\
&= -\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}\left\{e^{-(t-s)\lambda}\frac{\partial}{\partial t}(\lambda-A(t))^{-1}-e^{-(r-s)\lambda}\frac{\partial}{\partial r}(\lambda-A(r))^{-1}\right\}d\lambda \\
&\quad +\left\{\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)e^{-(t-s)A(t)}-\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)e^{-(r-s)A(r)}\right\} \\
&= -\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}e^{-(r-s)\lambda}[e^{-(t-r)\lambda}-1]\frac{\partial}{\partial t}(\lambda-A(t))^{-1}\,d\lambda \\
&\quad -\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}e^{-(r-s)\lambda}\left[\frac{\partial}{\partial t}(\lambda-A(t))^{-1}-\frac{\partial}{\partial r}(\lambda-A(r))^{-1}\right]d\lambda \\
&\quad +\left[\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)A(t)^\beta-\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)A(r)^\beta\right]A(t)^{-\beta}e^{-(t-s)A(t)} \\
&\quad +\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)A(r)^\beta[A(t)^{-\beta}e^{-(t-s)A(t)}-A(t)^{-\beta}e^{-(r-s)A(t)}] \\
&\quad +\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)A(r)^\beta[A(t)^{-\beta}e^{-(r-s)A(t)}-A(r)^{-\beta}e^{-(r-s)A(r)}].
\end{aligned}$$

The first term of the right hand side of the (4.84) is estimated as

$$\left\|\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}e^{-(r-s)\lambda}[e^{-(t-r)\lambda}-1]\frac{\partial}{\partial t}(\lambda-A(t))^{-1}\,d\lambda\right\| \tag{4.85}$$

$$\leqq C\int_0^\infty z^{-\beta}e^{-(r-s)z\cos\omega_1}(t-r)z\cdot\frac{dz}{z} \quad (\because\ (4.7))$$
$$=C_\beta(t-r)(r-s)^{\beta-1}.$$

The second term of the right hand side of (4.84) is estimated as

$$\left\|\frac{1}{2\pi\sqrt{-1}}\int_\Gamma \lambda^{-\beta}e^{-(r-s)\lambda}\left[\frac{\partial}{\partial t}(\lambda-A(t))^{-1}-\frac{\partial}{\partial r}(\lambda-A(r))^{-1}\right]d\lambda\right\| \tag{4.86}$$
$$\leqq C\int_0^\infty z^{-\beta}e^{-(r-s)z\cos\omega_1}(t-r)^\alpha\,dz \quad (\because\ (4.42))$$
$$=C_\beta(t-r)^\alpha(r-s)^{\beta-1}.$$

By considering the adjoint operators in (4.41) and (4.43) we get

$$\left\|\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)A(t)^\beta\right\|\leqq C_\beta \tag{4.41'}$$

and

$$\left\|\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)A(t)^\beta-\left(\frac{\partial}{\partial s}A(s)^{-\beta}\right)A(s)^\beta\right\|\leqq C_\beta|t-s|^\alpha. \tag{4.43'}$$

Therefore, the third term of the right hand side of (4.84) is estimated as

$$\left\|\left(\frac{\partial}{\partial t}A(t)^{-\beta}\right)A(t)^\beta-\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)A(r)^\beta\right\|\cdot\|A(t)^{-\beta}\|\cdot\|e^{-(t-s)A(t)}\| \tag{4.87}$$
$$\leqq C_\beta(t-r)^\alpha,$$

because of the known estimate

$$\|A(t)^{-\beta}\|\leqq C_\beta. \tag{4.88}$$

See Kato [10] for (4.88). On the other hand, we have

$$\|A(r)^\kappa[e^{-tA(r)}-e^{-sA(r)}]\|\leqq C_\kappa(t-s)s^{-\kappa-1} \quad (t\geqq s>0) \tag{4.89}$$

for $\kappa>-1$ and

$$\|A(t)^{-\beta}e^{-rA(t)}-A(s)^{-\beta}e^{-rA(s)}\|\leqq C_\beta|t-s|\cdot r^{\beta-1} \tag{4.90}$$

for $0<\beta<1$, which will be proved just later. Therefore, the fourth and final terms of the right hand side of (4.84) are estimated as

$$\left\|\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)A(r)^\beta\right\|\cdot\|A(t)^{-\beta}e^{-(t-s)A(t)}-A(t)^{-\beta}e^{-(r-s)A(t)}\|\leqq C_\beta(t-r)(r-s)^{\beta-1} \tag{4.91}$$

and

(4.92) $$\left\|\left(\frac{\partial}{\partial r}A(r)^{-\beta}\right)A(r)^{\beta}\right\|\cdot\|A(t)^{-\beta}e^{-(r-s)A(t)}-A(r)^{-\beta}e^{-(r-s)A(r)}\|\leqq C_{\beta}(t-r)(r-s)^{\beta-1},$$

respectively. Hence (4.82) follows. □

PROOF OF (4.89). We have

$$\begin{aligned}&\|A(r)^{\kappa}[e^{-tA(r)}-e^{-sA(r)}]\|\\&=\left\|\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\lambda^{\kappa}e^{-s\lambda}[e^{-(t-s)\lambda}-1](\lambda-A(r))^{-1}d\lambda\right\|\\&\leqq C\int_0^{\infty}z^{\kappa}e^{-sz\cos\omega_1}(t-s)z\frac{dz}{z}\\&=C_{\kappa}(t-s)s^{-\kappa-1}.\end{aligned}$$

□

PROOF OF (4.90). We have

$$\begin{aligned}&\|A(t)^{-\beta}e^{-rA(t)}-A(s)^{-\beta}e^{-rA(s)}\|\\&=\left\|\frac{1}{2\pi\sqrt{-1}}\int_{\Gamma}\lambda^{-\beta}e^{-r\lambda}[(\lambda-A(t))^{-1}-(\lambda-A(s))^{-1}]d\lambda\right\|\\&\leqq C\int_0^{\infty}z^{-\beta}e^{-rz\cos\omega_1}|t-s|dz\quad(\because\ (4.48))\\&=C_{\beta}|t-s|r^{\beta-1}.\end{aligned}$$

□

PROPOSITION 4.6. *The estimate*

(4.93) $$\|A(t)^{-\beta}R(t,s)-A(r)^{-\beta}R(r,s)\|\leqq C_{\beta}(t-r)^{\alpha}(r-s)^{\beta-1}\quad(T\geqq t>r>s\geqq 0)$$

holds true if $0<\beta<1/2$.

PROOF. By means of (4.29), we have

(4.94) $$\begin{aligned}A(t)^{-\beta}R(t,s)-A(r)^{-\beta}R(r,s)&=[A(t)^{-\beta}R_1(t,s)-A(r)^{-\beta}R_1(r,s)]\\&\quad+\int_r^t A(t)^{-\beta}R_1(t,z)R(z,s)dz\\&\quad+\int_s^r[A(t)^{-\beta}R_1(t,z)-A(r)^{-\beta}R_1(r,z)]R(z,s)dz.\end{aligned}$$

The estimate of the first term of the right hand side of (4.94) is given by (4.82). In the next place, we note the inequality

(4.31′) $$\|R_1(t,s)\|\leqq C,$$

which is proved in Kato-Tanabe [13], and get an estimate of the second term as

(4.95) $$\int_r^t\|A(t)^{-\beta}\|\cdot\|R_1(t,z)\|\cdot\|R(z,s)\|dz\leqq C_{\beta}(t-r).$$

Finally, the third term is estimated as

$$\int_s^r \|A(t)^{-\beta}R_1(t, z)-A(r)^{-\beta}R_1(r, z)\|\cdot\|R(z, s)\|dz \tag{4.96}$$
$$\leqq C_\beta\int_s^r (t-r)^\alpha(r-z)^{\beta-1}dz \quad (\because\ (4.82), (4.31))$$
$$=C_\beta(t-r)^\alpha(r-s)^\beta.$$

Thus, (4.93) has been established. □

Now, we give the

PROOF OF LEMMA 2′. By (4.28), we have

$$A(t)^{1-\beta}W(t, s)-A(r)^{1-\beta}W(r, s) \tag{4.97}$$
$$=\int_r^t A(t)^{1-\beta}e^{-(t-z)A(t)}R(z, s)dz$$
$$+\int_s^r [A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}]R(z, s)dz$$
$$=\int_r^t A(t)^{1-\beta}e^{-(t-z)A(t)}[R(z, s)-R(t, s)]dz$$
$$+\int_s^r [A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}]\cdot[R(z, s)-R(r, s)]dz$$
$$+\Big\{\int_r^t A(t)^{1-\beta}e^{-(t-z)A(t)}dzR(t, s)$$
$$+\int_s^r [A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}]dz\cdot R(r, s)\Big\}.$$

By (3.47) and (4.32), the first term of the right hand side of (4.97) is estimated as

$$\int_r^t \|A(t)^{1-\beta}e^{-(t-z)A(t)}\|\cdot\|R(z, s)-R(t, s)\|dz \tag{4.98}$$
$$\leqq C_{\beta,\gamma,\delta}\int_r^t (t-z)^{\beta-1}\{(t-z)^\gamma(z-s)^{-\gamma}+(t-z)^\delta(z-s)^{\alpha-\delta-1}\}dz$$
$$(0<\gamma<1, 0<\delta<\alpha)$$
$$=C_{\beta,\gamma}\int_r^t (t-z)^{\beta-1+\gamma}(z-s)^{-\beta+1-\gamma}\cdot(z-s)^{\beta-1}dz$$
$$+C_{\beta,\delta}\int_r^t (t-z)^{\beta-1+\delta}(z-s)^{-\beta+\alpha-\delta}\cdot(z-s)^{\beta-1}dz$$
$$\leqq C_{\beta,\gamma}\int_r^t (t-z)^{\beta-1+\gamma}(z-r)^{-\beta+1-\gamma}dz(r-s)^{\beta-1}$$
$$+C_{\beta,\delta}\int_r^t (t-z)^{\beta-1+\delta}(z-r)^{-\beta+\alpha-\delta}dz(r-s)^{\beta-1}$$

(in fact we can take γ and δ in $-\beta+1-\gamma<0$ and $-\beta+\alpha-\delta<0$, respectively)

$\leqq C_\beta(t-r)^\alpha(r-s)^{\beta-1}$.

In order to estimate the second term, we recall (3.54) and (4.89) and obtain

$$\begin{aligned}&\|A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}\|\\ &\leqq\|A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(t-z)A(r)}\|\\ &\quad+\|A(r)^{1-\beta}[e^{-(t-z)A(r)}-e^{-(r-z)A(r)}]\|\\ &\leqq C_\beta(t-r)(r-z)^{\beta-2}.\end{aligned}\tag{4.89'}$$

On the other hand we have

$$\begin{aligned}&\|A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}\|\\ &\leqq\|A(t)^{1-\beta}e^{-(t-z)A(t)}\|+\|A(r)^{1-\beta}e^{-(r-z)A(r)}\|\\ &\leqq C_\beta(r-z)^{\beta-1}\end{aligned}\tag{4.99}$$

by (3.47), hence

$$\begin{aligned}&\|A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}\|\\ &\leqq C_\beta(t-r)^\kappa(r-z)^{-\kappa+\beta-1}\quad(0\leqq\kappa\leqq1,\ 0\leqq\beta\leqq1).\end{aligned}\tag{4.100}$$

Therefore, the second term of the right hand side of (4.97) is estimated as

$$\begin{aligned}&\int_s^r\|A(t)^{1-\beta}e^{-(t-z)A(t)}-A(r)^{1-\beta}e^{-(r-z)A(r)}\|\cdot\|R(r,s)-R(z,s)\|dz\\ &\leqq C_{\beta,\gamma,\delta}\int_s^r(t-r)^\alpha(r-z)^{-\alpha+\beta-1}\{(r-z)^\gamma(z-s)^{-\gamma}+(r-z)^\delta(z-s)^{\alpha-\delta-1}\}dz\\ &\qquad\qquad(0<\gamma<1,\ 0<\delta<\alpha)\\ &\leqq C_\beta(t-r)^\alpha(r-s)^{\beta-1}\quad(\text{in fact we may take }\delta\text{ in }\alpha-\beta<\delta).\end{aligned}\tag{4.101}$$

The last term of the right hand side of (4.97) is rewritten as

$$\begin{aligned}&A(t)^{-\beta}e^{-(t-z)A(t)}\Big]_{z=r}^{z=t}\cdot R(t,s)\\ &\quad+\{A(t)^{-\beta}e^{-(t-z)A(t)}-A(r)^{-\beta}e^{-(r-z)A(r)}\}\Big]_{z=s}^{z=r}\cdot R(r,s)\\ &=A(t)^{-\beta}\{1-e^{-(t-r)A(t)}\}R(t,s)\\ &\quad+A(t)^{-\beta}\{e^{-(t-r)A(t)}-e^{-(t-s)A(t)}\}R(r,s)\\ &\quad-A(r)^{-\beta}\{1-e^{-(r-s)A(r)}\}R(r,s)\\ &=\{A(t)^{-\beta}R(t,s)-A(r)^{-\beta}R(r,s)\}\\ &\quad-e^{-(t-r)A(t)}[A(t)^{-\beta}R(t,s)-A(r)^{-\beta}R(r,s)]\\ &\quad-e^{-(t-r)A(t)}[A(r)^{-\beta}-A(t)^{-\beta}]R(r,s)\end{aligned}\tag{4.102}$$

$$-(A(t)^{-\beta}e^{-(t-s)A(t)}-A(t)^{-\beta}e^{-(r-s)A(t)})R(r, s)$$
$$-(A(t)^{-\beta}e^{-(r-s)A(t)}-A(r)^{-\beta}e^{-(r-s)A(r)})R(r, s).$$

The estimate of the first term of the right hand side of (4.102) is given by (4.93). Similarly, the second term is estimated as

$$\|e^{-(t-r)A(t)}\|\cdot\|A(t)^{-\beta}R(t, s)-A(r)^{-\beta}R(r, s)\| \leqq C_\beta(t-r)^\alpha(r-s)^{\beta-1}. \tag{4.103}$$

Since the estimate

$$\|A(t)^{-\beta}-A(s)^{-\beta}\|\leqq C_\beta|t-s| \tag{4.104}$$

is derived by (3.2) and (4.88), the third term of the right hand side of (4.102) is estimated as

$$\|e^{-(t-r)A(r)}\|\cdot\|A(r)^{-\beta}-A(t)^{-\beta}\|\cdot\|R(r, s)\|\leqq C_\beta(t-r). \tag{4.105}$$

By (4.89) the fourth term is estimated as

$$\|A(t)^{-\beta}e^{-(t-s)A(t)}-A(t)^{-\beta}e^{-(r-s)A(t)}\|\cdot\|R(r, s)\| \leqq C_\beta(t-r)(r-s)^{\beta-1}, \tag{4.106}$$

and by (4.90) the last term is estimated as

$$\|A(t)^{-\beta}e^{-(r-s)A(t)}-A(r)^{-\beta}e^{-(r-s)A(r)}\|\cdot\|R(r, s)\| \leqq C_\beta(t-r)(r-s)^{\beta-1}. \tag{4.107}$$

Thus, (4.40) follows. □

5. Proof of Theorem 2

Put

$$u_h(t)=U_h(t, 0)P_h a, \tag{5.1}$$

$$u_h^\tau(t)=U_h^\tau(t, 0)P_h a \quad (t=t_n) \tag{5.2}$$

and

$$e_h^\tau(t)=u_h^\tau(t)-u_h(t) \quad (t=t_n). \tag{5.3}$$

By means of (2.21) and (2.26) we have

$$\begin{aligned} e_h^\tau(t+\tau)-e_h^\tau(t) &=\int_t^{t+\tau}[A_h(r)u_h(r)-A_h(t+\tau)u_h^\tau(t+\tau)]dr \\ &=\int_t^{t+\tau}[A_h(r)u_h(r)-A_h(t+\tau)u_h(t+\tau)]dr-\tau A_h(t+\tau)e_h^\tau(t+\tau), \end{aligned} \tag{5.4}$$

hence

$$e_h^\tau(t+\tau)=(1+\tau A_h(t+\tau))^{-1}e_h^\tau(t)+(1+\tau A_h(t+\tau))^{-1} \times\int_t^{t+\tau}[A_h(r)u_h(r)-A_h(t+\tau)u_h(t+\tau)]dr \quad (t=t_n). \tag{5.5}$$

Since

$$e_h^\tau(0)=0, \tag{5.6}$$

we have

$$\begin{aligned} e_h^\tau(t_n)&=E_h^\tau(t_n)P_h a \\ &=-\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(1+\tau A_h(t_n))^{-1}(1+\tau A_h(t_{n-1}))^{-1} \\ &\quad\cdots(1+\tau A_h(t_k))^{-1}[A_h(t_k)U_h(t_k,0)-A_h(r)U_h(r,0)]drP_h a, \end{aligned} \tag{5.7}$$

$E_h^\tau(t)$ being the error operator:

$$E_h^\tau(t)=U_h^\tau(t,0)-U_h(t,0). \tag{5.8}$$

By virtue of Lemma 2, we have

$$\begin{aligned} -E_h^\tau(t_n)&=\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(1+\tau A_h(t_n))^{-1}\cdots(1+\tau A_h(t_k))^{-1}A_h(t_k)[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]dr \\ &\quad+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(1+\tau A_h(t_n))^{-1}\cdots(1+\tau A_h(t_k))^{-1}A_h(t_k)^\beta Z_{h,\beta}(t_k,r,0)dr \\ &=(1+\tau A_h(t_n))^{-1}A_h(t_n)\int_{t_{n-1}}^{t_n}[e^{-t_nA_h(t_n)}-e^{-rA_h(t_n)}]dr \\ &\quad+\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}U_h^\tau(t_n,t_{k-1})A_h(t_k)[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]dr \\ &\quad+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}U_h^\tau(t_n,t_{k-1})A_h(t_k)^\beta Z_{h,\beta}(t_k,r,0)dr \quad (0<\beta<1/2). \end{aligned} \tag{5.9}$$

Supposing $n\geqq 2$, the first term of the right hand side of (5.9) is estimated as

$$\begin{aligned} &\|(1+\tau A_h(t_n))^{-1}A_h(t_n)\|\int_{t_{n-1}}^{t_n}\|e^{-t_nA_h(t_n)}-e^{-rA_h(t_n)}\|dr \\ &\leqq C\tau^{-1}\int_{t_{n-1}}^{t_n}(t_n-r)r^{-1}dr \quad (\because\ (4.89)) \\ &\leqq C\tau^{-1}\cdot\tau^2(t_{n-1})^{-1}\leqq Cn^{-1}. \end{aligned} \tag{5.10}$$

By (2.32) and (2.34), the third term of the right hand side of (5.9) is estimated as

(5.11)
$$\begin{aligned}&\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\|U_h^\tau(t_n,t_{k-1})A_h(t_k)^\beta\|\cdot\|Z_{h,\beta}(t_k,r,0)\|dr\\ &\leqq C_\beta\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(t_n-t_{k-1})^{-\beta}(t_k-r)^\alpha r^{-1+\beta}dr\\ &\leqq C_\beta\sum_{k=1}^{n}(n-k+1)^{-\beta}\tau^{-\beta}\tau^{\alpha+1}(k\tau)^{\beta-1}\\ &=C_\beta\tau^\alpha\sum_{k=1}^{n}(n-k+1)^{-\beta}k^{-1+\beta}\leqq C\tau^\alpha\quad(\because\ (3.20)).\end{aligned}$$

Therefore, we have only to prove

(5.12)
$$\left\|\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}U_h^\tau(t_n,t_{k-1})A_h(t_k)[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]dr\right\|\leqq C/n.$$

To this end, we note

(5.13)
$$\begin{aligned}&n\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}U_h^\tau(t_n,t_{k-1})A_h(t_k)[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]dr\\ &=\sum_{k=1}^{n-1}(n-k+1)U_h^\tau(t_n,t_{k-1})A_h(t_k)^{1+\beta}\int_{t_{k-1}}^{t_k}A_h(t_k)^{-\beta}[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]dr\\ &\quad+\sum_{k=2}^{n-1}(k-1)U_h^\tau(t_n,t_{k-1})A_h(t_k)^{1-\beta}\int_{t_{k-1}}^{t_k}A_h(t_k)^{\beta}[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]dr\\ &\hspace{25em}(0<\beta<1/3).\end{aligned}$$

The first term of the right hand side of (5.13) is estimated as

(5.14)
$$\begin{aligned}&\sum_{k=1}^{n-1}(n-k+1)\|U_h^\tau(t_n,t_{k-1})A_h(t_k)^{1+\beta}\|\cdot\int_{t_{k-1}}^{t_k}\|A_h(t_k)^{-\beta}[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]\|dr\\ &\leqq C_\beta\sum_{k=1}^{n-1}(n-k+1)^{-\beta}\tau^{-1-\beta}\int_{t_{k-1}}^{t_k}(t_k-r)r^{\beta-1}dr\quad(\because\ (4.89),(2.32))\\ &\leqq C_\beta\sum_{k=1}^{n-1}(n-k+1)^{-\beta}\tau^{-1-\beta}\cdot\tau^2\cdot(k\tau)^{\beta-1}\\ &=C_\beta\sum_{k=1}^{n-1}(n-k+1)^{-\beta}k^{\beta-1}\leqq C\quad(\because\ (3.20)).\end{aligned}$$

The second term of the right hand side of (5.13) is estimated as

(5.15)
$$\begin{aligned}&\sum_{k=2}^{n-1}(k-1)\|U_h^\tau(t_n,t_{k-1})A_h(t_k)^{1-\beta}\|\int_{t_{k-1}}^{t_k}\|A_h(t_k)^{\beta}[e^{-t_kA_h(t_k)}-e^{-rA_h(t_k)}]\|dr\\ &\leqq C_\beta\sum_{k=2}^{n-1}(k-1)(t_n-t_{k-1})^{-1+\beta}\int_{t_{k-1}}^{t_k}(t_k-r)r^{-\beta-1}dr\quad(\because\ (4.89),(2.32))\\ &\leqq C_\beta\sum_{k=2}^{n-1}(k-1)(n-k+1)^{\beta-1}\tau^{\beta-1}\cdot\tau^2\cdot((k-1)\tau)^{-\beta-1}\\ &=C_\beta\sum_{k=2}^{n-1}(k-1)^{-\beta}(n-k+1)^{\beta-1}\leqq C\quad(\because\ (3.20)).\end{aligned}$$

Thus, (5.12) follows. □

6. Proof of Theorem 1

We put

$$E_h(t, s)=U_h(t, s)P_h-U(t, s). \tag{6.1}$$

By operating $\int_s^t \cdot dr$ on both sides of

$$-\frac{\partial}{\partial r}[U_h(t, r)P_hE_h(r, s)]=U_h(t, r)[A_h(r)P_h-P_hA(r)]U(r, s), \tag{6.2}$$

we obtain

$$P_hE_h(t, s)=\int_s^t U_h(t, r)[P_hA(r)-A_h(r)P_h]U(r, s)dr. \tag{6.3}$$

We introduce the so-called Ritz-projection $R_h(t)$: $V\to V_h$ through the identity

$$a_t(R_h(t)v, \chi)=a_t(v, \chi) \quad (v\in V, \chi\in V_h). \tag{6.4}$$

Lax-Milgram's theorem assures us of the well-definedness of $R_h(t)$. Since for $v\in D(A(t))$ and $\chi\in V_h$ the equality

$$\begin{aligned}(A_h(t)R_h(t)v, \chi)&=a_t(R_h(t)v, \chi)\\ &=a_t(v, \chi)=(A(t)v, \chi)\end{aligned}$$

holds, we have

$$A_h(t)R_h(t)v=P_hA(t)v. \tag{6.5}$$

Therefore, the equality

$$\begin{aligned}E_h(t, s)&=(1-P_h)E_h(t, s)+P_hE_h(t, s)\\ &=E_h^{(1)}(t, s)+E_h^{(2)}(t, s)+E_h^{(3)}(t, s)\end{aligned} \tag{6.6}$$

follows, where

$$E_h^{(1)}(t, s)=(1-U_h(t, s)P_h)(R_h(t)-1)U(t, s), \tag{6.7}$$

$$E_h^{(2)}(t, s)=\int_s^t U_h(t, r)A_h(r)[R_h(r)-R_h(t)]U(t, s)dr \tag{6.8}$$

and

$$E_h^{(3)}(t, s)=\int_s^t U_h(t, r)A_h(r)P_h(R_h(r)-1)[U(r, s)-U(t, s)]dr. \tag{6.9}$$

We want to prove $\|E_h^{(j)}(t, s)\| \leqq Ch^2/(t-s)$ for $j=1, 2, 3$.

Estimate of $E_h^{(1)}(t, s)$: The inequality (2.19) implies the following

LEMMA 3. *The estimates*

$$\|(R_h(t)-1)v\|_1 \leqq Ch\|v\|_2 \quad (v \in H^2(\Omega) \cap V) \tag{6.10}$$

and

$$\|(R_h(t)-1)v\|_0 \leqq Ch^2\|v\|_2 \quad (v \in H^2(\Omega) \cap V) \tag{6.11}$$

hold.

Therefore, by means of the elliptic estimate of Agmon-Douglis-Nirenberg [2], we have

$$\begin{aligned}\|E_h^{(1)}(t, s)\| &\leqq (1+\|U_h(t, s)\| \cdot \|P_h\|) \cdot \|(R_h(t)-1)A(t)^{-1}\| \cdot \|A(t)U(t, s)\| \\ &\leqq Ch^2/(t-s) \quad (\because\ (2.16),\ (6.11)).\end{aligned} \tag{6.12}$$

□

PROOF OF LEMMA 3. We put

$$z=(1-R_h(t))v \quad (v \in H^2(\Omega) \cap V).$$

Then we have

$$\begin{aligned}\delta\|z\|_1^2 &\leqq \mathrm{Re}\ a_t((1-R_h(t))v, (1-R_h(t))v) \\ &= \mathrm{Re}\ a_t((1-R_h(t))v, v) \quad (\because\ (6.4)) \\ &= \mathrm{Re}\ a_t((1-R_h(t))v, v-\chi) \quad (\because\ (6.4)) \\ &\leqq C\|(1-R_h(t))v\|_1\|v-\chi\|_1,\end{aligned}$$

χ being an arbitrary element in V_h. Hence we get

$$\|z\|_1 \leqq C \inf_{\chi \in V_h} \|v-\chi\|_1 \leqq Ch\|v\|_2$$

by (2.19). The estimate (6.11) is given as follows by virtue of Nitshe's trick: Taking $\chi \in V_h$, we have

$$\begin{aligned}\|z\|_0^2 &= a_t(z, A(t)^{*-1}z) \\ &= a_t((1-R_h(t))v, A(t)^{*-1}z-\chi) \quad (\because\ (6.4)) \\ &\leqq C\|(1-R_h(t))v\|_1\|A(t)^{*-1}z-\chi\|_1,\end{aligned}$$

so that

$$\begin{aligned}\|z\|_0^2 &\leqq Ch\|v\|_2 \inf_{\chi \in V_h} \|A(t)^{*-1}z-\chi\|_1 \\ &\leqq Ch^2\|v\|_2 \cdot \|A(t)^{*-1}z\|_2 \leqq Ch^2\|v\|_2\|z\|_0\end{aligned}$$

by the elliptic estimate. □

Estimate of $E_h^{(2)}(t, s)$:

LEMMA 4. *The estimate*

$$\|(R_h(t)-R_h(s))v\|_0 \leqq Ch^2|t-s|\,\|v\|_2 \quad (v \in H^2(\Omega)\cap V) \tag{6.13}$$

holds.

By this lemma and by the elliptic estimate, we obtain

$$\begin{aligned}\|E_h^{(2)}(t, s)\| &\leqq \int_s^t \|U_h(t, r)A_h(r)\|\cdot\|(R_h(r)-R_h(t))A(t)^{-1}\|\cdot\|A(t)U(t, s)\|dr \\ &\leqq C\int_s^t (t-r)^{-1+1}h^2 dr(t-s)^{-1} \\ &= Ch^2 \leqq Ch^2/(t-s),\end{aligned} \tag{6.14}$$

for

$$\|U_h(t, s)A_h(s)\| \leqq C(t-s)^{-1} \tag{2.25'}$$

is obvious by considering the adjoint operator in (2.25). □

PROOF OF LEMMA 4. Let $\hat{R}_h(t): V \to V_h$ be the Ritz-projection associated with the adjoint form of a_t; namely,

$$a_t^*(\hat{R}_h(t)v, \chi) = a_t^*(v, \chi) \quad (v \in V, \chi \in V_h), \tag{6.4'}$$

where $a_t^*(\,,\,)$ is the sesquilinear form given by (4.55). In the same way as in Lemma 3, the estimates

$$\|(\hat{R}_h(t)-1)v\|_1 \leqq Ch\|v\|_2 \quad (v \in H^2(\Omega)\cap V) \tag{6.10'}$$

and

$$\|(\hat{R}_h(t)-1)v\|_0 \leqq Ch^2\|v\|_2 \quad (v \in H^2(\Omega)\cap V) \tag{6.11'}$$

hold.

We put

$$z = (R_h(t)-R_h(s))v \in V_h$$

and obtain

$$\begin{aligned}\|z\|_0^2 &= a_t(z, A(t)^{*-1}z) \\ &= a_t(z, \hat{R}_h(t)A(t)^{*-1}z) \quad (\because\ (6.4')) \\ &= a_t((1-R_h(s))v, \hat{R}_h(t)A(t)^{*-1}z) \quad (\because\ (6.4)) \\ &= (a_t-a_s)((1-R_h(s))v, \hat{R}_h(t)A(t)^{*-1}z) \quad (\because\ (6.4)) \\ &= (a_t-a_s)((1-R_h(s))v, (\hat{R}_h(t)-1)A(t)^{*-1}z) \\ &\quad + (a_t-a_s)((1-R_h(s))v, A(t)^{*-1}z)\end{aligned}$$

$$
\begin{aligned}
&=(a_t-a_s)((1-R_h(s))v,\ (\hat{R}_h(t)-1)A(t)^{*-1}z)\\
&\quad+a_s((1-R_h(s))v,\ (A(s)^{*-1}-A(t)^{*-1})z)\\
&=(a_t-a_s)((1-R_h(s))v,\ (\hat{R}_h(t)-1)A(t)^{*-1}z)\\
&\quad+a_s((1-R_h(s))v,\ (1-R_h(s))(A(s)^{*-1}-A(t)^{*-1})z) \quad (\because\ (6.4))\\
&\leqq C|t-s|\ \|(1-R_h(s))v\|_1\|(\hat{R}_h(t)-1)A(t)^{*-1}z\|_1\\
&\quad+C\|(1-R_h(s))v\|_1\|(1-R_h(s))(A(s)^{*-1}-A(t)^{*-1})z\|_1\\
&\leqq C|t-s|h^2\|v\|_2\|z\|_0\\
&\quad+Ch^2\|v\|_2\|(A(s)^{*-1}-A(t)^{*-1})z\|_2 \quad (\because\ (6.10),(6.10')).
\end{aligned}
$$

From the elliptic estimate, the inequality

$$\|(A(s)^{*-1}-A(t)^{*-1})z\|_2\leqq C|t-s|\ \|z\|_0 \tag{6.15}$$

follows. See, Tanabe [22], for instance. Therefore, we get (6.13). □

Estimate of $E_h^{(3)}(t,s)$: We note the equality

$$(t-s)E_h^{(3)}(t,s)=\sum_{j=1}^{5}F_h^{(j)}(t,s), \tag{6.16}$$

where

$$F_h^{(1)}(t,s)=\int_s^t(r-s)U_h(t,r)A_h(r)P_h(R_h(r)-1)[U(r,s)-U(t,s)]dr, \tag{6.17}$$

$$F_h^{(2)}(t,s)=\int_s^t(t-r)U_h(t,r)A_h(r)(R_h(r)-R_h(s))[U(r,s)-U(t,s)]dr, \tag{6.18}$$

$$
\begin{aligned}
F_h^{(3)}(t,s)=\int_s^t&(t-r)[U_h(t,r)A_h(r)-U_h(t,s)A_h(s)]\\
&\times P_h(R_h(s)-1)[U(r,s)-U(t,s)]dr,
\end{aligned} \tag{6.19}
$$

$$F_h^{(4)}(t,s)=-U_h(t,s)A_h(s)P_h(R_h(s)-1)\cdot\int_s^t(r-s)[U(r,s)-U(t,s)]dr \tag{6.20}$$

and

$$F_h^{(5)}(t,s)=U_h(t,s)A_h(s)P_h(R_h(s)-1)(t-s)\cdot\int_s^t[U(r,s)-U(t,s)]dr. \tag{6.21}$$

The following inequalities will be proved later for $0\leqq\beta\leqq1$ and $0\leqq\kappa<\alpha$:

$$
\begin{aligned}
&\|U(t,s)-U(r,s)\|_{L^2(\Omega)\to H^2(\Omega)}\\
&\leqq C_\kappa\{(t-r)^\beta(r-s)^{-\beta-1}+(t-r)^\kappa(r-s)^{-1}\} \quad (T\geqq t>r>s\geqq0).
\end{aligned} \tag{6.22}
$$

$$
\begin{aligned}
&\|U_h(t,r)A_h(r)-U_h(t,s)A_h(s)\|\\
&\leqq C_\kappa\{(t-r)^{-1-\beta}(r-s)^\beta+(t-r)^{-1}(r-s)^\kappa\} \quad (T\geqq t>r>s\geqq0).
\end{aligned} \tag{6.23}
$$

(6.24)
$$\left\|\int_s^t [U(t,s)-U(r,s)]dr\right\|_{L^2(\Omega)\to H^2(\Omega)} \leqq C,$$

From these inequalities, we can derive the following estimates:

(6.25)
$$\begin{aligned}\|F_h^{(1)}(t,s)\| &\leqq \int_s^t (r-s)\|U_h(t,r)A_h(r)\| \\ &\quad \times \|(R_h(r)-1)\|_{H^2(\Omega)\to L^2(\Omega)}\|U(r,s)-U(t,s)\|_{L^2(\Omega)\to H^2(\Omega)}dr \\ &\leqq C_\kappa h^2\int_s^t (r-s)(t-r)^{-1}\{(t-r)^\beta(r-s)^{-\beta-1}+(t-r)^\kappa(r-s)^{-1}\}dr \quad (\because (6.22)) \\ &\leqq Ch^2 \quad (0<\beta<1).\end{aligned}$$

(6.26)
$$\begin{aligned}\|F_h^{(2)}(t,s)\| &\leqq \int_s^t (t-r)\|U_h(t,r)A_h(r)\| \\ &\quad \times \|(R_h(r)-R_h(s))\|_{H^2(\Omega)\to L^2(\Omega)}\|U(r,s)-U(t,s)\|_{L^2(\Omega)\to H^2(\Omega)}dr \\ &\leqq C_\kappa h^2\int_s^t (r-s)\{(t-r)^\beta(r-s)^{-\beta-1}+(t-r)^\kappa(r-s)^{-1}\}dr \quad (\because (6.13)) \\ &\leqq Ch^2 \quad (0<\beta<1).\end{aligned}$$

(6.27)
$$\begin{aligned}\|F_h^{(3)}(t,s)\| &\leqq \int_s^t (t-r)\|U_h(t,r)A_h(r)-U_h(t,s)A_h(s)\| \\ &\quad \times \|R_h(s)-1\|_{H^2(\Omega)\to L^2(\Omega)}\|U(t,s)-U(r,s)\|_{L^2(\Omega)\to H^2(\Omega)}dr \\ &\leqq C_\kappa h^2\int_s^t (t-r)\{(t-r)^{-1-\beta}(r-s)^\beta+(t-r)^{-1}(r-s)^\kappa\} \\ &\quad \times \{(t-r)^\gamma(r-s)^{-1-\gamma}+(t-r)^\kappa(r-s)^{-1}\}dr \quad (\because (6.23)) \\ &\leqq Ch^2 \quad (1\geqq\beta\geqq\kappa>\gamma>0).\end{aligned}$$

(6.28)
$$\begin{aligned}\|F_h^{(4)}(t,s)\| &\leqq \|U_h(t,s)A_h(s)\| \\ &\quad \times \|R_h(s)-1\|_{H^2(\Omega)\to L^2(\Omega)}\int_s^t \|U(t,s)-U(r,s)\|_{L^2(\Omega)\to H^2(\Omega)}\cdot(r-s)dr \\ &\leqq C_\kappa(t-s)^{-1}h^2\int_s^t \{(t-r)^\beta(r-s)^{-\beta-1}+(t-r)^\kappa(r-s)^{-1}\}\cdot(r-s)dr \\ &\leqq Ch^2 \quad (0<\beta<1).\end{aligned}$$

(6.29)
$$\begin{aligned}\|F_h^{(5)}(t,s)\| &\leqq \|U_h(t,s)A_h(s)\|\cdot\|R_h(s)-1\|_{H^2(\Omega)\to L^2(\Omega)}(t-s) \\ &\quad \times \left\|\int_s^t [U(r,s)-U(t,s)]dr\right\|_{L^2(\Omega)\to H^2(\Omega)} \\ &\leqq Ch^2 \quad (\because (6.24)).\end{aligned}$$

Summing up these estimates, we obtain

(6.30)
$$\|E_h^{(3)}(t,s)\| \leqq Ch^2/(t-s). \qquad \square$$

In order to complete the proof, we show (6.22), (6.23), (6.24). To this end we need

LEMMA 5. *The estimates*

(6.31)
$$\|A(t)U(t, s)-A(r)U(r, s)\| \leqq C_\kappa\{(t-r)^\beta(r-s)^{-\beta-1}+(t-r)^\kappa(r-s)^{-1}\} \quad (T\geqq t>r>s\geqq 0)$$

and

(6.32)
$$\|A_h(t)U_h(t, s)-A_h(r)U_h(r, s)\| \leqq C_\kappa\{(t-r)^\beta(r-s)^{-\beta-1}+(t-r)^\kappa(r-s)^{-1}\} \quad (T\geqq t>r>s\geqq 0)$$

hold for each β *and* κ *in* $0\leqq\beta\leqq 1$ *and* $0\leqq\kappa<\alpha$.

PROOF OF LEMMA 5. The proof is similar to that of Lemma 2. In fact we have

(6.33)
$$A(t)U(t, s)-A(r)U(r, s) =(A(t)e^{-(t-s)A(t)}-A(r)e^{-(r-s)A(r)})+(A(t)W(t, s)-A(r)W(r, s))$$

by (4.27). The first term of the right hand side of (6.33) is estimated by (4.100). That is

(6.34)
$$\|A(t)e^{-(t-s)A(t)}-A(r)e^{-(r-s)A(r)}\| \leqq C(t-r)^\beta(r-s)^{-\beta-1} \quad (0\leqq\beta\leqq 1).$$

By (4.28), the second term of the right hand side of (6.33) is equal to

(6.35)
$$\begin{aligned} &A(t)W(t, s)-A(r)W(r, s) \\ &=\int_r^t A(t)e^{-(t-z)A(t)}R(z, s)dz \\ &\quad+\int_s^r [A(t)e^{-(t-z)A(t)}-A(r)e^{-(r-z)A(r)}]R(z, s)dz \\ &=\int_r^t A(t)e^{-(t-z)A(t)}[R(z, s)-R(t, s)]dz \\ &\quad+\int_s^r [A(t)e^{-(t-z)A(t)}-A(r)e^{-(r-z)A(r)}][R(z, s)-R(r, s)]dz \\ &\quad+(1-e^{-(t-r)A(t)})[R(t, s)-R(r, s)] \\ &\quad-(e^{-(t-s)A(t)}-e^{-(r-s)A(r)})R(r, s). \end{aligned}$$

Each term except the second one of the right hand side of (6.35) is estimated in the same way as in §4. That is, we have

(6.36)
$$\begin{aligned}&\int_r^t \|A(t)e^{-(t-z)A(t)}\|\cdot\|R(z, s)-R(t, s)\|dz\\ &\leqq C_{\gamma,\delta}\int_r^t (t-z)^{-1}\{(t-z)^{\gamma}(z-s)^{-\gamma}+(t-z)^{\delta}(z-s)^{\alpha-\delta-1}\}dz\\ &\qquad\qquad (\because\ (4.32))\quad (0<\gamma<1, 0<\delta<\alpha)\\ &\leqq C(t-r)^{\alpha}(r-s)^{-1},\end{aligned}$$

(6.37)
$$\begin{aligned}&(1+\|e^{-(t-r)A(t)}\|)\|R(t, s)-R(r, s)\|\\ &\leqq C_{\gamma,\delta}\{(t-r)^{\gamma}(r-s)^{-\gamma}+(t-r)^{\delta}(r-s)^{\alpha-\delta-1}\}\quad (\because\ (4.32))\quad (0<\gamma<1, 0<\delta<\alpha),\end{aligned}$$

and

(6.38)
$$\begin{aligned}&\|e^{-(t-s)A(t)}-e^{-(r-s)A(r)}\|\cdot\|R(r, s)\|\\ &\leqq C(t-r)(r-s)^{-1}\quad (\because\ (4.100), (4.31)).\end{aligned}$$

Finally, the second term of the right hand side of (6.35) is estimated as

(6.39)
$$\begin{aligned}&\int_s^r \|A(t)e^{-(t-z)A(t)}-A(r)e^{-(r-z)A(r)}\|\cdot\|R(r, s)-R(z, s)\|dz\\ &\leqq C_{\gamma,\delta}\int_s^r (t-r)^{\kappa}(r-z)^{-\kappa-1}\{(r-z)^{\gamma}(z-s)^{-\gamma}+(r-z)^{\delta}(z-s)^{\alpha-\delta-1}\}dz\quad (\because\ (4.100))\\ &=C_{\kappa}(t-r)^{\kappa}(r-s)^{-\kappa}\{1+(r-s)^{\alpha-1}\}\quad (0<\kappa<\alpha-\delta)\\ &\leqq C_{\kappa}(t-r)^{\kappa}(r-s)^{-1},\end{aligned}$$

taking $0\leqq\kappa<\gamma, \delta$. □

PROOF OF (6.22). We shall only consider the case of the boundary condition (2.4′), for the proof is simpler in the case of the boundary condition (2.4). We take $a\in L^2(\Omega)$ and set

$$u(t, x)=(U(t, s)a)(x).$$

On $\partial\Omega$, we have

$$\mathscr{B}(t, x, D)u(t, x)=0,$$

hence

(6.40)
$$\begin{aligned}&\mathscr{B}(t, x, D)(u(t, x)-u(r, x))\\ &=(\mathscr{B}(r, x, D)-\mathscr{B}(t, x, D))u(r, x)\quad (x\in\partial\Omega).\end{aligned}$$

One the other hand, we have

(6.41)
$$\begin{aligned}&\mathscr{A}(t, x, D)(u(t, x)-u(r, x))\\ &=(A(t)U(t, s)a)(x)-(A(r)U(r, s)a)(x)\\ &\quad+[\mathscr{A}(r, x, D)-\mathscr{A}(t, x, D)]u(r, x).\end{aligned}$$

We extend $n=n(x)\in C^2(\partial\Omega)$, $\sigma=\sigma(x)\in C^0(\partial\Omega)$ to $n=\tilde{n}(x)\in C^2(\bar{\Omega})$, $\sigma=\tilde{\sigma}(x)\in C^0(\bar{\Omega})$, and put

$$(6.42)\qquad \tilde{\mathscr{B}}(t, x, D)=\sum_{i,j=1}^{2}\tilde{n}_i(x)a_{ij}(t, x)\frac{\partial}{\partial x_j}+\tilde{\sigma}(t, x)\quad (t\in[0, T], x\in\bar{\Omega}).$$

Then, by the elliptic estimate we obtain

$$\begin{aligned}(6.43)\qquad &\|[U(t, s)-U(r, s)]a\|_2=\|u(t, \cdot)-u(r, \cdot)\|_2\\ &\leqq C(\|\mathscr{A}(t, \cdot, D)[u(t, \cdot)-u(r, \cdot)]\|_0\\ &\quad+\|\mathscr{B}(t, \cdot, D)[u(t, \cdot)-u(r, \cdot)]\|_{H^{1/2}(\partial\Omega)})\\ &\leqq C(\|(A(t)U(t, s)-A(r)U(r, s))a\|_0\\ &\quad+\|(\mathscr{A}(r, \cdot, D)-\mathscr{A}(t, \cdot, D))u(r, \cdot)\|_0\\ &\quad+\|(\tilde{\mathscr{B}}(r, \cdot, D)-\tilde{\mathscr{B}}(t, \cdot, D))u(r, \cdot)\|_1)\\ &\leqq C(\|A(t)U(t, s)-A(r)U(r, s)\|\cdot\|a\|_0+(t-r)\|u(r, \cdot)\|_2)\\ &\leqq C(\|A(t)U(t, s)-A(r)U(r, s)\|\cdot\|a\|_0+(t-r)(r-s)^{-1}\|a\|_0)\quad (\because\ (2.16)).\end{aligned}$$

Thus, (6.22) is reduced to Lemma 5. □

PROOF OF (6.23). By considering the adjoint operator in (6.32), we obtain (6.23). □

PROOF OF (6.24). We shall only consider the case of the boundary condition (2.4′). We take $a\in L^2(\Omega)$ and set

$$u(t, x)=(U(t, s)a)(x).$$

Then, in the same way as in the proof of (6.22), we have

$$\begin{aligned}(6.44)\qquad &\mathscr{A}(s, x, D)[u(t, x)-u(r, x)]\\ &=(\mathscr{A}(s, x, D)-\mathscr{A}(t, x, D))u(t,x)\\ &\quad+(\mathscr{A}(r, x, D)-\mathscr{A}(s, x, D))u(r, x)\\ &\quad+([A(t)U(t, s)-A(r)U(r, s)]a)(x)\end{aligned}$$

in Ω, and

$$\begin{aligned}(6.45)\qquad &\mathscr{B}(s, x, D)[u(t, x)-u(r, x)]\\ &=(\mathscr{B}(s, x, D)-\mathscr{B}(t, x, D))u(t, x)+(\mathscr{B}(r, x, D)-\mathscr{B}(s, x, D))u(r, x)\end{aligned}$$

on $\partial\Omega$. By the elliptic estimate we get

$$\begin{aligned}(6.46)\qquad &\left\|\int_s^t[U(t, s)-U(r, s)]a\,dr\right\|_2\\ &=\left\|\int_s^t[u(t, \cdot)-u(r, \cdot)]dr\right\|_2\end{aligned}$$

$$
\begin{aligned}
&\leqq C\Big((t-s)\|(\mathscr{A}(s,\cdot,D)-\mathscr{A}(t,\cdot,D))u(t,\cdot)\|_0 \\
&\quad+\int_s^t\{\|(\mathscr{A}(r,\cdot,D)-\mathscr{A}(s,\cdot,D))u(r,\cdot)\|_0 \\
&\qquad\qquad+\|(\tilde{\mathscr{B}}(r,\cdot,D)-\tilde{\mathscr{B}}(s,\cdot,D))u(r,\cdot)\|_1\}dr \\
&\quad+(t-s)\|(\tilde{\mathscr{B}}(s,\cdot,D)-\tilde{\mathscr{B}}(t,\cdot,D))u(t,\cdot)\|_1\Big) \\
&\quad+C\left\|\int_s^t[A(t)U(t,s)-A(r)U(r,s)]dra\right\|_0 \\
&\leqq C\Big((t-s)^2\|u(t,\cdot)\|_2+\int_s^t(r-s)\|u(r,\cdot)\|_2dr\Big) \\
&\quad+C(\|(t-s)A(t)U(t,s)\|+\|(U(t,s)-1)\|\cdot\|a\|_0) \\
&\leqq C(t-s)\|a\|_0+C\|a\|_0 \\
&\leqq C\|a\|_0. \qquad \square
\end{aligned}
$$

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