On the semi-discrete finite element approximation for the nonstationary Stokes equation

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1. Introduction.

We are concerned in this paper and a forthcoming one mainly with the semi-discrete finite element approximation for the nonstationary Stokes or Navier-Stokes equation. Here "semi-discrete" means that we discretize only the space variables, keeping the time variable continuous.

Recent study shows that the accuracy of the finite element approximation for the stationary Stokes equation is analogous to that for the Poisson equation. Among various works Bercovier and Pironneau [2] is worthy of particular notice because of the following two characters; their elements are continuous and very simple (piecewise linear or piecewise quadratic), yet the error is of optimal order (compare it with the earlier work Crouzeix and Raviart [7]); the condition $\operatorname{div} u=0$ is indirectly approximated by means of the linear relation $(\operatorname{div} u, q)=0$ for all q in a certain class of scalar functions, and the discretized pressure arises as a Lagrange multiplier for some variational problem. Their work may be considered as a special and concrete application of a general theory developped in Girault and Raviart's book [12], where they constructed an abstract theory for the mixed finite element method, however, without obtaining optimal rate of convergence.

On the other hand, Fujita and Mizutani [10] gave another general method to derive error estimate committed by finite element approximations for evolution equations of parabolic type. Their error estimate for nonstationary solutions was given in terms of the error estimate in the stationary problem with the aid of the Dunford integral. Actually they succeeded in obtaining optimal rate of convergence for nonstationary solutions of parabolic equations without assuming smoothness of the initial value (in the case of piecewise linear elements).

Our aim in the present paper is to show for the nonstationary Stokes equation that we have optimal rate of convergence similar to that in Fujita and Mizutani [10] when we use Bercovier and Pironneau's finite element spaces. Our results are valid whenever the initial value belongs to H, which is the set of solenoidal L^2-vector functions (for exact meaning see Section 2). In particular, the existence and further regularity of the derivatives of the initial value is not required at all, which may be regarded as one of the merits of our analysis. In a forthcoming paper we shall deal with the Navier-Stokes equation, where the dependence of the error on the initial value will be studied in detail.

The present paper consists of six sections. In Section 2 we formulate the Stokes problem and the scheme of semi-discrete approximation for them within a framework of the operator theory. In Section 3 we derive an error estimate with respect to $H^1(\Omega)$-norm, while in Section 4 an error estimate in $L^2(\Omega)$ is given. Section 5 is devoted to error estimation for the pressure. Throughout the above five sections the time variable is left to be continuous, but in Section 6 we shall give some remarks about the full-discrete approximation. There we consider the backward or the forward finite difference schemes with respect to the time variable.

After finishing this work we have been informed of Heywood and Rannacher's recent work [18], which treats the same problem as ours for the Navier-Stokes equation. They have obtained an optimal rate of convergence in a way quite different from ours. Their method requires some estimates for $\partial u/\partial t$ or $\partial^2 u/\partial t^2$ $(0<t)$, where u is the solution of the nonstationary Navier-Stokes equation. On the contrary our method requires only $\|u(0)\|_{H^2(\Omega)}<\infty$ or even $\|u(0)\|_{H^1(\Omega)}<\infty$ to derive similar results. In particular, we need not estimate $\partial^2 u/\partial t^2$. For details, see Okamoto [20].

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2. Notations and formulations of the problem.

2.1. Continuous problems.

Let Ω be a convex polygonal (or polyhedral) domain in $\boldsymbol{R}^2$ (or $\boldsymbol{R}^3$). We introduce the following symbols; $X=H_0^1(\Omega)^k\cdots$ the space of all $\boldsymbol{R}^k$-valued functions defined in Ω, which vanish on the boundary $\partial\Omega$ and have all the first order derivatives in $L^2(\Omega)^k$ (k is 2 or 3 according as $\Omega\subset\boldsymbol{R}^2$ or $\Omega\subset\boldsymbol{R}^3$).

$$M=\left\{q\in L^2(\Omega);\ \int_\Omega q(x)dx=0\right\},$$
$$V=\{v\in X;\ \mathrm{div}\, v=0 \text{ in } \Omega\},$$
$$H=\{v\in L^2(\Omega)^k;\ \mathrm{div}\, v=0 \text{ in } \Omega,\ v\cdot n=0 \text{ on } \partial\Omega\}.$$

Here n is the outerward normal vector on $\partial\Omega$. As is obvious, X is a Hilbert space with respect to the inner-product $(u, v)_X=(u, v)+\sum_{j=1}^{k}(\partial u/\partial x_j, \partial v/\partial x_j)$. Here and hereafter $(\ ,\ )$ means the inner-product of $L^2(\Omega)^k$ or $L^2(\Omega)$. The norm of M or H is the L^2 norm i.e., $\|q\|_M^2=\int_\Omega |q(x)|^2dx$, $\|u\|_H^2=\sum_{i=1}^{k}\int_\Omega |u_i(x)|^2dx$. Furthermore H is the closure in $L^2(\Omega)^k$ of $C_{0,\sigma}^\infty(\Omega)=\{v=(v_1, \cdots, v_k)\in C_0^\infty(\Omega)^k;\ \mathrm{div}\, v=0 \text{ in } \Omega\}$ (see Fujita and Kato [9] or Temam [22]).

Under these notations we can formulate the nonstationary Stokes equation as follows:

Find $u\in C([0, \infty[: H)\cap C^1(]0, \infty[: H)$ *and* $p\in C(]0, \infty[: M)$ *such that*

$$(S)_1 \qquad \begin{cases} (2.1)\quad u(t)\in V & (0<t), \\ (2.2)\quad (du/dt, v)+(\nabla u(t), \nabla v)-(\mathrm{div}\, v, p(t))=0 & (0<t), \\ \qquad\quad \textit{for any } v\in X, & \\ (2.3)\quad u(0)=a. & \end{cases}$$

For simplicity we have assumed that the external force is absent and the boundary condition is homogeneous. We assume that the initial value a belongs to H.

Now we eliminate the pressure $p(t)$ and obtain the following formulation:

Find $u\in C([0, \infty[: H)\cap C^1(]0, \infty[: H)$ *such that*

$$(S)_2 \qquad \begin{cases} (2.4)\quad u(t)\in V & (0<t), \\ (2.5)\quad (du/dt, v)+(\nabla u(t), \nabla v)=0 & (0<t) \\ \qquad\quad \textit{for any } v\in V, & \\ (2.6)\quad u(0)=a. & \end{cases}$$

In order to study the problem $(S)_2$ from the view point of abstract evolution equations we make the following

DEFINITION. The self-adjoint operator in H associated with the quadratic form $(\nabla\cdot, \nabla\cdot)$ on V is called the Stokes operator and is denoted by A. Namely, A is characterized by

$$D(A)=\{v\in H;\ w\mapsto(\nabla v, \nabla w) \text{ can be extended to a bounded functional on } H.\}\cdots\cdots \text{(the domain of } A)$$

and

$$(Av, w)=(\nabla v, \nabla w) \qquad (v\in D(A),\ w\in V).$$

Then we see easily that the problem $(S)_2$ is equivalent to the following abstract Cauchy problem:

Find $u \in C([0, \infty[: H) \cap C^1(]0, \infty[: H)$ *such that*

$$(S)_3 \qquad \begin{cases} du/dt + Au = 0 & (0<t) \\ u(0) = a. \end{cases}$$

Since A is positive definite the problem $(S)_3$ has a unique solution $u(t)=e^{-tA}a$ according to the theory of semi-groups of operators (see, e.g., Yosida [23]). It is well-known that for this $u(t)=e^{-tA}a$ we can find out a unique pressure $p \in C(]0, \infty[: M)$ such that $\{u, p\}$ is a unique solution of the problem $(S)_1$ (see, e.g., Temam [22] or Ladyzhenskaya [19]).

2.2. Semi-discretization of the problem $(S)_1$.

From now on we consider mainly the two dimensional problem for simplicity. As for the three dimensional case, see the remark in Section 4.

Let us triangulate Ω as usual and let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of the triangulations of Ω with the size parameter h (see, e.g., Ciarlet [6]). For each $h>0$, $\mathcal{T}_h$ is composed of element triangles. Furthermore we introduce a refinement $\tilde{\mathcal{T}}_h$ of $\mathcal{T}_h$ in the following manner. Element triangles of $\tilde{\mathcal{T}}_h$ are obtained from those of $\mathcal{T}_h$ by dividing each element $K \in \mathcal{T}_h$ into four equal triangles by segments connecting midpoints of the side of K (see Figure I). Following Bercovier and Pironneau [2], we introduce finite element spaces below.

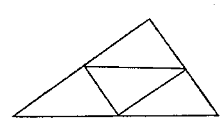

Figure I.

X_h is composed of all the continuous functions v_h; $\bar{\Omega} \to \boldsymbol{R}^2$ vanishing on $\partial\Omega$ such that their restrictions $v_h|K$ onto each triangle $K \in \tilde{\mathcal{T}}_h$ are polynomials of degree$\leqq 1$.

M_h is composed of all the continuous functions q_h; $\bar{\Omega} \to \boldsymbol{R}$ with $\int_\Omega q_h(x)dx=0$ such that $q_h|K$ for each $K \in \mathcal{T}_h$ is polynomials of degree$\leqq 1$.

$$V_h = \{v_h \in X_h;\ (v_h, \nabla q_h)=0 \quad \text{for all } q_h \in M_h\}.$$

We note that $X_h \subset X$, $M_h \subset M \cap H^1(\Omega)$, however, in general $V_h \not\subset V$. We regard X_h, M_h and V_h as Hilbert spaces equipped with the L^2-inner product.

Denoting the L^2-projection from $L^2(\Omega)$ onto V_h by Q_h, we formulate the approximate problem as follows:

Find $u_h \in C^1([0, \infty[: V_h)$ *and* $p_h \in C(]0, \infty[: M_h)$ *such that*

$$(S)_1^h \qquad \begin{cases} (2.7) \quad (du_h/dt, v_h) + (\nabla u_h, \nabla v_h) + (v_h, \nabla p_h) = 0 \quad (0<t) \\ \qquad\quad \textit{for any } v_h \in X_h, \\ (2.8) \quad u_h(0) = Q_h a. \end{cases}$$

Similarly to the case of the continuous problem, we can eliminate p_h from (2.7) and obtain a formulation involving u_h alone.

Find $u_h \in C^1([0, \infty[: V_h)$ *such that*

$$(S)_2^h \quad \begin{cases} (2.9) & (du_h/dt, v_h)+(\nabla u_h, \nabla v_h)=0 \quad (0<t), \\ & \textit{for any } v_h \in V_h, \\ (2.10) & u_h(0)=Q_h a. \end{cases}$$

The corresponding analogue of $(S)_3$ is given by

$$(S)_3^h \quad \begin{cases} (2.11) & du_h/dt+A_h u_h(t)=0 \quad (0<t), \\ (2.12) & u_h(0)=Q_h a, \end{cases}$$

where the operator A_h is an approximation of A defined below.

DEFINITION. We define the self-adjoint operator A_h in V_h by

$$(2.13) \qquad (A_h v_h, w_h)=(\nabla v_h, \nabla w_h) \qquad (v_h, w_h \in V_h),$$

and we call A_h the discrete Stokes operator. Now it is easy to see that the problem $(S)_2^h$, which is equivalent to $(S)_3^h$, has a unique solution $u_h(t)=\exp\{-tA_h\}Q_h a$.

Here we refer to the question whether $(S)_2^h$ is equivalent to $(S)_1^h$, i.e., whether we can find the discrete pressure p_h which satisfies (2.7). We can answer to this question affirmatively with the aid of Lemma 2.1 below due to Bercovier and Pironneau, which requires the following assumption.

Assumption B-P: All triangles of $\mathscr{T}_h$ have at least one vertex in Ω (not on $\partial\Omega$).

This assumption is assumed throughout this paper.

LEMMA 2.1. *Under Assumption* B-P *there exists a positive constant* β *which is independent of* h *such that*

$$(2.14) \qquad \sup_{v_h \in X_h} \frac{|(v_h, \nabla q_h)|}{\|v_h\|_0} \geqq \beta \|\nabla q_h\|_0 \qquad (q_h \in M_h).$$

(*Here and hereafter* $\| \ \|_0$ *means the* L^2*-norm.*)

As for the proof of Lemma 2.1 we refer to Bercovier and Pironneau [2] or Glowinski and Pironneau [13].

The importance of inequalities of the type (2.14) for mixed finite element methods was earlier indicated by Brezzi [4] and Kikuchi [17]. Essential use will be made of Lemma 2.1 when we derive order estimates of the error as $h \to 0$ in later sections. It should be noted that in Lemma 2.1 the convexity of Ω is not required.

With the aid of Lemma 2.1 we can show in the same way as in Bercovier and Pironneau [2] that $(S)_1^h$ and $(S)_2^h$ are equivalent. In view of completeness let us sketch the proof of the unique existence of the discrete pressure $p_h(t)$.

The uniqueness of $p_h(t)$ is immediate from Lemma 2.1. The existence of $p_h(t)$ is assured as follows.

For each $t>0$ let φ_t be an element of X_h defined by

$$(\varphi_t, v_h) = -(du_h/dt, v_h) - (\nabla u_h, \nabla v_h) \qquad (v_h \in X_h),$$

in terms of the solution $u_h(t)$ of $(S)_2^h$. Then $\varphi_t \perp V_h$. φ_t is continuous in $t \in [0, \infty[$, since $u_h(t) = \exp\{-tA_h\}Q_h a$ is a C^1-function. On the other hand, we define a bounded linear operator B_h; $X_h \to M_h$ by

$$(B_h w_h, q_h) = (w_h, \nabla q_h) \qquad (w_h \in X_h,\ q_h \in M_h),$$

and denote its adjoint operator by B_h^*; $M_h \to X_h$. Then we have

$$\begin{aligned}\varphi_t \in V_h^{\perp} &(= \text{the orthogonal complement of } V_v \text{ in } X_h)\\ &= (\operatorname{Ker} B_h)^{\perp}\\ &= R(B_h^*) \quad (= \text{the range of } B_h^*)\end{aligned}$$

by the closed range theorem (see, e.g., Yosida [23]). Hence there exists $p_h(t) \in M_h$ such that for all $v_h \in X_h$ we have

$$\begin{aligned}(du_h/dt, v_h) + (\nabla u_h, \nabla v_h) &= -(\varphi_t, v_h)\\ &= -(B_h^* p_h(t), v_h)\\ &= -(p_h(t), B_h v_h)\\ &= -(\nabla p_h(t), v_h),\end{aligned}$$

which implies (2.7). Since $(B_h^*)^{-1}$ exists by (2.14), the continuity of $p_h(t) = (B_h^*)^{-1}\varphi_t$ in t is obvious. Hence $(S)_1^h$ and $(S)_2^h$ are equivalent. Q.E.D.

Henceforth, in addition to Assumption B-P, we assume what is called the inverse assumption, i.e., there exists a positive constant c independent of h such that

$$ch < \min_{K \in \mathscr{T}_h} h(K)$$

where $h(K)$ is the length of the greatest side of $K \in \mathscr{T}_h$ (note that $h = \max_{K \in \mathscr{T}_h} h(K)$).

As a consequence of this assumption we obtain the inverse estimate (see, Ciarlet [6])

(2.15) $$\|w_h\|_1 < ch^{-1}\|w_h\|_0 \qquad (w_h \in X_h),$$

which will be used in Sections 3, 4 and 5.

Here and hereafter we denote the usual norm of the Sobolev space $H^s(\Omega)$ by $\|\ \|_s$. Furthermore we use the symbol c to denote various constants independent of h which may be different in different contexts.

3. Error estimate in $H^1(\Omega)^2$.

The aim of this section is to prove the following

THEOREM 3.1. *There exists a positive constant c which depends only on Ω such that*

(3.1) $$\|u(t)-u_h(t)\|_1 \leqq ch\|a\|_0/t \qquad (0<t),$$

where $u(t)$ or $u_h(t)$ is the solution of $(S)_1$ or $(S)_1^h$, respectively.

The proof of this theorem will be given after we have prepared several lemmas.

Following Fujita and Mizutani [10], we start with the integral representation,

(3.2) $$\begin{aligned} u(t)-u_h(t) &= e^{-tA}a - e^{-tA_h}Q_h a \\ &= \frac{1}{2\pi i}\int_\Gamma e^{-tz}\{(z-A)^{-1}-(z-A_h)^{-1}Q_h\}a\,dz \end{aligned}$$

where the path Γ in the complex z-plane is taken as shown in Figure II.

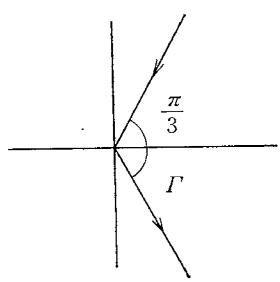

Figure II.

We put $\Sigma=\{z\in \boldsymbol{C};\ \pi/3\leqq|\arg z|\leqq\pi\}$. The following lemma was proved in Fujita and Mizutani [10].

LEMMA 3.1. *There exists a positive constant c such that for any $z\in\Sigma$ and any $v\in X$ we have*

$$|z|\,\|v\|_0^2+\|v\|_1^2 \leqq c|z(v, v)-(\nabla v, \nabla v)|. \tag{3.3}$$

Now we see from the formula (3.2) that the problem is reduced to estimate $(z-A)^{-1}a-(z-A_h)^{-1}Q_h a$. Therefore we put $w=(z-A)^{-1}g$ and $w_h=(z-A_h)^{-1}Q_h g$ for arbitrary $z \in \Sigma$ and $g \in H$. w and w_h are characterized as follows: $w \in V$ and it satisfies the condition

$$z(w, v)-(\nabla w, \nabla v)-(v, \nabla \rho)=(g, v) \qquad (v \in X) \tag{3.4}$$

for some $\rho \in M$, while $w_h \in V_h$ and it satisfies

$$z(w_h, v_h)-(\nabla w_h, \nabla v_h)-(v_h, \nabla \rho_h)=(g, v_h) \qquad (v_h \in X_h) \tag{3.5}$$

for some $\rho_h \in M_h$.

Here $\rho=\rho(z) \in M$ and $\rho_h=\rho_h(z) \in M_h$ are uniquely determined by the well-known properties of V and by Lemma 2.1 (see Bercovier and Pironneau [2], Temam [22] and the proof of the existence of $p_h(t)$ in Section 2). Concerning $w=w(z)$ and $w_h=w_h(z)$, we need two lemmas below, the proof of which are established by means of Bercovier and Pironneau's method for the special case $z=0$.

LEMMA 3.2. *There exists a positive constant c such that for any $g \in H$ and any $z \in \Sigma$ we have*

$$\begin{aligned} &|z|\,\|w-w_h\|_0^2+\|w-w_h\|_1^2 \\ &\leqq c \inf_{v_h \in V_h}\{|z|\,\|w-v_h\|_0^2+\|w-v_h\|_1^2\} \\ &\quad + c \inf_{q_h \in M_h}\|\rho-q_h\|_0^2. \end{aligned} \tag{3.6}$$

PROOF. Let v_h be an arbitrary element of V_h. Substituting w_h-v_h in (3.4) and (3.5) for v and v_h, respectively, and subtracting them, we obtain

$$z(w-w_h, w_h-v_h)-(\nabla(w-w_h), \nabla(w_h-v_h))+(\operatorname{div}(w_h-v_h), \rho-\rho_h)=0.$$

or equivalently,

$$\begin{aligned} &z\|w_h-v_h\|_0^2-\|\nabla(w_h-v_h)\|_0^2 \\ &=z(w-v_h, w_h-v_h)-(\nabla(w-v_h), \nabla(w_h-v_h)) \\ &\quad +(\operatorname{div}(w_h-v_h), \rho-\rho_h). \end{aligned} \tag{3.7}$$

Since w_h-v_h belongs to V_h, we have for any $q_h \in M_h$

$$(\operatorname{div}(w_h-v_h), \rho-\rho_h)=(\operatorname{div}(w_h-v_h), \rho-q_h). \tag{3.8}$$

From (3.3), (3.7) and (3.8), it follows that

$$
\begin{aligned}
&|z|\,\|w_h-v_h\|_0^2+\|w_h-v_h\|_1^2 \\
&\leqq c|z(w-v_h,\, w_h-v_h)-(\nabla(w-v_h),\, \nabla(w_h-v_h))| \\
&\quad +c|(\operatorname{div}(w_h-v_h),\, \rho-q_h)| \\
&\leqq c|z|^{1/2}\|w-v_h\|_0|z|^{1/2}\|w_h-v_h\|_0+c'\|w-v_h\|_1\|w_h-v_h\|_1 \\
&\quad +c'\|w_h-v_h\|_1\|\rho-q_h\|_0,
\end{aligned}
$$

which, together with the inequality

$$
st\leqq \varepsilon s^2/2+t^2/2\varepsilon \qquad (s,\, t>0,\, \varepsilon>0),
$$

yields

$$
\begin{aligned}
&|z|\,\|w_h-v_h\|_0^2+\|w_h-v_h\|_1^2 \\
&\leqq c\{|z|\,\|w-v_h\|_0^2+\|w-v_h\|_1^2\}+c\|\rho-q_h\|_0^2.
\end{aligned} \tag{3.9}
$$

Now (3.6) follows immediately from (3.9) and the obvious inequality $\|w-w_h\|_j\leqq \|w-v_h\|_j+\|v_h-w_h\|_j$ $(j=0,1)$. Q.E.D.

LEMMA 3.3. *There exists a positive constant c such that for any $z\in\Sigma$ and $g\in H$ we have*

$$
\begin{aligned}
&\inf_{v_h\in V_h}\{|z|\,\|w-v_h\|_0^2+\|w-v_h\|_1^2\} \\
&\leqq c\inf_{y_h\in X_h}\{(|z|+h^{-2})\|w-y_h\|_0^2+\|w-y_h\|_1^2\}.
\end{aligned} \tag{3.10}
$$

PROOF. Let y_h be an arbitrary element of X_h, and consider the solution $\{v_h,\, r_h\}\in X_h\times M_h$ of the following problem:

$$
\begin{aligned}
&-z(v_h,\, \varphi_h)+(\nabla v_h,\, \nabla\varphi_h)-(\operatorname{div}\varphi_h,\, r_h) \\
&=-z(y_h,\, \varphi_h)+(\nabla y_h,\, \nabla\varphi_h) \qquad (\varphi_h\in X_h),
\end{aligned} \tag{3.11}
$$

$$
v_h\in V_h. \tag{3.12}
$$

The solution of this problem exists uniquely since $z\in\Sigma$. We substitute v_h-y_h for φ_h in (3.11). Then we have

$$
-z\|v_h-y_h\|_0^2+\|\nabla(v_h-y_h)\|_0^2=-(v_h-y_h,\, \nabla r_h). \tag{3.13}
$$

From (3.3) and (3.13) we have

$$
\begin{aligned}
|z|\,\|v_h-y_h\|_0^2+\|\nabla(v_h-y_h)\|_0^2&\leqq c|(\nabla r_h,\, v_h-y_h)| \\
&=c|(\nabla r_h,\, w-y_h)|,
\end{aligned} \tag{3.14}
$$

since $\operatorname{div} w=0$ and $v_h\in V_h$.

On the other hand, we have

$$\|\nabla r_h\|_0 \leqq c\{|z|\,\|v_h-y_h\|_0+h^{-1}\|v_h-y_h\|_1\}, \tag{3.15}$$

which is a consequence of Lemma 2.1, (3.11) and (2.15). Namely,

$$\begin{aligned}\|\nabla r_h\|_0 &\leqq \beta^{-1}\sup_{\varphi_h\in X_h}\frac{|(\varphi_h, \nabla r_h)|}{\|\varphi_h\|_0}\\ &=\beta^{-1}\sup_{\varphi_h\in X_h}\frac{|z(v_h-y_h, \varphi_h)-(\nabla(v_h-y_h), \nabla\varphi_h)|}{\|\varphi_h\|_0}\\ &\leqq c\{|z|\,\|v_h-y_h\|_0+h^{-1}\|v_h-y_h\|_1\}.\end{aligned}$$

Now we obtain by (3.14) and (3.15)

$$\begin{aligned}&|z|\,\|v_h-y_h\|_0^2+\|v_h-y_h\|_1^2\\ &\leqq c\{|z|\,\|v_h-y_h\|_0+h^{-1}\|v_h-y_h\|_1\}\|w-y_h\|_0\\ &\leqq \frac{1}{2}|z|\,\|v_h-y_h\|_0^2+c'|z|\,\|w-y_h\|_0^2\\ &\quad+\frac{1}{2}\|v_h-y_h\|_1^2+c'h^{-2}\|w-y_h\|_0^2,\end{aligned}$$

which implies

$$|z|\,\|v_h-y_h\|_0^2+\|v_h-y_h\|_1^2 \leqq c(|z|+h^{-2})\|w-y_h\|_0^2. \tag{3.16}$$

Then we have (3.10) by means of (3.16) and the obvious inequality

$$\|w-v_h\|_j \leqq \|w-y_h\|_j+\|y_h-v_h\|_j \qquad (j=0, 1)$$ Q.E.D.

In order to obtain the optimal rate of convergence, the regulality result

$$D(A)=H^2(\Omega)\cap V,\ \|Av\|_0 \leqq c\|v\|_2,\ \|v\|_2 \leqq c\|Av\|_0 \qquad (v\in D(A)) \tag{3.17}$$

is required. In the case where Ω is a convex polygonal domain in $\boldsymbol{R}^2$, which we are considering, (3.17) is true due to Kellogg and Osborn [16]. On the other hand, we have from the general theory

$$\|A^\alpha(z-A)^{-1}\| \leqq c|z|^{-1+\alpha} \qquad (0\leqq\alpha\leqq 1,\ z\in\Sigma),$$

where $\|\ \|$ is the operator norm in H. In this way we obtain

$$\|w\|_0=\|(z-A)^{-1}g\|_0 \leqq c|z|^{-1}\|g\|_0, \tag{3.18}$$

$$\|w\|_1 \leqq c|z|^{-1/2}\|g\|_0, \tag{3.19}$$

$$\|w\|_2 \leqq c\|g\|_0. \tag{3.20}$$

REMARK 3.1. (3.17) assures that

$$u\in C(]0, \infty[;\ H^2(\Omega)\cap V),\ \text{ and hence } p\in C(]0, \infty[;\ H^1(\Omega)).$$

With the aid of the foregoing lemmas we can now estimate the error for the resolvent, i.e., we have the following

THEOREM 3.2. *There exists a positive constant c such that for any* $g\in H$ *and any* $z\in\Sigma$, $\|w-w_h\|_1\leqq ch\|g\|_0$ *holds true, i.e.,*

$$\|(z-A)^{-1}g-(z-A_h)^{-1}Q_hg\|_1\leqq ch\|g\|_0. \tag{3.21}$$

PROOF. From LEMMAS 3.2 and 3.3 we obtain

$$\|w-w_h\|_1\leqq c\inf_{y_h\in M_h}\{|z|^{1/2}\|w-y_h\|_0+h^{-1}\|w-y_h\|_0+\|w-y_h\|_1\} \tag{3.22}$$
$$+c\inf_{q_h\in M_h}\|\rho-q_h\|_0.$$

In order to estimate right hand side we recall the following well-known results (see, e.g., Bramble and Hilbert [3] or Ciarlet [6])

$$\inf_{q_h\in M_h}\|\eta-q_h\|_0\leqq ch\|\nabla\eta\|_0 \qquad (\eta\in H^1(\Omega)\cap M) \tag{3.23}$$

$$\inf_{y_h\in X_h}\|\zeta-y_h\|_j\leqq ch^{2-j}\|\zeta\|_2 \qquad (\zeta\in H^2(\Omega)\cap H_0^1(\Omega),\ j=0,1) \tag{3.24}$$

$$\inf_{y_h\in X_h}\|\zeta-y_h\|_0\leqq ch\|\zeta\|_1 \qquad (\zeta\in H_0^1(\Omega)). \tag{3.24$'$}$$

The standard estimates (3.23), (3.24) and (3.24)′ allow us to estimate the right-hand side of (3.22) as

$$\inf_{q_h\in M_h}\|\rho-q_h\|_0\leqq ch\|\nabla\rho\|_0,$$
$$\inf_{y_h\in X_h}\{|z|^{1/2}\|w-y_h\|_0+h^{-1}\|w-y_h\|_0+\|w-y_h\|_1\}$$
$$\leqq ch|z|^{1/2}\|w\|_1+h^{-1}ch^2\|w\|_2+ch\|w\|_2$$
$$\leqq c'h\|g\|_0.$$

Here use has been made of (3.19) and (3.20). Thus we have $\|w-w_h\|_1\leqq ch\|g\|_0+ch\|\nabla\rho\|_0$. On the other hand, we have

$$\|\nabla\rho\|_0\leqq c\|g\|_0 \tag{3.25}$$

by the equation $\nabla\rho=-g+\Delta w+zw$, (3.18) and (3.20). Therefore (3.21) has been established. Q.E.D.

PROOF OF THEOREM 3.1. By virtue of the formula (3.2) and the estimate (3.21) we have

$$
\begin{aligned}
&\|u(t)-u_h(t)\|_1 \\
&\leqq \frac{1}{2\pi}\int_\Gamma |e^{-tz}|\,\|(z-A)^{-1}a-(z-A_h)^{-1}Q_h a\|_1 |dz| \\
&\leqq ch\int_0^\infty e^{-tr/2}\|a\|_0 dr = 2ch\|a\|_0 t^{-1}. \qquad \text{Q.E.D.}
\end{aligned}
$$

4. Error estimate in $L^2(\Omega)^2$.

Notations introduced in the preceding section will be used here and in Section 5. The aim of this section is to prove the following

THEOREM 4.1. *There exists a positive constant c which depends only on Ω such that the inequality*

$$\|u(t)-u_h(t)\|_0 \leqq ch^2 t^{-1}\|a\|_0 \qquad (0<t) \tag{4.1}$$

holds true.

This theorem is a direct consequence of Theorem 4.2 below. In fact the same argument as in the proof of Theorem 3.1 is applicable to the present case. Therefore we have only to show the following

THEOREM 4.2. *There exists a positive constant c such that for any $g\in H$ and any $z\in\Sigma$ we have $\|w-w_h\|_0 \leqq ch^2\|g\|_0$, i.e.,*

$$\|(z-A)^{-1}g-(z-A_h)^{-1}Q_h g\|_0 \leqq ch^2\|g\|_0. \tag{4.2}$$

We begin with the discrete version of (3.18), (3.19) and (3.25):

LEMMA 4.1. *There exists a positive constant c independent of h or $z\in\Sigma$ such that*

$$\|w_h\|_0 \leqq c|z|^{-1}\|g\|_0, \tag{4.3}$$

$$\|w_h\|_1 \leqq c|z|^{-1/2}\|g\|_0, \tag{4.4}$$

$$\|\nabla\rho_h\|_0 \leqq c\|g\|_0. \tag{4.5}$$

PROOF OF LEMMA 4.1. (4.3) and (4.4) are proved in the same way as in Fujita and Mizutani [10], p. 757, Corollary 3.4. Therefore we can omit the proof. (4.5) is proved as follows. By means of Lemma 2.1 we have

$$\|\nabla\rho_h\|_0 \leqq \beta^{-1}\sup_{v_h\in X_h}\frac{|(\nabla\rho_h, v_h)|}{\|v_h\|_0}. \tag{4.6}$$

By making use of (3.4) and (3.5), we see easily that

$$(\nabla\rho_h, v_h)=-(g, v_h)-(\nabla w_h, \nabla v_h)+z(w_h, v_h)$$
$$=-z(w-w_h, v_h)+(\nabla(w-w_h), \nabla v_h)+(v_h, \nabla\rho).$$

Then we obtain

$$\|\nabla\rho_h\|_0\leqq c\{|z|\,\|w-w_h\|_0+h^{-1}\|w-w_h\|_1+\|\nabla\rho\|_0\}\leqq c'\|g\|_0$$

by (2.15), (3.18), (3.19), (3.25), (4.3) and (4.6). Q.E.D.

PROOF OF THEOREM 4.2. Our proof is based on Nitsche's trick (see Bercovier and Pironneau [2] Proposition 3).

Let $\{\eta, \pi\}\in X\times M$ be the solution of the problem

$$-z(\eta, v)+(\nabla\eta, \nabla v)+(v, \nabla\pi)=(w-w_h, v) \qquad (v\in X), \tag{4.7}$$

$$\eta\in V. \tag{4.8}$$

Then we know

$$\eta\in H^2(\Omega)\cap V,\ \pi\in H^1(\Omega),\ \|\eta\|_2+\|\nabla\pi\|_0\leqq c\|w-w_h\|_0. \tag{4.9}$$

Substituting $w-w_h$ for v in (4.7), we have

$$\begin{aligned}\|w-w_h\|_0^2&=-z(\eta, w-w_h)+(\nabla\eta, \nabla(w-w_h))\\&\quad+(w-w_h, \nabla\pi)\\&=-z(\eta-v_h, w-w_h)+(\nabla(\eta-v_h), \nabla(w-w_h))\\&\quad+(w-w_h, \nabla\pi)-z(v_h, w-w_h)+(\nabla v_h, \nabla(w-w_h))\end{aligned} \tag{4.10}$$

for any $v_h\in X_h$.

By making use of (3.4) and (3.5), we can rewrite (4.10) as follows:

$$\begin{aligned}\|w-w_h\|_0^2&=-z(\eta-v_h, w-w_h)+(\nabla(\eta-v_h), \nabla(w-w_h))\\&\quad+(w-w_h, \nabla\pi)-(v_h, \nabla(\rho-\rho_h)).\end{aligned} \tag{4.11}$$

On the other hand, by virtue of $\operatorname{div} w=0$, $w_h\in V_h$ and $\operatorname{div}\eta=0$, we have for any $q_h\in M_h$

$$(v_h, \nabla(\rho-\rho_h))=-(\eta-v_h, \nabla(\rho-\rho_h)), \tag{4.12}$$

$$(w-w_h, \nabla\pi)=-(\operatorname{div}(w-w_h), \pi-q_h). \tag{4.13}$$

From (4.11), (4.12), (4.13) and the Schwarz inequality, we obtain

$$\begin{aligned}\|w-w_h\|_0^2\leqq \inf_{v_h\in X_h}&\{|z|\,\|\eta-v_h\|_0\|w-w_h\|_0\\&+\|\nabla(\eta-v_h)\|_0\|\nabla(w-w_h)\|_0\\&+\|\eta-v_h\|_0\|\nabla(\rho-\rho_h)\|_0\}\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
&+\|\operatorname{div}(w-w_h)\|_0 \inf_{q_h\in M_h}\|\pi-q_h\|_0 \\
&\leqq\{ch^2|z|\,\|w-w_h\|_0+ch\|w-w_h\|_1 \\
&\quad +ch^2\|\nabla(\rho-\rho_h)\|_0\}\|\eta\|_2 \\
&\quad +c\|w-w_h\|_1 ch\|\nabla\pi\|_0 \\
&\leqq c'h^2\|g\|_0(\|\eta\|_2+\|\nabla\pi\|_0) \\
&\leqq c''h^2\|g\|_0\|w-w_h\|_0.
\end{aligned}
$$

Here we have used Theorem 3.2, (4.3) and (4.5). Thus we obtain (4.2). Q.E.D.

REMARK 4.1. So far we have dealt with the two dimensional problem. However, even in the three dimensional problem we obtain the same result if we assume that

$$D(A)=H^2(\Omega)^3\cap V,\ \|v\|_2\leqq c\|Av\|_0,\ \|Av\|_0\leqq c\|v\|_2 \tag{4.15}$$

for any $v\in D(A)$.

Unfortunately (4.15) is not proved yet even for the case of a convex polyhedral domain in $\boldsymbol{R}^3$.

REMARK 4.2. In order to obtain optimal rates of convergence we need (3.17) (or (4.15)) which is not valid for non-convex polygonal (polyhedral) domain Ω. However, even when Ω is not convex, we can obtain strong convergence in $H_0^1(\Omega)$, i.e.,

$$\|u(t)-u_h(t)\|_1\longrightarrow 0 \quad \text{as} \quad h\longrightarrow 0, \tag{4.16}$$

where the convergence is uniform in $t\in[\varepsilon,\infty[$ for any $\varepsilon>0$. To show (4.16), it is sufficient to prove that

$$\|(z-A)^{-1}g-(z-A_h)^{-1}Q_hg\|_1\longrightarrow 0 \quad \text{as} \quad h\longrightarrow 0, \tag{4.17}$$

where the convergence is uniform in z on each compact subset $K\subset\Sigma$.

Let us show (4.17) briefly. Even if Ω is not convex, Lemmas 2.1, 3.1, 3.2 and 3.3 are still valid. Hence we have

$$
\begin{aligned}
\|w-w_h\|_1 &\leqq c\inf_{v_h\in V_h}\{|z|^{1/2}\|w-v_h\|_0+\|w-v_h\|_1\}+c\inf_{q_h\in M_h}\|\rho-q_h\|_0 \\
&\leqq c\{|z|^{1/2}\|w-v\|_0+\|w-v\|_1+\|\rho-r\|_0\} \\
&\quad +c\inf_{v_h\in V_h}\{|z|^{1/2}\|v-v_h\|_0+\|v-v_h\|_1\}+c\inf_{q_h\in M_h}\|r-q_h\|_0
\end{aligned}
$$

for any $r\in H^1(\Omega)$ and any $v\in H^2(\Omega)^k\cap V$ ($k=2$ or 3).

Applying the same argument as in the proof of Lemma 3.3 to the second term of the right side, we have

(4.18) $$\|w-w_h\|_1 \leqq c\{|z|^{1/2}\|w-v\|_0+\|w-v\|_1+\|\rho-r\|_0\} \\ +c \inf_{y_h \in X_h}\{(|z|^{1/2}+h^{-1})\|v-y_h\|_0+\|v-y_h\|_1\}+c \inf_{q_h \in M_h}\|r-q_h\|_0.$$

Let $\varepsilon>0$ be an arbitrary fixed number. By virtue of the compactness of K, we can take $L>0$ such that for all $z \in \Sigma$ there exists $v_z \in H^2(\Omega) \cap V$ and $r_z \in H^1(\Omega)$ satisfying

(4.19) $$\|v_z\|_2 \leqq L, \ \|v_z-w\|_1=\|v_z-w(z)\|_1 \leqq \varepsilon \quad \text{and} \quad \|\nabla r_z\|_0 \leqq L, \ \|\rho-r_z\|_0 \leqq \varepsilon.$$

Then the first term of the right side of (4.18) is majorized by $c\varepsilon$. The second and the third term tend to zero as $h \to 0$. Therefore, we obtain (4.17). Furthermore, the convergence of (4.17) is uniform in $z \in K$, since L in (4.19) can be chosen independently of $z \in K$.

5. Error estimate for the pressure.

The aim of this section is to show that $\|p(t)-p_h(t)\|_0$ is $O(h)$ as $h \to 0$. Ω is still assumed to be a convex polygon in $\boldsymbol{R}^2$.

LEMMA 5.1. *There exists a positive constant c such that*

(5.1) $$\|du/dt-du_h/dt\|_0 \leqq cht^{-3/2}\|a\|_0 \qquad (0<t),$$

(5.2) $$\|\nabla p_h(t)\|_0 \leqq ct^{-1}\|a\|_0 \qquad (0<t).$$

PROOF. To show (5.1) we make use of

(5.3) $$\|w-w_h\|_0=\|(z-A)^{-1}g-(z-A_h)^{-1}Q_h g\|_0 \leqq ch|z|^{-1/2}\|g\|_0,$$

which is a consequence of Lemmas 3.2 and 3.3. In fact, from these lemmas we have

(5.4) $$|z|^{1/2}\|w-w_h\|_0 \leqq c \inf_{y_h \in X_h}\{(|z|^{1/2}+h^{-1})\|w-y_h\|_0 \\ +\|w-y_h\|_1\}+c \inf_{q_h \in M_h}\|\rho-q_h\|_0.$$

In the proof of Theorem 3.2 we have already shown that the right hand side of (5.4) is majorized by $ch\|g\|_0$. Hence we have (5.3).

Now differentiation of the formula (3.2) with respect to t yields

$$du/dt-du_h/dt=\frac{-1}{2\pi i}\int_\Gamma ze^{-tz}\{(z-A)^{-1}-(z-A_h)^{-1}Q_h\}a\,dz,$$

whence we obtain by (5.3)

$$\begin{aligned}&\|du/dt-du_h/dt\|_0\\ &\leqq\frac{1}{2\pi}\int_\Gamma |z|^{1/2}|e^{-tz}|\,|z|^{1/2}\|(z-A)^{-1}a-(z-A_h)^{-1}Q_h a\|_0|dz|\\ &\leqq cht^{-3/2}\|a\|_0.\end{aligned}$$

To show (5.2) we note an inequality obvious from Lemma 2.1:

$$\|p_h(t)\|_0\leqq\beta^{-1}\sup_{v_h\in X_h}\frac{|(v_h, \nabla p_h(t))|}{\|v_h\|_0}. \tag{5.5}$$

On the other hand, we have from the defining equations of $u(t)$ and $u_h(t)$

$$(v_h, \nabla p_h(t))=(du/dt-du_h/dt, v_h)+(\nabla(u-u_h), \nabla v_h)+(v_h, \nabla p(t)) \tag{5.6}$$

for arbitrary $v_h\in X_h$. Then from (2.15), (3.23), (5.5) and (5.6) it follows that

$$\begin{aligned}\|\nabla p_h(t)\|_0&\leqq c\{\|du/dt\|_0+\|du_h/dt\|_0+h^{-1}\|u-u_h\|_1+\|\nabla p(t)\|_0\}\\ &\leqq ct^{-1}\|a\|_0.\end{aligned}$$

Here use has been made of

$$\|du_h/dt\|_0=\|A_h\exp\{-tA_h\}Q_h a\|_0\leqq t^{-1}\|Q_h a\|_0\leqq t^{-1}\|a\|_0$$

which is obvious in view of the self-adjointness of A_h and $A_h>0$. Q.E.D.

By means of the above lemma we can prove that the error $\|p(t)-p_h(t)\|_0$ is of optimal order. Namely, we have

THEOREM 5.1. *There exists a positive constant c which depends only on Ω such that*

$$\|p(t)-p_h(t)\|_0\leqq ch(t^{-1}+t^{-3/2})\|a\|_0 \qquad (0<t). \tag{5.7}$$

PROOF. We again use Nitsche's trick. We start with the equality

$$\|p(t)-p_h(t)\|_0=\sup_{\xi\in M}\frac{|(\xi, p(t)-p_h(t))|}{\|\xi\|_0}. \tag{5.8}$$

It is well-known that div; $V^\perp\to M$ is an isomorphism, where $V^\perp$ is the orthogonal complement of V in X with respect to the inner-product $(\nabla\cdot, \nabla\cdot)$. (See, for instance, Temam [22] or Girault and Raviart [12].) Hence for any $\xi\in M$ we can choose $\phi\in X$ which satisfies

$$\operatorname{div}\phi=\xi, \quad \|\phi\|_1\leqq c\|\xi\|_0. \tag{5.9}$$

Then we have for any $\phi_h\in X_h$

$$(5.10)\qquad (\xi,\, p(t)-p_h(t))=(\operatorname{div}\phi,\, p(t)-p_h(t)) \\ =-(\phi-\phi_h,\, \nabla(p(t)-p_h(t))) \\ -(\phi_h,\, \nabla(p(t)-p_h(t))).$$

By means of (3.4), (3.5) and (5.10), we obtain

$$(\xi,\, p(t)-p_h(t))=-(\phi-\phi_h,\, \nabla(p(t)-p_h(t))) \\ +(du/dt-du_h/dt,\, \phi_h) \\ +(\nabla(u(t)-u_h(t)),\, \nabla\phi_h).$$

Then, by the arbitrariness of $\phi_h\in X_h$, it follows that

$$(5.11)\qquad |(\xi,\, p(t)-p_h(t))|\leqq ch\|\phi\|_1\{\|\nabla p(t)\|_0+\|\nabla p_h(t)\|_0\} \\ +\|du/dt-du_h/dt\|_0 c\|\phi\|_1+c\|u(t)-u_h(t)\|_1\|\phi\|_1 \\ \leqq c'h\|\xi\|_0\|a\|_0(t^{-1}+t^{-3/2}).$$

Here use has been made of (3.1), (5.1) and (5.2).

Now (5.7) follows immediately from (5.8) and (5.11). Q.E.D.

6. Convergence rate of the full-discrete scheme.

In this section we make a brief study of the full-discrete scheme where the time variable is also discretized.

In the first place, we deal with the backward difference scheme for time-discretization combined with the finite element approximation for space-discretization considered so far. Namely, let $\tau>0$ be the time mesh, and consider the problem

$$(6.1)\qquad u_h(t+\tau)-u_h(t)+\tau A_h u_h(t+\tau)=0 \qquad (t=n\tau,\ n=0, 1, 2, \cdots),$$

$$(6.2)\qquad u_h(0)=Q_h a,$$

or equivalently

$$(6.3)\qquad u_h(t)=(1+\tau A_h)^{-n}Q_h a \qquad (t=n\tau,\ n=0, 1, \cdots).$$

Then we obtain the following theorem, the proof of which is the same as that of Theorems 6.1 and 6.3 in Fujita and Mizutani [10], and so is omitted.

THEOREM 6.1. *Under the above situation we have*

$$(6.4)\qquad \|u_h(t)\|_0\leqq 1 \qquad (t=n\tau,\ n=0, 1, \cdots),$$

$$(6.5)\qquad \|u(t)-u_h(t)\|_0\leqq c(h^2+\tau)t^{-1}\|a\|_0 \qquad (t=n\tau,\ n=0, 1, \cdots).$$

where $u(t)=e^{-tA}a,\ u_h(t)=(I+\tau A_h)^{-n}Q_h a.$

In the next place, we consider the forward difference scheme, i.e.,

$$u_h(t+\tau)-u_h(t)+\tau A_h u_h(t)=0 \qquad (t=n\tau,\ n=0,1,2,\cdots), \tag{6.6}$$

$$u_h(0)=Q_h a. \tag{6.7}$$

Then the approximate solution is $u_h(t)=(1-\tau A_h)^n Q_h a$ $(n=0,1,\cdots)$. The following theorem is essentially an application of Theorem of Fujita and Mizutani [10].

THEOREM 6.2.

i) *If the condition*

$$\tau\|A_h\|\leqq 2 \qquad (\tau>0,\ h>0) \tag{6.8}$$

is satisfied, then we have the stability in the sense that

$$\|(1-\tau A_h)^n\|_0\leqq 2 \qquad (n=0,1,2,\cdots). \tag{6.9}$$

ii) *If the condition*

$$\sup_{\tau,h>0}\tau\|A_h\|<2 \tag{6.10}$$

is satisfied, then we have the error estimate

$$\|e^{-tA}a-(1-\tau A_h)^n Q_h a\|_0\leqq c(h^2+\tau)t^{-1}\|a\|_0 \qquad (t=n\tau,\ n=0,1,2,\cdots). \tag{6.11}$$

REMARK 6.1. The stability condition (6.8) is equivalent to

$$\sup_{\phi_h\in V_h}\frac{(\nabla\phi_h,\nabla\phi_h)}{\|\phi_h\|_0^2}\leqq 2/\tau. \tag{6.12}$$

According to the inverse estimate (2.15), there exists a constant c_0 such that

$$\|\nabla\phi_h\|_0\leqq c_0 h^{-1}\|\phi_h\|_0 \qquad (\phi_h\in X_h),$$

and, therefore, the stability condition is satisfied if

$$\tau/h^2\leqq 2c_0^{-2}. \tag{6.13}$$

REMARK 6.2. More sophisticated time-discretization, for instance, the Crank-Nicolson scheme, can be dealt with similarly (see, e.g., Fujita and Mizutani [11]).

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