On the integrals of certain singular theta-functions

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To the memory of Takuro Shintani

Introduction

In a previous paper [4] we constructed a certain class of theta-functions associated to rational quadratic forms of signature $(n, 1)$. These functions, which are holomorphic Siegel modular forms of weight $(n+1)/2$ and genus n, are direct generalizations of Hecke's elliptic modular cusp forms of weight 1 associated to real quadratic fields (the case $n=1$). When the quadratic form in question is anisotropic over $\boldsymbol{Q}$, our Siegel modular forms are once again cusp forms, as in Hecke's case. However, when the quadratic form is isotropic (e.g. $n \geqq 4$), we were only able to obtain our functions under a certain technical condition [4, p. 521] which excluded certain "singular" theta-series, whose integral is not termwise absolutely convergent. Unfortunately this technical restriction did not allow us to determine the behavior of our functions at an arbitrary cusp and was inconvenient for several applications. In this paper we will use a method, employed by Shintani [8], to compute the integral of the "singular" theta series and thereby eliminate the technical condition of [4]. As a consequence, we will show that our functions are not, in general, cusp forms in the isotropic case.

We now give a more precise description of our result.

Let $V(\boldsymbol{Q})$ be a rational vector space with $\dim_{\boldsymbol{Q}} V(\boldsymbol{Q})=n+1$, and let $(\,,)$ be a non-degenerate symmetric bilinear form on $V(\boldsymbol{Q})$ with signature $(n, 1)$. Let $L \subset V(\boldsymbol{Q})$ be a $\boldsymbol{Z}$-lattice and assume that the dual lattice

$$L^* = \{v \in V(\boldsymbol{Q}) \mid (v, v') \in \boldsymbol{Z},\ \forall v' \in L\}$$

contains L, i.e. $L^* \supset L$. Let

$$G(\boldsymbol{Q}) = \{g \in SL(V(\boldsymbol{Q})) \mid (gv, gv) = (v, v) \quad \forall v \in L\}$$

be the special orthogonal group of $V(\boldsymbol{Q})$, $(\,,)$, viewed as an algebraic group over $\boldsymbol{Q}$. Let

$$\Gamma(L) = \{g \in G(\boldsymbol{Q}) \mid gL = L \quad \text{and} \quad g \text{ acts trivially in } L^*/L\}.$$

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We fix a subgroup of finite index $\Gamma\subset\Gamma(L)\cap G^0(\boldsymbol{R})$ where $G^0(\boldsymbol{R})$ is the connected component of the identity of $G(\boldsymbol{R})$.

For $\tau\in\mathfrak{H}_n$, the Siegel space of genus n, and for $h\in(L^*/L)^n$, let

$$\vartheta^+(\tau;h,L;\Gamma)=\sum_{\substack{X\in h+L^n\\(X,X)>0\\\bmod\Gamma}}\varepsilon(X)|\Gamma_X|^{-1}e\Big(\frac{1}{2}\mathrm{tr}\,(\tau(X,X))\Big)\tag{0.1}$$

where $\varepsilon(X)=\pm1$ is a certain "sign character" defined by (3.1) and $|\Gamma_X|$ is the order of the stabilizer in Γ of the frame X. This series is absolutely convergent and defines a holomorphic function of $\tau\in\mathfrak{H}_n$. Moreover, if $V(\boldsymbol{Q})$, $(\,,)$ is anisotropic, or if $h\in(L^*/L)^n$ is non-singular as defined in §2 below, then $\vartheta^+(\tau;h,L)$ is, in fact, a holomorphic Siegel modular form of weight $(n+1)/2$ with respect to a certain congruence subgroup of $Sp(n,\boldsymbol{Z})$. This was proved in [4] by introducing a non-holomorphic theta-form $\Theta(\tau;h,L)$ given by (2.3) below. The series (2.3), whose transformation behavior with respect to the action of $Sp(n,\boldsymbol{Z})$ on τ was given in [4], determines a Γ-invariant n-form on the space B of majorants of $(\,,)$ on $V(\boldsymbol{R})$. For *any* $V(\boldsymbol{Q})$, $(\,,)$, $L^*\supset L$ and h, the form Θ is L^1 on a fundamental domain for Γ in B, and so the integral

$$I(\tau;h,L;\Gamma)=\int_{\Gamma\backslash B}\Theta(\tau;h,L)\tag{0.2}$$

is well defined. On the other hand, when $V(\boldsymbol{Q})$, $(\,,)$ is anisotropic or h is non-singular, the integral of the series (2.3) is actually termwise absolutely convergent so that we may compute it term by term to obtain

$$I(\tau;h,L;\Gamma)=2^{-n/2}\vartheta^+(\tau;h,L;\Gamma).\tag{0.3}$$

In general the integral of the series (2.3) is *not* termwise absolutely convergent. More precisely, let $l_0,\cdots,l_r$ be a set of representatives for the Γ-orbits in the set of isotropic lines in $V(\boldsymbol{Q})$. Then we can write

$$\Theta=\Theta^++\sum_{i=0}^{r}\Theta^{(i)}\tag{0.4}$$

where Θ^+ and $\Theta^{(i)}$ are Γ-invariant n-forms on B given by (2.18) and (2.19) of §2 below. If we let θ_*^+ and $\theta_*^{(i)}$ be the series (2.20) and (2.21), obtained by summing the pointwise norms of the terms in Θ^+ and $\Theta^{(i)}$, then θ_*^+ is rapidly decreasing at every cusp of $\Gamma\backslash B$ and $\theta_*^{(i)}$ is slowly increasing at the cusp corresponding to l_i and rapidly decreasing at all other cusps. Thus, as before, we can integrate Θ^+ termwise to obtain

$$I^+(\tau,h,L,\Gamma)=2^{-n/2}\vartheta^+(\tau,h,L,\Gamma)\tag{0.5}$$

where $I^+(\tau,h,L,\Gamma)=\int_{\Gamma\backslash B}\Theta^+$. However, the "singular" integrals

(0.6) $$I^{(i)}(\tau, h, L, \Gamma)=\int_{\Gamma\backslash B}\Theta^{(i)}$$

must be computed by another method.

To compute $I^{(i)}$ we introduce the Eisenstein series $E(z, s, L)$ defined, for $\mathrm{Re}(s)>n$, by (4.1). This series has a holomorphic analytic continuation to the half-plane $\mathrm{Re}(s)>n-3/2$ except for a simple pole with non-vanishing residue at $s=n-1$. We then define the wave packet

(0.7) $$\mathcal{E}(z, s)=\frac{1}{2\pi i}\int_{\rho_0-i\infty}^{\rho_0+i\infty}\frac{e^{\rho^2}}{s-\rho}E(z, \rho, L)d\rho$$

where $n-1<\rho_0<\mathrm{Re}(s)$. Then

(0.8) $$\kappa=\lim_{\sigma\to(n-1)^+}(\sigma-n+1)\mathcal{E}(z, \sigma)$$

is a finite constant and

(0.9) $$\int_{\Gamma\backslash B}\Theta^{(i)}=\kappa^{-1}\cdot\lim_{\sigma\to(n-1)^+}(\sigma-n+1)\int_{\Gamma\backslash B}\mathcal{E}(z, \sigma)\Theta^{(i)}.$$

On the other hand, for $\mathrm{Re}(s)>n+3$ the integral on the right hand side of (0.9) can be computed term-by-term. By analytically continuing the resulting expression—Propositions 4.5 and 4.6—we obtain an explicit formula for $I^{(i)}(\tau, h, L, \Gamma)$—Theorem 4.7. As a consequence we obtain our main result—Theorem 3.2—which describes the "singular" term $\vartheta^0(\tau, h, L, \Gamma)$ which must be added to $\vartheta^+(\tau, h, L, \Gamma)$ to obtain a Siegel modular form when h is "singular".

The functions $\vartheta(\tau, h, L, \Gamma)$, for general h, have several applications. In particular we hope, in a later paper, to give a geometric interpretation of the singular term $\vartheta^0(\tau, h, L, \Gamma)$ in terms of the boundary behavior of the totally geodesic cycles of [5] in the non-compact case.

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§ 1. Preliminaries.

We retain the notation of the introduction.

1.1. Let

$$B=\{Z\in V(\boldsymbol{R})\mid (Z, Z)=-1\}^0$$

be one component of the hyperboloid of two sheets in $V(\boldsymbol{R})$. We fix an orientation of V and determine an orientation of B by requiring that, for every property oriented basis $\{w_1, \cdots, w_n\}$ for $T_Z(B)\cong Z^{\perp}$ (see (1.18)) the basis $\{w_1, \cdots, w_n, Z\}$ is properly oriented for V. Observe that the action of $G^0(\boldsymbol{R})$ on B preserves this orientation. We also identify B with the space of majorants of $(,)$ by defining, for $Z\in B$, the majorant

$$(1.1)\qquad (,)_Z=\begin{cases} (,) & \text{on } Z^{\perp} \\ -(,) & \text{on } \boldsymbol{R}Z. \end{cases}$$

1.2. As in the introduction, let $l_0, \cdots, l_r$ be a set of representatives for the Γ-orbits in the set of isotropic lines in $V(\boldsymbol{Q})$. Choose vectors $u_0, \cdots, u_r$ with $u_i\in L$ primitive and such that

$$(1.2)\qquad l_i=\boldsymbol{Q}u_i$$

and

$$(1.2)'\qquad (u_i, Z)<0 \qquad \forall Z\in B.$$

Also choose $g_i\in G^0(\boldsymbol{Q})=G(\boldsymbol{Q})\cap G^0(\boldsymbol{R})$ such that

$$(1.3)\qquad g_i u_0=u_i$$

and with $g_0=1$.

We choose a properly oriented Witt basis $u_0, v_1, \cdots, v_{n-1}, u_0'$ for $V(\boldsymbol{Q})$ with $(u_0, u_0')=-1/2$, $(u_0', u_0')=(u_0', v_i)=(u_0, v_i)=0$ and set

$$Q=((v_i, v_j))\in M_{n-1}(\boldsymbol{Q}).$$

Then $Q={}^tQ>0$ and

$$(1.4)\qquad (,)\sim\begin{pmatrix} & & -1/2 \\ & Q & \\ -1/2 & & \end{pmatrix}.$$

Let P be the $\boldsymbol{Q}$-parabolic subgroup of G defined by

$$P(\boldsymbol{Q})=\{g\in G(\boldsymbol{Q})\mid gl_0=l_0\}.$$

Then

$$(1.5)\qquad N(\boldsymbol{Q})=\left\{n(x)=\begin{pmatrix} 1 & 2{}^txQ & Q[x] \\ & 1_{n+1} & x \\ & & 1 \end{pmatrix}\,\middle|\, x\in\boldsymbol{Q}^{n-1}\right\}$$

is the unipotent radical of $P(\boldsymbol{Q})$ and

$$(1.6)\qquad A(\boldsymbol{Q})=\left\{a(t)=\begin{pmatrix} t & & \\ & 1_{n-1} & \\ & & t^{-1} \end{pmatrix}\,\middle|\, t\in\boldsymbol{Q}^{\times}\right\}$$

is a maximal $\boldsymbol{Q}$-split torus in $P(\boldsymbol{Q})$. Here $Q[x]={}^t xQx$.

Since we assumed that $(Z, u_0)<0$ for all $Z\in B$, fix

$$Z_0=\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\in B. \tag{1.7}$$

Let

$$K=\{g\in G(\boldsymbol{R})\mid gZ_0=Z_0\}$$

be the corresponding maximal compact subgroup of $G^0(\boldsymbol{R})$. Let

$$M=\left\{\begin{pmatrix} 1 & & \\ & m & \\ & & 1 \end{pmatrix}\middle| m\in SO(Q)(\boldsymbol{R})\right\}$$

so that

$$M=P(\boldsymbol{R})\cap K.$$

Then we have decompositions

$$G^0(\boldsymbol{R})=N(\boldsymbol{R})A^0(\boldsymbol{R})K \tag{1.8}$$

and

$$P^0(\boldsymbol{R})=N(\boldsymbol{R})A^0(\boldsymbol{R})M \tag{1.9}$$

where $P^0(\boldsymbol{R})=P(\boldsymbol{R})\cap G^0(\boldsymbol{R})$ and $A^0(\boldsymbol{R})=A(\boldsymbol{R})\cap G^0(\boldsymbol{R})$.

For $t\in \boldsymbol{R}_+^\times$, let

$$A_t=\{a(t')\in A^0(\boldsymbol{R})\mid t'>t\} \tag{1.10}$$

and for an open, relatively compact subset $\omega\subset N(\boldsymbol{R})$, define the Siegel set

$$\mathfrak{S}_t=\omega A_t K\subset G^0(\boldsymbol{R}). \tag{1.11}$$

Then by [1, Théorème 13.1] there exists a Siegel set $\mathfrak{S}\subset G^0(\boldsymbol{R})$ such that

$$G^0(\boldsymbol{R})=\bigcup_i \Gamma g_i \mathfrak{S} \tag{1.12}$$

and

$$B=\bigcup_i \Gamma g_i \mathfrak{S}' \tag{1.13}$$

where $\mathfrak{S}'=\mathfrak{S}\cdot Z_0$.

1.3. Let

$$\boldsymbol{B}=\{z=(z_0, z_1)\in \boldsymbol{R}_+^\times\times \boldsymbol{R}^{n-1}\mid z_0-Q(z_1)>0\}$$

and define an isomorphism

$$\iota : \boldsymbol{B} \xrightarrow{\sim} B \qquad (1.14)$$

$$z \longmapsto P_-(z)(z_0 - Q(z_1))^{-1/2}$$

where

$$P_-(z) = \begin{pmatrix} z_0 \\ z_1 \\ 1 \end{pmatrix} \in V(\boldsymbol{R}). \qquad (1.15)$$

For $g \in G^0(\boldsymbol{R})$ and $z \in \boldsymbol{B}$ we define the automorphy factor $\mu(g, z) \in \boldsymbol{R}_+^\times$ by

$$gP_-(z) = P_-(gz)\mu(g, z) \qquad (1.16)$$

where we define the action of $G^0(\boldsymbol{R})$ on $\boldsymbol{B}$ via ι. Also define

$$t : G^0(\boldsymbol{R}) \longmapsto \boldsymbol{R}_+^\times$$

by the condition

$$g = n(x)a(t)k \qquad (1.17)$$

with $t = t(g)$ for the decomposition (1.8). Finally, for $z \in \boldsymbol{B}$ and $Z = \iota(z) \in B$ we let

$$t(Z) = t(z) = t(g)$$

where $z = g(1, 0)$. Note that then $Z = gZ_0$. By an easy calculation we obtain:

LEMMA 1.1.

i) $t(g) = \mu(g, (1, 0))^{-1}$

ii) $t(gz) = \mu(g, z)^{-1} t(z)$

iii) $t(z)^2 = z_0 - Q(z_1)$ *when* $z = (z_0, z_1) \in \boldsymbol{B}$

iv) $\mu(g, z) = 2|(u_0, gP_-(z))|$.

1.4. For $Z \in B$ we observe that the tangent space $T_Z(B)$ to B at Z can be canonically identified with $Z^\perp$ i.e.

$$T_Z(B) \cong Z^\perp. \qquad (1.18)$$

The inner product $(\,,\,)$ then determines a $G^0(\boldsymbol{R})$-invariant metric on B since its restriction to each $Z^\perp$ is positive definite. Let $d\mu$ denote the corresponding invariant volume form on B.

Let $C(B)$ denote the space of continuous functions on B. For $g_i \in G^0(\boldsymbol{Q})$ as in 1.2, and for $Z \in B$, let

$$t_i(Z) = t(g_i^{-1} Z). \qquad (1.19)$$

Then for $r \in \boldsymbol{R}$ we define a semi-norm $\rho_i(r)$ on $C(B)$ by setting

$$\begin{aligned}(1.20)\qquad \rho_i(r)(f) &= \sup_{Z \in \mathfrak{S}'} t(Z)^r |f(g_i Z)| \\ &= \sup_{Z \in g_i \mathfrak{S}'} t_i(Z)^r |f(Z)|,\end{aligned}$$

where $\mathfrak{S}'$ is the Siegel set in B chosen in 1.2. We also let

$$(1.21)\qquad \rho(r)(f) = \sup_i \rho_i(r)(f)$$

and, identifying $C(\Gamma\backslash B)$ with $C(B)^{\Gamma}$, the Γ-invariant functions on B, we define

$$(1.22)\qquad C_i(\Gamma\backslash B\,; r) = \{f \in C(\Gamma\backslash B) \mid \rho_i(r)(f) < \infty\}$$

and

$$(1.23)\qquad C(\Gamma\backslash B\,; r) = \{f \in C(\Gamma\backslash B) \mid \rho(r)(f) < \infty\}.$$

LEMMA 1.2. *If $r > 1-n$, then*

$$C(\Gamma\backslash B\,; r) \subset L^1(\Gamma\backslash B, d\mu).$$

§ 2. A theta series and its singular terms.

For $Z \in B$, $X \in V(\boldsymbol{Q})^n$ and $\tau = u + iv \in \mathfrak{H}_n$ we let

$$(2.1)\qquad (X, X)_{\tau, Z} = u(X, X) + iv(X, X)_Z$$

where $(\,,)_Z$ is the majorant associated to Z, defined by (1.1). We then define an n-form $\Phi(\tau, X)$ on B by

$$(2.2)\qquad \Phi(\tau, X)_Z(W) = \det v^{1/2} \det(X, W) e\left(\frac{1}{2} \operatorname{tr}(X, X)_{\tau, Z}\right)$$

where $W = (W_1, \cdots, W_n)$ with $W_i \in Z^{\perp} \cong T_Z(B)$. Finally, for $h \in (L^*/L)^n$ we define a Γ-invariant n-form on B by

$$(2.3)\qquad \Theta(\tau\,; h, L) = \sum_{X \in h + L^n} \Phi(\tau, X).$$

For convenience we shall sometimes omit the dependence on τ and write Θ or $\Theta(h, L)$ for $\Theta(\tau\,; h, L)$ and $\Phi(X)$ for $\Phi(\tau, X)$.

We want to consider the convergence of the integral (0.2) of the introduction. Write

$$(2.4)\qquad \Theta(h, L) = \theta(h, L) d\mu$$

where $\theta(h, L)$ is a Γ-invariant function on B.

PROPOSITION 2.1. *The n-form $\Theta(h, L)$ defined by* (2.3) *is L^1 on $\Gamma\backslash B$, i.e.*

$$\theta(h, L)\in L^1(\Gamma\backslash B, d\mu).$$

PROOF. First note that for $g\in G^0(\boldsymbol{Q})$,

$$\theta(h, L)(gZ)=\theta(g^{-1}h, g^{-1}L)(Z). \tag{2.5}$$

Therefore, by (1.13), it will be sufficient to show that, for arbitrary h and L, $\theta(h, L)$ is in $L^1(\mathfrak{S}')$.

Next choose $M\in \boldsymbol{Z}_{>0}$ such that, for the Witt basis chosen in 1.2, we have

$$(L')^*\supset L^*\supset L\supset L'$$

where $L'=M\boldsymbol{Z}^{n+1}$. We then have

$$\Theta(h, L)=\sum_{\substack{h'\equiv h(L)\\ \mathrm{mod}\, L'}}\Theta(h', L') \tag{2.6}$$

and

$$\theta(h, L)=\sum_{\substack{h'\equiv h(L)\\ \mathrm{mod}\, L'}}\theta(h', L'). \tag{2.7}$$

Thus it is sufficient to prove that each $\theta(h', L')$ is in $L^1(\mathfrak{S}')$.

Let W_0 be the properly oriented orthonormal n-frame in Z_0^+ given by

$$W_0=-\begin{pmatrix} 0 & 1 \\ T & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

where $T\in GL_{n-1}^+(\boldsymbol{R})$ such that ${}^tTQT=1_{n-1}$. Then for $Z=n(y)^{-1}a(t)Z_0\in\mathfrak{S}'$ we have

$$\theta(h', L')(Z)=\det v^{1/2}\sum_{X\equiv h'(L')}\det(Y, W_0)e\left(\frac{1}{2}\operatorname{tr}(Y, Y)_{\tau, Z_0}\right) \tag{2.9}$$

where $Y=a(t)^{-1}n(y)X$. Note that if

$$X=\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

then

$$Y=\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} t^{-1}(x_0+2{}^tyQx_1+Q[y]x_2) \\ x_1+yx_2 \\ tx_2 \end{pmatrix}, \tag{2.10}$$

where $x_0=h_0'+x_0'$ and ${}^tx_0'$ runs over $M\boldsymbol{Z}^n$. We now apply Poisson summation to the sum on x_0' and obtain, after a straightforward but lengthy calculation which we omit:

$$(2.11)\qquad \theta(h', L')(Z)$$

$$=-C\det v^{-1}t^{n+1}\sum_{\substack{{}^t w\in M^{-1}Z^n\\ x_1\equiv h_1'(M)\\ x_2\equiv h_2'(M)}}\det\begin{pmatrix}2w-x_2\bar{\tau}\\ y_1\end{pmatrix}e\Big(\frac{1}{2}\operatorname{tr}(Q[y_1]+\frac{1}{2}it^2v^t x_2x_2)\Big)$$

$$\times e(-\alpha^t w)\exp\Big(-2\pi t^2\Big(w-\frac{1}{2}x_2u\Big)v^{-1\,t}\Big(w-\frac{1}{2}x_2u\Big)\Big)$$

where

$$(2.12)\qquad \alpha=h_0'+2^t yQx_1+Q[y]x_2$$

and

$$(2.13)\qquad C=iM^{-n}2^{n/2-1}\det Q^{1/2}.$$

To estimate $|\theta(h', L')|$ on $\mathfrak{S}'$ note that all terms in (2.11) are exponentially decreasing as $t\to\infty$ except those for which $x_2=w=0$. But the coefficient

$$\det\begin{pmatrix}2w-x_2\bar{\tau}\\ y_1\end{pmatrix}$$

vanishes for such terms, and so $\theta(h', L')$ is in $L^1(\mathfrak{S}')$ as claimed.

Next we write

$$(2.14)\qquad \Phi(X)=\varphi(X)d\mu$$

and consider the series

$$(2.15)\qquad \theta_*(h, L)=\sum_{X\in h+L^n}|\varphi(X)|.$$

It was shown in [4], Proposition A, that $\theta_*(h, L)$ is not, in general, in $L^1(\Gamma\backslash B)$. We need more precise information.

First note that $\Phi(X)\equiv 0$ if $\dim_{\boldsymbol{Q}}\operatorname{span} X<n$. Let

$$Fr_n=\{X\in V(\boldsymbol{Q})^n\mid \dim_{\boldsymbol{Q}}\operatorname{span} X=n\}$$

and for each isotropic line $l\subset V(\boldsymbol{Q})$ let

$$S_n(l)=\{X\in Fr_n\mid l_\perp \operatorname{span} X\}.$$

Also let

$$S_n=\bigcup_l S_n(l)$$

and, for any subset Λ of the set of isotropic lines in $V(\boldsymbol{Q})$, let

$$S_n(\Lambda)=\bigcup_{l\in\Lambda} S_n(l).$$

We will call S_n the set of "singular" frames in $V(\boldsymbol{Q})$.

LEMMA 2.2.

i) $Fr_n=Fr_n^+\cup S_n$

where

$$Fr_n^+=\{X\in Fr_n \mid (\ ,\ )|_{\operatorname{span} X} \text{ is non-degenerate}\}.$$

ii) *If* $l\neq l'$, *then*

$$S_n(l)\cap S_n(l')=\varnothing$$

so that

$$S_n=\coprod_l S_n(l).$$

PROOF. To prove (i) observe that if $X\in Fr_n^+$, then $(\operatorname{span} X)^\perp$ is a line on which $(\ ,\ )$ is definite, either positive or negative. Therefore $Fr_n^+\cap S_n(l)=\varnothing$ for all l. On the other hand, if $X\in Fr_n-Fr_n^+$, then $(\ ,\ )|_{\operatorname{span} X}$ is degenerate and

$$l_X=\operatorname{span} X\cap(\operatorname{span} X)^\perp$$

is, by signature considerations, precisely the isotropic line such that $X\in S_n(l)$. This proves (ii) as well.

If $l_0, \cdots, l_r$ are a set of null lines as in Section 1.2, then

$$S_n=\coprod_i S_n(\Gamma\cdot l_i). \tag{2.16}$$

We then obtain a decomposition of $\Theta(h, L)$ into Γ-invariant n-forms

$$\Theta(h, L)=\Theta^+(h, L)+\sum_i \Theta^{(i)}(h, L) \tag{2.17}$$

given by:

$$\Theta^+(h, L)=\sum_{X\in(h+L^n)\cap Fr_n^+} \Phi(X) \tag{2.18}$$

and

$$\Theta^{(i)}(h, L)=\sum_{X\in(h+L^n)\cap S_n(\Gamma\cdot l_i)} \Phi(X). \tag{2.19}$$

We also consider the series

$$\theta_*^+(h, L)=\sum_{X\in(h+L^n)\cap Fr_n^+} |\varphi(X)| \tag{2.20}$$

and

$$\theta_*^{(i)}(h, L)=\sum_{X\in(h+L^n)\cap S_n(\Gamma\cdot l_i)} |\varphi(X)|. \tag{2.21}$$

PROPOSITION 2.3.

i) $\theta_*^{\ddagger}(h, L)\in C(\Gamma\backslash B; r)$ $\forall r$

ii) $\theta_*^{(i)}(h, L)\in C_i(\Gamma\backslash B; r)$ *for* $r<-(n+3)$

and

$\theta_*^{(i)}(h, L)\in C_j(\Gamma\backslash B; r)$ $\forall r$ *if* $i\neq j$.

PROOF. If $Z\in\mathfrak{S}'$ and if $g_j\in G^0(\boldsymbol{Q})$ is as in Section 1.2, then

$$\theta_*^{(i)}(h, L)(g_jZ)=\sum_{X\in g_j^{-1}S_n(\Gamma\cdot l_i)\cap g_j^{-1}(h+L^n)}|\varphi(X)(Z)|.$$

Now the proof of Proposition A of [4] actually shows that, for arbitrary h' and L',

$$\sum_{X\in S_n(l)\cap(h'+(L')^n)}|\varphi(X)(Z)|=\begin{cases}0(t^{n+3}) & \text{if } l=l_0\\ 0(e^{-ct}) & \text{otherwise}\end{cases}$$

on $\mathfrak{S}'$. Since

$$S_n(l_0)\subset g_j^{-1}S_n(\Gamma\cdot l_i)$$

if and only if $g_jS_n(l_0)=S_n(l_j)\subset S_n(\Gamma\cdot l_i)$, i.e. $i=j$, we obtain (ii). The first statement follows immediately from Proposition A of [4] and its proof.

We can now make the following:

DEFINITION. $h\in(L^*/L)^n$ is *non-singular* if $(h+L^n)\cap S_n(l)=\varnothing$ for all isotropic lines l in $V(\boldsymbol{Q})$. Note that any h which is non-singular in the sense of [4] is necessarily *non-singular* in the sense just defined.

Combining Propositions 2.1 and 2.3 with the results of [4] we obtain the following:

COROLLARY 2.4. *If h is non-singular in the sense defined above, then the integral* (0.2) *can be computed term by term and the function*

$$I(\tau; h, L; \Gamma)=2^{-n/2}\vartheta^+(\tau; h, L; \Gamma)$$

is a holomorphic Siegel modular form of weight $(n+1)/2$ *whose precise transformation law is the same as that of the function* $\vartheta(\tau; h, L; \Gamma)$, *given in Theorem* 3.2 *below.*

§ 3. The main theorem

Before stating our main result we recall and extend the definition of the sign function $\varepsilon(X)$. If $X\in Fr_n^+$, then there is a unique point $Z=B\cap(\text{span } X)^{\perp}$. In this case we define, as in [4],

$$\varepsilon(X)=\begin{cases}+1 \text{ if } X \text{ is properly oriented in } T_Z(B)\cong\text{span } X\\ -1 \text{ otherwise.}\end{cases} \tag{3.1}$$

Next for $X\in S_n$, let

(3.2) $$l_X=(\text{span } X)\cap(\text{span } X)^{\perp}$$

so that $X\in S_n(l_X)$, as in the proof of Lemma 2.2. Now l_X, and in fact any null line in $V(\boldsymbol{Q})$, has a distinguished basis vector u_X determined by the conditions (i) $l_X=\boldsymbol{Q}u_X$ (ii) $(u_X, Z)<0$ $\forall Z\in B$, and (iii) u_X is primitive in L. Choose any null line $l'\subset V(\boldsymbol{R})$ such that $l'+l_X=H$ is a hyperbolic plane, and note that

(3.3) $$\text{span } X=l_X+H^{\perp}.$$

Let $u'\in l'$ be a basis vector for l' satisfying (ii). We then obtain a distinguished orientation for $H^{\perp}$ by requiring that, for any properly oriented basis $v_1, \cdots, v_{n-1}$ for $H^{\perp}$, the basis $u_X, v_1, \cdots, v_{n-1}, u'$ is properly oriented for $V(\boldsymbol{R})$, with respect to the orientation of $V(\boldsymbol{R})$ fixed in Section 1.1. Then define

(3.4) $$\varepsilon(X)=\begin{cases}+1 \text{ if } X \text{ and } \{u_X, v_1, v_1, \cdots, v_{n-1}\} \text{ determine the} \\ \qquad\qquad \text{same orientation of span } X \\ -1 \text{ otherwise.}\end{cases}$$

Note that $\varepsilon(X)$ is independent of the choice of l' since the stabilizer $G^0(\boldsymbol{R})_{u_X}$ of u_X in $G^0(\boldsymbol{R})$ acts transitively on the set of such l''s and $G^0(\boldsymbol{R})_{u_X}\cong N(\boldsymbol{R})M$ is connected.

If $X\in S_n$ there exists an $a\in \boldsymbol{Z}^n$, $a\neq 0$ primitive, such that

(3.5) $$X\cdot a=\nu u_X$$

with $\nu\in\boldsymbol{Q}$. Note that X determines a uniquely up to ± 1.

DEFINITION. A frame $X\in S_n$ is *reduced* if there exists a choice of $a\in\boldsymbol{Z}^n$, $a\neq 0$ primitive, such that $\nu\in[0, 1)$. If X is reduced this choice of a is unique and we let

(3.6) $$\nu(X)=\nu.$$

Note that if X is reduced and $\gamma\in\Gamma(L)\cap G^0(\boldsymbol{R})$, then γX is also reduced and $\nu(X)=\nu(\gamma X)$.

The following is then easily checked:

LEMMA 3.1. *If $a\in\boldsymbol{Z}^n$, $a\neq 0$ primitive, choose $a'\in\boldsymbol{Z}^n$ such that ${}^t a' a=1$. If $X\in S_n$ is reduced with respect to a, then*

$$\tilde{X}=X-u_X{}^t a'$$

is reduced with respect to $-a$,

$$\nu(\tilde{X})=1-\nu(X)$$

and

$$\boldsymbol{B}_1(\nu(X))\varepsilon(X)=\boldsymbol{B}_1(\nu(\tilde{X}))\varepsilon(\tilde{X}),$$

where

$$\boldsymbol{B}_1(\alpha)=\begin{cases}\alpha-1/2 & \text{if } \alpha\in(0,1)\\ 0 & \text{if } \alpha=0.\end{cases}$$

The main result of this paper is the following theorem, whose proof will be given in Sections 4 and 5.

THEOREM 3.2. *Assume that Γ is torsion free. Let*

$$\vartheta^0(\tau, h, L, \Gamma)=\frac{1}{2}(-1)^n \sum_{\substack{X\in(h+L^n)\cap S_n\\ X \text{ reduced}\\ \bmod \Gamma}} \boldsymbol{B}_1(\nu(X))\varepsilon(X)e\Big(\frac{1}{2}\operatorname{tr}(\nu(X, X))\Big),$$

where $\boldsymbol{B}_1$ is as in Lemma 3.1 and $\varepsilon(X)$ is defined by (3.4). Let

$$\vartheta(\tau, h, L, \Gamma)=\vartheta^0(\tau, h, L, \Gamma)+\vartheta^+(\tau, h, L, \Gamma)$$

where $\vartheta^+(\tau, h, L, \Gamma)$ is given by (0.1). Then $\vartheta(\tau, h, L, \Gamma)$ is a holomorphic Siegel modular form of weight $(n+1)/2$.

More precisely, if $N\in \boldsymbol{Z}_{>0}$ is such that $NL^*\subset L$ and $N(v, v')\in 2\boldsymbol{Z}$ for all $v, v'\in L^*$, then for all $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)\subset Sp(n, \boldsymbol{Z})$ such that $\operatorname{tr}(Xa, Xb)\in 2\boldsymbol{Z}$ and $\operatorname{tr}(Yc, Yd)\in 2\boldsymbol{Z}$ for all $X, Y\in L^n$:

$$\vartheta(\gamma\tau, h, L, \Gamma)=\lambda(\gamma)\mathfrak{g}(b, d\,; L)\det(c\tau+d)^{(n+1)/2}$$
$$\times e\Big(\frac{1}{2}\operatorname{tr}((h, h)b^t a)\Big)\vartheta(\tau, ha, L, \Gamma)$$

where $\lambda(\gamma)^8=1$ and

$$\mathfrak{g}(b, d\,; L)=|\det d|^{(n+1)/2}\sum_{X\in L/L^t d} e\Big(\frac{1}{2}\operatorname{tr}((X, X)bd^{-1})\Big).$$

Moreover, if n is odd and $(v, v')\in 2\boldsymbol{Z}$ for all $v, v'\in L$, then

$$\vartheta(\gamma\tau, h, L, \Gamma)=\chi(d)\det(c\tau+d)^{(n+1)/2}e\Big(\frac{1}{2}\operatorname{tr}((h, h)b^t a)\Big)\vartheta(\tau, ha, L, \Gamma)$$

where

$$\chi(d)=\Big(\frac{(-1)^{(n+1)/2}\Delta}{|\det d|}\Big)(\operatorname{sgn}\det d)^{(n-1)/2}.$$

Here $\Delta=\det((v_i, v_j))$ where $\{v_1, \cdots, v_{n+1}\}$ is a $\boldsymbol{Z}$-basis for L.

REMARK. 1) When $n=1$ and h is singular $\vartheta(\tau, h, L, \Gamma)$ is precisely Hecke's Eisenstein series of weight 1 [4, Theorem 3.2].

2) The Fourier coefficients of $\vartheta(\tau, h, L, \Gamma)$ are rational and in fact have bounded denominators. To see this observe that if $M\in \boldsymbol{Z}_{>0}$ such that $Mh\in L^n$, then for any reduced X in $h+L^n$

$$M\nu(X)u_X=MX\cdot a\in L^n\cdot a=L$$

so that $M\nu(X)\in \boldsymbol{Z}$.

§ 4. Eisenstein series.

In this section we summarize certain facts about the simplest type of Eisenstein series for $SO(n, 1)$.

For $z\in \boldsymbol{B}$ and $s\in \boldsymbol{C}$ define

$$E(z, s, L)=\sum_{\substack{u\in L \text{ primitive}\\ (u,u)=0\\ u\neq 0}} |(u, P_-(z))|^{-s}t(z)^s \tag{4.1}$$

where the notation is as in Section 1.3. By Proposition 5.4 of [7] this series converges absolutely in the half-plane $\mathrm{Re}(s)>n$. We note that, by Lemma 1.1,

$$\begin{aligned} E(z, s, L)&=2\sum_{i=1}^{r}\sum_{\gamma\in\Gamma_i\backslash\Gamma}|(\gamma^{-1}u_i, P_-(z))|^{-s}t(z)^s \\ &=2^{s+1}\sum_i\sum_{\gamma\in\Gamma_i\backslash\Gamma}t_i(\gamma z)^s \end{aligned}\tag{4.2}$$

where $\Gamma_i=\{\gamma\in\Gamma \mid \gamma u_i=u_i\}$. Thus, up to the factor 2^{s+1}, $E(z, s, L)$ is the sum over the various inequivalent cusps of Γ of the standard Eisenstein series for the constant function. It therefore follows from the standard theory of such series, [6], [2], that $E(z, s, L)$ has a meromorphic analytic continuation to the whole s-plane with at most simple poles lying in the interval $[1-n, n-1]$, and has a functional equation relating $E(z, s, L)$ and $E(z, n-1-s, L)$. In fact we will only be interested in the behavior of $E(z, s, L)$ in the half-plane $\mathrm{Re}(s)>n-3/2$ and the facts which we need can be proved by direct calculations which we omit.

For convenience we assume that

$$\Gamma_i=\Gamma\cap g_iPg_i^{-1}. \tag{4.3}$$

Then there exists a lattice $\Lambda_i\subset \boldsymbol{Q}^{n-1}$ such that

$$\Gamma_i=\{g_in(\lambda')g_i^{-1} \mid \lambda'\in\Lambda_i\}. \tag{4.4}$$

Let

$$\Lambda_i^*=\{\lambda\in \boldsymbol{Q}^{n-1} \mid {}^t\lambda'\lambda\in \boldsymbol{Z} \quad \text{for all } \lambda'\in\Lambda_i\} \tag{4.5}$$

be the dual lattice of Λ_i.

PROPOSITION 4.1. *The Fourier expansion of* $E(z, s, L)$ *with respect to the translations of* Γ_i *is*

$$E(z, s, L)=2^{s+1}t^s+C_i\frac{\pi^{-\nu}\Gamma(\nu)}{\pi^{-s}\Gamma(s)}\zeta(s)^{-1}D_i(s, 0, L)t^{n-1-s}$$

$$+C_i\frac{2\pi^s}{\Gamma(s)}t^{n-1-s}\sum_{\substack{\lambda\in\Lambda_i^*\\ \lambda\neq 0}}D_i(s, \lambda, L)\xi_i(t, s, \lambda)e({}^t\lambda x)$$

where $z=g_in(x)a(t)(1, 0)\in\boldsymbol{B}$ *and* $t=t_i(z)$. *Here*

$$\nu=s-\frac{1}{2}(n-1)$$

$$C_i=\det Q^{-1/2}\operatorname{vol}(\boldsymbol{R}^{n-1}/\Lambda_i)^{-1}$$

and

$$\xi_i(t, s, \lambda)=t^\nu Q^{-1}(\lambda)^{\nu/2}K_\nu(2\pi t\sqrt{Q^{-1}(\lambda)})$$

with K_ν *the usual Bessel function. Finally the Dirichlet series* $D_i(s, \lambda, L)$ *in the* λ*-the coefficient is given by*

$$D_i(s, \lambda, L)=\sum_{\substack{u\in L\\ (u,u)=0\\ (u,u_i)\neq 0\\ \mathrm{mod}\,\Gamma_i}}|(u, u_i)|^{-s}\phi_i(\lambda, u)$$

where, if

$$g_i^{-1}u=\begin{pmatrix} a\\ b\\ c\end{pmatrix}$$

with respect to the Witt basis of Section 1.2,

$$\phi_i(\lambda, u)=e(-c^{-1}\,{}^t\lambda b).$$

An analysis of the Dirichlet series $D_i(s, \lambda, L)$ yields the following:

PROPOSITION 4.2. (i) *The function* $\zeta(s)^{-1}D_i(s, 0, L)$ *is holomorphic in the half-plane* $\mathrm{Re}(s)>n-3/2$ *except for a simple pole with non-vanishing residue at* $s=n-1$.

(ii) *For* $\lambda\neq 0$ *the function* $\zeta(s)^{-1}D_i(s, \lambda, L)$ *is holomorphic in the half-plane* $\mathrm{Re}(s)>n-3/2$ *and there exist a constant* $C>0$ *independent of* λ *and* s *and a function* $C(s)>0$ *bounded uniformly in vertical strips such that*

$$|\zeta(s)^{-1}D_i(s, \lambda, L)|\leqq C(s)Q^{-1}(\lambda)^{\mathrm{Re}(s)+C}.$$

The analytic continuation of $E(z, s, L)$ follows.

COROLLARY 4.3. (i) *The function* $E(z, s, L)$ *has a holomorphic analytic con-*

tinuation to the half-plane $\mathrm{Re}(s)>n-3/2$ *except for a simple pole with non-zero residue at* $s=n-1$.

(ii) *If*

$$E_i'(z, s, L)=C_i\frac{2\pi^s}{\Gamma(s)}t^{n-1-s}\sum_{\substack{\lambda\neq 0\\ \lambda\in\Lambda_i^*}}\zeta(s)^{-1}D_i(s, \lambda, L)\xi_i(t, s, \lambda)e({}^t\lambda x)$$

in the notation of Proposition 4.1, *then for* $\mathrm{Re}(s)>n-3/2$ *and* z *such that* $t_i(z)>\varepsilon>0$ *for some* ε,

$$|E_i'(z, s, L)|\leqq\frac{C(s, \varepsilon)}{|\Gamma(\nu+1/2)\Gamma(s)|}t_i(z)^{n/2-1}e^{-ct_i(z)}$$

where $C(s, \varepsilon)$ *is bounded uniformly in vertical strips and* $C>0$ *is a constant.*

We now state the analogue of Lemma 2.9 of Shintani [8]. Let $\psi(\rho)=e^{\rho^2}$ and define $\mathcal{E}(z, s)$ by (0.7).

LEMMA 4.4. (i) $\mathcal{E}(z, s)\in C(\Gamma\backslash B, \mathrm{Re}(s)-n+1)$.

(ii) *For all* i,

$$\sup_{\substack{n-1\leqq\sigma\leqq M\\ z\in g_i\mathfrak{S}'}}|(\sigma-n+1)\mathcal{E}(z, \sigma)|<\infty.$$

(iii) *The limit*

$$\kappa=\lim_{\sigma\to(n-1)^+}(\sigma-n+1)\mathcal{E}(z, \sigma)$$

is finite with

$$\kappa=\psi(n-1)\operatorname*{Res}_{s=n-1}E(z, s, L).$$

In particular, $\kappa\neq 0$.

The resulting integral formula will be the key to our computation of $\int_{\Gamma\backslash B}\Theta^{(i)}$.

COROLLARY 4.5. *If* $f\in L^1(\Gamma\backslash B, d\mu)$, *then*

$$\lim_{\sigma\to(n-1)^+}(\sigma-n+1)\int_{\Gamma\backslash B}f(z)\mathcal{E}(z, s)d\mu(z)=\kappa\int_{\Gamma\backslash B}f(z)d\mu(z).$$

§ 5. The integral of the singular terms.

In this section we apply the integral formula of Section 4, Corollary 4.5 to the form $\Theta^{(i)}$ defined by (2.19). This gives:

$$\int_{\Gamma\backslash B}\Theta^{(i)}=\kappa^{-1}\lim_{\sigma\to(n-1)^+}(\sigma-n+1)\int_{\Gamma\backslash B}\mathcal{E}(\sigma)\Theta^{(i)}. \tag{5.1}$$

On the other hand, by Propositions 2.3 and 4.4 and Lemma 1.2, the n-form $\mathcal{E}(s)\Theta^{(i)}$ can be integrated term by term provided $\mathrm{Re}(s)>n+3$. Let

$$\Theta_0^{(i)}(h, L)=\sum_{X\in(h+L^n)\cap S_n(l_i)}\Phi(X). \tag{5.2}$$

By Lemma 2.2 and the fact that $g^*\Phi(X)=\Phi(g^{-1}X)$ for $g\in G^0(\boldsymbol{R})$,

$$\Theta^{(i)}(h, L)=\sum_{\gamma\in\Gamma_i\backslash\Gamma}\gamma^*\Theta_0^{(i)}(h, L). \tag{5.3}$$

Thus, for $\mathrm{Re}(s)>n+3$,

$$\int_{\Gamma\backslash B}\mathcal{E}(s)\Theta^{(i)}=\int_{\Gamma\backslash B}\mathcal{E}(s)\sum_{\gamma\in\Gamma_i\backslash\Gamma}\gamma^*\Theta_0^{(i)} \tag{5.4}$$

$$=\int_{\Gamma_i\backslash B}\mathcal{E}(s)\Theta_0^{(i)}$$

by the usual unfolding argument.

A fundamental domain for Γ_i in $\boldsymbol{B}$ is given by

$$F_i=\{z=g_i n(x)a(t)(1, 0)\,|\,x\in\boldsymbol{R}^{n-1}/\Lambda_i\}. \tag{5.5}$$

As in Section 2 write

$$\Theta_0^{(i)}=\theta_0^{(i)}d\mu \tag{5.6}$$

where

$$\theta_0^{(i)}=\sum_{X\in(h+L^n)\cap S_n(l_i)}\varphi(X) \tag{5.7}$$

and $\varphi(X)$ is given by (2.14). Viewing $\theta_0^{(i)}$ and $\mathcal{E}(s)$ as functions on $\boldsymbol{B}$ via (1.14), we obtain

$$\int_{\Gamma_i\backslash B}\mathcal{E}(s)\Theta_0^{(i)}=(-1)^n\det Q^{1/2}\int_0^\infty\int_{R^{n-1}/\Lambda_i}\mathcal{E}(z, s)\theta_0^{(i)}(z)t^{-n}dx\,dt \tag{5.8}$$

where $z=g_i n(x)a(t)(1, 0)\in\boldsymbol{B}$ and we note that in these coordinates

$$\iota^*(d\mu)=(-1)^n\det Q^{1/2}t^{-n}dx_1\wedge\cdots\wedge dx_{n-1}\wedge dt. \tag{5.9}$$

Let

$$\theta_0^{(i)}(z)=\sum_{\lambda\in\Lambda_i^*}A_i(t, \lambda)e({}^t\lambda x) \tag{5.10}$$

and

$$\mathcal{E}(z, s)=\sum_{\lambda\in\Lambda_i^*}B_i(t, s, \lambda)e(-{}^t\lambda x) \tag{5.11}$$

be the Fourier expansions of $\theta_0^{(i)}$ and $\mathcal{E}(z, s)$ with respect to the translations of Γ_i. Let

$$I_0(s)=\int_0^\infty A_i(t, 0)B_i(t, s, 0)t^{-n}dt \tag{5.12}$$

and

$$I_1(s)=\int_0^{\infty}\sum_{\substack{\lambda\in\Lambda_i^*\\ \lambda\neq 0}}A_i(t,\ \lambda)B_i(t,\ s,\ \lambda)t^{-n}dt\,. \tag{5.13}$$

Then, as in Shintani [8] p. 179, Parseval's identity applied to (5.8) yields

$$\operatorname{vol}(\boldsymbol{R}^{n-1}/\Lambda_i)^{-1}\det Q^{-1/2}\int_{\Gamma\backslash B}\mathcal{E}(s)\Theta_0^{(i)}=I_0(s)+I_1(s)\,. \tag{5.14}$$

LEMMA 5.1. *For* $z=g_i n(x)a(t)(1,\ 0)$,

$$A_i(t,\ \lambda)=\operatorname{vol}(\boldsymbol{R}^{n-1}/\Lambda_i)^{-1}\sum_{\substack{X\in(h+L^n)\cap S_n(l_i)\\ \bmod \Gamma_i}}\int_{R^{n-1}}\varphi(X)(z)e(-{}^t\lambda x)dx\,.$$

PROOF. By Lemma 2.2

$$\theta_0^{(i)}=\sum_{\substack{X\in(h+L^n)\cap S_n(l_i)\\ \bmod \Gamma_i}}\ \sum_{\gamma\in\Gamma_i}\varphi(\gamma^{-1}X)$$

so that the integral

$$A_i(t,\ \lambda)=\operatorname{vol}(\boldsymbol{R}^{n-1}/\Lambda_i)^{-1}\int_{R^{n-1}/\Lambda_i}\theta_0^{(i)}(z)e(-{}^t\lambda x)dx$$

can be unfolded in the usual way.

LEMMA 5.2. *If* $X\in S_n(l_i)$ *and* $z=g_i n(x)a(t)(1,\ 0)\in\boldsymbol{B}$, *then*

$$\varphi(X)=\frac{(-1)^n}{2}\det v^{1/2}\det Q^{1/2}t^{-1}\det\begin{pmatrix} y_1\\ y_0\end{pmatrix}e\Big(\frac{1}{2}\operatorname{tr}(\tau(X,\ X))\Big)$$

$$\times\exp\Big(-\frac{1}{2}\pi t^{-2}\operatorname{tr}(v^t(y_0-2^t xQy_1)(y_0-2^t xQy_1))\Big)\,,$$

where

$$y=g_i^{-1}X=\begin{pmatrix} y_0\\ y_1\\ y_2\end{pmatrix}.$$

PROOF. Since $\varphi(g^{-1}X)(z)=\varphi(X)(gz)$ for $g\in G^0(\boldsymbol{R})$, it is sufficient to compute $\varphi(X)(Z_0)$ for arbitrary X. By (2.2)

$$\varphi(X)(Z_0)=\det v^{1/2}\det(X,\ W_0)e\Big(\frac{1}{2}\operatorname{tr}(X,\ X)_{\tau,Z_0}\Big)$$

where W_0 is a properly oriented orthonormal n-frame in $T_{Z_0}(B)\cong Z_0^{\perp}$. Explicitly let

$$W_0=-\begin{pmatrix} 0 & 1\\ T & 0\\ 0 & -1\end{pmatrix}$$

where $T\in GL_{n-1}^{+}(\boldsymbol{R})$ such that ${}^tTQT=1_{n-1}$. Also

$$(X, X)_{\tau,Z}=\tau(X, X)-2iv(X, Z)(Z, Z)^{-1}(Z, X)$$

so that, if X is given by (2.10)′,

$$(X, X)_{\tau,Z_0}=\tau(X, X)-\frac{1}{2}iv^t(x_0+x_2)(x_0+x_2).$$

Next observe that if $X\in S_n(l_i)$, then $g_i^{-1}X=Y\in S_n(l_0)$ so that $y_2=0$.

LEMMA 5.3. (i) *If* $\lambda=0$,

$$A_i(t, 0)=2^{-(n+1)/2}C_it^{n-2}\sum_{\substack{X\in(h+L^n)\cap S_n(l_i)\\ \mathrm{mod}\,\Gamma_i}}(-1)^n\,\mathrm{sgn}\det\begin{pmatrix}y_1\\ y_0\end{pmatrix}$$
$$\times e\Big(\frac{1}{2}\mathrm{tr}(\tau(X, X))\Big)\xi^{-1/2}\exp\Big(-\frac{1}{2}\pi t^{-2}\xi^{-1}\Big)$$

where $Y=g_i^{-1}X$ *as in Lemma* 5.2 *and*

$$\xi=\det v^{-1}\det\begin{pmatrix}y_1\\ y_0\end{pmatrix}^{-2}\det v[{}^ty_1],$$

with $v[{}^ty_1]=y_1v^ty_1$. *Also* C_i *is as in Proposition* 4.1.

(ii) *If* $\lambda\neq0$

$$|A_i(t, \lambda)|\leqq Ct^{n-2}\sum_{\substack{X\in(h+L^n)\cap S_n(l_i)\\ \mathrm{mod}\,\Gamma_i}}\xi^{-1/2}\exp\Big(-\frac{1}{2}\pi t^{-2}\xi^{-1}-\pi\,\mathrm{tr}(v(X, X))\Big).$$

PROOF. By Lemma 5.1 and Lemma 5.2 it is sufficient to consider

$$f(t, \lambda)=\int_{R^{n-1}}\exp\Big(-\frac{1}{2}\pi t^{-2}\,\mathrm{tr}(v^t(y_0-2^txQy_1)(y_0-2^txQy_1))\Big)e(-{}^t\lambda x)dx.$$

Note that

$$|f(t, \lambda)|\leqq f(t, 0)$$

so that (ii) follows from (i). Let

$$S=\begin{pmatrix}-2Qy_1\\ y_0\end{pmatrix}v\,{}^t\!\begin{pmatrix}-2Qy_1\\ y_0\end{pmatrix}$$

so that

$$f(t, 0)=\int_{R^{n-1}}\exp\Big(-\frac{1}{2}\pi t^{-2}S\Big[\begin{pmatrix}x\\ 1\end{pmatrix}\Big]\Big)dx.$$

Then we find easily that

$$f(t, 0)=2^{(n-1)/2}t^{n-1}\det S^{-1/2}\xi^{-1/2}\exp(-1/2\,\pi t^{-2}\xi)$$

where

$$\xi=S^{-1}\Big[\begin{pmatrix}0\\ 1\end{pmatrix}\Big].$$

Since

$$\det S = 2^{2(n-1)} \det Q^2 \det \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^2 \det v$$

and ξ is as in (i) we obtain the expression claimed.

Next we consider $B_i(t, s, \lambda)$. Let

$$b_i(t, s, \lambda) = C_i \frac{2\pi^s}{\Gamma(s)} t^{n-1-s} \zeta(s)^{-1} D_i(s, \lambda, L) \xi_i(t, s, \lambda) \tag{5.15}$$

so that

$$B_i(t, s, \lambda) = \frac{1}{2\pi i} \int_{\rho_0 - i\infty}^{\rho_0 + i\infty} \frac{\phi(\rho)}{s-\rho} b_i(t, \rho, \lambda) d\rho .$$

For $\lambda \neq 0$, $b_i(t, \rho, \lambda)$ is holomorphic in the half-plane $\mathrm{Re}(s) > n-3/2$ and we may move the contour of integration in (5.16) to $\mathrm{Re}(\rho) = \rho_0'$ with $n-3/2 < \rho_0' < n-1-\varepsilon$ for some ε with $0 < \varepsilon < 1/2$.

LEMMA 5.4. *Let $\alpha = Q^{-1}(\lambda)^{1/2}$. Then for any $k>0$, there exist a constant $C(s, k) > 0$, depending on s, such that*

$$|b_i(t, s, \lambda)| \leqq C(s, k) t^{n-1-\sigma} \alpha^{C'+\sigma} (1+\alpha t)^{-k}$$

and

$$|B_i(t, s, \lambda)| \leqq C(k) t^{n-1-\sigma} \alpha^{C'+\sigma} (1+\alpha t)^{-k}$$

where $C' > 0$ is as in Corollary 4.6 and $\sigma = \mathrm{Re}(s)$.

PROOF. This follows easily from Corollary 4.6 and standard estimates on $|K_\nu(x)|$.

PROPOSITION 5.5. *$I_1(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > n-1-\varepsilon$ for some $\varepsilon > 0$. In particular*

$$\lim_{\sigma \to (n-1)^+} (\sigma - n + 1) I_1(\sigma) = 0 .$$

PROOF. By Lemma 5.4 and (ii) of Lemma 5.3 we have, for $\mathrm{Re}(s) > n-1-\varepsilon$,

$$\begin{aligned} |I_1(s)| &\leqq \int_0^\infty \sum_{\lambda \neq 0} |A_i(t, \lambda)| \, |B_i(t, s, \lambda)| t^{-n} dt \\ &\leqq C_k \sum_{\lambda \neq 0} \sum_{\substack{X \in (h+L^n) \cap S_n(l_i) \\ \mathrm{mod}\, \Gamma_i}} \xi^{-1/2} \exp(-\pi \,\mathrm{tr}\,(v(X, X))) \alpha^{C'+\sigma} \\ &\quad \times \int_0^\infty \exp\left(-\frac{1}{2}\pi t^{-2} \xi^{-1}\right) t^{n-3-\sigma} (1+\alpha t)^{-k} dt \\ &\leqq C_k \sum_{\lambda \neq 0} \sum_X \alpha^{C'+\sigma-k} \xi^{-(n-2-\sigma-k)/2} \exp(-\pi \,\mathrm{tr}\,(v(X, X))) . \end{aligned}$$

The series $\sum_{\lambda\neq 0}\alpha^{C'+\sigma-k}$ is convergent for sufficiently large k, so we need only consider the sum on X. Note that if $Y=g_i^{-1}X$ for $X\in(h+L^n)\cap S_n(l_i)$, then $(X, X)=(Y, Y)=Q[y_1]$, Choose $M\in \boldsymbol{Z}_{>0}$ such that $M\boldsymbol{Z}^{n-1}\subset \Lambda_i$ and $g_i^{-1}(h+L^n)\subset (M^{-1}\boldsymbol{Z}^{n+1})^n$. Then we have

$$\sum_{X \bmod \Gamma_i}\xi^{-(n-2-\sigma-k)/2}\exp(-\pi\,\mathrm{tr}\,(v(X, X)))$$

$$\leqq C\det v^{(n-2-\sigma-k)/2}\sum_{y_1\in M^{-1}M_{n-1,n}(\boldsymbol{Z})}\det v[{}^t y_1]^{(\sigma+k+2-n)/2}\exp(-\pi\,\mathrm{tr}\,(vQ[y_1]))$$

$$\times \sum_{\substack{{}^t y_0\in M^{-1}\boldsymbol{Z}^n \\ \det\binom{y_1}{y_0}\neq 0 \\ \bmod \{2^t x Q y_1 | x\in M\boldsymbol{Z}^{n-1}\}}}\left|\det\begin{pmatrix} y_1 \\ y_0\end{pmatrix}\right|^{n-2-\sigma-k}.$$

For fixed y_1 the inner sum is convergent for $\sigma+k>n-1$ and the proposition is proved.

PROPOSITION 5.6. *For* $\mathrm{Re}(s)>n+3$ *let*

$$\Omega_i(s, \tau, h, L)=\sum_{\substack{X\in(h+L^n)\cap S_n(l_i) \\ \bmod \Gamma_i}} f(s, g_i^{-1}X)e\left(\frac{1}{2}\mathrm{tr}\,(\tau(X, X))\right)$$

where, for $Y=g_i^{-1}X$,

$$f(s, Y)=(-1)^n\,\mathrm{sgn}\det\begin{pmatrix} y_1 \\ y_0\end{pmatrix}\left|\det\begin{pmatrix} y_1 \\ y_0\end{pmatrix}\right|^{n-1-s}\det v[{}^t y_1]^{(s+1-n)/2}.$$

Then the function

$$I_0(s)-C_i(s)\zeta(s)^{-1}D_i(s, 0, L)\Omega_i(s, \tau, h, L)$$

has a holomorphic analytic continuation to the half-plane $\mathrm{Re}(s)>n-1-\varepsilon$ *for some* ε *with* $0<\varepsilon<1/2$. *Here*

$$C_i(s)=C_i^2\phi(s)\frac{\pi^{-\nu}\Gamma(\nu)}{\pi^{-s}\Gamma(s)}2^{n-(s-1)/2}\pi^{(n-2-s)/2}\Gamma\left(\frac{1}{2}(s-n+2)\right)\det v^{(n-1-s)/2}.$$

PROOF. Let

$$b_i'(t, s, 0)=2^{s+1}t^s$$

and

$$b_i''(t, s, 0)=C_i\frac{\pi^{-\nu}\Gamma(\nu)}{\pi^{-s}\Gamma(s)}\zeta(s)^{-1}D_i(s, 0, L)t^{n-1-s},$$

and define

$$B_i'(t, s, 0)=\frac{1}{2\pi i}\int_{\rho_0-i\infty}^{\rho_0+i\infty}\frac{\phi(\rho)}{s-\rho}b_i'(t, \rho, 0)d\rho$$

and

$$B_i''(t, s, 0)=\frac{1}{2\pi i}\int_{\rho_0-i\infty}^{\rho_0+i\infty}\frac{\phi(\rho)}{s-\rho}b_i''(t, \rho, 0)d\rho$$

Then

$$B_i(t, s, 0)=B_i'(t, s, 0)+B_i''(t, s, 0)$$

and there is a corresponding decomposition

$$I_0(s)=I_0'(s)+I_0''(s).$$

First consider

$$I_0'(s)=\int_0^\infty A_i(t, 0)\frac{1}{2\pi i}\int_{\rho_0-i\infty}^{\rho_0+i\infty}\frac{2^{\rho+1}t^\rho\phi(\rho)}{s-\rho}d\rho\, t^{-n}dt$$

where we may take any $\rho_0<\mathrm{Re}(s)$. But then

$$\begin{aligned}|I_0'(s)|&\leqq C\int_0^\infty |A_i(t, 0)|t^{\rho_0-n}dt\\ &\leqq C\sum_{\substack{X\in(h+L^n)\cap S_n(l_i)\\ \mathrm{mod}\,\Gamma_i}}\exp(-\pi\,\mathrm{tr}(v(X, X)))\xi^{-\rho_0/2}\end{aligned}$$

which is finite for $\rho_0<-1$ as in the proof of Proposition 5.5.

Next write

(5.17) $$B_i''(t, s, 0)=b_i''(t, s, 0)\phi(s)+\frac{1}{2\pi i}\int_{\rho_0''-i\infty}^{\rho_0''+i\infty}\frac{\phi(\rho)}{s-\rho}b_i''(t, \rho, 0)d\rho$$

where $\rho_0''>\mathrm{Re}(s)$. The same argument as that given for $I_0'(s)$ shows that for $\rho_0''>n-1$, the contribution of the second term on the right hand side of (5.17) is holomorphic for $\mathrm{Re}(s)>n-1-\varepsilon$. Finally we find that for $\mathrm{Re}(s)>n+3$,

$$\int_0^\infty A_i(t, 0)b_i''(t, s, 0)\phi(s)t^{-n}dt=C_i(s)\zeta(s)^{-1}D_i(s, 0, L)\Omega_i(s, \tau, h, L).$$

We can now, finally, compute the integral of the singular term $\Theta^{(i)}$. For $a\in\boldsymbol{Z}^n$, $a\neq 0$, let

(5.18) $$S_n(l_i, a)=\{X\in S_n(l_i)\,|\,X\cdot a\in l_i\}$$

and let

(5.19) $$\Xi_i(h, L)=\{a\in\boldsymbol{Z}^n\,|\,a\neq 0 \text{ is primitive in } \boldsymbol{Z}^n \text{ and } (h+L^n)\cap S_n(l_i, a)\neq\varnothing\}.$$

For $a\in\Xi_i(h, L)$ and $X\in(h+L^n)\cap S_n(l_i, a)$ write

$$X\cdot a=\nu u_i$$

with $\nu\in\boldsymbol{Q}$. Then since a and u_i are both primitive the value

(5.20) $$\nu(a)=\langle\nu\rangle\in\boldsymbol{Q}/\boldsymbol{Z}$$

is well defined, independent of the choice of X. Here for $x\in\boldsymbol{R}$, $\langle x\rangle$ is the fractional part of x.

THEOREM 5.7.

$$\int_{\Gamma\backslash B}\Theta^{(i)}=(-1)^{n}2^{-1-n/2}\sum_{a\in\Xi_{i}(h,L)}\boldsymbol{B}_{1}(\nu(a))\sum_{\substack{X\in(h+L^{n})\cap S_{n}(l_{i})\\ X\cdot a=\nu(a)u_{i}\\ \bmod \Gamma_{i}}}\varepsilon(X)e\Big(\frac{1}{2}\operatorname{tr}(\tau(X,X))\Big)$$

where $\varepsilon(X)$ *is given by* (3.4) *and* $\boldsymbol{B}_{1}$ *is as in Lemma* 3.1.

PROOF. We first show that $\Omega_{i}(s,\tau,h,L)=\Omega_{i}(s)$ has a holomorphic analytic continuation to the half-plane $\operatorname{Re}(s)>n-3/2$. For convenience let $\tilde{L}=h+L^{n}$. For $a\in\Xi_{i}(h,L)$, let

$$\Omega_{i}(s,a)=\sum_{\substack{X\in\tilde{L}\cap S_{n}(l_{i},a)\\ \bmod \Gamma_{i}}}f(s,g_{i}^{-1}X)e\Big(\frac{1}{2}\operatorname{tr}(\tau(X,X))\Big).\tag{5.21}$$

Now for $X\in S_{n}(l_{i})$ there exists an $a\in\boldsymbol{Z}^{n}$ with $a\neq 0$ primitive and unique up to ±1 such that $X\in S_{n}(l_{i},a)$. Thus

$$\Omega_{i}(s)=\frac{1}{2}\sum_{a\in\Xi_{i}(h,L)}\Omega_{i}(s,a).\tag{5.22}$$

For each $a\in\Xi_{i}(h,L)$ choose $g\in SL_{n}(\boldsymbol{Z})$ such that $ge_{0}=a$ where $e_{0}={}^{t}(1,0,\cdots,0)\in\boldsymbol{Z}^{n}$. Then

$$g_{i}^{-1}S_{n}(l_{i},a)g=S_{n}(l_{0},e_{0})\tag{5.23}$$

and so

$$\Omega_{i}(s,a)=\sum_{\substack{Y\in g_{i}^{-1}\tilde{L}g\cap S_{n}(l_{0},e_{0})\\ \bmod g_{i}^{-1}\Gamma_{i}g_{i}}}f(s,Yg^{-1})e\Big(\frac{1}{2}\operatorname{tr}(\tau'(Y,Y))\Big)$$

where $\tau'={}^{t}g^{-1}\tau g^{-1}$. Now any $Y\in S_{n}(l_{0},e_{0})$ has the form

$$Y=(y^{1},y')\tag{5.24}$$

with

$$y^{1}=\begin{pmatrix}y_{0}^{1}\\0\\0\end{pmatrix}\in l_{0}\quad\text{and}\quad y'=\begin{pmatrix}y_{0}'\\y_{1}'\\0\end{pmatrix}\in V(\boldsymbol{Q})^{n-1}.\tag{5.25}$$

Then

$$\begin{aligned}f(s,Yg^{-1})=&-\operatorname{sgn}y_{0}^{1}|y_{0}^{1}|^{n-1-s}\operatorname{sgn}\det y_{1}'|\det y_{1}'|^{n-1-s}\\&\times\det\left(v'\left[\begin{pmatrix}0\\{}^{t}y_{1}'\end{pmatrix}\right]\right)^{(s+1-n)/2},\end{aligned}\tag{5.26}$$

where $v'={}^{t}g^{-1}vg^{-1}$. If $g_{i}^{-1}\tilde{L}g\cap S_{n}(l_{0},e_{0})\neq\varnothing$ we may write

$$g_i^{-1}\tilde{L}g=k+g_i^{-1}L^n$$

with $k=(k^1, k')\in S_n(l_0, e_0)$. Then since $g_i^{-1}L\cap l_0=\boldsymbol{Z}u_0$ we obtain

(5.27)
$$\Omega_i(s, a)=\sum_{\substack{y'\in k'+g_i^{-1}L^{n-1}\\ (y',u_0)=0,\det y_1'\neq 0\\ \bmod g_i^{-1}\Gamma_i g_i}} -\operatorname{sgn}\det y_1'|\det y_1'|^{n-1-s}$$
$$\times\det v'[{}^t(0, y_1')]^{(s+1-n)/2}e\Big(\frac{1}{2}\operatorname{tr}(\tau' Q[(0, y_1')])\Big)$$
$$\times\sum_{\substack{y_0^1\in k_0^1+\boldsymbol{Z}\\ y_0^1\neq 0}}\operatorname{sgn} y_0^1|y_0^1|^{n-1-s}.$$

The inner sum vanishes if $k_0^1\in\boldsymbol{Z}$ while, if $k_0^1\notin\boldsymbol{Z}$

(5.28)
$$\sum_{\substack{y_0^1\in k_0^1+\boldsymbol{Z}\\ y_0^1\neq 0}}\operatorname{sgn} y_0^1|y_0^1|^{n-1-s}=H(\langle k_0^1\rangle, s+1-n)-H(1-\langle k_0^1\rangle, s+1-n)$$

where $H(x, s)=\sum_{n=0}^{\infty}(x+n)^{-s}$ is the Hurwitz zeta function, in the notation of Weil [9] p. 58. The standard properties of $H(x, s)$ imply that (5.28) has an analytic continuation to an entire function of s and that its value at $s=n-1$ is

$$-2\boldsymbol{B}_1(\langle k_0^1\rangle).$$

The sum on y' in (5.27) is absolutely convergent for all s and the claimed analytic continuation of $\Omega_i(s)$ follows immediately.

Now setting $s=n-1$ yields:

$$\Omega_i(n-1)=\sum_{a\in\Xi_i(h,L)}\boldsymbol{B}_1(\langle k_0^1\rangle)\sum_{\substack{y'\in k'+g_i^{-1}L^{n-1}\\ (y',u_0)=0,\det y_1'\neq 0\\ \bmod g_i^{-1}\Gamma_i g_i}}\operatorname{sgn}\det y_1'\, e\Big(\frac{1}{2}\operatorname{tr}(\tau' Q[(0, y_1')]\Big).$$

For each y' in the inner sum let $Y=(y^1, y')$ with $y^1=\langle k_0^1\rangle u_0$, and let

$$X=g_iYg^{-1}.$$

Then $X\in(h+L^n)\cap S_n(l_i)$ with

$$X\cdot a=\langle k_0^1\rangle u_i$$

so that $\langle k_0^1\rangle=\nu(a)$ according to (5.20). Finally note that

$$\operatorname{sgn}\det y_1'=\operatorname{sgn}\det\begin{pmatrix}\nu(a) & y_0'\\ 0 & y_1'\end{pmatrix}$$
$$=\operatorname{sgn}\det\begin{pmatrix}y_0\\ y_1\end{pmatrix}$$

where $g_i^{-1}X=\begin{pmatrix}y_0\\ y_1\\ 0\end{pmatrix}$. We then apply Proposition 5.6 and the expression just found for $\Omega_i(n-1)$ to obtain the claimed expression for $\int_{\Gamma\backslash B}\Theta^{(i)}$ via (5.1).

We are now, at last, in a position to finish the

PROOF OF THEOREM 3.2.
Summing the result of Theorem 5.7 over i and recalling (0.3) we obtain

$$2^{n/2}\int_{\Gamma\backslash B}\Theta=\vartheta^{+}(\tau,\ h,\ L,\ \Gamma)+\vartheta^{0}(\tau,\ h,\ L,\ \Gamma),$$

so that this integral is a holomorphic function of $\tau\in\mathfrak{H}_n$. But $\vartheta(\tau, h, L, \Gamma)$ then inherits the transformation law of $\Theta(\tau, h, L)$ given in Proposition 1.1 of [4] and the theorem is proved.

REMARK. It was erroneously stated in the introduction to [4] that the integral of Siegel's analogue of Hecke's theta-series vanishes for indefinite forms of signature $(n, 1)$ with $n>1$. In fact these integrals vanish for $n>2$, but can be non-zero for $n=2$ as shown by Raghavan and Rangachari [10].

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