

L and ε Factors for $GS\mathfrak{p}(4)$

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To the memory of Takuro Shintani

Introduction.

The theory of L -functions for holomorphic Siegel modular forms of genus 2 was introduced by Andrianov [12]. A generalization of this theory to some types of nonanalytic modular forms was suggested by Kurokawa. However these constructions did not give an opportunity to define L and ε factors for every cuspidal automorphic form for $GS\mathfrak{p}(4)$. Even in the case of Siegel modular forms they could not give the exact L -factor for the case of a congruence subgroup, since they did not have the local definition of L factors. The definition of L and ε factors following Jacquet-Langlands pattern was given in [1]¹⁾. It is natural that our case is more complicated than the classical $GL(n)$ case. For instance, for some cuspidal representations the L -factor is not necessarily trivial. In comparison with the $GL(n)$ case, the main new difficulty is that we have to consider an infinite number of Whittaker models and this produces a new problem: To prove that the L and ε factors do not depend on the choice of the Whittaker model. One of the goals of this paper is to prove it for all representations of the principal series.

Let k be a local nonarchimedean field. We compute L in §2 (Technical reasons made us exclude some nonimportant cases).

The main aim of this paper is to compute ε . We use the methods of [2], where we computed ε in the finite field case, and proved that the Weil lifting from (any) $GO(4)$ to $GS\mathfrak{p}(4)$ preserves ε . In this paper we consider only a split $GO(4, k)$, the connected component of which is isomorphic to $H = GL(2, k)^2 / \{(tI_2, t^{-1}I_2)\}$. Let σ be an irreducible admissible representation of H . It is given by a pair (σ_1, σ_2) of irreducible admissible representations of $GL(2, k)$ having the same central character. Denote by $\Pi(\sigma)$ the Weil lifting of σ to $GS\mathfrak{p}(4, k)$. We prove (in §3) that

$$\varepsilon(\Pi(\sigma), \mu, s, \phi) = \varepsilon(\sigma_1, \mu, s, \phi) \varepsilon(\sigma_2, \mu, s, \phi)$$

¹⁾ According to this definition, for any irreducible cuspidal automorphic representation π , $L(\pi, \mu, s)$ is meromorphic, having at most two poles at $-1/2$, $3/2$, and satisfies the usual functional equation.

where $\varepsilon(\sigma_i, \mu, s, \phi)$ is the Jacquet-Godement ε -factor for $GL(2, k)$ (cf. [3]).

The notion of "Weil lifting" was given by Howe in [4] in terms of distributions. However our use of "Weil lifting" is different. It is given by an explicit integral expression. Unfortunately we can do it only for a special choice of a dual pair (see 1.e). According to our definition the Weil lifting depends on the generalized Whittaker model. It has a unique generalized Whittaker model and a trivial commutant. Our definition is motivated by our finite field computation of ε (cf. [2]), and is also much better compatible with the global definition. It is known that even for the simplest pair (O_3, \widetilde{SL}_2) , the global definition via the θ -kernel is not always compatible with the local definition via distributions. In other words, what comes from the global definition is not always a product of what comes from the local definition.

As a corollary of our computations in §§ 2, 3 we get that (in the cases considered) L and ε are independent of the choice of the Whittaker model.

Notations

— Let π be a representation (of a group G). We denote its space by V_π . If the restriction of π to the center of G acts by scalars, we denote the central character by ω_π .

— Let k be a field. Let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, then

$$GSp(2n, k) = \{g \in GL(2n, k) \mid {}^t g J g = \lambda(g) J, \lambda(g) \in k^*\}.$$

— Let Q be a quadratic form on a space of dimension m over k then

$$GO(m, Q) = \{g \in \text{End}_k(V) \mid Q(gv) = \lambda(g)Q(v), \lambda(g) \in k^*, v \in V\}.$$

— For a locally compact abelian group X , we denote by $S(X)$ the space of Schwarz-Bruhat functions on X .

§ 1. Definitions and preliminaries.

In this section we give the definitions of a generalized Whittaker model for $GSp(4)$ (1.a), and of the local factors L , ε , γ (1.b), and explain how to compute these factors, (1.d). We give the definition of Weil lifting in (1.e).

Throughout this section, k denotes a local or finite field of characteristic $\neq 2$. ϕ —a fixed nontrivial additive character of k . $G = GSp(4, k)$.

(a) *Generalized Whittaker models.* Let $S = \left\{ u(s) = \begin{pmatrix} I_2 & s \\ 0 & I_2 \end{pmatrix} \middle| \begin{matrix} s \in M(2, k) \\ s = {}^t s \end{matrix} \right\}$.

S is an abelian subgroup of G . It is the unipotent radical of the parabolic $P =$

$\left\{ \begin{pmatrix} A & * \\ 0 & x^t A^{-1} \end{pmatrix} \middle| \begin{matrix} A \in GL(2, k) \\ x \in k^* \end{matrix} \right\}$. $P = M.S : M = \left\{ \begin{pmatrix} A & 0 \\ 0 & x^t A^{-1} \end{pmatrix} \right\}$ — the reductive part.

The application $u(s) \mapsto \phi(\text{tr}(\phi s))$, where ${}^t\phi = \phi \in M(2, k)$, defines a character ϕ_ϕ of S . All characters of S can be obtained in this way. ϕ_ϕ is called non-degenerate if $\phi \in GL(2, k)$.

Let ϕ_ϕ be nondegenerate, and let \tilde{D} be the stabilizer of ϕ_ϕ in M . There exists a unique semisimple algebra K over k , with $(K:k)=2$, such that $\tilde{D} \cong K^* Z_2$. Denote by D the connected component of \tilde{D} , then $D \cong K^*$. K is either a quadratic extension of k , $K = K_1 = k(\sqrt{\rho})$, $(\rho \notin (k^*)^2)$, or $K = K_2 = k \oplus k$ (k embedded diagonally). We take in the first case $\phi = \phi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}$, and in the second case $\phi = \phi_2 = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. In both cases the isomorphism $K_i^* \cong D_i = D$ is given by $r \mapsto$

$\begin{pmatrix} f_i(r) & 0 \\ 0 & (\det f_i(r)) {}^t f_i(r)^{-1} \end{pmatrix}$, where $r \mapsto f_i(r)$ is the following embedding of K_i^* in

$GL(2, k)$. $f_1(x + y\sqrt{\rho}) = \begin{pmatrix} x & y\rho \\ y & x \end{pmatrix}$; $f_2(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Denote $\phi_i = \phi_{\phi_i}$, $i=1, 2$.

Define $R_i = D_i S$. Each character ν of K_i^* defines, together with ϕ_i , a character of R_i , which we denote by $\nu \otimes \phi_i$. The following theorem is well known.

THEOREM 1.1. *Let k be a local field and $i=1, 2$. Let π be an irreducible admissible preunitary representation of G . Then up to a scalar there exists at most one nonzero linear functional $l: V_\pi \rightarrow \mathbb{C}$ satisfying*

$$(1.1) \quad l(\pi(r)v) = \nu \otimes \phi_i(r) l(v) : r \in R_i, v \in V_\pi.$$

(If k is archimedean l is required to be continuous in the C^∞ topology.)

A functional satisfying (1.1) is called a *Whittaker functional* with respect to (ν, ϕ_i) . (cf. [5, 6, 7].) Theorem 1.1 is “essentially” true where k is finite. That is, it is true for $i=1$, and for $i=2$ it is true for “almost all” characters ν of K_2^* (cf. [2]).

Let π have a nonzero Whittaker functional l with respect to (ν, ϕ_i) . Let $v \in V_\pi$. Put $W_v(g) = l(\pi(g)v)$, ($g \in G$). W_v is called the *Whittaker function* of v . It satisfies $W_v(rg) = \nu \otimes \phi_i(r) W_v(g)$, ($r \in R_i, g \in G$). Denote by W_π^{ν, ϕ_i} the space of all these functions. G acts on W_π^{ν, ϕ_i} by right translations, and the representation of G in W_π^{ν, ϕ_i} is equivalent to π . W_π^{ν, ϕ_i} is called the *generalized Whittaker model* of π with respect to (ν, ϕ_i) . We can say more when k is finite, there is a vector $\xi_\pi \in V_\pi$, which is unique up to a scalar, (in “most” of the cases) such that $\pi(r)\xi_\pi = \nu \otimes \phi_i(r)\xi_\pi$, ($r \in R_i$). ξ_π is called a *Whittaker vector* with respect to (ν, ϕ_i) . Denote by J_π its Whittaker function, normalized by the condition $J_\pi(I)$

$=1$. J_π is called the *Bessel function* of π with respect to (ν, ϕ_i) . It satisfies $J_\pi(r_1 g r_2) = \nu \otimes \phi_i(r_1 r_2) J_\pi(g)$, $(r_1, r_2 \in R_i, g \in G)$. There is no such smooth function on G when k is local.

(b) *The functional equation and the factors L, ε, γ .* Denote by $\bar{}$ the unique nontrivial k automorphism of K_i . Put $\text{tr} = \text{tr}_{K_i/k}$ and $N = NK_i/k$. Let $V_i = K_i^2$. We write vectors in V_i in row form. Define for $x = (x_1, x_2), y = (y_1, y_2)$ in V_i , $\tau_i(x, y) = \text{tr}(x_1 y_2 - x_2 y_1)$. τ_i is a nondegenerate antisymmetric form on V_i . Regard V_i as a 4-dimensional vector space over k . Let

$$GSp(\tau_i) = \{g \in GL(4, k) \mid \tau_i(xg, yg) = \lambda(g)\tau_i(x, y); x, y \in V_i, \lambda(g) \in k^*\}.$$

Consider the group $G_i = \{g \in GL(2, K_i) \mid \det g \in k^*\}$. G_i acts on V_i from the right, preserving τ_i up to a factor, and so we get a natural embedding $G_i \subset GSp(\tau_i)$. Let $N_i = \{u(s) \in S \mid \text{tr}(\phi_i s) = 0\}$. There exists an isomorphism $\varphi_i: GSp(\tau_i) \rightarrow G$ such that $\varphi_i(G_i) \cap R_i = D_i N_i$. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_i$ (for $i=1, a=a_1+a_2\sqrt{\rho}$ etc. and for $i=2, a=(a_1, a_2)$ etc.) then

$$(1.2) \quad \varphi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{cc|cc} a_1 & a_2\rho & \frac{1}{2}b_1 & \frac{1}{2}b_2 \\ a_2 & a_1 & \frac{1}{2}b_2 & \frac{1}{2\rho}b_1 \\ \hline 2c_1 & 2c_2\rho & d_1 & d_2 \\ 2c_2\rho & 2c_1\rho & d_2\rho & d_1 \end{array} \right); \quad \varphi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{cc|cc} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ \hline c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{array} \right).$$

From now on we shall identify G_i with $\varphi_i(G_i)$. Note that $\varphi_i(U_i) = N_i$ where $U_i = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid z \in K_i \right\}$. Let $\alpha_i(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$, $((x_1, x_2) \in V_i)$. α_i defines an element of $GSp(\tau_i)$ whose image in G is

$$(1.3) \quad \alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \alpha_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Note that if $W_v(g)$ is a Whittaker function with respect to (ν, ϕ_i) , then $\tilde{W}_v(g) = W_v(\alpha_i g)$ is a Whittaker function with respect to $(\bar{\nu}, \phi_i)$. $(\bar{\nu}(t) = \nu(\bar{t}))$.

Let π have a unique Whittaker model with respect to (ν, ϕ_i) . Let μ be a character of k^* . Define for $\phi \in S(V_i)$, (the Schwarz-Bruhat functions on V_i), $w \in W_\pi^{\nu, \phi_i}$, $s \in C$,

$$(1.4) \quad L_\nu^{(i)}(w, \phi, \mu, s) = \int_{N_i \backslash G_i} w(g) \phi((0, 1)g) \mu(\det g) |\det g|^{s+1/2} dg$$

(when k is finite we understand $|x|=1$ for $x \in k^*$). The integral (1.4) converges in a half plane $\operatorname{Re}(s) > s_0$ and has a meromorphic continuation to the whole plane. There is an Euler factor $L_\nu^{(i)}(\pi, \mu, s)$ such that $L_\nu^{(i)}(w, \phi, \mu, s) / L_\nu^{(i)}(\pi, \mu, s)$ is entire for all w, ϕ . (For finite k , $L_\nu^{(i)}(\pi, \mu, s) \equiv 1$.) It is easy to see that for a fixed i , $L_\nu^{(i)}(\pi, \mu, s)$ does not depend on ϕ . We will show that in many important cases, $L_\nu^{(i)}(\pi, \mu, s)$ does not depend on i and ν . $L_\nu^{(i)}(\pi, \mu, s)$ is called the L -factor associated to (π, μ) .

THEOREM 1.2 (The functional equation). *Let π, μ, ν be as above, then there is an entire function $\varepsilon_\nu^{(i)}(\pi, \mu, s, \phi)$ without zeros satisfying*

$$(1.5) \quad \varepsilon_\nu^{(i)}(\pi, \mu, s, \phi) \frac{L_\nu^{(i)}(w, \phi, \mu, s)}{L_\nu^{(i)}(\pi, \mu, s)} = \frac{L_\nu^{(i)}(\tilde{w}, \hat{\phi}, \mu^{-1} \omega_\pi^{-1}, 1-s)}{L_\nu^{(i)}(\hat{\pi}, \mu^{-1}, 1-s)}.$$

$\hat{\pi}$ — the contragredient representation of π , $\hat{\phi}$ — the Fourier transform of ϕ with respect to $\phi^{-1} \circ \tau_i$, (with a self dual measure). $\varepsilon_\nu^{(i)}(\pi, \mu, s, \phi)$ is called the ε -factor associated to (π, μ) . cf. [1]. We introduce the γ -function associated to (π, μ)

$$(1.6) \quad \gamma_\nu^{(i)}(\pi, \mu, s, \phi) = \varepsilon_\nu^{(i)}(\pi, \mu, s, \phi) \frac{L_\nu^{(i)}(\hat{\pi}, \mu^{-1}, 1-s)}{L_\nu^{(i)}(\pi, \mu, s)}.$$

When k is finite Theorem 1.2 reads as follows.

THEOREM 1.2'. *Let π, ν, μ be as above. Assume that for $i=1$, $\bar{\mu}\nu \neq 1$ (where $\bar{\mu}(t) = \mu(t\bar{t})$, $t \in K_1^*$), and for $i=2$, $\mu\nu_1, \mu\nu_2 \neq 1$, where $\nu(x, y) = \nu_1(x)\nu_2(y)$ for $(x, y) \in K_2^*$. Then there exists a complex number $\gamma_\nu^{(i)}(\pi, \mu, \phi)$, which may still be undefined for at most four characters μ of k^* , such that*

$$(1.7) \quad \gamma_\nu^{(i)}(\pi, \mu, \phi) \sum_{N_i \backslash G_i} w(g) \phi((0, 1)g) \mu(\det g) = \sum_{N_i \backslash G_i} \tilde{w}(g) \hat{\phi}((0, 1)g) \mu^{-1} \omega_\pi^{-1}(\det g).$$

(c) *Weil representation.* Put $G = \operatorname{GSp}(2n, k)$. Let Y_{2m} be a $2m$ dimensional vector space over k , with a nondegenerate symmetric form B , and Q the related quadratic form. Put $Z = Y_{2m} \otimes k^n$. Each symmetric matrix $s \in M(n, k)$ defines a symmetric form on Z by

$$[y_1 \otimes v_1, y_2 \otimes v_2]_s = B(y_1, y_2)(v_1, v_2)_s, \quad [\sum y_i \otimes v_j, \sum y_t \otimes v_l]_s = \sum [y_i \otimes v_j, y_t \otimes v_l]_s$$

where $(v_1, v_2)_s = {}^t v_1 s v_1$, $(v_1, v_2 \in k^n)$, is the symmetric form on k^n corresponding to s . $M(n, k)$ acts on Z from the right by $(\sum y_i \otimes v_i)A = \sum y_i \otimes (v_i A)$ ($A \in M(n, k)$). Let X be the set of nontrivial additive characters of k . For $\theta \in X$ and $t \in k^*$, we denote $\theta^t(a) = \theta(ta)$, ($a \in k$).

We now define the Weil representation ω of G (corresponding to Q). The space of ω is $S(Z \times X)$, the Schwarz-Bruhat functions on $Z \times X$. (We identify X and k^* , fixing $\phi \in X$), ω is defined on the generators of G as follows:

$$(1.8) \quad \omega \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} f(z, \theta) = \tilde{Q}(Q, A) |\det A|^m f(zA, \theta^{v^{-1}}); \quad A \in GL(n, k), \quad y \in k^*$$

$$\omega \begin{pmatrix} I_n & s \\ 0 & I_n \end{pmatrix} f(z, \theta) = \theta([z, z]_s) f(z, \theta); \quad {}^t s = s \in M(n, k)$$

$$\omega \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} f(z, \theta) = P(Q)^{-1} \int_Z \theta(\langle z, z' \rangle) f(z', \theta) d_{\theta, Q} z'$$

where $\langle z, z' \rangle = [z, z']_{2I_n}$; $d_{\theta, Q} z'$ is the self dual measure with respect to $\theta \cdot \langle, \rangle$. The values of $\tilde{Q}(Q, A)$ and $P(Q)$ are computed in [8]. See also [9].

The general orthogonal group $GO(2m, Q)$ acts on Z from the left by $h(y \otimes v) = (hy) \otimes v$, ($h \in GO(2m, Q)$). This gives a representation τ of $GO(2m, Q)$ in $S(Z \times X)$, $\tau(h)f(z, \theta) = \alpha(h)f(h^{-1}z, \theta^{\lambda(h)})$ $\lambda(h)$ denotes the multiplier of h . ($Q(hy) = \lambda(h)Q(y)$, $y \in Y_{2m}$). $\alpha(h)$ is a positive constant which makes $\tau(h)$ unitary. τ and ω commute. From now on we shall denote $\omega(g, h) = \omega(g)\tau(h)$ (for $g \in G$, $h \in GO(2m, Q)$). In this paper we study the case $n=m=2$.

(d) *Computation of the γ -function when k is finite.* Let k be a finite field (of characteristic $\neq 2$). Put $G = GSp(4, k)$. Let Q be a nondegenerate quadratic form on a four dimensional space E over k . B — the related bilinear form. Denote by H the connected component of $GO(4, Q)$. Let σ be an irreducible representation of H which appears in $\text{Res}_H \omega$. Consider the space

$$V(\sigma) = \{f: Z \times X \longrightarrow V_\sigma \mid f(hz, \theta^{\lambda(h)^{-1}}) = \sigma(h)f(z, \theta), \quad h \in H\}.$$

G acts on $V(\sigma)$ according to the formulas (1.8). It is known that each irreducible representation of G appears in one of the $V(\sigma)$'s. (cf. [10, 11].) Denote the representation of G in $V(\sigma)$ by $\Pi(\sigma)$. Up to equivalence, there are two nondegenerate quadratic forms Q on E . In the first case, E is a sum of hyperbolic planes (put $\varepsilon(Q)=1$). In the second case E is not a sum of hyperbolic planes (put $\varepsilon(Q)=-1$). In (1.8) we have $\tilde{Q}(Q, A)=1$, $P(Q)^{-1}=\varepsilon(Q)^2=1$.

We have the following realizations of H . If $\varepsilon(Q)=1$, then $E=M(2, k)$ and $Q=\det$. There is a homomorphism $\varphi: GL(2, k) \times GL(2, k) \rightarrow GO(4, Q)$ given by

$$\varphi(g_1, g_2)(x) = g_1 x {}^t g_2, \quad (x \in E; \quad g_1, g_2 \in GL(2, k)); \quad \lambda(\varphi(g_1, g_2)) = \det g_1 g_2.$$

$\text{Ker } \varphi = \{(tI_2, t^{-1}I_2) \mid t \in k^*\}$ and $H \cong GL(2, k) \times GL(2, k) / \text{ker } \varphi$.

If $\varepsilon(Q)=-1$, let L be the quadratic extension of k , then we take $E = \{g \in M(2, L) \mid g^* = g\}$ and $Q = \det$. ($g^* = {}^t \bar{g}$ for $g \in M(2, L)$, $\bar{}$ denotes the conjugation of L/k .) There is a homomorphism $\varphi: GL(2, L) \rightarrow GO(4, Q)$ given by

$\varphi(g)(x)=gxg^*, (x \in E; g \in GL(2, L)); \lambda(\varphi(g))=N_{L/k}(\det g); \text{Ker } \varphi=\{uI_2 | N_{L/k}(u)=1\}$ and $H \cong GL(2, L)/\text{ker } \varphi$.

We have $Z=E \otimes k^2 \cong E \times E$. From now on ω will be written through this isomorphism and the above realizations of H . The action of $M(2, k)$ on $E \times E$ is by $(e_1, e_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ae_1 + ce_2, be_1 + de_2)$.

In the case $\varepsilon(Q)=1$, the irreducible representations of H are given by pairs $\sigma=(\sigma_1, \sigma_2)$ of irreducible representations of $GL(2, k)$ such that $\omega_{\sigma_1}=\omega_{\sigma_2}$ and in the case $\varepsilon(Q)=-1$, the irreducible representations of H are given by irreducible representations σ of $GL(2, L)$ such that $\omega_\sigma(u)=1$ whenever $N_{L/k}(u)=1$.

Now we look for generalized Whittaker models in $V(\sigma)$. In the majority of the cases, for every character ν of K_i^* , except of at most four, $V(\sigma)$ contains a unique Whittaker vector (up to a scalar) f_0 with respect to (ν, φ_i) . (For the other ν 's, there are in $V(\sigma)$ no more than four linearly independent Whittaker vectors with respect to (ν, ϕ_i)). In order to find this Whittaker vector, we solve the following equations in $V(\sigma)$

$$(1.9) \quad \Pi(\sigma)(u(s))f(y)=\phi_i(u(s))f(y); \quad {}^t s=s$$

$$(1.10) \quad \Pi(\sigma) \begin{pmatrix} r & 0 \\ 0 & (\det r){}^t r^{-1} \end{pmatrix} f(y)=\nu(r)f(y); \quad r \in K_i^*.$$

If $f(y) \neq 0$, $y=(M_1, M_2, \phi^a)$, then (using (1.8)), (1.9) reads as follows,

$$(1.11) \quad a \begin{pmatrix} Q(M_1) & B(M_1, M_2) \\ B(M_2, M_1) & Q(M_2) \end{pmatrix} = \phi_i.$$

It then follows that the support of a solution f_0 is a single orbit $\{hy_0 | h \in H\}$, and from (1.10) it follows that $f_0(y_0)=\xi_\nu$, where $\xi_\nu \in V_\sigma$ is an eigenvector of σ with respect to a certain subgroup $\tilde{H}_i \subset H$ ("approximately" $K_i^* \times K_i^*$), $\sigma(h)\xi_\nu = \varphi_\nu(h)\xi_\nu$, ($h \in \tilde{H}_i$). φ_ν is the related character of \tilde{H}_i . This guarantees the uniqueness (in general) up to a scalar of f_0 . The Whittaker functional l_ν is then given by $l_\nu(f)=P_\nu[f(y_0)]$ ($f \in V(\sigma)$) where $P_\nu: V_\sigma \rightarrow \mathbb{C}$ is the linear functional satisfying $P_\nu(\sigma(h)\xi)=\varphi_\nu(h)P_\nu(\xi)$ ($h \in \tilde{H}_i$, $\xi \in V_\sigma$) and $P_\nu(\xi_\nu)=1$. Thus we get a generalized Whittaker model with respect to (ν, ϕ_i) of (the appropriate component of) $\Pi(\sigma)$.

We now give the list of the various $y_0, \tilde{H}_i, \xi_\nu, \varphi_\nu$. Recall the embeddings

$$f_i: K_i^* \subset GL(2, k): x+y\sqrt{\rho} \xrightarrow{f_1} \begin{pmatrix} x & y\rho \\ y & x \end{pmatrix}; (x, y) \xrightarrow{f_2} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

(1) $\varepsilon(Q)=1; \sigma=(\sigma_1, \sigma_2), \omega_{\sigma_1}=\omega_{\sigma_2}$. We have $y_0=(u_0, \phi)$. For $i=1, u_0=(I_2, \begin{pmatrix} 0 & \rho \\ 1 & 0 \end{pmatrix})$, and for $i=2, u_0=(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})$. The stabilizer of u_0 in $GL(2, k)$

$\times GL(2, k)$ is $\{(\delta, {}^t\delta^{-1}) \mid \delta \in K_i^* \} \cong K_i^*$. The inverse image of \tilde{H}_i in $GL(2, k) \times GL(2, k)$ is $\{(\delta_1, {}^t\delta_2) \mid \delta_1, \delta_2 \in K_i^* \} \cong K_i^* \times K_i^*$. $\xi_\nu = \xi_\nu^{(1)} \otimes \xi_\nu^{(2)}$, where $\sigma_1(\delta)\xi_\nu^{(1)} = \nu(\delta)\xi_\nu^{(1)}$ and $\sigma_2({}^t\delta)\xi_\nu^{(2)} = \nu(\delta)\xi_\nu^{(2)}$, for $\delta \in K_i^*$.

(2) $\varepsilon(Q) = -1$; $\omega_\sigma(u) = 1$ for $N_{L/k}(u) = 1$; $i = 1$. We have $y_0 = (u_0, \phi^{-1})$. $u_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{\rho} \\ -\sqrt{\rho} & 0 \end{pmatrix} \right)$. The stabilizer of u_0 in $GL(2, L)$ is $\left\{ \begin{pmatrix} a & c \\ c & a \end{pmatrix} \mid (a-c)(\bar{a}+\bar{c}) = 1 \right\}$. The inverse image of \tilde{H}_1 in $GL(2, L)$ is $\left\{ \begin{pmatrix} a & c \\ c & a \end{pmatrix} \in GL(2, L) \right\} \cong L^* \times L^*$.

ξ_ν satisfies $\sigma \begin{pmatrix} a & c \\ c & a \end{pmatrix} \xi_\nu = \nu((a-c)(\bar{a}+\bar{c}))\xi_\nu$.

(3) $\varepsilon(Q) = -1$; $\omega_\sigma(u) = 1$ for $N_{L/k}(u) = 1$; $i = 2$. We have $y_0 = (u_0, \phi)$. $u_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$. The stabilizer of u_0 in $GL(2, L)$ is $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid N_{L/k}(a) = N_{L/k}(b) \right\}$. The inverse image of \tilde{H}_2 in $GL(2, L)$ is $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL(2, L) \right\}$. ξ_ν satisfies $\sigma \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \xi_\nu = \nu(N_{L/k}a, N_{L/k}b)\xi_\nu$.

We denote for a character η of K_i^* and a character θ of K_i , $\gamma_{K_i}(\eta, \theta) = \sum_{x \in K_i^*} \eta^{-1}(x)\theta^{-1}(x)$. Also for a character μ of k^* , we denote $\bar{\mu} = \mu \circ N_{K_i/k} = \mu \circ N$, and for a character ϕ of k , we denote $\tilde{\phi} = \phi \circ \text{tr}_{K_i/k} = \phi \circ \text{tr}$. The following proposition says that the γ -function is (up to a factor) the Mellin transform of a "Bessel function".

PROPOSITION 1.3. *Let π be an irreducible representation of G , having a Whittaker model with respect to (ν, ϕ_i) . Let μ be a character of k^* satisfying the assumptions of Theorem 1.2', then*

$$(1.12) \quad \gamma_\nu^{(i)}(\pi, \mu, \phi) = q^2 \gamma_{K_i}(\nu \bar{\mu}, \tilde{\phi}) \sum_{x \in k^*} J_\pi(\delta_i(x)) \mu^{-1} \omega_\pi^{-1}(x)$$

where $\delta_i(x) = \alpha_i \begin{pmatrix} 0 & -x \\ 1 & 0 \end{pmatrix}$, and $J_\pi(g)$ is the Bessel function introduced in (a).

PROOF. Substitute in the functional equation (1.7) $w(g) = J_\pi(g)$ and $\phi = \phi_0$, where $\phi_0(x) = \delta_{(0, 1), x}$, ($x \in V_i = K_i^?$). (1.12) follows by a simple calculation. \square

In the local field case ϕ_0 is not a smooth function and we shall have to choose approximating sequences of functions.

Now we use (1.12) to compute $\gamma_\nu^{(i)}(\Pi(\sigma), \mu, \phi)$ for an irreducible representation σ of H . We have $J_{\Pi(\sigma)}(g) = P_\nu[\Pi(\sigma)(g)f_0(y_0)]$, and (1.12) implies that we have to compute $\Pi(\sigma)(\delta_i(x))f_0(y_0)$. We write $\delta_i(x) = \begin{pmatrix} A_i & 0 \\ 0 & {}^tA_i^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & xI_2 \end{pmatrix}$.

The formulas (1.8) give the specific action of $\delta_i(x)$. The action of $\begin{pmatrix} A_i & 0 \\ 0 & {}^t A_i^{-1} \end{pmatrix}$ and of $\begin{pmatrix} I_2 & 0 \\ 0 & x I_2 \end{pmatrix}$ is very simple, and so, practically, the action of $\delta_i(x)$ is as that of $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$, which acts as a Fourier transform. Write $y_0 = (\tilde{x}_0, \tilde{y}_0, \theta_0)$ and $(\tilde{x}_0, \tilde{y}_0)A_i = (X_0^{(i)}, Y_0^{(i)})$ then

$$\Pi(\sigma)(\delta_i(x))f_0(y_0) = q^{-4} \sum_{M_1, M_2 \in E} \theta_0[2B(X_0^{(i)}, M_1) + 2B(Y_0^{(i)}, M_2)] f_0(M_1, M_2, \theta_0^{x^{-1}}).$$

Since the support of f_0 is the orbit of y_0 under H , the summation is only over $(M_1, M_2, \theta_0^{x^{-1}})$ of the form $h y_0$, where $h \in H$ is such that $\lambda(h) = x$, and so the last sum is of the form $q^{-4} \sum_{h \in H, \lambda(h) = x} \theta_0(\beta(y_0, h)) \sigma(h) \xi_\nu$. This together with (1.12) gives $\gamma_\nu^{(i)}(\Pi(\sigma), \gamma, \phi)$ "in the language of H " which is a " $GL(2)$ language". Now we can use the γ -function theory for $GL(2)$ in order to compute $\gamma_\nu^{(i)}(\Pi(\sigma), \mu, \phi)$.

THEOREM 1.4.

(1) Let σ_1 and σ_2 be irreducible representations of $GL(2, k)$ such that $\dim \sigma_1, \dim \sigma_2 > 1$ and $\omega_{\sigma_1} = \omega_{\sigma_2}$. Let $\Pi(\sigma)$ have a unique component which has a Whittaker model with respect to (ν, ϕ_i) (which is generally the case) then

$$(1.13) \quad \gamma_\nu^{(i)}(\Pi(\sigma), \mu, \phi) = \gamma(\sigma_1, \mu, \phi) \gamma(\sigma_2, \mu, \phi) \quad (i=1, 2).$$

(2) Let σ be an irreducible representation of $GL(2, L)$ (L — the quadratic extension of k) such that $\dim \sigma > 1$ and $\omega_\sigma(u) = 1$ whenever $N_{L/k}(u) = 1$. Let $\Pi(\sigma)$ have a unique component which has a Whittaker model with respect to (ν, ϕ_i) , then

$$(1.14) \quad \gamma_\nu^{(i)}(\Pi(\sigma), \mu, \phi) = \gamma(\sigma, \tilde{\mu}, \tilde{\phi}) \quad (i=1, 2).$$

The γ -functions in (1.13), (1.14) are γ -functions for $GL(2, k)$. A fully detailed account of the above can be found in [2]. In [2] we get similar results for generic representations of G .

(e) *Weil lifting.* From now on k is a local nonarchimedean field; \mathfrak{P} — the maximal ideal; q — the number of elements in the residue field and $G = GSp(4, k)$.

We consider the quadratic space $(E, Q) = (M(2, k), \det)$. We take the realization $H = GL(2, k) \times GL(2, k) / \{(tI_2, t^{-1}I_2)\}$ of the connected component of $GO(4, Q)$. Denote by B the bilinear form related to \det . The formulas (1.8) read as follows.

$$\begin{aligned}
 (1.15) \quad & \omega \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} f(M_1, M_2, \phi^x) = |\det A|^2 f((M_1, M_2)A, \phi^{x\gamma^{-1}}) \\
 & \omega \begin{pmatrix} I_2 & s \\ 0 & I_2 \end{pmatrix} f(M_1, M_2, \phi^x) = \phi^x \left(\operatorname{tr} s \begin{pmatrix} \det M_1 & B(M_1, M_2) \\ B(M_2, M_1) & \det M_2 \end{pmatrix} \right) f(M_1, M_2, \phi^x) \\
 & \omega \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} f(M_1, M_2, \phi^x) = |x|^4 \int_{E \times E} \phi^x(2B(M_1, W_1) + 2B(M_2, W_2)) f(W_1, W_2, \phi^x) \\
 & \quad \cdot dW_1 dW_2.
 \end{aligned}$$

The action of H is by

$$(1.16) \quad \omega(h)f(M_1, M_2, \phi^x) = |\lambda(h)|^{-2} f(h^{-1}(M_1, M_2), \phi^{x\lambda(h)}).$$

Let σ be an irreducible admissible representation of H . σ is given by a pair of irreducible admissible representations of $GL(2, k)$ satisfying $\omega_{\sigma_1} = \omega_{\sigma_2}$. We want to construct a representation $\Pi(\sigma)$ of G in analogy with (1.d). The formula for l_ν in (1.d) motivates the following. Let $i=1, 2$, and let ν be a character of K_i^* such that $\nu|_{k^*} = \omega_{\sigma_1} = \omega_{\sigma_2}$. Let $u_0 = \begin{pmatrix} I_2 & \begin{pmatrix} 0 & \rho \\ 1 & 0 \end{pmatrix} \end{pmatrix}$ for $i=1$, and $u_0 = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$ for $i=2$. Denote by F_i the stabilizer of u_0 in H . The inverse image of F_i in $GL(2, k) \times GL(2, k)$ is $\{(\delta, {}^t\delta^{-1}) | \delta \in K_i^*\} \cong K_i^*$. Let the spaces $V_{\sigma_1}, V_{\sigma_2}$ be spaces of smooth functions on $GL(2, k)$ which satisfy the following. For $\xi_1 \in V_{\sigma_1}$, $\xi_1(\delta A) = \nu(\delta)\xi_1(A)$, ($A \in GL(2, k)$, $\delta \in K_1^*$), and for $\xi_2 \in V_{\sigma_2}$, $\xi_2({}^t\delta A) = \nu(\delta)\xi_2(A)$, ($A \in GL(2, k)$, $\delta \in K_2^*$). Define for $\xi \in V_\sigma$ and for $f \in S(Z \times X)$

$$(1.17) \quad l_\nu^{(i)}(f, \xi) = \int_{F_i \backslash H} \omega(h)f(u_0, \phi)\xi(h)d\tilde{h} \quad (d\tilde{h} \text{ is } H \text{ right invariant on } F_i \backslash H).$$

LEMMA 1.5.

(1) The integral (1.17) converges absolutely.

(2) $l_\nu^{(i)}$ defines a linear functional on $S(Z \times X) \otimes V_\sigma$ satisfying

$$(1.18) \quad l_\nu^{(i)}(\omega(r)f, \xi) = \nu \otimes \phi_i(r) l_\nu^{(i)}(f, \xi), \quad r \in R_i.$$

PROOF. (1) It suffices to consider f with a support in a small closed neighbourhood $U(e_0)$ of a point $e_0 = (M_0, M'_0, \phi^{x_0})$, and then one checks that in both cases $i=1, 2$, $\{h \in H | h^{-1}(u_0, \phi) \in U(e_0)\}$ is compact modulo F_i .

(2) This is immediate. The choice of u_0 was made so that (1.18) holds true. \square

Define for each f, ξ as above and $g \in G$

$$(1.19) \quad W_{f, \xi}(g) = l_\nu^{(i)}(\omega(g)f, \xi) = \int_{F_i \backslash H} \omega(g, h)f(u_0, \phi)\xi(h)d\tilde{h}$$

then for $r \in R_i$, $W_{f, \xi}(rg) = \nu \otimes \phi_i(r)W_{f, \xi}(g)$. Thus $W_{f, \xi}$ is a Whittaker function

with respect to (ν, ϕ_i) . Denote by $V(\sigma)$ the space generated by all functions (1.19). G acts on $V(\sigma)$ by right translations, and $V(\sigma)$ is comprised of Whittaker functions with respect to (ν, ϕ_i) . Denote the representation of G in $V(\sigma)$ by $\Pi(\sigma)$.

THEOREM 1.6. $\Pi(\sigma)$ has a unique (up to scalars) nonzero Whittaker functional with respect to (ν, ϕ_i) and a trivial commutant.

PROOF. We first show that there is a unique (up to a scalar) nonzero linear functional l on $\Pi(\sigma)$ satisfying $l(\Pi(\sigma)(r)w) = \nu \otimes \phi_i(r) l(w)$, ($r \in R_i$, $w \in V(\sigma)$), and that functional is $w \rightarrow w(I)$. Let l be such a nonzero functional. l induces a bilinear form \langle, \rangle on $S(Z \times X) \times V_\sigma$ defined by $(f, \xi) \rightarrow w_{f, \xi} \rightarrow l(w_{f, \xi}) = \langle f, \xi \rangle$. We have

- (1) $\langle \omega(h)f, \sigma(h)\xi \rangle = \langle f, \xi \rangle \quad (h \in H)$
- (2) $\langle \omega(u(s))f, \xi \rangle = \phi_i(u(s)) \langle f, \xi \rangle \quad (s \in M(2, k))$
- (3) $\langle \omega(d)f, \xi \rangle = \nu(d) \langle f, \xi \rangle \quad (d \in D_i)$.

We shall show that there is a unique such form on $S(Z \times X) \times V_\sigma$. From (2) it follows that $\langle f, \xi \rangle$ is determined by the restriction of f to the orbit under H of (u_0, ϕ) . Put $\alpha_f(h) = f(h^{-1}u_0, \phi^{\lambda(h)})$ and define $\langle \alpha_f, \xi \rangle = \langle f, \xi \rangle$. This gives a bilinear form on $S(F_i \backslash H) \times V_\sigma$ and it satisfies

- (1') $\langle \omega(h)\alpha, \sigma(h)\xi \rangle = \langle \alpha, \xi \rangle, \quad (h \in H)$
- (2') $\langle \omega(d)\alpha, \xi \rangle = \nu(d) \langle \alpha, \xi \rangle, \quad (d \in D_i)$.

Consider the linear functional $l_\xi(\alpha) = \langle \alpha, \xi \rangle$. Since ξ is a smooth function there exists a smooth function φ_ξ on $F_i \backslash H$ such that $l_\xi(\alpha) = \int_{F_i \backslash H} \varphi_\xi(h) \alpha(h) d\tilde{h}$. From (1'), (2') it follows that $\varphi_{\sigma(h_0)\xi}(h) = |\lambda(h_0)|^2 \varphi_\xi(hh_0)$, ($h, h_0 \in H$) and $\varphi_\xi((d, I)h) = |Nd|^{-2} \nu(d) \varphi_\xi(h)$, ($d \in D_i$). Let $P(\xi) = \varphi_\xi(I)$. P is a linear functional on V_σ satisfying $P(\sigma(d, I)\xi) = \nu(d)P(\xi)$ such a functional is unique up to a scalar, and hence there is $c_0 \neq 0$ such that $P(\xi) = c_0 \xi(I)$. This implies that $\varphi_\xi(h) = c_0 |\lambda(h)|^{-2} \xi(h)$, and $\langle f, \xi \rangle = c_0 w_{f, \xi}(I)$. Now let $A: \Pi(\sigma) \rightarrow \Pi(\sigma)$ be an equivalence. The form $\langle f, \xi \rangle = A(w_{f, \xi})(I)$ satisfies the conditions (1)–(3) and hence $A(w_{f, \xi})(I) = c_0 w_{f, \xi}(I)$. This implies $A = c_0 \cdot id$. \square

We shall refer to $\Pi(\sigma)$ as a *Weil lifting* of σ to G . Note that it may depend on ν .

§ 2. Computation of the L -factor.

In this section we compute $L_\nu^{(t)}(\Pi(\sigma), \mu, s)$ for the most important cases. (We exclude the other cases because of technical reasons.) We get (in these

cases) that $L_\nu^{(i)}(\Pi(\sigma), \mu, s)$ doesn't depend on ν, i . We use the notations of 1.e.

(a) *Poles of $L_\nu^{(i)}(\pi, \mu, s)$.* We summarize the results of [1] about L -functions. Let π be an irreducible admissible representation of G having a Whittaker model with respect to (ν, ϕ_i) . Then it is known that there exists a finite set Δ_π of characters of k^* , such that for each $w \in W_\pi^{\nu, \phi_i}$, we have the following asymptotic expansion

$$(2.1) \quad w \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = |x|^{s/2} \sum_{\eta \in \Delta_\pi} \sum_{a=0}^{N(\eta)-1} l_{\eta, a}(w) \eta(x) \nu^a(x)$$

for $|x|$ small enough; It is known that $\sum_{\eta \in \Delta_\pi} N(\eta) \leq 4$. ($\nu(x)$ denotes the discrete valuation of k).

A pole of $L_\nu^{(i)}(\pi, \mu, s)$ is called a *regular pole* if it is a pole of some $L_\nu^{(i)}(w, \phi, \mu, s)$ with $\phi \in S_0(V_i)$, where $S_0(V_i) = \{\phi \in S(V_i) \mid \phi(0, 0) = 0\}$. The notations are as in (1.b). Any other pole of $L_\nu^{(i)}(\pi, \mu, s)$ is called an *exceptional pole*. Let $\phi \in S_0(V_i)$ and $w \in W_\pi^{\nu, \phi_i}$. A pole of $L_\nu^{(i)}(w, \phi, \mu, s)$ comes from the asymptotic expansion (2.1). Indeed

$$L_\nu^{(i)}(w, \phi, \mu, s) = \sum_j b \int_{k^*} w_j \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} \mu(x) |x|^{s-3/2} d^*x$$

where w_j are translates of w . It is now clear that all the poles of $L_\nu^{(i)}(w, \phi, \mu, s)$ come from the asymptotic behaviour of the w_j 's. Moreover

THEOREM 2.1. *The regular part of $L_\nu^{(i)}(\pi, \mu, s)$ is $\prod_{\eta \in \Delta_\pi} L^{N(\eta)}(\eta\mu, s)$, ($L(\chi, s)$ is the standard Tate L -function).*

THEOREM 2.2. *If s_0 is an exceptional pole of $L_\nu^{(i)}(\pi, \mu, s)$, then there exists a linear functional l on V_π such that*

$$l(\pi(g)\xi) = \mu^{-1}(\det g) |\det g|^{-(s_0+1/2)} l(\xi)$$

for $\xi \in V_\pi$, $g \in G_i$.

THEOREM 2.3. *If π is generic or a subquotient of an induced representation from the parabolic P , then $L_\nu^{(i)}(\pi, \mu, s)$ has no exceptional pole.*

COROLLARY. *If $L_\nu^{(i)}(\pi, \mu, s)$ has regular poles, then it has no exceptional poles.*

Indeed, if $L_\nu^{(i)}(\pi, \mu, s)$ has regular poles then $\Delta_\pi \neq \emptyset$, and hence π is a subquotient of an induced representation from P .

(b) *Computations for $L_\nu^{(i)}(\Pi(\sigma), \mu, s)$.* The notations are as in 1.e. Let $f \in S(E \times E \times k^*)$ and $\xi = \xi_1 \otimes \xi_2 \in V_\sigma$, then

$$(2.2) \quad w_{f, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = \omega_\sigma(x) \int_{F_i \setminus H} |\lambda(h)|^{-2} f(h^{-1}u_0, \phi^{x\lambda(h)}) \xi(h) d\tilde{h}.$$

We are going to exhibit functions f_0 , such that the characters of the asymptotic behaviour of elements of V_σ appear in (2.2) for (f_0, ξ) .

We note that if $f| \{(M_1, M_2, \phi^t) | (B(M_i, M_j))=0, t \in k^*\} = 0$, then the integral (2.2) vanishes for $|x|$ small enough. Denote $\mathcal{O} = \{(M_1, M_2) | (B(M_i, M_j))=0\}$. \mathcal{O} is an H invariant set. Here are representatives for the different orbits.

$$\begin{aligned} \mathcal{O}_1: & \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right), \quad \mathcal{O}_2: \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right), \quad \mathcal{O}_{3,t}: \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} \right), \\ \mathcal{O}_3: & \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right), \quad \mathcal{O}_4: (0, 0). \end{aligned}$$

Let $\tilde{\mathcal{O}}$ be an orbit and \tilde{u} — the related representative. We take $f_0 = f_n$, the characteristic function of the neighbourhood of (\tilde{u}, ϕ) having a radius q^{-n} around each coordinate.

Case $\tilde{\mathcal{O}} = \mathcal{O}_1, i=1$: Taking n large enough (depending on the smoothness of ξ) we get from (2.2)

$$(2.3) \quad w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = c_n |x| \xi_1 \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \xi_2(I).$$

Let W_{σ_1} be the Whittaker model of σ_1 with respect to $\left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\}$. Denote by w_{ξ_1} the Whittaker function of ξ_1 . We have

$$\begin{aligned} \xi_1 \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} &= \int_{k^* \setminus K_1^*} \nu^{-1}(\delta) w_{\xi_1} \left(\delta \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \right) d\tilde{\delta} \\ &= \int_k \nu^{-1}(1+t\sqrt{\rho}) \phi(t) w_{\xi_1} \begin{pmatrix} 1 & 0 \\ 0 & x(1-t^2\rho) \end{pmatrix} \frac{dt}{|1-t^2\rho|}. \end{aligned}$$

(The last integral stabilizes for large $|t|$.) Thus from (2.3),

$$(2.4) \quad \begin{aligned} w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} \\ = c_n |x| \int_{|t| \leq M_\xi} \nu^{-1}(1+t\sqrt{\rho}) \phi(t) w_{\xi_1} \begin{pmatrix} 1 & 0 \\ 0 & x(1-t^2\rho) \end{pmatrix} \frac{dt}{|1-t^2\rho|} \xi_2(I). \end{aligned}$$

Case $\tilde{\mathcal{O}} = \mathcal{O}_1, i=2$. Here, using the formula $\xi_1(g) = \int_{k^*} \nu^{-1}(t) w_{\xi_1} \left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g \right) d^*t$ ($g \in GL(2, k)$), where $\nu = (\nu_1, \nu_2)$, we get

$$(2.5) \quad w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = c_n |x| \int_{|t| \leq M_\xi} \nu_2^{-1}(t) \phi(t) w_{\xi_1} \begin{pmatrix} 1 & 0 \\ 0 & xt \end{pmatrix} d^*t \cdot \xi_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Case $\tilde{O}=O_2$, $i=1$. Here we get

$$(2.6) \quad w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = c_n |x| \int_{|t| \leq M_\xi} \nu^{-1}(1+t\sqrt{\rho}) \phi(t\rho) w_{\xi_2} \begin{pmatrix} 1 & 0 \\ 0 & x(1-t^2\rho) \end{pmatrix} \frac{dt}{|1-t^2\rho|} \xi_1(I).$$

Case $\tilde{O}=O_2$, $i=2$. Here we get

$$(2.7) \quad w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = c_n |x| \int_{|t| \leq M_\xi} \nu_2^{-1}(t) \phi(t) w_{\xi_2} \begin{pmatrix} 1 & 0 \\ 0 & xt \end{pmatrix} d^*t \cdot \xi_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Case $\tilde{O}=O_{3,1}$, $i=1$. Here we get

$$(2.8) \quad w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = c_n \int_{|a|, |x\alpha^{-1} \det^{-1} B - 1| \leq q^{-n}, B - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathfrak{P}^n)} \xi_1 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \cdot \xi_2 \begin{pmatrix} 1 & 0 \\ 0 & \det B \end{pmatrix} da dB.$$

Case $\tilde{O}=O_{3,1}$, $i=2$. Here we get for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$(2.9) \quad w_{f_n, \xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = \int_{|x^{-1}yz \det A - 1| \leq q^n, A - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, A \begin{pmatrix} y & 0 \\ 0 & z \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathfrak{P}^n)} \begin{pmatrix} 1 & 0 \\ yc & y \det A \end{pmatrix} \xi_2 \begin{pmatrix} 1 & 0 \\ 1 & z \end{pmatrix} dA dy dz.$$

Now substitute in formulas (2.4)-(2.9) the asymptotic expansion of the w_{ξ_j} . It is easy to see that for the functions related to the orbits O_1 , O_2 , we get all the characters involved in the asymptotic expansions of elements of V_σ . (Note that the coefficient of the asymptotic expansions for $i=1$ contain

$\int_k \nu^{-1}(1+t\sqrt{\rho}) \phi(t) \frac{dt}{|1-t^2\rho|}$ which is essentially a Gauss sum and hence nonzero.)

We "need" the functions related to $O_{3,1}$ for the case $\sigma_1 = \sigma_2 = \sigma_{\eta_1, \eta_2}$. It is possible to see that in this case in the asymptotic expansion of (2.8), (2.9) we can find $\eta_1(x)v(x)$ and $\eta_2(x)v(x)$, and in the case $\eta_1 = \eta_2 = \eta$ we can find $\eta(x)v^2(x)$, $\eta(x)v^3(x)$. (The orbits O_3 , $O_{3,i}$ give nothing new.) Thus we proved

THEOREM 2.4. *Let none of σ_1 , σ_2 be supercuspidal or a special representation, then*

$$(1) \quad L_v^{(i)}(\Pi(\sigma), \mu, s) = L(\sigma_1, \mu, s) L(\sigma_2, \mu, s).$$

$$(2) \quad L_v^{(i)}(\Pi(\sigma), \mu, s) \text{ does not depend on the choice of the Whittaker model.}$$

§3. Computation of the γ -factor.

In this section we carry out the computation of $\gamma_v^{(i)}(\Pi(\sigma), \mu, \phi, s)$. The notations are as in 1.e. We put $\gamma_v^{(i)}(\Pi(\sigma), \mu, s)$ instead of $\gamma_v^{(i)}(\Pi(\sigma), \mu, \phi, s)$. Our main theorem is

THEOREM 3.1. *Let $\sigma=(\sigma_1, \sigma_2)$ be a pair of irreducible admissible representations of $GL(2, k)$, such that $\omega_{\sigma_1}=\omega_{\sigma_2}$. Then*

$$(3.1) \quad \gamma_v^{(i)}(\Pi(\sigma), \mu, s) = \gamma(\sigma_1, \mu, s) \gamma(\sigma_2, \mu, s)$$

where $\gamma(\sigma_j, \mu, s)$ is the Jacquet-Godement γ -function of σ_j (cf. [3]).

COROLLARY. $\gamma(\Pi(\sigma), \mu, s)$ is independent of the choice of the Whittaker model.

The proof of (3.1) will be an imitation of its finite field version [2]. Write the functional equation (1.5) via gamma,

$$(3.2) \quad \gamma_v^{(i)}(\Pi(\sigma), \mu, s) L_v^{(i)}(w, \phi, \mu, s) = L_v^{(i)}(\tilde{w}, \hat{\phi}, \mu^{-1} \omega_{\sigma}^{-1}, 1-s)$$

where $\omega_{\sigma} = \omega_{\Pi(\sigma)} = \omega_{\sigma_1} = \omega_{\sigma_2}$. We understand the integral $L_v^{(i)}(w, \phi, \mu, s)$ as a power series $\sum_{r=-\infty}^{\infty} q^{r(s+1/2)} \int_{g \in N_i \backslash G_i, |\det g|=q^r} w(g) \phi((0, 1)g) \mu(\det g) dg$.

Step I. Let ξ be any function in V_{σ} . Let $n \in \mathbb{Z}$ be large enough and $m \gg n$. We substitute $\phi = \phi_m$ and $w = w_{f_n, \xi}$ and prove

$$(3.3) \quad \gamma_v^{(i)}(\Pi(\sigma), \mu, s) \xi(I) = q^{4m+7n} L_v^{(i)}(\tilde{w}_{f_n, \xi}, \hat{\phi}_m, \mu^{-1} \omega_{\sigma}^{-1}, 1-s).$$

ϕ_m is the characteristic function of the neighbourhood of $(0, 1)$ having a radius q^{-m} around each k -coordinate, (similar to the choice of the δ -function of $(0, 1)$ in Proposition 1.3). f_n approximates the δ -function of (u_0, ϕ) . (cf. (1.19).) For $i=1$, each element in the orbit (under H) of u_0 can be uniquely written in the form $(A, \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix}) u_0$ where $(A, \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix}) \in GL(2, k)^2$. For $i=2$, except for a set of measure zero, any other element in the orbit of u_0 can be written in the form $(A, \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}) u_0$ where $(A, \begin{pmatrix} 1 & x \\ y & 1 \end{pmatrix}) \in GL(2, k)^2$. We choose f_n to be the characteristic function of the following neighbourhood O_i of (u_0, ϕ) .

$$O_1 = \left\{ \left(A \begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, A \begin{pmatrix} 0 & -u_2 \\ 1 & v \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}; \phi^t \right) \middle| A - I_2 \in M_2(\mathcal{P}^n) \right. \\ \left. \alpha, v, u_1 - 1, u_2 + \rho, t - 1 \in \mathcal{P}^n \right\}$$

$$O_2 = \left\{ \left(A \begin{pmatrix} 1 & x \\ 0 & u_1 \end{pmatrix}, A \begin{pmatrix} u_2 & 0 \\ y & 1 \end{pmatrix}; \phi^t \right) \middle| A - I_2 \in M_2(\mathcal{P}^n) \right. \\ \left. x, y, u_1, u_2, t - 1 \in \mathcal{P}^n \right\}.$$

Note that the Jacobian of the transformation

$$(M_1, M_2) = \begin{cases} \left(A \begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, A \begin{pmatrix} 0 & -u_2 \\ 1 & v \end{pmatrix} \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \right), & i=1 \\ \left(A \begin{pmatrix} 1 & x \\ 0 & u_1 \end{pmatrix}, A \begin{pmatrix} u_2 & 0 \\ y & 1 \end{pmatrix} \right), & i=2 \end{cases}$$

is 1 for n large enough. Put $w_{n,\xi} = w_{f_n,\xi}$.

LEMMA 3.2. *Let η be a character of k^* and let $\phi \in S(V_i)$. The integral*

$$(3.4) \quad \int_{N_i \setminus G_i^c} \int_{F_i \setminus H} \phi((0, 1)g) \eta(\det g) \omega(g, h) f_n(u_0, \phi) \xi(h) d\check{h} dg$$

converges absolutely. Here $G_i^c = \{g \in G_i \mid |\det g| = c\}$.

PROOF. It suffices to consider integration with respect to g of a diagonal form and then, using the fact that ϕ has a compact support, we have only to consider integration with respect to $g = \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix}$ where $|x| \geq M_0$. We have $\omega(g, h) f_n(u_0, \phi) = |\lambda(h)|^{-2} |x|^4 f_n(h^{-1}(xu_0), \phi^{x^{-1}\lambda(h)})$ which vanishes when $|x|$ is large enough. \square

Lemma 3.2 permits us to reverse the order of integration in (3.4). Now we compute $L_v^{(i)}(w_{n,\xi}, \phi_m, \mu, s)$ (in (3.2)). It is enough to integrate over the set $\left\{ \begin{pmatrix} xy^{-1} & 0 \\ c & y \end{pmatrix} \in G_i \right\}$, the invariant measure being $d^*x d^*y dc |N(x^{-1}y)|$. We get

$$L_v^{(i)}(w_{n,\xi}, \phi, \mu, s) = \int_{(c,y) \in \text{Supp}(\phi_m)} w_{n,\xi} \begin{pmatrix} xy^{-1} & 0 \\ c & y \end{pmatrix} \mu(x) |x|^{s+1/2} |N(x^{-1}y)| d^*x d^*y dc.$$

For m large enough (depending on $w_{n,\xi}$) we have for $(c, y) \in \text{Supp}(\phi_m)$

$$w_{n,\xi} \begin{pmatrix} xy^{-1} & 0 \\ c & y \end{pmatrix} = w_{n,\xi} \left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ c & y \end{pmatrix} \right) = w_{n,\xi} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

and hence

$$L_v^{(i)}(w_{n,\xi}, \phi, \mu, s) = q^{-4m} \int w_{n,\xi} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \mu(x) |x|^{s-3/2} d^*x.$$

Considering the support of f_n , we see that $|x-1| \leq q^{-n}$. Taking n large enough (depending on ξ) we get (using (2.2))

$$w_{n,\xi} \begin{pmatrix} xI_2 & 0 \\ 0 & I_2 \end{pmatrix} = \begin{cases} q^{-\epsilon_n \xi(I)}, & |x-1| \leq q^{-n} \\ 0, & \text{otherwise.} \end{cases}$$

Let n be also such that $\mu(x) = 1$ for $|x-1| \leq q^{-n}$. Thus $L_v^{(i)}(w_{n,\xi}, \phi_m, \mu, s) = q^{-4m-7n\xi(I)}$. This proves (3.3).

Step II. The Fourier transform of $\phi \in S(V_i)$ is given by

$$\hat{\phi}(x) = d_i \int_{V_i} \phi^{-1}(\tau_i(x, y)) \phi(y) dy; \quad d_i = \begin{cases} |4\rho|, & i=1 \\ 1, & i=2. \end{cases}$$

We have $\hat{\phi}_m(c, y) = q^{-4m} d_i \tilde{\phi}^{-1}(c)$. Recall that we denote $\tilde{\phi} = \phi \circ \text{tr}$ and $\bar{\mu} = \mu \circ N$, where for $i=1$, $c = c_1 + c_2 \sqrt{\rho}$, $y = y_1 + y_2 \sqrt{\rho}$ satisfy $|2c_1|, |2\rho c_2|, |2y_1|, |2\rho y_2| \leq q^m$, and for $i=2$, $c = (c_1, c_2)$, $y = (y_1, y_2)$ satisfy $|c_1|, |c_2|, |y_1|, |y_2| \leq q^m$. Otherwise, $\hat{\phi}_m(c, y) = 0$. In order to compute $L_\nu^{(i)}(\tilde{w}_{n,\xi}, \hat{\phi}_m, \mu^{-1}\omega_\sigma^{-1}, 1-s)$, it is enough to integrate over the set $\left\{ \begin{pmatrix} 0 & -xc^{-1} \\ c & y \end{pmatrix} \in G_i \right\}$, the measure being $d^*x d^*c dy |N(x^{-1}c)|$. We get

$$(3.5) \quad L_\nu^{(i)}(\tilde{w}_{n,\xi}, \hat{\phi}_m, \mu^{-1}\omega_\sigma^{-1}, 1-s) \\ = q^{-4m} d_i \int_{(c,y) \in \text{Supp}(\hat{\phi}_m)} \tilde{w}_{n,\xi} \begin{pmatrix} 0 & -x \\ 1 & yc^{-1} \end{pmatrix} \tilde{\phi}^{-1}(c) \nu(c) \mu \tilde{\omega}_\sigma^{-1}(c) \\ \cdot |Nc|^{1/2-s} \mu^{-1}\omega_\sigma(x) |x|^{-1/2-s} d^*c dy d^*x.$$

It is clear that in (3.5), it is enough to take c , the coordinates of which are bounded by a number q^{l_0} which depends only on ν, μ .

Step III. We compute the integral $T = \int_{(c,y) \in \text{Supp}(\hat{\phi}_m)} \tilde{w}_{n,\xi} \begin{pmatrix} 0 & -x \\ 1 & yc^{-1} \end{pmatrix} dy$. (Note

that in the finite field case $J_{H(\sigma)} \begin{pmatrix} 0 & -x \\ 1 & z \end{pmatrix} = J_{H(\sigma)} \begin{pmatrix} 0 & -x \\ 1 & 0 \end{pmatrix}$ for any $z \in K_i$.)

The case $i=1$. Using (1.2), (1.3), $\alpha_1 \begin{pmatrix} 0 & -x \\ 1 & yc^{-1} \end{pmatrix}$ represents in G the element

$$\begin{pmatrix} -\frac{1}{2} & & & \\ & \frac{1}{2\rho} & & \\ & & -2 & \\ & & & 2\rho \end{pmatrix} \begin{pmatrix} & & & I_2 \\ & & -I_2 & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} I_2 & & & \\ & I_2 & & \\ & & & \\ & & & xI_2 \end{pmatrix} u(yc^{-1}),$$

where $u(yc^{-1})$ is the image of $\begin{pmatrix} 1 & yc^{-1} \\ 0 & 1 \end{pmatrix}$ in G . Put $A = \begin{pmatrix} 0 & \rho \\ 1 & 0 \end{pmatrix}$. Substituting (1.19) for $w_{n,\xi}$ and using (1.15), (1.16) we obtain

$$T = |4\rho|^{-2} \int_{|2y_1|, |2\rho y_2| \leq q^m} \int_{F_1 \setminus H} \int_{E \times E} |\lambda(h)|^2 \phi^{-1/2}(\text{tr}(hM_1) + \text{tr} A^{-1}(hM_2))$$

$$\begin{aligned} & \times \phi^{x^{-1}\lambda(h)} \left(\frac{1}{2Nc} y_1 \left(c_1 \left(\det M_1 + \frac{1}{\rho} \det M_2 \right) - 2c_2 B(M_1, M_2) \right) + \frac{1}{2Nc} y_2 \left(2c_1 B(M_1, M_2) \right. \right. \\ & \left. \left. - \rho c_2 \left(\det M_1 + \frac{1}{\rho} \det M_2 \right) \right) \right) f_n(M, \phi^{x^{-1}\lambda(h)} \xi(h) dM d\tilde{h} dy. \end{aligned}$$

Now perform an integration with respect to (y_1, y_2) (Lemma 3.2). We get

$$\begin{aligned} T = & |4\rho|^{-3} q^{2m} \int_{F_1 \setminus H} \int_{A(m, c)} |\lambda(h)|^2 \phi^{-1/2} (\text{tr}(hM_1) + \text{tr} A^{-1}(hM_2)) \\ & \times f_n(M, \phi^{x^{-1}\lambda(h)} \xi(h) dM d\tilde{h} \end{aligned}$$

here

$$A(m, c) = \left\{ M \in E \times E \left| \begin{aligned} & \left| \frac{1}{Nc} \left(c_1 \left(\det M_1 + \frac{1}{\rho} \det M_2 \right) - 2c_2 B(M_1, M_2) \right) \right| \leq |4|q^{-m} \\ & \left| \frac{1}{Nc} \left(2c_1 B(M_1, M_2) - \rho c_2 \left(\det M_1 + \frac{1}{\rho} \det M_2 \right) \right) \right| \leq |4\rho|q^{-m} \end{aligned} \right. \right\}.$$

Now substitute the support of f_n ; $(M_1, M_2) = (A, \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}) \cdot (\begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix}, \begin{pmatrix} 0 & -u_2 \\ 1 & v \end{pmatrix})$ where $A - I \in M_2(\mathcal{P}^n)$ and $\alpha, v, u_1 - 1, u_2 + \rho \in \mathcal{P}^n$. The related Jacobian is 1 (n is large enough). Put $h_1 = (A, \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix})$ and $dh'_1 = d^* A d\alpha$. We get

$$\begin{aligned} T = & |4\rho|^{-3} q^{2m} \int_{F_1 \setminus H} \int_{\tilde{A}(n, m, c)} \int_{h_1 \sim I, |x^{-1}\lambda(h) - 1| \leq q^{-n}} |\lambda(h)|^2 \phi^{-1/2} (\text{tr}(hh_1 \begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix}) \\ & + \text{tr} A^{-1} (hh_1 \begin{pmatrix} 0 & -u_2 \\ 1 & v \end{pmatrix})) \xi(h) dh'_1 du_1 du_2 dv d\tilde{h} \\ \tilde{A}(n, m, c) = & \left\{ (u_1, u_2, v) \left| \begin{aligned} & |v|, |u_1 - 1|, |u_2 + \rho| \leq q^{-n} \\ & \left| \frac{1}{Nc} \left(c_1 \left(u_1 + \frac{1}{\rho} u_2 \right) - c_2 v \right) \right| \leq |4|q^{-m} \\ & \left| \frac{1}{Nc} \left(c_1 v - \rho c_2 \left(u_1 + \frac{1}{\rho} u_2 \right) \right) \right| \leq |4\rho|q^{-m} \end{aligned} \right. \right\}. \end{aligned}$$

Using the smoothness of ξ , we can make the change of variables $hh_1 \rightarrow h$ and get

$$\begin{aligned} T = & |4\rho|^{-3} q^{2m-5n} |x|^2 \int_{F_1 \setminus H, |x^{-1}\lambda(h) - 1| \leq q^{-n}} \int_{\tilde{A}(n, m, c)} \phi^{-1/2} (\text{tr}(h \begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix}) \\ & + \text{tr} A^{-1} (h \begin{pmatrix} 0 & -u_2 \\ 1 & v \end{pmatrix})) \xi(h) du_1 du_2 dv d\tilde{h}. \end{aligned}$$

We realize $F_1 \setminus H$ as $\{(A, \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix}) \in GL(2, k) \times GL(2, k)\}$. The right invariant

measure being $d^*A d^*\delta d\alpha$. Put for $h = \left(A, \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} \right)$, $\text{tr} \left(h \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \alpha(h)a + \beta(h)b + \gamma(h)c + \delta(h)d$; $\text{tr} A^{-1} \left(h \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \alpha'(h)a + \beta'(h)b + \gamma'(h)c + \delta'(h)d$. Let $u = u_1 + \frac{1}{\rho} u_2$ and $(v + u\sqrt{\rho})c^{-1} = \tilde{v} + \tilde{u}\sqrt{\rho}$, then for $(u_1, u_2, v) \in \tilde{A}(n, m, c)$ we have $|\rho u| \leq q^{-n}$, $|v| \leq q^{-n}$, $|\tilde{v}| \leq 4\rho|q^{-m}$, $|\tilde{u}| \leq 4|q^{-m}$. Since the coordinates of c are bounded independently of m , we get for m large enough that the conditions $\{|\tilde{v}| \leq 4\rho|q^{-m}$, $|\tilde{u}| \leq 4|q^{-m}\}$ imply that $|\rho u|, |v| \leq q^{-n}$. With this change of variables we get

$$(3.6) \quad T = |4\rho|^{-1} q^{-6n} |Nc| |x|^2 \int_{B_1(m, c) \cap B_2(n), |x^{-1}\lambda(h) - 1| \leq q^{-n}} \phi^{-1/2}(\text{tr}(h \cdot I) + \text{tr} A^{-1}(h \cdot A)) \xi(h) d\tilde{h}$$

here

$$B_1(m, c) = \left\{ h = \left(A, \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} \right) \left| \begin{array}{l} |c_1 \delta'(h) - \rho c_2 \beta'(h)| \leq |2\rho|^{-1} q^m \\ |\rho c_2 \delta'(h) - \rho c_1 \beta'(h)| \leq |2|^{-1} q^m \end{array} \right. \right\}$$

$$B_2(n) = \left\{ h = \left(A, \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} \right) \left| |\delta(h) + \rho \beta'(h)| \leq |2| q^n \right. \right\}.$$

The case $i=2$. Using (1.2), (1.3), $\alpha_2 \begin{pmatrix} 0 & -x \\ 1 & yc^{-1} \end{pmatrix}$ represents in G the element

$$\left(\begin{array}{c|c} -1 & \\ \hline -1 & -1 \\ \hline & -1 \end{array} \right) \left(\begin{array}{cc} I_2 & \\ -I_2 & \end{array} \right) \left(\begin{array}{cc} I_2 & \\ & xI_2 \end{array} \right) u(yc^{-1}), \text{ where } u(yc^{-1}) \text{ is the image of}$$

$$\begin{pmatrix} 1 & yc^{-1} \\ 0 & 1 \end{pmatrix} \text{ in } G. \text{ Put } A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \text{ We have}$$

$$T = \int_{|y_1|, |y_2| \leq q^m} \int_{F_2 \setminus H} \int_{\mathcal{E} \times E} |\lambda(h)|^2 \phi^{-1}(\text{tr}(A_1(h \cdot M_1)) + \text{tr}(A_2(h \cdot M_2)))$$

$$\times \phi^{x^{-1}\lambda(h)} (y_1 c_2^{-1} \det M_1 + y_2 c_1^{-1} \det M_2) f_n(M, \phi^{x^{-1}\lambda(h)}) \xi(h) dM d\tilde{h} dy.$$

Now perform an integration with respect to $y = (y_1, y_2)$. We get

$$T = q^{2m} \int_{F_2 \setminus H} \int_{A(m, c)} |\lambda(h)|^2 \phi^{-1}(\text{tr}(A_1(h \cdot M_1)) + \text{tr}(A_2(h \cdot M_2)))$$

$$\times f_n(M, \phi^{x^{-1}\lambda(h)}) \xi(h) dM d\tilde{h}$$

here $A(m, c) = \{M \in E \times E \mid |c_2^{-1} \det M_1|, |c_1^{-1} \det M_2| \leq q^{-m}\}$. Substitute the support of f_n . $(M_1, M_2) = (A, I) \left(\begin{pmatrix} 1 & x \\ 0 & u_1 \end{pmatrix}, \begin{pmatrix} u_2 & 0 \\ y & 1 \end{pmatrix} \right)$. Here $A - I \in M_2(\mathcal{P}^n)$ and $x, y, u_1, u_2 \in \mathcal{P}^n$. The related Jacobian is 1 (n is large enough). Put $h_1 = (A, I)$ and $dh'_1 = d^*A$. We get,

$$\begin{aligned}
T &= q^{2m} \int_{F_2 \setminus H} \int_{\tilde{A}(n, m, c)} \int_{h_1 \sim I, |x^{-1}\lambda(h)-1| \leq q^{-n}} |\lambda(h)|^2 \psi^{-1} \left(\text{tr} \left(A_1 \left(h h_1 \begin{pmatrix} 1 & x \\ 0 & u_1 \end{pmatrix} \right) \right) \right. \\
&\quad \left. + \text{tr} \left(A_2 \left(h h_1 \begin{pmatrix} u_2 & 0 \\ y & 1 \end{pmatrix} \right) \right) \right) \xi(h) d h_1' d(x, y, u_1, u_2) d\tilde{h} \\
\tilde{A}(n, m, c) &= \left\{ (x, y, u_1, u_2) \left| \begin{array}{l} |x|, |y|, |u_1|, |u_2| \leq q^{-n} \\ |c_2^{-1}u_1|, |c_1^{-1}u_2| \leq q^{-m} \end{array} \right. \right\} \\
&= \left\{ (x, y, u_1, u_2) \left| \begin{array}{l} |x|, |y| \leq q^{-n} \\ |c_2^{-1}u_1|, |c_1^{-1}u_2| \leq q^{-m} \end{array} \right. \right\}
\end{aligned}$$

(since c_1, c_2 are bounded independently of m , and m is large enough). Change variables as follows: $h h_1 \rightarrow h$, $c_2^{-1}u_1 \rightarrow u_1$, $c_1^{-1}u_2 \rightarrow u_2$ and integrate with respect to x, y, u_1, u_2 . (The smoothness of ξ allows it.) We get

$$(3.7) \quad T = q^{-6n} |Nc| |x|^2 \int_{\substack{B_1(m, c) \cap B_2(n) \\ |x^{-1}\lambda(h)-1| \leq q^{-n}}} \psi^{-1}(\text{tr}(A_1(h \cdot A_1)) + \text{tr}(A_2(h \cdot A_2))) \xi(h) d\tilde{h}.$$

It is enough to integrate in (3.7) over the set

$$\left\{ \left(A, \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \right) \in GL(2, k)^2 \right\}$$

the right invariant measure being $d^* A d z_1 d z_2$. Put for $h = \left(A, \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \right)$,

$$\text{tr} \left(A_1 \left(h \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \right) = \alpha(h)a + \beta(h)b + \gamma(h)c + \delta(h)d;$$

$$\text{tr} \left(A_2 \left(h \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \right) = \alpha'(h)a + \beta'(h)b + \gamma'(h)c + \delta'(h)d$$

then

$$B_1(m, c) = \left\{ h = \left(A, \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \right) \left| |c_2 \delta(h)| \leq q^m, |c_1 \alpha'(h)| \leq q^m \right\};$$

$$B_2(n) = \left\{ h = \left(A, \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \right) \left| |\beta(h)| \leq q^n, |\gamma'(h)| \leq q^n \right\}.$$

Step IV. We substitute (3.6) ((3.7)) in (3.5).

The case $i=1$.

$$\begin{aligned}
(3.8) \quad \gamma_\nu^{(1)}(H(\sigma), \mu, s) \xi(I) &= \sum_{r, l=-\infty}^{\infty} q^{(3/2-s)(r+l)} \int_{|x|=q^r} \mu^{-1} \omega_\sigma^{-1}(x) \int_{\substack{Nc|=q^l \\ |2c_1|, |2\rho c_2| \leq q^{l_0}}} \nu \mu \tilde{\omega}_\sigma^{-1}(c) \tilde{\phi}^{-1}(c) \\
&\quad \times q^n \int_{B_1(m, c) \cap B_2(n), |x^{-1}\lambda(h)-1| \leq q^{-n}} \psi^{-1/2}(\text{tr}(h \cdot I) + \text{tr} A^{-1}(h \cdot A)) \xi(h) d\tilde{h} d^* c d^* x.
\end{aligned}$$

Since the left hand side of (3.8) is independent of n, m , the integral stabilizes for $m \geq M_0, n \geq N_0$. Changing variable $t = x\lambda(h)^{-1}$ we get

$$\begin{aligned} \gamma_v^{(1)}(\Pi(\sigma), \mu, s)\xi(I) &= \gamma_{K_1}\left(\nu\bar{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \sum_{r=-\infty}^{\infty} q^{(3/2-s)r} \\ &\times \int_{F_1 \setminus H, |\lambda(h)|=q^r} \mu^{-1}\omega_{\sigma}^{-1}(\lambda(h))\phi^{-1/2}(\text{tr}(h \cdot I) + \text{tr}(A^{-1}(h \cdot A)))\xi(h)d\tilde{h} \\ (3.9) \quad \gamma_v^{(1)}(\Pi(\sigma), \mu, s)\xi(I) &= \gamma_{K_1}\left(\nu\bar{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int_{F_1 \setminus H} \phi^{-1/2}(\text{tr}(h \cdot I) \\ &+ \text{tr}(A^{-1}(h \cdot A)))\xi(h)\mu^{-1}\omega_{\sigma}^{-1}(\lambda(h))|\lambda(h)|^{3/2-s}d\tilde{h} \end{aligned}$$

here $\gamma_{K_1}(\alpha, s, \phi) = \int_{K_1} \phi^{-1}(x)\alpha^{-1}(x)|x|^{1-s}d^*x$.

The case $i=2$. Similar arguments as those for $i=1$ give us

$$\begin{aligned} (3.10) \quad \gamma_v^{(2)}(\Pi(\sigma), \xi, s)\xi(I) &= \gamma_{K_2}\left(\nu\bar{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int_{F_2 \setminus H} \phi^{-1}(\text{tr}(A_1(h \cdot A_1)) \\ &+ \text{tr}(A_2(h \cdot A_2)))\xi(h)\mu^{-1}\omega_{\sigma}^{-1}(\lambda(h))|\lambda(h)|^{3/2-s}dh \end{aligned}$$

here $\gamma_{K_2}(\alpha, s, \phi) = \int_{K_2^*} \phi^{-1}(x_1, x_2)\alpha^{-1}(x_1, x_2)|x_1x_2|^{1-s}d^*(x_1, x_2)$.

Formulas (3.9), (3.10) are analogous to (1.12).

PROOF OF THEOREM 3.1 for $i=1$. We carry on from (3.9) formally. The justifications and meaning of the integrals will soon follow. Take $\xi = \xi_1 \otimes \xi_2$. We have

$$\begin{aligned} \gamma_v^{(1)}(\Pi(\sigma), \mu, s)\xi(I) &= \gamma_{K_1}\left(\nu\bar{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int_{GL(2, k) \times k^* \times k^*} \phi^{-1}\left(\text{tr}\left(\frac{1}{2}\begin{pmatrix} 1+\delta & \alpha \\ \rho^{-1}\alpha & 1+\delta \end{pmatrix}A\right)\right) \\ &\times \xi_1(A)\xi_2\begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} \mu^{-1}\omega_{\sigma}^{-1}(\delta \det A)|\delta \det A|^{3/2-s}d^*Ad^*\delta d\alpha \\ &= \gamma_{K_1}\left(\nu\bar{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int \phi^{-1}(\text{tr } A)\xi_1(A)\mu^{-1}\omega_{\sigma}^{-1}(\det A)|\det A|^{3/2-s}\nu^{-1}\left(\frac{1}{2}\left(1+\delta+\frac{\alpha}{\sqrt{\rho}}\right)\right) \\ &\times \mu\omega_{\sigma}\left(\delta^{-1}N\left(\frac{1}{2}\left(1+\delta+\frac{\alpha}{\sqrt{\rho}}\right)\right)\right)\left|\delta^{-1}N\left(\frac{1}{2}\left(1+\delta+\frac{\alpha}{\sqrt{\rho}}\right)\right)\right|^{s-3/2}\xi_2\begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix}d^*Ad^*\delta d\alpha. \end{aligned}$$

In the above we changed variables $\frac{1}{2}\begin{pmatrix} 1+\delta & \alpha \\ \rho^{-1}\alpha & 1+\delta \end{pmatrix}A \rightarrow A$, and used the fact that $\xi_1(rg) = \nu(r)\xi_1(g)$ for $r \in K_1^*$. We need the following lemma.

LEMMA 3.3. Let μ be a character of k^* . Let σ be an irreducible admissible

representation of $GL(2, k)$. Then for any matrix coefficient $\xi(g)$ of σ

$$(3.11) \quad \gamma(\sigma, \mu, s)\xi(I) = \int_{GL(2, k)} \phi^{-1}(\text{tr } g)\xi(g^{-1})\mu^{-1}(\det g)|\det g|^{3/2-s}d^*g.$$

The integral in (3.11) is understood in the sense of power series in $q^{3/2-s}$.

PROOF. We have the Jacquet Godement functional equation (cf. [3])

$$\begin{aligned} \gamma(\sigma, \mu, s) \int_{GL(2, k)} \phi(g)\xi(g)\mu(\det g)|\det g|^{s+1/2}d^*g \\ = \int_{GL(2, k)} \hat{\phi}(g)\xi(g^{-1})\mu^{-1}(\det g)|\det g|^{3/2-s}d^*g \end{aligned}$$

here ϕ is a Schwarz-Bruhat function on $M(2, k)$ and $\hat{\phi}$ is its Fourier transform with respect to $\phi^{-1} \cdot \text{tr}$. The integrals are understood in the sense of power series. Substitute $\phi = \phi_{\bar{n}}$, the characteristic function of the neighbourhood of I , having a radius q^{-n_i} around the i -th coordinate ($\bar{n} = (n_1, \dots, n_4)$). Using the smoothness of ξ , and taking n_i large enough gives

$$\begin{aligned} \gamma(\sigma, \mu, s)\xi(I) &= \int_{M_{\bar{n}}} \phi^{-1}(\text{tr } g)\xi(g^{-1})\mu^{-1}(\det g)|\det g|^{3/2-s}d^*g \\ &= \sum_{l=-\infty}^{\infty} q^{(3/2-s)l} \int_{g \in M_{\bar{n}}, |\det g|=q^l} \phi^{-1}(\text{tr } g)\xi(g^{-1})\mu^{-1}(\det g)d^*g. \end{aligned}$$

Here $M_{\bar{n}} = \left\{ \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in M(2, k) \mid |a_i| \leq q^{n_i}, i=1, 2, 3, 4 \right\}$ \square .

COROLLARY 1.

$$(3.12) \quad \gamma(\sigma, \mu, s)\xi(I) = \int_{GL(2, k)} \phi^{-1}(\text{tr } g)\xi(g)\mu^{-1}\omega_{\sigma}^{-1}(\det g)|\det g|^{3/2-s}d^*g.$$

COROLLARY 2. Let σ be an irreducible admissible representation of $GL(2, k)$ acting by right translations in a space V_{σ} of smooth functions on $GL(2, k)$, then (3.12) holds true for any $\xi \in V_{\sigma}$.

PROOF. Define for a small compact open subgroup U of $GL(2, k)$, $\xi_U(g) = \frac{1}{m(U)} \int_U \xi(gu)du$ ($\xi \in V_{\sigma}$). $\xi_U(g)$ is a matrix coefficient, and so (3.12) is true for $\xi_U(g)$. Since ξ is smooth then there is U such that $\xi_U = \xi$. \square

We continue the proof of the theorem ($i=1$). It follows from (3.12) that

$$(3.13) \quad \gamma_v^{(1)}(\Pi(\sigma), \mu, s)\xi(I) = \gamma(\sigma_1, \mu, s)\xi_1(I)\gamma_{K_1}\left(\nu\bar{n}, s - \frac{1}{2}, \tilde{\phi}\right)A(\sigma_2, \nu, \mu, s)$$

here

$$A = A(\sigma_2, \nu, \mu, s) = \int \nu^{-1} \left(\frac{1}{2} \left(1 + \delta + \frac{\alpha}{\sqrt{\rho}} \right) \right) \mu \omega_{\sigma_2} \left(\delta^{-1} N \left(\frac{1}{2} \left(1 + \delta + \frac{\alpha}{\sqrt{\rho}} \right) \right) \right) \\ \times \left| \delta^{-1} N \left(\frac{1}{2} \left(1 + \delta + \frac{\alpha}{\sqrt{\rho}} \right) \right) \right|^{s-3/2} \xi_2 \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} d^* \delta d\alpha.$$

The integral A has the following sense

$$A = \sum_{m, n=-\infty}^{\infty} q^{(m+n)(s-3/2)} \int_{\substack{|\delta|=q^m \\ |N((1/2)(1+\delta^{-1}+\alpha\delta^{-1}/\sqrt{\rho}))|=q^n}} \nu^{-1} \left(\frac{1}{2} \left(1 + \delta + \frac{\alpha}{\sqrt{\rho}} \right) \right) \mu \omega_{\sigma_2} \left(\delta^{-1} N \left(\frac{1}{2} \left(1 + \delta + \frac{\alpha}{\sqrt{\rho}} \right) \right) \right) \\ \times \xi_2 \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} d^* \delta d\alpha.$$

LEMMA 3.4.

$$A = \left(r_{K_1} \left(\nu \bar{\mu}, s - \frac{1}{2}, \bar{\psi} \right) \right)^{-1} \gamma(\sigma_2, \mu, s) \xi_2(I).$$

PROOF. Change variables in (3.12), $g = \begin{pmatrix} x & y \\ y\rho & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix}$ for g in a compact set of $M(2, k)$. We get

$$\gamma(\sigma_2, \mu, s) \xi_2(I) = \sum_{n=-\infty}^{\infty} q^{(s/2-s)n} \int_{\substack{|\delta N(x+y\sqrt{\rho})|=q^n, |\delta x|, |\delta y| \leq q^{n_1} \\ |x(1+\delta)+\alpha y| \leq q^{n_2}, |\alpha x+\rho y(1+\delta)| \leq q^{n_3}}} \phi^{-1} \left(\text{tr}(x+y\sqrt{\rho}) \right) \\ \times \frac{1}{2} \left(1 + \delta + \frac{\alpha}{\sqrt{\rho}} \right) \nu(x+y\sqrt{\rho}) \mu^{-1} \omega_{\sigma_2}^{-1}(\delta N(x+y\sqrt{\rho})) \\ \times \xi_2 \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} d^*(x+y\sqrt{\rho}) d^* \delta d\alpha$$

where $n_i \geq N_i$; N_2, N_3 depend only on (ν, μ) . Change variables $\delta x \rightarrow x, \delta y \rightarrow y$ and integrate over a different sequence of increasing compacts

$$\gamma(\sigma_2, \mu, s) \xi_2(I) = \sum_{m, n=-\infty}^{\infty} q^{(s-3/2)(m+n)} \int_{\substack{|N((1/2)(1+\delta^{-1}+\alpha\delta^{-1}/\sqrt{\rho}))|=q^n, |\alpha|=q^m}} \mu(\delta) \bar{\nu} \bar{\mu} \left(\frac{1}{2} \left(1 + \delta^{-1} + \frac{\alpha\delta^{-1}}{\sqrt{\rho}} \right) \right) \xi_2 \begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} \\ \times \int_{\substack{|x|, |y| \leq q^{n_1} \\ |x(1+\delta^{-1})+y\alpha\delta^{-1}| \leq q^{n_2} \\ |x\alpha\delta^{-1}+\rho y(1+\delta^{-1})| \leq q^{n_3}}} \phi^{-1} \left(\text{tr}(x+y\sqrt{\rho}) \right) \frac{1}{2} \left(1 + \delta^{-1} + \frac{\alpha\delta^{-1}}{\sqrt{\rho}} \right) \bar{\nu}^{-1} \bar{\mu}^{-1} \\ \times \left((x+y\sqrt{\rho}) \frac{1}{2} \left(1 + \delta^{-1} + \frac{\alpha\delta^{-1}}{\sqrt{\rho}} \right) \right) \\ \times \left| N(x+y\sqrt{\rho}) \frac{1}{2} \left(1 + \delta^{-1} + \frac{\alpha\delta^{-1}}{\sqrt{\rho}} \right) \right|^{s/2-s} d^*(x+y\sqrt{\rho}) d^* \delta d\alpha.$$

Since the last sum stabilizes for n_1, n_2, n_3 we get

$$\begin{aligned} \gamma(\sigma_2, \mu, s)\xi_2(I) &= \gamma_{K_1}\left(\nu\tilde{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \sum_{m, n=-\infty}^{\infty} q^{(s-3/2)(m+n)} \\ &\quad \times \int_{\substack{|\delta|=q^m \\ |N((1/2)(1+\delta^{-1}+\alpha\delta^{-1}/\sqrt{\rho}))|=q^n}} \nu^{-1}\left(\frac{1}{2}\left(1+\delta+\frac{\alpha}{\sqrt{\rho}}\right)\right) \mu\omega_{\sigma_2}\left(\delta^{-1}N\left(\frac{1}{2}\left(1+\delta+\frac{\alpha}{\sqrt{\rho}}\right)\right)\right) \xi_2\left(\begin{pmatrix} 1 & 0 \\ \alpha & \delta \end{pmatrix} d^* \delta d\alpha\right) \\ &= \gamma_{K_1}\left(\nu\tilde{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) A. \quad \square \end{aligned}$$

Combining Lemma 3.4 and (3.13) gives $\gamma_v^{(1)}(\Pi(\sigma), \mu, s) = \gamma(\sigma_1, \mu, s)\gamma(\sigma_1, \mu, s)$. \square

PROOF OF THEOREM 3.1 for the case $i=2$. We carry on from (3.10) formally. The justifications are similar to those for $i=1$. Take $\xi = \xi_1 \otimes \xi_2$. We integrate in (3.10) over the set $\left\{ \left(A, \begin{pmatrix} 1 & 0 \\ z_2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z_1 & 1 \end{pmatrix} \in LG(2, k)^2 \right\}$ the right H invariant measure being $d^* A dz_1 dz_2$. We get

$$\begin{aligned} \gamma_v^{(2)}(\Pi(\sigma), \mu, s)\xi(I) &= \gamma_{K_2}\left(\nu\tilde{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int_{GL(2, k) \times k^* \times k^*} \phi^{-1}\left(\text{tr} \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} A\right) \\ &\quad \times \xi_1(A) \xi_2\left(\begin{pmatrix} z_1 & 1 \\ 1+z_1 z_2 & z_2 \end{pmatrix} \mu^{-1} \omega_{\sigma}^{-1}(-\det A) |\det A|^{3/2-s} d^* A dz_1 dz_2\right) \\ &= \gamma_{K_2}\left(\nu\tilde{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int \phi^{-1}(\text{tr } A) \xi_1(A) \mu^{-1} \omega_{\sigma}^{-1}(\det A) |\det A|^{3/2-s} \nu^{-1}(z_1, z_2) \\ &\quad \times \mu\omega_{\sigma}(-z_1 z_2) |z_1 z_2|^{s-3/2} \xi_2\left(\begin{pmatrix} z_1 & 1 \\ 1+z_1 z_2 & z_2 \end{pmatrix} d^* A dz_1 dz_2\right) \\ &= \gamma(\sigma_1, \mu, s) \xi_1(I) \gamma_{K_2}\left(\nu\tilde{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int \nu^{-1}(z_1, z_2) \mu\omega_{\sigma}(-z_1 z_2) |z_1 z_2|^{s-3/2} \\ &\quad \times \xi_2\left(\begin{pmatrix} z_1 & 1 \\ 1+z_1 z_2 & z_2 \end{pmatrix} dz_1 dz_2\right). \end{aligned}$$

(We used Corollary 2 to Lemma 3.3.) The next lemma is analogous to Lemma 3.4.

LEMMA 3.5.

$$\begin{aligned} \gamma(\sigma_2, \mu, s)\xi_2(I) &= \gamma_{K_2}\left(\nu\tilde{\mu}, s - \frac{1}{2}, \tilde{\phi}\right) \int \nu^{-1}(z_1, z_2) \mu\omega_{\sigma}(-z_1 z_2) |z_1 z_2|^{s-3/2} \\ &\quad \times \xi_2\left(\begin{pmatrix} z_1 & 1 \\ 1+z_1 z_2 & z_2 \end{pmatrix} dz_1 dz_2\right). \end{aligned}$$

PROOF. Change variables in (3.12) $g = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} z_1 & 1 \\ 1+z_1z_2 & z_2 \end{pmatrix}$. We get

$$\begin{aligned} \gamma(\sigma_2, \mu, s)\xi_2(I) &= \int \phi^{-1}(x_1z_1+x_2z_2)\xi_2\begin{pmatrix} z_1 & 1 \\ 1+z_1z_2 & z_2 \end{pmatrix} \nu(x_1, x_2)\mu^{-1}\omega_{\sigma_2}^{-1}(-x_1, x_2)|x_1, x_2|^{3/2-s} \\ &\quad \times d^*(x_1, x_2)dz_1dz_2 \\ &= \int \phi^{-1}(x_1+x_2)\nu(x_1, x_2)\mu^{-1}\omega_{\sigma_2}^{-1}(x_1, x_2)|x_1x_2|^{3/2-s}d^*(x_1, x_2) \\ &\quad \times \int \mu\omega_{\sigma_2}(-z_1z_2)\nu^{-1}(z_1, z_2)|z_1z_2|^{s-3/2}\xi_2\begin{pmatrix} z_1 & 1 \\ 1+z_1z_2 & z_2 \end{pmatrix} dz_1dz_2 \\ &= \gamma_{K_2}\left(\nu\tilde{\mu}, s-\frac{1}{2}, \tilde{\phi}\right) \int \nu^{-1}(z_1, z_2)\mu\omega_{\sigma_2}(-z_1z_2)|z_1z_2|^{s-3/2} \\ &\quad \times \xi_2\begin{pmatrix} z_1 & 1 \\ 1+z_1z_2 & z_2 \end{pmatrix} dz_1dz_2. \quad \square \end{aligned}$$

Finally $\gamma_v^{(i)}(\Pi(\sigma), \mu, s) = \gamma(\sigma_1, \mu, s)\gamma(\sigma_2, \mu, s)$.

The proof of Theorem 3.1 is complete. \square

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