On the irreducibility of Schottky's divisor[*)]

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To the memory of Takuro Shintani

Introduction. In [13] Schottky introduced a homogeneous polynomial J of degree 16 in the Thetanullwerte of genus or degree 4 which vanishes at every jacobian point of the Siegel upper-half space $\mathfrak{S}_4$; and he proved a certain invariance property of J. This invariance property implies that J is a modular form, in fact a cusp form, relative to $\Gamma_4(1)=Sp_8(\boldsymbol{Z})$. Therefore $J=0$ defines a positive divisor of the quotient variety $\Gamma_4(1)\backslash\mathfrak{S}_4$, which is quasi-projective. In this paper we shall show that the divisor defined by $J=0$ is irreducible, i.e., it has only one component with multiplicity one. We have also included a justification of Schottky's proof of the fact that J vanishes at every jacobian point. We have emphasized a precise formulation of the "bekannten algebraischen Satz" in [14], p. 256 and the verification of a subtle condition in that formulation.

The above irreducibility was announced more than 13 years ago in [3], p. 246 and it has been considered by some as a "folklore". We would like to dedicate this paper in fondest memory of Takuro Shintani.

1. We recall that Schottky introduced his "J" and examined its basic properties in [13]; he later gave a clearer presentation of his idea in [14]. We further recall that his method is closely related to Riemann's approach which he explained in his lectures of 1861-62, especially in those on 2-28 and 3-3, 3-4; cf. [12], pp. 19-23. By extracting J out of certain relations Schottky used, implicitly in [13] and explicitly in [14], a loosely formulated lemma. We shall first give its correct formulation with proof:

LEMMA 1. *Let $x_1, x_2, \cdots, x_n$ denote n elements of a vector space V over an arbitrary field K satisfying*

$$\sum_{i=1}^{n} a_i(x_i\otimes_K x_i)=0$$

for some $a_1, a_2, \cdots, a_n\neq 0$ in K; then the dimension, say p, of the K-span of $x_1, x_2, \cdots, x_n$ in V is at most equal to $n/2$. Assume that $n=2p$ and let $b_1, \cdots, b_n$,

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$c_1, \cdots, c_n$ *denote* $2n$ *elements of* K *satisfying*

$$\sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} c_i x_i = 0\,;$$

then necessarily

$$\sum_{i=1}^{n} a_i^{-1} b_i c_i = 0\,.$$

PROOF. We may assume that $x_1, \cdots, x_p$ are linearly independent over K and we put $n-p=q$; then we can write

$$x_{p+j} = \sum_{i=1}^{p} d_{ji} x_i$$

with d_{ji} in K for $1 \leqq j \leqq q$. Furthermore the p^2 elements $x_i \otimes_K x_j$ of $V \otimes_K V$ for $1 \leqq i,\ j \leqq p$ are linearly independent over K. Therefore if we put

$$A' = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_p \end{bmatrix}, \quad A'' = \begin{bmatrix} a_{p+1} & & \\ & \ddots & \\ & & a_n \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} \cdots d_{1p} \\ \vdots \quad\ \vdots \\ d_{q1} \cdots d_{qp} \end{bmatrix},$$

the first condition becomes $A' + {}^tDA''D = 0$, and this implies

$$\det({}^tDA''D) = \det(-A') = (-1)^p a_1 \cdots a_p.$$

Since $a_1 \cdots a_p \neq 0$ by assumption, we get $p \leqq q$, i.e., $2p \leqq n$.

Assume that $n=2p$; then not only A', A'' but also D are invertible, hence $A' + {}^tDA''D = 0$ can be rewritten as $(A')^{-1} + D^{-1}(A'')^{-1}\,{}^tD^{-1} = 0$. On the other hand if we put

$$b' = {}^t(b_1 \cdots b_p)\,, \quad b'' = {}^t(b_{p+1} \cdots b_n)\,, \quad \text{etc.,}$$

the conditions on b_i, c_i become $b' + {}^tDb'' = c' + {}^tDc'' = 0$; hence

$$\sum_{i=1}^{n} a_i^{-1} b_i c_i = {}^tb'(A')^{-1}c' + {}^tb''(A'')^{-1}c''$$
$$= {}^tb'((A')^{-1} + D^{-1}(A'')^{-1}\,{}^tD^{-1})c' = 0\,. \qquad \text{q. e. d.}$$

2. In our later application of Lemma 1 the verification of the condition "$n=2p$" becomes a problem. We shall determine the number of "tritangents" of a canonical curve of genus 4 by a method which we shall use in that verification; the universal field can have any characteristic other than 2:

LEMMA 2. *Let* C *denote a smooth complete non-hyperelliptic curve of genus* 4 *and identify* C *with its canonical curve; then* C *is contained in a unique quadratic surface* F_2, *which is irreducible. Furthermore* C *has* 120 *isolated tritangents; in the case where* F_2 *is a cone and only in that case* C *has a* 1-*parameter family of tritangents, and they are the tangent planes to* F_2.

PROOF. The first part is an immediate consequence of the Riemann-Roch theorem. The second and the third parts, except for the number of tritangents, follow from Weil [16], Hilfssatz 6, p. 49; we shall determine that number:

In general if C is a smooth complete curve of genus $g \geqq 1$ and (A, ϕ) is its jacobian variety, we shall denote by W the subvariety of A consisting of all $\phi(\mathfrak{a})$, in which $\mathfrak{a}$ is a positive divisor of C of degree $g-1$; then $\phi(\mathfrak{a})$ is in W_{smooth}, i.e., $\phi(\mathfrak{a})$ is a smooth point of W, if and only if $l(\mathfrak{a})=1$; cf. [16], Satz 4, p. 46. If $X=W_c$ is a symmetric polar divisor, i.e., invariant under $x \to -x$, and if A_2 is the subgroup of A of points of order 2, a point r of $X \cap A_2$ can be written as $r=\phi(\mathfrak{a})+c$ such that $2\mathfrak{a}$ is a canonical divisor; the converse is also true. Furthermore $\mathfrak{a}$ is unique if and only if r is in X_{smooth}. The point is that $N_g=\text{card}(X_{\text{smooth}} \cap A_2)$ remains constant under specialization. More precisely if (C', A', X') is a specialization of (C, A, X) with reference to some field, then $(X')_{\text{smooth}} \cap (A')_2$ is the unique specialization of $X_{\text{smooth}} \cap A_2$; cf., e.g., [4], p. 832. Since the space of moduli of curves of genus g is irreducible by Deligne and Mumford [1], we see that N_g depends only on g; and N_4 gives the number of isolated tritangents of a canonical curve of genus 4.

After this observation we take as C' a hyperelliptic curve of genus 4; for the sake of simplicity we shall drop the prime. Then X_{sing}, the complement of X_{smooth} in X, consists of $\phi(\mathfrak{c}+P)+c$, in which $\mathfrak{c}$ is a hyperelliptic divisor and P is a point of C; cf. [16], p. 49. Therefore $X_{\text{sing}} \cap A_2$ consists of $3\phi(P)+c$, in which $2P$ is one of the 10 hyperelliptic divisors of that form. On the other hand we know that

$$\text{card}(X \cap A_2)=2^8-\binom{9}{4}=130;$$

cf. [4], Lemma 3, p. 827. Therefore we get $N_4=130-10=120$. q.e.d.

We recall that there exists an irreducible cubic surface F_3, unique modulo F_2, such that $C=F_2 \cdot F_3$; that consequently the space of moduli of curves of genus 4 is of dimension $(9+15)-15=9$. We also recall a geometric form of Riemann's vanishing theorem in the general case: let $\mathfrak{k}$ denote a canonical divisor of C and x a point of A; then $\phi^{-1}(W_x)$ is defined (as a proper intersection on $C \times A$ and its projection to C) if and only if the equation $x+\phi(\mathfrak{k})=\phi(\mathfrak{m})$ in the unknown positive divisor $\mathfrak{m}$ of degree g has a unique solution; and in that case $\phi^{-1}(W_x)=\mathfrak{m}$; cf. Weil [15], Théorème 20, p. 76.

3. We shall recall some theta relations for $g=4$; we shall first recall a symbol introduced by Schottky: for a general g we shall denote by a, b, c, m, n, etc. column vectors in $\boldsymbol{Z}^{2g}$ and e.g., by a' and a'' the first and the second entry vectors of a in $\boldsymbol{Z}^g$; then

$$(a, b, c)=e\left(\frac{1}{2}\cdot\sum_{i=1}^{g}(a_i'b_i''c_i''+a_i''b_i'c_i''+a_i''b_i''c_i')\right)$$

is the Schottky symbol; cf. [13], p. 310. It is clear that (a, b, c) depends only on a, b, c mod 2 and it gives rise to a symmetric tricharacter of $(\boldsymbol{Z}/2\boldsymbol{Z})^{2g}$; further the usual symbols $e(a)$, $e(a, b)$ become (a, a, a), $(a, b, a+b)$, respectively. We also introduce, after Schottky, the following derived symbols:

$$(a/b)_c=(a, b, b+c), \qquad (a)_{b,c}=(a, a+b, a+c),$$

and $(a/b)=(a/b)_0$. We say that a is even or odd according as $e(a)=1$ or $e(a)=-1$; that a triplet $\{a, b, c\}$ is azygetic if $e(a)e(b)e(c).\ e(a+b+c)=-1$. We say that a sequence $\{m_1, \cdots, m_k\}$ is even or odd according as $m_1, \cdots, m_k$ are all even or all odd; that $\{m_1, \cdots, m_k\}$ is azygetic if all triplets in the sequence are azygetic; that $\{m_1, \cdots, m_k\}$, where k is even, is closed if $m_1+\cdots+m_k\equiv 0$ mod 2. An azygetic sequence with $2g+2$ terms is necessarily closed and it is called a fundamental system. The number of odd terms in any fundamental system is congruent to g mod 4; conversely for any such number there exists a fundamental system with that many odd terms.

We shall denote by $\mathfrak{S}_g$ the Siegel upper-half space of degree g and for every τ in $\mathfrak{S}_g$ and z in $\boldsymbol{C}^g$ we define the theta function of characteristic m and of modulus τ as

$$\theta_m(\tau, z)=\sum_{p\in \boldsymbol{Z}^g} e\left(\frac{1}{2}\cdot\tau\left[p+\frac{1}{2}m'\right]+{}^t\left(p+\frac{1}{2}m'\right)\left(z+\frac{1}{2}m''\right)\right),$$

in which $\tau[x]={}^tx\tau x$. The theta function is a holomorphic function on $\mathfrak{S}_g\times\boldsymbol{C}^g$ and it obviously has the following properties:

$$\theta_m(\tau, -z)=e(m)\theta_m(\tau, z), \qquad \theta_{m+2n}(\tau, z)=(m/n)\theta_m(\tau, z).$$

Consequently if m is even or odd, the function $\theta_m(\tau, z)$ of z is even or odd. If m is even, we put $\theta_m(\tau)=\theta_m(\tau, 0)$; the holomorphic function θ_m on $\mathfrak{S}_g$ is different from the constant 0 and it is called a Thetanullwert.

In view of the second property above we have only to consider those m with coefficients 0, 1; we shall denote the set so defined by M and convert M into a group isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^{2g}$ by taking as the product mn of m, n in M the unique element of M satisfying $mn\equiv m+n$ mod 2. If N is a subgroup of M of rank 2, i.e., of order 4, and if $(a)_{b,c}$ for any b, c generating N depends only on a and N, we shall denote the common value by $(a)_N$.

There are three types of biquadratic relations between the 136 Thetanullwerte for $g=4$; and they are as follows:

(Type 1) *Let* $\{m_0, \cdots, m_9\}$ *denote an even fundamental system in* M *and put* $\theta_i=\theta_{m_i}$ *for* $0\leqq i\leqq 9$; *then we have*

$$\theta_0^4=\sum_{i=1}^{9}(i0/i0)\theta_i^4\,;$$

(Type 2) *Let* $\{m_1, \cdots, m_6\}$ *denote a closed even azygetic sequence in* M; *then there exist* 5 *elements* $a\neq 0$ *of* M *such that* $m_1a, \cdots, m_6a$ *are even and for each* a *we have*

$$(\theta_6\theta_{6a})^2=\sum_{i=1}^{5}(i6/i6)_a(\theta_i\theta_{ia})^2\,,$$

in which i, ia *stand for* m_i, m_ia;

(Type 3) *Let* $\{m_1, \cdots, m_4\}$ *denote an even azygetic quadruplet in* M; *then there exist* 15 *subgroups* N *of* M *of rank* 2 *such that the elements of* $m_1N, \cdots, m_4N$ *are all even and for each* N *we have*

$$\prod_{m\in m_4N}\theta_m=\sum_{i=1}^{3}(i4)_N\cdot\prod_{m\in m_iN}\theta_m\,.$$

We shall recall one more theta relation or a type of theta relations which is really fundamental:

(Type $2^{\#}$) *Let* $\{m_1, \cdots, m_6\}$ *denote a closed odd azygetic sequence in* M; *then there exist* 9 *elements* $a\neq 0$ *of* M *such that* $m_1a, \cdots, m_6a$ *are odd. Furthermore for each* a *there exists a unique pair* $\{ab, b\}$ *in* M *such that* $m_1ab, m_1b, \cdots, m_6ab, m_6b$ *are all even and if we put*

$$p_m(\tau, z)=\theta_m(\tau, z)\theta_{ma}(\tau, z)\,,\qquad p_m(\tau)=p_m(\tau, 0)\,,$$

we have

$$p_{6b}(\tau)p_6(\tau, z)=\sum_{i=1}^{5}(i6b/i6)_a p_{ib}(\tau)p_i(\tau, z)\,.$$

All these theta relations are well known and they can be either proved directly or derived from Riemann's theta formula; cf. [10], pp. 289–291.

4. We shall recall some more basic facts on theta functions: we have

$$\theta_m(\tau, z+(\tau 1_g)n)=e(m, n)e\Big(-\frac{1}{2}\cdot\tau[n']-{}^tn'z\Big)\theta_m(\tau, z)$$

for every m, n in $\boldsymbol{Z}^{2g}$. If we put $\Lambda=(\tau 1_g)\boldsymbol{Z}^{2g}$ and for any m, a in M if we define $p_m(\tau, z)$ as $\theta_m(\tau, z)\theta_{ma}(\tau, z)$, the above property implies that $p_m(\tau, z)/p_n(\tau, z)$ is single valued on the complex torus $\boldsymbol{C}^g/\Lambda$. We know that $\boldsymbol{C}^g/\Lambda$ is biholomorphic to an abelian variety A in a projective space; and the equation $\theta_m(\tau, z)=0$ defines a positive divisor of $\boldsymbol{C}^g/\Lambda$, hence a positive divisor of A. If we denote the positive divisor of A corresponding to $m=0$ by Θ and the image of $(\tau 1_g)m/2$ in A by r, the one corresponding to this m is Θ_r.

We take a smooth complete curve C of genus g over $\boldsymbol{C}$; then a "canonical basis" for the 1-dimensional integral homology group of C determines a $\boldsymbol{C}$-basis, say $dz={}^t(dz_1 \cdots dz_g)$, for the vector space of holomorphic 1-forms on C such that the corresponding period matrix becomes $(\tau 1_g)$ with τ in $\mathfrak{S}_g$. We call such a τ a *jacobian point* corresponding to C; it is unique up to

$$\tau \longrightarrow \sigma \cdot \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1},$$

in which σ is in $\Gamma_g(1) = Sp_{2g}(\boldsymbol{Z})$. If we choose a point P_0 of C and compose the holomorphic map $C \to \boldsymbol{C}^g/\Lambda$ defined by

$$P \longrightarrow \int_{P_0}^{P} dz \mod \Lambda$$

and the biholomorphic map $\boldsymbol{C}^g/\Lambda \to A$, we get a function ϕ on C with values in A; and (A, ϕ) becomes the jacobian variety of C. Furthermore if we define a subvariety W of A as before, we get $\Theta = W_c$ for some c satisfying $\phi(\mathfrak{k}) + 2c = 0$.

We shall assume that C is a canonical curve of genus 4 such that F_2 in Lemma 2 is smooth or, equivalently, no Thetanullwerte vanish at τ. Let $\mathfrak{a}$ denote a positive divisor of C for which $2\mathfrak{a}$ is a canonical divisor and m the element of M such that $(\tau 1_4)m/2$ is mapped to $r = \phi(\mathfrak{a}) + c$; then $\theta_m(\tau, 0) = 0$, hence m is odd. Since the two sets under consideration have the same cardinality, the correspondence $m \to \mathfrak{a}$ gives a bijection. If Q is a point of C not among the components of $\mathfrak{a}$, the divisor of the meromorphic function

$$\Phi(P) = \theta_m\left(\tau, \int_Q^P dz\right)$$

on the universal covering surface of C becomes the preimage of $\mathfrak{a} + Q$; this follows from Riemann's vanishing theorem. On the other hand if m, n, ma, na are odd elements of M and if $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{a}'$, $\mathfrak{b}'$ are the corresponding divisors of C, there exists a rational function, say $f_{m,n}$, on C with $(\mathfrak{a} + \mathfrak{a}') - (\mathfrak{b} + \mathfrak{b}')$ as its divisor. Furthermore we can normalize the constant factor in $f_{m,n}$ so that we get the following classical identity:

$$p_m\left(\tau, \int_Q^P dz\right) \Big/ p_n\left(\tau, \int_Q^P dz\right) = f_{m,n}(P) f_{m,n}(Q)$$

valid for every P, Q in C not among the components of $\mathfrak{a} + \mathfrak{a}' + \mathfrak{b} + \mathfrak{b}'$; we observe that $f_{m,n}$ is unique up to sign. Furthermore if m_1, m_2, m_3 are elements of M such that m_i, $m_i a$ for $1 \leqq i \leqq 3$ are odd, we can take $f_{1,2} f_{2,3}$ as $f_{1,3}$.

5. We shall rewrite Type $2^{\#}$ relation in a form suitable for our purpose: for any given $a \neq 0$ in M we can find a closed azygetic sequence $\{m_1, \cdots, m_8\}$ in M such that m_i, $m_i a$ for $1 \leqq i \leqq 7$ are odd hence m_8, $m_8 a$ are even. We can then use $m_1, \cdots, m_5, m_1 \cdots m_5$ as $m_1, \cdots, m_6$ and $m_6 m_7$ as b in that relation; and we get

$$p_8(\tau)p_{678}(\tau, z)=\sum_{i=1}^{5}(i8/i678)_a p_{i67}(\tau)p_i(\tau, z),$$

in which τ is an arbitrary point of $\mathfrak{S}_4$. If now τ corresponds to a canonical curve with smooth F_2, by the classical identity we get

$$p_8(\tau)=\sum_{i=1}^{5}(i8/i678)_a p_{i67}(\tau)f_{i,678}(P)f_{i,678}(Q).$$

Since P and Q are independent variable points of C, we can replace $f_{i,678}(P)f_{i,678}(Q)$ above by $f_{i,678}\otimes_C f_{i,678}$. Since no Thetanullwerte vanish at τ, by the first part of Lemma 1 we get

$$(*) \qquad \dim_C\left(C+\sum_{i=1}^{5}Cf_{i,678}\right)\leqq 3,$$

hence

$$\sum_{i=1}^{4}\lambda_i f_{i,678}=0$$

for some $\lambda_1, \cdots, \lambda_4$ in C not all 0.

Actually $(*)$ for a general g is an immediate consequence of the Riemann-Roch theorem; cf. [12], p. 20. What is important is that the equality holds in $(*)$ if C is general; in fact we shall later show that $f_{i,678}$ for $1\leqq i\leqq 3$ are linearly independent. We might mention that Schottky did not clarify this point by simply saying in [13], p. 327 the following: "Wir nehmen an, dass nicht schon zwischen drei dieser Grössen eine lineare homogene Gleichung mit constanten Coefficienten besteht; $\cdots$." At any rate by the second part of Lemma 1 we get

$$\sum_{i=1}^{4}(i8/i678)_a p_{i67}(\tau)^{-1}\lambda_i^2=0.$$

By replacing $f_{i,678}$ by $f_{i,568}f_{568,678}$ we also have

$$\sum_{i=1}^{4}\lambda_i f_{i,568}=0.$$

On the other hand if we start from $m_1, \cdots, m_4, m_7$ instead of $m_1, \cdots, m_5$, we get

$$p_8(\tau)=\sum_{i=1,\cdots,4,7}(i8/i568)_a p_{i56}(\tau)f_{i,568}\otimes_C f_{i,568};$$

hence

$$\sum_{i=1}^{4}(i8/i568)_a p_{i56}(\tau)^{-1}\lambda_i^2=0.$$

In the same way we get

$$\sum_{i=1}^{4}(i8/i578)_a p_{i57}(\tau)^{-1}\lambda_i^2=0.$$

As Schottky showed in [13], pp. 332-333, the coefficient-matrix for the above three linear equations in $\lambda_1^2, \cdots, \lambda_4^2$ has rank 3. In view of the following Type 3 relations

$$\sum_{i=1}^{4}(i8/i678)_a(p_{i56}p_{i57})(\tau)=\sum_{i=1}^{4}(i8/i568)_a(p_{i57}p_{i67})(\tau)$$

$$=\sum_{i=1}^{4}(i8/i578)_a(p_{i56}p_{i67})(\tau)=0\,,$$

therefore, we get

$$\lambda_i^2=\text{common factor}\cdot(p_{i56}p_{i57}p_{i67})(\tau)$$

for $1\leqq i\leqq 4$. In the same way we get

$$\sum_{i=1,2,3,5}\mu_i f_{i,678}=0\,,$$

in which

$$\mu_i^2=\text{common factor}\cdot(p_{i46}p_{i47}p_{i67})(\tau)$$

for $i=1, 2, 3, 5$. Therefore by using Lemma 1 again we get

$$\sum_{i=1}^{3}(i8/i678)_a p_{i67}(\tau)^{-1}\lambda_i\mu_i=\sum_{i=1}^{3}\pm(p_{i46}p_{i47}p_{i56}p_{i57})(\tau)^{1/2}=0\,.$$

LEMMA 3. *We choose an even azygetic triplet* $\{m_1, m_2, m_3\}$ *in* M, *a subgroup* N *of* M *of rank* 3 *such that the elements of* m_1N, m_2N, m_3N *are all even, and put*

$$\pi_i=\prod_{m\in m_iN}\theta_m$$

for $1\leqq i\leqq 3$; *then*

$$J=\pi_1^2+\pi_2^2+\pi_3^2-2(\pi_2\pi_3+\pi_3\pi_1+\pi_1\pi_2)$$

depends neither on the triplet $\{m_1, m_2, m_3\}$ *nor on* N.

We refer to Schottky [13], pp. 345–348 for an elegant proof of this remarkable fact. By putting these together we get the following theorem:

THEOREM 1. *The Schottky invariant* J *vanishes at every jacobian point corresponding to a general curve of genus* 4.

We refer to Mumford [7] for publications concerning the above theorem and its generalizations.

6. We shall prove the existence of an azygetic triplet $\{m_1, m_2, m_3\}$ and $a\neq 0$ in M such that m_i, m_ia for $1\leqq i\leqq 3$ are odd and such that if $\mathfrak{a}_i$, $\mathfrak{b}_i$ are the corresponding positive divisors of a general curve C of genus 4 and if f_i for $1\leqq i\leqq 3$ are the rational functions on C satisfying $(f_i)=(\mathfrak{a}_i+\mathfrak{b}_i)-(\mathfrak{a}_1+\mathfrak{b}_1)$, they are linearly independent. Since we can embed $\{m_1, m_2, m_3\}$ in a closed azygetic sequence $\{m_1, \cdots, m_8\}$ in M such that m_i, m_ia are odd for $1\leqq i\leqq 7$, that will settle a

crucial point in the proof of Theorem 1.

By passing to a different projective model, we may assume that C has a smooth hyperelliptic curve C' as a specialization. We take 120 positive divisors of C such that twice of each is a canonical divisor; then not only the set is unique but also it has a unique specialization over $C \to C'$. This follows from Lemma 2 or rather from its proof. Furthermore if $2P_0, 2P_1, \cdots, 2P_9$ are the 10 hyperelliptic divisors of C' of the form $2P$, the specialized set consists of $(ijk) = P_i + P_j + P_k$ for distinct i, j, k among $0, 1, \cdots, 9$; this can be proved, e.g., as follows: since $2(ijk)$ is a canonical divisor and since the number of (ijk)'s is 120, in view of the proof of Lemma 2, we have only to show that $l(ijk)=1$ for every i, j, k. If this is not the case, a certain (ijk) is linearly equivalent to the sum of a hyperelliptic divisor and a point, hence to $2P_k + P$ for some P; cf. [16], p. 49. Then $P_i + P_j$ is linearly equivalent to $P + P_k \neq P_i + P_j$, hence $P_i + P_j$ is a hyperelliptic divisor, a contradiction.

We put $\mathfrak{a}'_i = (123) + P_4 - P_i$, $\mathfrak{b}'_i = (123) + P_5 - P_i$ for $1 \leqq i \leqq 3$ and denote by $\mathfrak{a}_i$, $\mathfrak{b}_i$ the positive divisors of C which specialize to $\mathfrak{a}'_i$, $\mathfrak{b}'_i$ over $C \to C'$. Let (A', ϕ') denote the jacobian variety of C'; put $r_i = \phi(\mathfrak{a}_i) + c$, $s_i = \phi(\mathfrak{b}_i) + c$ and define r'_i, s'_i similarly for C'. Since A_2 specializes isomorphically to $(A')_2$ over $C \to C'$ and since $r'_i + s'_i$ for $1 \leqq i \leqq 3$ are equal, so are $r_i + s_i$. Therefore the odd elements of M which correspond to $\mathfrak{a}_1, \mathfrak{b}_1, \cdots, \mathfrak{a}_3, \mathfrak{b}_3$ can be written as $m_1, m_1 a, \cdots, m_3, m_3 a$. We shall show that the triplet $\{m_1, m_2, m_3\}$ is azygetic:

We observe that $\{m_1, m_2, m_3\}$ is azygetic if and only if $m_1 m_2 m_3$ is even. Since $(\tau 1_4) m_1 m_2 m_3 / 2$ is mapped to $\phi(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 - \mathfrak{k}) + c$, therefore, the above condition is equivalent to $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 - \mathfrak{k}$ not linearly equivalent to a positive divisor $\mathfrak{a}$ for which $l(\mathfrak{a}) = 1$. Suppose that $\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 - \mathfrak{k}$ is linearly equivalent to such a divisor; then $\mathfrak{a}'_1 + \mathfrak{a}'_2 + \mathfrak{a}'_3 - \mathfrak{k}'$, where $\mathfrak{k}'$ is a canonical divisor of C', is linearly equivalent to a similar divisor. This follows from what we have said and from the invariance of linear equivalence under specialization; cf., "Arithmetic genera of normal varieties in an algebraic family," Proc. Nat. Acad. Sci. 41 (1955), 34–37. However $\mathfrak{a}'_1 + \mathfrak{a}'_2 + \mathfrak{a}'_3 - \mathfrak{k}'$ is linearly equivalent to $3P_4$ and $l(3P_4) \geqq 2$, a contradiction.

Let f'_i for $1 \leqq i \leqq 3$ denote rational functions on C' such that $(f'_i) = (\mathfrak{a}'_i + \mathfrak{b}'_i) - (\mathfrak{a}'_1 + \mathfrak{b}'_1)$; then they are linearly independent in view of $(f'_i) = 2(P_1 - P_i)$ for $1 \leqq i \leqq 3$. This implies that the f_i satisfying $(f_i) = (\mathfrak{a}_i + \mathfrak{b}_i) - (\mathfrak{a}_1 + \mathfrak{b}_1)$ for $1 \leqq i \leqq 3$ are linearly independent, a fact which can easily be proved, e.g., by the method in op. cit.

7. We have presented Schottky's theory for its own sake and also for making preparations for the proof of the irreducibility of $J=0$. In the following lemma the universal field is arbitrary:

LEMMA 4. *Let* (A, X) *denote a principally polarized abelian variety; suppose* *that X is reducible. Then (A, X) is a product of principally polarized abelian varieties (A_1, X_1), (A_2, X_2), i.e., A is isomorphic to $A_1\times A_2$ under which X is mapped to $X_1\times A_2+A_1\times X_2$.*

We have learned from G.R. Kempf a proof of this lemma depending on Nishi's observations in [9], pp. 6-7. Since we understand that the lemma is well known, we pass to the next lemma:

LEMMA 5. *Let τ denote a point of $\mathfrak{S}_4$ where exactly 10 Thetanullwerte with their characteristics forming a fundamental system vanish; then τ is a hyperelliptic point, i.e., τ is a jacobian point corresponding to a hyperelliptic curve.*

This remarkable fact was proved by Pringsheim [11] in 1877, i.e., 11 years earlier than Schottky's classical paper. We might also mention that Mumford has found a proof of a more general statement depending on Neumann's dynamical system; we understand that his proof will be included in Chap. III, § 10 of his lecture note [8].

LEMMA 6. *Let τ denote an arbitrary point of $\mathfrak{S}_4$ where three Thetanullwerte θ_1, θ_2, θ_3 with $\{m_1, m_2, m_3\}$ forming an azygetic triplet vanish; then either τ is reducible, i.e., τ is $\Gamma_4(1)$-equivalent to*

$$\begin{bmatrix} \tau' & 0 \\ 0 & \tau'' \end{bmatrix}$$

for some τ' in $\mathfrak{S}_{g'}$ and τ'' in $\mathfrak{S}_{g''}$, where $g'\geqq g''\geqq 1$, or τ is a hyperelliptic point.

PROOF. We may assume that m_1, m_2, m_3 are in M and we shall only use characteristics in M. We choose m_4 so that $m_1, \cdots, m_4$ form an even azygetic quadruplet; then by Type 3 relation we get $\theta_4(\tau)=0$ for at least one such m_4. We embed $\{m_1, \cdots, m_4\}$ for that m_4 in an even fundamental system $\{m_1, \cdots, m_9, m_{10}=m_0\}$. If we put

$$c=m_1\cdots m_4=m_5\cdots m_{10},$$

up to a permutation we only have one alternative, which is $\{m_5c, \cdots, m_{10}c\}$ instead of $\{m_5, \cdots, m_{10}\}$. We take $m_im_7m_8$ as m_i for $1\leqq i\leqq 5$ and m_5m_6c, m_7m_8 as a, b in Type $2^{\#}$ relation; in that way we get

$$p_{5c}(\tau)p_{690}(\tau, z)=\sum_{i=1}^{5}\pm p_i(\tau)p_{i78}(\tau, z),$$

in which $5c$, 690, etc. stand for m_5c, $m_6m_9m_{10}$, etc. Since $\theta_i(\tau)=0$ for $1\leqq i\leqq 4$, we get $p_i(\tau)=0$ also for $1\leqq i\leqq 4$. Therefore the above relation becomes the following simple relation:

$$(\theta_{5c}\theta_6)(\tau)(\theta_{678}\theta_{690})(\tau, z)=\pm(\theta_5\theta_{6c})(\tau)(\theta_{578}\theta_{590})(\tau, z).$$

In the following we shall denote by A an abelian variety biholomorphic to C^4/Λ for $\Lambda=(\tau 1_4)Z^8$ and by Θ the positive divisor of A defined by $\theta_m(\tau, z)=0$ for $m=0$.

Suppose that $(\theta_{5c}\theta_6)(\tau)\neq 0$ or, equivalently, $(\theta_5\theta_{6c})(\tau)\neq 0$; let r_1, r_2, r_3, r_4 denote the images of $(\tau 1_4)m/2$ in A for $m=m_6m_7m_8,\ m_6m_9m_{10},\ m_5m_7m_8,\ m_5m_9m_{10}$; then we get

$$\Theta_{r_1}+\Theta_{r_2}=\Theta_{r_3}+\Theta_{r_4}.$$

Since $r_1, \cdots, r_4$ are distinct, so are $\Theta_{r_1}, \cdots, \Theta_{r_4}$. Therefore Θ has to be reducible, hence by Lemma 4 the principally polarized abelian variety (A, Θ) becomes a product, and hence τ is reducible.

We shall pass to the case where both $\theta_{5c}\theta_6$ and $\theta_5\theta_{6c}$ vanish at τ. Since otherwise we will have the reducibility as above, we may actually assume that $(\theta_{ic}\theta_j)(\tau)=(\theta_i\theta_{jc})(\tau)=0$ for every distinct i, j among $5, \cdots, 9, 0$. We shall show that either $\theta_5(\tau)=\cdots=\theta_0(\tau)=0$ or $\theta_{5c}(\tau)=\cdots=\theta_{0c}(\tau)=0$. We have only to show that if, e.g., $\theta_{5c}(\tau)\neq 0$, then $\theta_j(\tau)=0$ for $j=5, \cdots, 9, 0$. Since $(\theta_{5c}\theta_j)(\tau)=0$, we get $\theta_j(\tau)=0$ for $j=6, \cdots, 9, 0$, hence for all $j\neq 5$; then Type 1 relation shows that $\theta_j(\tau)=0$ also for $j=5$. If no other Thetanullwert vanishes at τ, it is a hyperelliptic point by Lemma 5.

We shall show that if all θ_i and one more Thetanullwert vanish at τ, then τ is reducible. We observe that the 126 Thetanullwerte other than $\theta_1, \cdots, \theta_9, \theta_0$ can be written as $\theta_{i_1\cdots i_5}=\theta_{j_1\cdots j_5}$, in which the 10 subscripts form a permutation of $1, \cdots, 9, 0$. We may assume that $\theta_{12345}(\tau)=0$. Since not all Thetanullwerte vanish at τ, cf., e.g., Lemma 7 below, we have $\theta_{i_1\cdots i_5}(\tau)\neq 0$ for some $i_1, \cdots, i_5$. By applying a permutation and also by passing to the complementary set of subscripts we may assume that either $\theta_{12346}(\tau)\neq 0$ or $\theta_{12367}(\tau)\neq 0$. In the genuine second case we have $\theta_{12346}(\tau)=0$ and $\theta_{12367}(\tau)\neq 0$, hence we are in the first case after a permutation of the subscripts. Therefore we have only to consider the case where $\theta_{12345}(\tau)=\theta_{5c}(\tau)=0$ and $\theta_{12346}(\tau)=\theta_{6c}(\tau)\neq 0$. Since $\{m_2, m_3, m_4, m_5c, \cdots, m_{10}c, m_1\}$ is an even fundamental system and $m_2m_3m_4m_5c=m_1m_5$, we see that

$$\{m_2, m_3, m_4, m_5c, m_1m_5m_6c, \cdots, m_1m_5m_{10}c, m_5\}$$

is also an even fundamental system with m_1m_5 as the new c. Since $\theta_2(\tau)=\cdots=\theta_{5c}(\tau)=0$, $\theta_{6c}(\tau)\neq 0$, if τ is not reducible, we get $\theta_{156c}(\tau)=\cdots=\theta_{150c}(\tau)=0$ as above. In the same way we get $\theta_{i56c}(\tau)=\cdots=\theta_{i50c}(\tau)=0$ also for $i=2, 3, 4$. On the other hand the even fundamental system $\{m_7, \cdots, m_{10}, m_1, \cdots, m_6\}$ gives rise to the following even fundamental system:

$$\{m_7, \cdots, m_{10}, m_1m_5m_6c, \cdots, m_4m_5m_6c, m_6c, m_5c\}.$$

We have shown that except for θ_{6c} the corresponding Thetanullwerte vanish at τ; this contradicts Type 1 relation. Therefore τ is reducible. q.e.d.

We might mention that a statement similar to Lemma 6 was formulated, although incorrectly, by M. Noether in [10], pp. 291-293. At any rate by using Lemma 6 we shall reduce the global irreducibility of $J=0$ to the irreducibility of J at a particular point.

8. We recall that a holomorphic function $f(\tau)$ on $\mathfrak{S}_g$ which satisfies $f(\sigma\cdot\tau)=\det(c\tau+d)^k f(\tau)$ for some integer k and for every σ in $\Gamma_g(1)$ composed of a, b, c, d and remains bounded or vanishes as $\mathrm{Im}(\tau_{gg})\to\infty$ is called a modular form or a cusp form of weight k relative to $\Gamma_g(1)$. If f is not the constant 0, it defines a positive divisor (f) on the quotient variety $\Gamma_g(1)\backslash\mathfrak{S}_g$, which is quasi-projective by Baily. On the other hand if $\Gamma_g(4, 8)$ denotes the subgroup of $\Gamma_g(1)$ defined by $\sigma\equiv 1_{2g} \bmod 4$, $\mathrm{diag}(a^t b)\equiv\mathrm{diag}(c^t d)\equiv 0 \bmod 8$, the canonical map $\mathfrak{S}_g\to\Gamma_g(4, 8)\backslash\mathfrak{S}_g$ is locally biholomorphic. About this quotient variety we have the following result in [6], which is more than enough for our purpose:

LEMMA 7. *The correspondence*

$$\tau \longrightarrow (\theta_m(\tau))_{m\in M,\,\mathrm{even}}$$

gives rise to a biholomorphic map of $\Gamma_g(4, 8)\backslash\mathfrak{S}_g$ *to its image, say* $\mathfrak{X}$, *in the projective space* $\boldsymbol{P}_d$ *for* $d=2^{g-1}(2^g+1)-1$ *and it extends to a bicontinuous morphism of the standard compactification of* $\Gamma_g(4, 8)\backslash\mathfrak{S}_g$ *to the closure* $\bar{\mathfrak{X}}$ *of* $\mathfrak{X}$ *in* $\boldsymbol{P}_d$. *In particular* $bd(\mathfrak{X})=\bar{\mathfrak{X}}\backslash\mathfrak{X}$ *is a Zariski closed subset of* $\bar{\mathfrak{X}}$ *of codimension* g.

We shall pass to the special case where $g=4$: we know that the invariance property of J in Lemma 3 implies that J is a cusp form of weight 8 relative to $\Gamma_4(1)$; we have discussed this and the expression of J by "analytic class invariants" already in [5]; we are ready to prove the following theorem:

THEOREM 2. *The positive divisor* (J) *is irreducible.*

PROOF. For the sake of simplicity we put $\mathfrak{Y}=\Gamma_4(1)\backslash\mathfrak{S}_4$; we keep in mind that the morphism $\alpha:\mathfrak{X}\to\mathfrak{Y}$ is a covering, in fact a Galois covering. Let Y denote an arbitrary component of (J) and Z a component of $\alpha^{-1}(Y)$; then the induced morphism $Z\to Y$ is a covering and in particular $\dim(Y)=\dim(Z)=9$. We shall denote the homogeneous coordinates in the ambient space $\boldsymbol{P}_{135}$ of $\bar{\mathfrak{X}}$ by X_m, etc. in an obvious manner; then two distinct hyperplanes $X_1=X_2=0$ for $m=m_1$, m_2 intersect the closure $\bar{Z}$ of Z at a non-empty closed set each component of which is of dimension at least $9-2=7$. Since $bd(\mathfrak{X})$ is of dimension 6, no component is contained in $bd(\mathfrak{X})$; hence every component is the closure in $\bar{\mathfrak{X}}$ of a component of the intersection of Z and $X_1=X_2=0$. We choose m_3 so that m_1, m_2, m_3 form an even azygetic triplet and then a subgroup N of M of rank 3 such that the elements of m_1N, m_2N, m_3N are all even; and we denote by π_i the product of θ_m for all m in m_iN as in Lemma 3.

We take a component, say Z_1, of the intersection of Z and $X_1=X_2=0$, choose an arbitrary point of Z_1, and a point τ of $\mathfrak{S}_4$ which is mapped to that point. By definition we have $J(\tau)=\theta_1(\tau)=\theta_2(\tau)=0$, hence $\pi_3(\tau)=0$, and hence $\theta_3(\tau)=0$ if m_3 is replaced by a suitable element of m_3N. We know by Lemma 6 that τ is either reducible or a hyperelliptic point. We shall restate this fact slightly differently: let $R_{g'g''}$ denote the image in $\mathfrak{Y}$ of the set of reducible points in Lemma 6, i.e., the image in $\mathfrak{Y}$ of $\mathfrak{S}_{g'}\times\mathfrak{S}_{g''}$; also let H denote the image in $\mathfrak{Y}$ of the set of hyperelliptic points. Then the image Y_1 of Z_1 under α is contained in $R_{31}\cup R_{22}\cup H$.

We observe that R_{31}, R_{22} are closed in $\mathfrak{Y}$ while the closure, say $\bar{H}^*$, of H in $\mathfrak{Y}$ is contained in $R_{31}\cup R_{22}\cup H$ and further

$$\dim(R_{31})=\dim(H)=7\,,\qquad \dim(R_{22})=6\,.$$

Since $\dim(Y_1)\geqq 7$, therefore, we either have $Y_1=R_{31}$ or $Y_1=\bar{H}^*$, hence Y contains either R_{31} or $\bar{H}^*$. Since R_{31} and $\bar{H}^*$ both contain R_{1111}, the image of $(\mathfrak{S}_1)^4$ in $\mathfrak{Y}$, so does Y. We recall that Y was an arbitrary component of (J).

We take a point τ_0 of $(\mathfrak{S}_1)^4$ with $\omega_1, \omega_2, \omega_3, \omega_4$ as its diagonal coefficients and denote by $\mathfrak{o}$ the holomorphic local ring of $\mathfrak{S}_4$ at τ_0; it is enough to show that J is irreducible in $\mathfrak{o}$. We shall use

$$\tau_{ii}-\omega_i \quad (1\leqq i\leqq 4)\,,\qquad 2\pi\sqrt{-1}\,\tau_{ij} \quad (1\leqq i<j\leqq 4)$$

as a minimal set of generators of the maximal ideal $\mathfrak{m}$ of $\mathfrak{o}$. We arrange $2\pi\sqrt{-1}\,\tau_{ij}$ in the order $(ij)=(12), (34), (13), (24), (14), (23)$ and call them $x_1, x_2, \cdots, x_6$; also we denote by δ the unique cusp form of weight 12 relative to $\Gamma_1(1)$ normalized as $\delta(\omega)=e(\omega)+\cdots$. Then J is in $\mathfrak{m}^8$ and its image in $\mathfrak{m}^8/\mathfrak{m}^9$ is given by $2^{16}\cdot\delta(\omega_1)\cdots\delta(\omega_4)P(x)$, in which

$$P(x)=(x_1x_2-x_3x_4)^2(x_5x_6)^2-2(x_1x_2+x_3x_4)\cdot\sum_{i=1}^{6}x_i+\Big(\sum_{i=1}^{4}x_i\Big)^2\,.$$

We observe that $P(x)$ is a quadratic polynomial in x_6 with relatively prime coefficients and that its discriminant $D=4(x_1x_2x_3x_4)^3x_5^2$ is not a square. Therefore $P(x)$ is irreducible in $\boldsymbol{C}[x]$, hence J is irreducible in $\mathfrak{o}$, and hence (J) is irreducible. q.e.d.

9. We shall conclude this paper by a few remarks. Firstly the closure $\bar{H}^*$ of H in $\mathfrak{Y}$ can be made explicit: let H' denote the image in $\mathfrak{Y}$ of the subset of $\mathfrak{S}_3\times\mathfrak{S}_1$ defined by the condition that the $\mathfrak{S}_3$-coordinate of its point is hyperelliptic; then $\bar{H}^*=H\cup H'\cup R_{22}$. This can be proved by incorporating [4], Lemma 11, p. 851. Secondly analogues of $P(x)$ at various points τ of $\mathfrak{S}_4$ can easily be determined; for instance if τ corresponds to a general point of H or R_{31}, it is a non-degenerate *ternary quadratic form* or the *ternary quartic form* defining the canonical curve with τ' as its jacobian point. Thirdly we can describe the classification or the decomposition of the space of moduli of 4-dimensional principally polarized abelian varieties in terms of modular forms:

We put

$$\chi_{68}=\prod_{m\in M,\text{ even}}\theta_m, \qquad \chi_{540}=(\chi_{68})^8\cdot\sum_{m\in M,\text{ even}}(\theta_m)^{-8};$$

then we see that χ_k is a cusp form of weight k relative to $\Gamma_4(1)$ for $k=68, 540$. On the other hand by Hoyt [2] if a principally polarized abelian variety is a specialization of a jacobian variety, it is either a jacobian variety or a product of jacobian varieties. Therefore we can state the following corollary:

COROLLARY. *Let τ denote an arbitrary point of $\mathfrak{S}_4$; then it is not a jacobian point if $J(\tau)\neq 0$. It is a jacobian point of a canonical curve with smooth F_2 if $J(\tau)=0$, $\chi_{68}(\tau)\neq 0$ and with singular F_2 if $J(\tau)=\chi_{68}(\tau)=0$, $\chi_{540}(\tau)\neq 0$. Furthermore it is either a hyperelliptic point or a point corresponding to the product of jacobian varieties if $J(\tau)=\chi_{68}(\tau)=\chi_{540}(\tau)=0$.*

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