On the values at non-positive integers of Siegel's zeta functions of Q-anisotropic quadratic forms with signature (1, $n-1$)

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To the memory of Takuro Shintani

§ 0. Introduction.

In his paper [5], T. Shintani gave a method to compute values at non-positive integers of zeta functions arising from totally real algebraic number fields of finite degree, generalizing the method to express the values at non-positive integers of the Riemann zeta function by the Bernoulli numbers by using the contour integral of Hankel's type (cf. Whittaker-Watson [7], p. 266).

Let $Q(x)$ be a rational coefficient quadratic form of n variables with signature (1, $n-1$) such that $Q(x)$ does not express zero nontrivially in $\boldsymbol{Q}^n$. Hence $n \leqq 4$. For a lattice M in $\boldsymbol{Q}^n$, by Siegel [6], one can attach the zeta function $\zeta(s\,;Q, M)$ to the cone defined by $Q(x)>0$. In this paper, by a point of view similar to that of Shintani [5], we present a method to compute the values of the Siegel zeta function $\zeta(s\,;Q, M)$ at non-positive integers.

We can assume that $Q(x)$ is diagonal,

$$Q(x)=a_1x_1^2-a_2x_2^2-\cdots-a_nx_n^2 \qquad (a_i \in \boldsymbol{Q},\ a_i>0).$$

Let Ω be the cone in $\boldsymbol{R}^n$ defined by $Q(x)>0$ and $x_1>0$. For linearly independent vectors $v_1, \cdots, v_l$ in Ω and positive numbers $\xi_1, \cdots, \xi_l$, we put

$$\zeta(s\,;v_1, \cdots, v_l\,;\xi_1, \cdots, \xi_l)=\sum_{m_1,\cdots,m_l=0}^{\infty} Q\Big(\sum_{i=1}^{l}(\xi_i+m_i)v_i\Big)^{-s}. \tag{0.1}$$

We put $\Gamma=\{\gamma \in \mathrm{GL}_n(\boldsymbol{Q})\,;\gamma\Omega=\Omega, \gamma M=M\}$. Then, by reduction theory (cf. Ash et al. [1], Chap. II and Satake [3a]), there exists a Γ-equivariant decomposition of Ω by rational open simplicial cones, and $\zeta(s:Q, M)$ can be expressed as a $\boldsymbol{Q}$-linear sum of finitely many functions of the form (0.1) such that $v_i \in M$, $\xi_i \in \boldsymbol{Q}$ and $0<\xi_i\leqq 1$. In § 1, we shall show that the Dirichlet series (0.1) is absolutely convergent if $\mathrm{Re}\, s>l/2$ and has an analytic continuation to a meromorphic function in the whole complex plane and the value $\zeta(1-m\,;v_1, \cdots, v_l\,;\xi_1, \cdots, \xi_l)$ for a positive integer m is equal to

$$(-1)^{m-1}2^{2m-2}(m-1)!\sum_{\substack{k_1,\cdots,k_l\geq 0\\ k_1+\cdots+k_l=2m-2+l}}\frac{\phi_{k_1}(\xi_1)\cdots\phi_{k_l}(\xi_l)}{k_1!\cdots k_l!}$$
$$\times N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l).$$

Here ϕ_k denotes the k-th Bernoulli polynomial and $N(s;k_1,\cdots,k_l;v_1,\cdots,v_l)$ is an entire function of s given by

$$N(s;k_1,\cdots,k_l;v_1,\cdots,v_l)$$
$$=\frac{\sqrt{a_1\cdots a_n}}{\pi^{n/2-1}\Gamma(s-n/2+1)}\int_{a_1>a_2x_2^2+\cdots+a_nx_n^2}B(v_1,x)^{k_1-1}\cdots B(v_l,x)^{k_l-1}Q(x)^{s-n/2}dx_2\cdots dx_n$$
$$(\mathrm{Re}\, s>n/2-1),$$

where B is the bilinear form associated with $Q(x)$ and formally we put $x_1=1$.

In § 2, we shall prepare certain formulas for the calculation of $N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l)$. In § 3, we shall show that, by those formulas, one can calculate explicitly the values $N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l)$ *inductively on* l. Assuming $v_1,\cdots,v_l\in M$, the value $N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l)$ turns out to be a $\boldsymbol{Q}$-linear combination of 1 and

$$(0.2)\qquad \frac{1}{\sqrt{B(v_i,v_j)^2-Q(v_i)Q(v_j)}}\log\frac{B(v_i,v_j)+\sqrt{B(v_i,v_j)^2-Q(v_i)Q(v_j)}}{B(v_i,v_j)-\sqrt{B(v_i,v_j)^2-Q(v_i)Q(v_j)}}$$

for i and j such that $1\leqq i<j\leqq l$ and $k_i=k_j=0$. Hence, the value $N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l)$ is not necessarily a rational number. However, we shall conclude that $\zeta(1-m;Q,M)$ is rational for $n\leqq 3$, by using a property of ϕ_k. The author does not know whether the value $\zeta(1-m;Q,M)$ is rational for all positive integers m and for all Q and M as above with $n=4$.

In § 4, for a special quadratic form $Q(x)=x_1^2-7x_2^2-7x_3^2-7x_4^2$ and $M=\boldsymbol{Z}^4$, by using different two decompositions of Ω by rational open simplicial cones, we shall show that the values $\zeta(1-m;Q,M)$ are rational numbers for *all* positive integers m. Here the use of the theorem of Baker ([2], Theorem 2.1) is effective.

A similar result was also obtained by Satake [3b] for zeta functions associated with self-dual cones, whose talk on this subject at Hokkaido University in August, 1979 was helpful to the author. The author also takes this opportunity to thank Professor I. Satake for sending a copy of his notes to the author.

Notation. For a matrix A, we denote by A' the transposed matrix of A. For a finite set X, we denote by Card X the cardinality of X.

§ 1. Analogue of Shintani's method.

1. Let $Q(x)$ be a rational coefficient quadratic form of n variables $x_1, \cdots, x_n$ such that $n \geqq 2$ and the signature of $Q(x)$ is $(1, n-1)$. We assume that $Q(x)$ does not express zero nontrivially in $\boldsymbol{Q}^n$. Hence $n \leqq 4$. We regard the variable $x=(x_1, \cdots, x_n)'$ as a column vector. Let M be a lattice in $\boldsymbol{Q}^n$.

After Siegel [6], we define the zeta function $\zeta(s; Q, M)$ associated with Q and M as follows. The set $\{x \in \boldsymbol{R}^n; Q(x)>0\}$ is a disjoint union of two open convex cones Ω and $-\Omega$. We put

$$G=\{g \in GL_n(\boldsymbol{R}); g\Omega=\Omega\},$$

$$\Gamma=\{\gamma \in G; \gamma M=M\}.$$

Then, for $g \in G$, we have $Q(gx)=\nu(g)Q(x)$ for a positive number $\nu(g)$. For $\gamma \in \Gamma$, we have $\nu(\gamma)=1$. For every $x \in \Omega \cap M$, we put $\Gamma_x=\{\gamma \in \Gamma; \gamma x=x\}$. Then, Γ_x is a finite group. We put $\mu(x)=(\mathrm{Card}\,\Gamma_x)^{-1}$. We put

$$\zeta(s; Q, M)=\sum_{x \in \Gamma \backslash (\Omega \cap M)} \mu(x) Q(x)^{-s}, \tag{1.1}$$

where x runs over all Γ-equivalence classes in $\Omega \cap M$. Then, the Dirichlet series (1.1) is absolutely convergent if $\mathrm{Re}\, s>n/2$ and has an analytic continuation to a meromorphic function in the whole complex s-plane. When $M=\boldsymbol{Z}^n$, if one puts $Q(x)=x'\mathfrak{S}x$ with a rational symmetric matrix $\mathfrak{S}$, the relation between our notation and that employed by Siegel [6] is given by

$$2\zeta(s; Q, \boldsymbol{Z}^n)=\zeta_1(\mathfrak{S}, s)$$

for $n \geqq 3$.

The purpose of this paper is to evaluate the values of $\zeta(s; Q, M)$ at $s=1-m$ $(m=1, 2, \cdots)$ by a method analogous to Shintani [5]. In fact, if $n=2$, the calculation of $\zeta(1-m; Q, M)$ is contained in the result of Shintani [5], corresponding to the real quadratic field case.

We remark that, for $h \in GL_n(\boldsymbol{Q})$, we have

$$\zeta(s; Q(x), M)=\zeta(s; Q(h^{-1}x), hM).$$

Hence, for our purpose, we can assume that $Q(x)$ is a diagonal quadratic form,

$$Q(x)=a_1x_1^2-a_2x_2^2-\cdots-a_nx_n^2,$$

where a_i is a positive rational number. Furthermore, we put

$$\Omega=\{x \in \boldsymbol{R}^n; Q(x)>0, x_1>0\}.$$

2. It is known that there exists a Γ-equivariant decomposition of Ω by rational open simplicial cones. Such a decomposition can be obtained by using the Hariko of Satake [3a] as follows. Let Σ be the boundary of the convex closure of $\Omega\cap M$ in $\boldsymbol{R}^n$. Then, Σ is called a Hariko and can be regarded as a locally finite cell complex such that each cell is a convex polygon. By taking a suitable subdivision of Σ, we can obtain a Γ-equivariant decomposition of Ω by rational open *simplicial* cones (cf. also Ash et al. [1], Chap. II).

We fix a Γ-equivariant decomposition of Ω by rational open simplicial cones once and for all. Then, the cardinality of Γ-equivalence classes in the set of all open simplicial cones in that decomposition is finite. We take a representative system $C^{(1)}, \cdots, C^{(r)}$ for that Γ-equivalence classes. We remark that $C^{(1)}\cup\cdots\cup C^{(r)}$ is not necessarily a fundamental domain of $\Gamma\backslash\Omega$. We define $\mu(C^{(j)})$ by $\mu(C^{(j)})^{-1}=\mathrm{Card}\{\gamma\in\Gamma;\gamma C^{(j)}=C^{(j)}\}$. We put

$$\zeta(s;C^{(j)})=\sum_{x\in C^{(j)}\cap M}Q(x)^{-s}. \tag{1.2}$$

Then, the Dirichlet series (1.2) is absolutely convergent if $\mathrm{Re}\,s>n/2$, and we have

$$\zeta(s;Q,M)=\sum_{j=1}^{r}\mu(C^{(j)})\zeta(s;C^{(j)}). \tag{1.3}$$

Let $v_1^{(j)},\cdots,v_{l(j)}^{(j)}$ be generators of the closure of $C^{(j)}$. We assume $v_i^{(j)}\in M$. We put

$$\Xi^{(j)}=\left\{(\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)})\in\boldsymbol{Q}^{l(j)};\ 0<\xi_i^{(j)}\leqq 1,\ \sum_{i=1}^{l(j)}\xi_i^{(j)}v_i^{(j)}\in M\right\}.$$

Then, $\Xi^{(j)}$ is a finite set. We put

$$\zeta(s;v_1^{(j)},\cdots,v_{l(j)}^{(j)};\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)})=\sum_{m_1,\cdots,m_{l(j)}=0}^{\infty}Q\Big(\sum_{i=1}^{l(j)}(m_i+\xi_i^{(j)})v_i^{(j)}\Big)^{-s}. \tag{1.4}$$

Then, we have

$$\zeta(s;C^{(j)})=\sum_{(\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)})\in\Xi^{(j)}}\zeta(s;v_1^{(j)},\cdots,v_{l(j)}^{(j)};\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)}). \tag{1.5}$$

By (1.3) and (1.5), to evaluate the value of $\zeta(s;Q,M)$ at $s=1-m$ for positive integers m, it is sufficient to evaluate the value of (1.4) at $s=1-m$ for all $j=1,\cdots,r$ and $(\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)})\in\Xi^{(j)}$. We fix j and $(\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)})$, and we denote by l, $v_1,\cdots,v_l$ and $\xi_1,\cdots,\xi_l$ instead of $l(j)$, $v_1^{(j)},\cdots,v_{l(j)}^{(j)}$ and $\xi_1^{(j)},\cdots,\xi_{l(j)}^{(j)}$. We shall evaluate the value of $\zeta(s;v_1,\cdots,v_l;\xi_1,\cdots,\xi_l)$ at $s=1-m$.

3. The following formula (1.6) is well known (cf. Whittaker-Watson [7], p. 258) and convenient to us. Let R be a positive number, f a continuous function on the open interval $(0, R^2)$ and $\alpha_1,\cdots,\alpha_k$ complex numbers such that $\mathrm{Re}\,\alpha_i>0$. Then we have

$$\int_{y_1^2+\cdots+y_k^2<R^2} |y_1|^{\alpha_1-1}\cdots|y_k|^{\alpha_k-1} f(y_1^2+\cdots+y_k^2)dy_1\cdots dy_k \tag{1.6}$$
$$=\frac{\Gamma(\alpha_1/2)\cdots\Gamma(\alpha_k/2)}{\Gamma((\alpha_1+\cdots+\alpha_k)/2)}\int_0^R 2f(t^2)t^{\alpha_1+\cdots+\alpha_k-1}dt\,,$$

where $\Gamma(s)$ denotes the gamma function. Here (1.6) means that, if at least one integral of the two integrals in (1.6) is absolutely convergent, then so is the other integral and the equality (1.6) holds.

We define *the gamma function* $\Gamma_\Omega(s)$ *of* Ω by

$$\Gamma_\Omega(s)=\frac{1}{\sqrt{a_1\cdots a_n}}\pi^{n/2-1}2^{2s-1}\Gamma(s)\Gamma(s-n/2+1)\,.$$

Let $B(x, y)$ be the bilinear form on $\boldsymbol{R}^n$ given by

$$2B(x, y)=Q(x+y)-Q(x)-Q(y)\,.$$

The following lemma is the Hilfssatz 1 of Siegel [6] (cf. also Satake [3b], Lemma 1).

LEMMA 1. *For* $x\in\Omega$, *we have*

$$\Gamma_\Omega(s)Q(x)^{-s}=\int_\Omega e^{-B(x,t)}Q(t)^{s-n/2}dt_1\cdots dt_n\,, \tag{1.7}$$

if $\mathrm{Re}\, s>n/2-1$, *where we put* $t=(t_1, \cdots, t_n)'$.

PROOF. The volume element $Q(t)^{-n/2}dt_1\cdots dt_n$ on Ω is G-invariant. Take $g\in G$ such that $\nu(g)=1$, $x=gx_0$ and $x_0=(u, 0, \cdots, 0)'$, where $a_1u^2=Q(x)$. By the substitution $\sqrt{a_i}\,t_i=\mu y_i$ $(i=1, \cdots, n)$ and $y_1=1$, we see, by (1.6);

$$\int_\Omega e^{-a_1ut_1}Q(t)^{s-n/2}dt_1\cdots dt_n$$
$$=\frac{1}{\sqrt{a_1\cdots a_n}}\int_0^\infty e^{-\sqrt{a_1}u\mu}\mu^{2s-1}d\mu\int_{y_2^2+\cdots+y_n^2<1}(1-y_2^2-\cdots-y_n^2)^{s-n/2}dy_2\cdots dy_n$$
$$=\frac{1}{\sqrt{a_1\cdots a_n}}Q(x)^{-s}\Gamma(2s)\frac{\Gamma(1/2)^{n-1}}{\Gamma((n-1)/2)}\,\frac{\Gamma(s-n/2+1)\Gamma((n-1)/2)}{\Gamma(s+1/2)}$$
$$=\Gamma_\Omega(s)Q(x)^{-s}\,.$$

q. e. d.

We denote by Δ the subset of $\boldsymbol{R}^{n-1}$ given by $a_1\geqq a_2y_2^2+\cdots+a_ny_n^2$. For a function $f=f(y)=f(y_2, \cdots, y_n)$ on Δ, we put

$$I(s\,;f)=\frac{\sqrt{a_1\cdots a_n}}{\pi^{n/2-1}\Gamma(s-n/2+1)}\int_\Delta f(y)Q(y)^{s-n/2}dy_2\cdots dy_n\,, \tag{1.8}$$

where formally we put $y_1=1$. If the function f is continuous on Δ, by (1.6), we see that the integral of (1.8) is absolutely convergent if $\operatorname{Re} s>n/2-1$. If the function f is of the form $f=f(\mu, y)$ for some other variable μ, we often denote by $I(s; f(\mu, \cdot))$ instead of $I(s; f)$.

By (1.7), by the substitution $t_i=\mu y_i$ $(i=1, \cdots, n)$ and $y_1=1$, we have if $\operatorname{Re} s>n/2$,

$$\begin{aligned}
&\Gamma_\Omega(s)\zeta(s; v_1, \cdots, v_l; \xi_1, \cdots, \xi_l)\\
&=\int_\Omega \sum_{m_1, \cdots, m_l=0}^{\infty} \exp\Big[-B\Big(\sum_{i=1}^{l}(m_i+\xi_i)v_i, t\Big)\Big] Q(t)^{s-n/2} dt_1 \cdots dt_n\\
&=\int_\Omega \prod_{i=1}^{l} \frac{e^{-\xi_i B(v_i, t)}}{1-e^{-B(v_i, t)}} Q(t)^{s-n/2} dt_1 \cdots dt_n\\
&=\int_0^\infty \mu^{2s-l-1} d\mu \int_\Delta \prod_{i=1}^{l} \frac{\mu e^{-\mu\xi_i B(v_i, y)}}{1-e^{-\mu B(v_i, y)}} Q(y)^{s-n/2} dy_2 \cdots dy_n .
\end{aligned}$$

Hence, if we put

$$F(\mu, y)=\prod_{i=1}^{l} \frac{\mu e^{-\mu\xi_i B(v_i, y)}}{1-e^{-\mu B(v_i, y)}},$$

we have

$$2^{2s-1}\Gamma(s)\zeta(s; v_1, \cdots, v_l; \xi_1, \cdots, \xi_l)=\int_0^\infty \mu^{2s-l-1} I(s; F(\mu, \cdot)) d\mu, \tag{1.9}$$

if $\operatorname{Re} s>n/2$.

Now we need the following

LEMMA 2. *Let D be an open subset of $\boldsymbol{C}$ and $\tilde{\Delta}$ an open subset of $\boldsymbol{R}^{n-1}$ containing Δ. Let $f=f(\mu, y)$ be a function of $(\mu, y)\in D\times\tilde{\Delta}$ such that*

(i) *$f(\mu, y)$ is a C^∞ function on $D\times\tilde{\Delta}$, and*
(ii) *$f(\mu, y)$ is a holomorphic function of μ on D for every fixed value $y\in\Delta$.*

Then, the function $I(s; f(\mu, \cdot))$ defined by (1.8) *has an analytic continuation to a holomorphic function of (s, μ) on $\boldsymbol{C}\times D$. Furthermore, for i $(2\leqq i\leqq n)$, we have*

$$I\Big(s+1; \frac{\partial f}{\partial y_i}\Big)=2a_i I(s; y_i f) \tag{1.10}$$

PROOF. By (1.6), the integral of (1.8) for $f(\mu, y)$ is absolutely convergent if $\operatorname{Re} s>n/2-1$. Hence, $I(s; f(\mu, \cdot))$ is a holomorphic function on $\{s\in\boldsymbol{C}; \operatorname{Re} s>n/2-1\}\times D$. If $\operatorname{Re} s>n/2$, by the theorem of Stokes, we have

$$\int_{\Delta}\left\{\frac{\partial f}{\partial y_i}Q(y)^{s-n/2+1}-2a_i(s-n/2+1)y_i fQ(y)^{s-n/2}\right\}dy_2\wedge\cdots\wedge dy_n$$

$$=\int_{\Delta}d((-1)^i fQ(y)^{s-n/2+1}d\check{y}_2\wedge\cdots\wedge dy_i\wedge\cdots\wedge dy_n)$$

$$=\int_{\partial\Delta}(-1)^i fQ(y)^{s-n/2+1}d\check{y}_2\wedge\cdots\wedge dy_i\wedge\cdots\wedge dy_n$$

$$=0,$$

where $\partial\Delta$ denotes the boundary of Δ and $d\check{y}_i$ means the cancellation of dy_i. By multiplying this with $\sqrt{a_1\cdots a_n}/\pi^{n/2-1}\Gamma(s-n/2+2)$, we obtain (1.10).

Now we shall prove that $I(s;f(\mu,\cdot))$ has an analytic continuation, by the induction on n. First, we remark that, if we put

$$g(\mu, y_2, \cdots, y_n)=\frac{f(\mu, y_2, \cdots, y_n)-f(\mu, y_2, \cdots, y_{n-1}, 0)}{y_n}, \tag{1.11}$$

then $g(\mu, y_2, \cdots, y_n)$ can be uniquely extended to a function on $D\times\tilde{\Delta}$ satisfying (i) and (ii). By (1.10), we have,

$$I(s;f(\mu,\cdot))=I(s;f(\mu, y_2, \cdots, y_{n-1}, 0))+\frac{1}{2a_n}I\left(s+1;\frac{\partial g}{\partial y_n}\right). \tag{1.12}$$

Assume $n=2$. Then, by (1.6), we have

$$I(s;f(\mu, 0))=f(\mu, 0)\frac{\sqrt{\pi}\,a_1^s}{\Gamma(s+1/2)}, \tag{1.13}$$

which is a holomorphic function on $C\times D$. By repeating (1.12), we see that $I(s;f(\mu,\cdot))$ has an analytic continuation to a holomorphic function of $(s,\mu)\in C\times D$. Next, assume $n>2$. Then, by integrating with y_n, we have

$$\begin{aligned}(1.14)\quad & I(s;f(\mu, y_2, \cdots, y_{n-1}, 0))\\ &=\frac{\sqrt{a_1\cdots a_{n-1}}}{\pi^{(n-1)/2-1}\Gamma(s-(n-1)/2+1)}\int_{a_1\geq a_2y_2^2+\cdots+a_{n-1}y_{n-1}^2}f(\mu, y_2, \cdots, y_{n-1}, 0)\\ &\qquad\times(a_1-a_2y_2^2-\cdots-a_{n-1}y_{n-1}^2)^{s-(n-1)/2}dy_2\cdots dy_{n-1}.\end{aligned}$$

This is of the form $I(s;f(\mu,\cdot))$ for lower n. Hence, by the induction assumption, $I(s;f(\mu, y_2, \cdots, y_{n-1}, 0))$ is a holomorphic function on $C\times D$. Hence, repeating (1.12), $I(s;f(\mu,\cdot))$ is a holomorphic function on $C\times D$. q.e.d.

For a positive number δ, we put $D(\delta)=\{\xi+\sqrt{-1}\,\eta\in C;\ |\eta|<\delta\}$. The value of $B(v_i, y)$ with $y_1=1$ is positive on Δ for all $i=1, \cdots, l$. Hence, there exist a positive number δ and an open neighborhood $\tilde{\Delta}$ of Δ such that the conditions (i) and (ii) in Lemma 2 hold for $F(\mu, y)$ and $D(\delta)\times\tilde{\Delta}$. Then Lemma 2 implies that $I(s; F(\mu, \cdot))$ is a holomorphic function of $(s, \mu)\in \boldsymbol{C}\times D(\delta)$. For a positive number $\varepsilon<\delta$, we denote by $C(\varepsilon)$ the integral path in $\boldsymbol{C}$ consisting of the interval $[\varepsilon, +\infty)$ taken in the opposite direction, counterclockwise circle of radius ε around the origin and of the interval $[\varepsilon, +\infty)$. Then, by the same argument as Shintani [5] (cf. also Whittaker-Watson [7], p. 266), (1.9) is equal to

$$\frac{1}{e^{(2s-l-1)\pi\sqrt{-1}}-e^{-(2s-l-1)\pi\sqrt{-1}}}\int_{C(\varepsilon)}(-\mu)^{2s-l-1}I(s; F(\mu, \cdot))d\mu, \tag{1.15}$$

where $(-\mu)^{2s-l-1}$ takes the principal branch when μ is real and $\mu<0$.

Now we need the following

LEMMA 3. *Let D, $\tilde{\Delta}$ and $f(\mu, y)$ be as in Lemma 2, assuming* (i) *and* (ii). *We assume that $[0, +\infty)\subset D$ and* (iii) *for every multi-index $k=(k_2, \cdots, k_n)$ with $k_i\geqq 0$ $(2\leqq i\leqq n)$, there exists a real valued continuous function $\phi_k(\mu)$ on $[0, +\infty)$ such that $\phi_k(\mu)$ is a C^∞ function on an open interval $(r, +\infty)$ for some $r>0$, rapidly decreasing when $\mu\to+\infty$ (rapidly decreasing at $+\infty$, in short) and*

$$\left|\frac{\partial^{|k|}f}{\partial y^k}(\mu, y)\right|\leqq\phi_k(\mu)$$

holds for $\mu\in[0, +\infty)$ and $y\in\Delta$, where $|k|=k_2+\cdots+k_n$ and $\partial y^k=\partial y_2^{k_2}\cdots\partial y_n^{k_n}$.

Then, for every compact subset K of $\boldsymbol{C}$, there exists a real valued continuous function $\phi(\mu)$ on $[0, +\infty)$ rapidly decreasing at $+\infty$ such that

$$|I(s; f(\mu, \cdot))|\leqq\phi(\mu)$$

holds for $\mu\in[0, +\infty)$ and $s\in K$.

PROOF. We define $g(\mu, y)$ by (1.11). Then, $g(\mu, y)$ satisfies (iii). Actually, we have

$$\left|\frac{\partial^{|k|}g}{\partial y^k}(\mu, y)\right|\leqq\frac{1}{k_n+1}\phi_{k'}(\mu)$$

for $\mu\in[0, +\infty)$ and $y\in\Delta$, where we put $k'=(k_2, \cdots, k_{n-1}, k_n+1)$.

If K is contained in $\{s\in\boldsymbol{C}; \operatorname{Re}s>n/2-1\}$, the assertion is immediate by (1.8). We shall prove the assertion by the induction on n. Assume $n=2$. Then, by (1.13), repeating (1.12) we have the assertion. Next, assume $n>2$. Then, the function $f(\mu, y_2, \cdots, y_{n-1}, 0)$ satisfies (iii) for lower n. Hence, by (1.14), repeating (1.12) we have the assertion. q.e.d.

It is immediate to see that $F(\mu, y)$ and $D(\delta)$ satisfy the condition (iii) in

Lemma 3 with a suitable $\tilde{\Delta}$. Hence, the integral in (1.15) is an entire function of s. Hence, by (1.9), $\zeta(s;v_1,\cdots,v_l;\xi_1,\cdots,\xi_l)$ is a meromorphic function of s in the whole complex plane.

We define the Bernoulli polynomial $\phi_k(\xi)$ $(k=0,1,\cdots)$ by

$$\frac{\mu e^{-\xi\mu}}{1-e^{-\mu}}=\sum_{k=0}^{\infty}\frac{\phi_k(\xi)}{k!}\mu^k. \tag{1.16}$$

Then, the Taylor expansion of $I(s;F(\mu,\cdot))$ at $\mu=0$ is given by

$$\sum_{k_1,\cdots,k_l=0}^{\infty}\frac{\phi_{k_1}(\xi_1)\cdots\phi_{k_l}(\xi_l)}{k_1!\cdots k_l!}I(s;B(v_1,y)^{k_1-1}\cdots B(v_l,y)^{k_l-1})\mu^{k_1+\cdots+k_l},$$

where, by Lemma 2, $I(s;B(v_1,y)^{k_1-1}\cdots B(v_l,y)^{k_l-1})$ is an entire function of s. We put $s=1-m$ in (1.9) and (1.15). Then, since $(-\mu)^{2s-l-1}$ for $s=1-m$ is a single valued function, the integral along $C(\varepsilon)$ can be replaced by the integral along the counterclockwise circle of radius ε. Hence, by calculating the residue of $(-\mu)^{2(1-m)-l-1}I(1-m;F(\mu,\cdot))$ at $\mu=0$, if we put

$$N(s;k_1,\cdots,k_l;v_1,\cdots,v_l)=I(s;B(v_1,y)^{k_1-1}\cdots B(v_l,y)^{k_l-1}),$$

i.e.,

$$N(s;k_1,\cdots,k_l;v_1,\cdots,v_l)=\frac{\sqrt{a_1\cdots a_n}}{\pi^{n/2-1}\Gamma(s-n/2+1)} \tag{1.17}$$

$$\times\int_{a_1>a_2y_2^2+\cdots+a_ny_n^2}B(v_1,y)^{k_1-1}\cdots B(v_l,y)^{k_l-1}Q(y)^{s-n/2}dy_2\cdots dy_n$$

$$(\operatorname{Re}s>n/2-1),$$

we have obtained

THEOREM 1. *The Dirichlet series* $\zeta(s;v_1,\cdots,v_l;\xi_1,\cdots,\xi_l)$ *defined by* (1.4) *is absolutely convergent if* $\operatorname{Re}s>l/2$ *and has an analytic continuation to a meromorphic function in the whole complex plane, and the value at* $s=1-m$ *for a positive integer* m *is given by*

$$(-1)^{m-1}2^{2m-2}(m-1)!\sum_{\substack{k_1,\cdots,k_l\geq 0\\ k_1+\cdots+k_l=2m-2+l}}\frac{\phi_{k_1}(\xi_1)\cdots\phi_{k_l}(\xi_l)}{k_1!\cdots k_l!}N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l),$$

where $\phi_k(\xi)$ *is the Bernoulli polynomial given by* (1.16) *and* $N(s;k_1,\cdots,k_l;v_1,\cdots,v_l)$ *is the entire function of* s *given by* (1.17).

In the following two sections, we shall be concerned with the calculation of the value $N(1-m;k_1,\cdots,k_l;v_1,\cdots,v_l)$.

§ 2. Formulas for the calculation of $N(1-m; k_1, \cdots, k_l; v_1, \cdots, v_l)$.

In this section, we shall prove some formulas, by which we shall be able to calculate the value of $N(s; k_1, \cdots, k_l; v_1, \cdots, v_l)$ at $s=1-m$ for positive integers m. We fix $v_1, \cdots, v_l$ of § 1 throughout this section.

LEMMA 4. *Let $k_1, \cdots, k_l$ be arbitrary integers and g an element of G. We put $v_0=(1/a_1, 0, \cdots, 0)'\in\Omega$. Then we have*

$$(2.1)\qquad N(s; k_1, \cdots, k_l; v_1, \cdots, v_l)$$
$$=\nu(g)^s N(s; l+1-2s-(k_1+\cdots+k_l), k_1, \cdots, k_l; gv_0, gv_1, \cdots, gv_l).$$

Especially, if we define m by $k_1+\cdots+k_l=2m-2+l$, we have

$$(2.2)\qquad N(1-m; k_1, \cdots, k_l; v_1, \cdots, v_l)=\nu(g)^{1-m}N(1-m; k_1, \cdots, k_l; gv_1, \cdots, gv_l).$$

PROOF. For $y=(y_2, \cdots, y_n)'\in\Delta$, we put $\tilde{y}=(1, y_2, \cdots, y_n)'\in\Omega$. The group G acts on Δ by $g\tilde{y}=J(g, y)\tilde{\eta}$, where the action is denoted by $g: y\mapsto\eta$ and we put $g=(g_{ij})\in G\subset GL_n(\boldsymbol{R})$ and $J(g, y)=g_{11}+g_{12}y_2+\cdots+g_{1n}y_n$. We have

$$d\eta_2\cdots d\eta_n=\frac{|\det(g)|}{J(g, y)^n}dy_2\cdots dy_n.$$

Hence, by $J(g^{-1}, \eta)=\nu(g)^{-1}B(gv_0, \eta)$, we see

$$\int_\Delta B(v_1, y)^{k_1-1}\cdots B(v_l, y)^{k_l-1}Q(y)^{s-n/2}dy_2\cdots dy_n$$

$$=\int_\Delta J(g^{-1}, \eta)^\kappa B(v_1, g^{-1}\eta)^{k_1-1}\cdots B(v_l, g^{-1}\eta)^{k_l-1}Q(g^{-1}\eta)^{s-n/2}|\det(g^{-1})|d\eta_2\cdots d\eta_n$$

$$=\nu(g)^s\int_\Delta B(gv_0, \eta)^\kappa B(gv_1, \eta)^{k_1-1}\cdots B(gv_l, \eta)^{k_l-1}Q(\eta)^{s-n/2}d\eta_2\cdots d\eta_n,$$

where we put $\kappa=l-2s-(k_1+\cdots+k_l)$. By multiplying these with $\sqrt{a_1\cdots a_n}/\pi^{n/2-1}\Gamma(s-n/2+1)$, we obtain (2.1). If $k_1+\cdots+k_l=2m-2+l$ and $s=1-m$, we have $\kappa=0$. Hence we obtain (2.2). q. e. d.

We put $v_i=(v_{i1}, \cdots, v_{in})'$ for $1\leqq i\leqq l$.

LEMMA 5. *We assume $v_{ij}=0$ for $1\leqq i\leqq l$ and $l<j\leqq n$. Then we have*

$$(2.3)\qquad N(s; k_1, \cdots, k_l; v_1, \cdots, v_l)$$
$$=\frac{\sqrt{a_1\cdots a_l}}{\pi^{l/2-1}\Gamma(s-l/2+1)}\int_{a_1>a_2y_2^2+\cdots+a_ly_l^2}B(v_1, y)^{k_1-1}\cdots B(v_l, y)^{k_l-1}$$
$$\times(a_1-a_2y_2^2-\cdots-a_ly_l^2)^{s-l/2}dy_2\cdots dy_l,$$

i.e., in (1.17), *n can be replaced formally by l. When $l=1$, the above integral is regarded as* $(a_1 v_{11})^{k_1-1} a_1^{s-1/2}$.

PROOF. The formula (2.3) is obtained from (1.17) by integrating with $y_{l+1}, \cdots, y_n$ by using (1.6). q.e.d.

Fix an index j $(1 \leqq j \leqq l)$. Then, since $v_1, \cdots, v_l$ are linearly independent over $\boldsymbol{R}$ and the bilinear form B is non-degenerate, there exist unique rational numbers $C_1, \cdots, C_{j-1}, C_{j+1}, \cdots, C_l$ and a unique vector $v_j^* \in \boldsymbol{Q}^n$ such that

$$v_j = C_1 v_1 + \cdots + C_{j-1} v_{j-1} + v_j^* + C_{j+1} v_{j+1} + \cdots + C_l v_l ,$$

and $B(v_j^*, v_i)=0$ for $1 \leqq i \leqq l$, $i \neq j$. When $l=1$, we regard as $v_1 = v_1^*$.

PROPOSITION 6. *Let $k_1, \cdots, k_l$ be arbitrary integers such that $k_1 + \cdots + k_l \equiv l$* (mod 2) *and $k_j \geqq 1$ for a fixed index j $(1 \leqq j \leqq l)$. Let $C_1, \cdots, C_{j-1}, C_{j+1}, \cdots, C_l$ and v_j^* be as above. We define m by $k_1 + \cdots + k_l = 2m-2+l$. Then we have*

$$\begin{aligned}
(2.4)\qquad & N(1-m\,;\, k_1, \cdots, k_l\,;\, v_1, \cdots, v_l) \\
&= \sum \frac{(k_j-1)!}{p_1! \cdots p_{j-1}!\,(2q)!\,p_{j+1}! \cdots p_l!} \frac{(2q-1)!!}{2^q} C_1^{p_1} \cdots C_{j-1}^{p_{j-1}} (-Q(v_j^*))^q C_{j+1}^{p_{j+1}} \cdots C_l^{p_l} \\
&\quad \times N(1-m+q\,;\, k_1+p_1, \cdots, k_{j-1}+p_{j-1}, k_{j+1}+p_{j+1}, \cdots, k_l+p_l\,; \\
&\qquad\qquad v_1, \cdots, v_{j-1}, v_{j+1}, \cdots, v_l),
\end{aligned}$$

where the summation is taken for non-negative integers $p_1, \cdots, p_{j-1}, q, p_{j+1}, \cdots, p_l$ such that

$$(2.5)\qquad p_1 + \cdots + p_{j-1} + 2q + p_{j+1} + \cdots + p_l = k_j - 1 .$$

When $l=1$, we regard $N(1-m+q\,;\cdots)$ as 1.

PROOF. By Lemma 4, we can assume $v_{ik}=0$ for $1 \leqq i \leqq l$, $i \neq j$, $l \leqq k \leqq n$ and $v_{jk}=0$ for $l < k \leqq n$. Then we have $v_{jk}^*=0$ for $1 \leqq k \leqq n$, $k \neq l$. First we assume $l \geqq 2$. Then we have $B(v_j^*, y) = -a_l v_{jl}^* y_l$, and

$$\begin{aligned}
B(v_j, y)^{k_j-1} &= \sum \frac{(k_j-1)!}{p_1! \cdots p_{j-1}!\, r!\, p_{j+1}! \cdots p_l!} C_1^{p_1} \cdots C_{j-1}^{p_{j-1}} C_{j+1}^{p_{j+1}} \cdots C_l^{p_l} \\
&\quad \times B(v_1, y)^{p_1} \cdots B(v_{j-1}, y)^{p_{j-1}} B(v_j^*, y)^r B(v_{j+1}, y)^{p_{j+1}} \cdots B(v_l, y)^{p_l},
\end{aligned}$$

where the summation is taken for non-negative integers $p_1, \cdots, p_{j-1}, r, p_{j+1}, \cdots, p_l$ such that the sum of these is $k_j - 1$. In (2.3) for this case, we can integrate with y_l by using (1.6), and the terms with odd r vanish. Replacing r by $2q$, we obtain (2.4).

The case $l=1$ is immediate by (1.6). q. e. d.

PROPOSITION 7. *If $l=2$, we have*

$$N(1;0,0;v_1,v_2) = \frac{1}{\sqrt{B(v_1,v_2)^2-Q(v_1)Q(v_2)}} \log \frac{B(v_1,v_2)+\sqrt{B(v_1,v_2)^2-Q(v_1)Q(v_2)}}{B(v_1,v_2)-\sqrt{B(v_1,v_2)^2-Q(v_1)Q(v_2)}}. \tag{2.6}$$

PROOF. By Lemma 4, we can assume $v_1=(v_{11}, 0, \cdots, 0)'$ and $v_2=(v_{21}, v_{22}, 0, \cdots, 0)'$. The integral of (2.3) for this case is absolutely convergent if $\mathrm{Re}\, s>0$. The formula (2.6) is obtained by integrating with y_2 from (2.3). q. e. d.

In the next section, we shall show that one can calculate the value $\zeta(1-m; v_1, \cdots, v_l; \xi_1, \cdots, \xi_l)$ explicitly by making use of Propositions 6 and 7.

§3. The values $N(1-m; k_1, \cdots, k_l; v_1, \cdots, v_l)$ and $\zeta(1-m; Q, M)$.

In this section, at first, we shall give a method to calculate

$$N(1-m; k_1, \cdots, k_l; v_1, \cdots, v_l) \tag{3.1}$$

for $m\geqq 1$ and $k_1, \cdots, k_l\geqq 0$ such that $k_1+\cdots+k_l=2m-2+l$,

by the induction on l, by using only Propositions 6 and 7.

Assume $l=1$. Then the value (3.1) is given by Proposition 6, which is a rational number.

Next, assume $l\geqq 2$. Then there exists an index j $(1\leqq j\leqq l)$ such that $k_j\geqq 1$. We apply Proposition 6 to this j. Then $N(1-m+q; \cdots)$ in (2.4) is of the form (3.1) if and only if $m-q\geqq 1$. When $l=2$, the condition $m-q\geqq 1$ holds by (2.5). Hence, when $l=2$, the value (3.1) can be calculated by (2.4), which is a rational number. Assume $l=3$. Then the condition $m-q\geqq 1$ breaks when and only when

$$k_j=2m+1, \quad k_i=0 \quad (1\leqq i\leqq 3,\ i\neq j),$$

$$q=m, \quad p_i=0 \quad (1\leqq i\leqq 3,\ i\neq j),$$

and we have $N(1-m+q; \cdots)=N(1; 0, 0; v_{i_1}, v_{i_2})$, where we put $\{i_1, i_2, j\}=\{1, 2, 3\}$. This term is calculated by Proposition 7. Hence, when $l=3$, the value (3.1) can be calculated by (2.4) and (2.6). The value (3.1) for $l=3$ is a rational number except the case $k_j=2m+1$ for some $1\leqq j\leqq 3$. In the case $k_j=2m+1$, we have

$$N(1-m; k_1, k_2, k_3; v_1, v_2, v_3) \equiv \frac{(2m-1)!!}{2^m}(-Q(v_j^*))^m N(1; 0, 0; v_{i_1}, v_{i_2}) \qquad (\mathrm{mod}\ \boldsymbol{Q}), \tag{3.2}$$

where $\{i_1, i_2, j\}=\{1, 2, 3\}$ and $k_{i_1}=k_{i_2}=0$.

Next assume $l=4$. We can assume that k_j is minimal among positive integers in $\{k_1, k_2, k_3, k_4\}$. Then the condition $m-q\geqq 1$ breaks when and only when

$$k_j=2m+2,\quad k_i=0\quad (1\leqq i\leqq 4,\ i\neq j),$$

$$q=m,$$

$$p_{i_1}=1 \text{ for some } i_1\neq j\quad (1\leqq i_1\leqq 4),$$

$$p_i=0\quad (1\leqq i\leqq 4,\ i\neq j,\ i_1),$$

and we have

$$N(1-m+q\,;\cdots)=N(1\,;\ p_{i_1}, p_{i_2}, p_{i_3}\,;\ v_{i_1}, v_{i_2}, v_{i_3})$$
$$=N(1\,; 0, 0\,; v_{i_2}, v_{i_3}),$$

where we put $\{i_1, i_2, i_3, j\}=\{1, 2, 3, 4\}$.

Thus we have obtained

THEOREM 2. *The value* (3.1) *can be calculated explicitly by making use of* (2.4) *and* (2.6) *inductively on* l. *We put* $\rho=\mathrm{Card}\{1\leqq i\leqq l\,;\ k_i=0\}$. *Then, if* $\rho\leqq 1$, *the value* (3.1) *is a rational number, and if* $\rho\geqq 2$, *the value* (3.1) *is a* $\boldsymbol{Q}$*-linear combination of* 1 *and* $N(1\,;0,0\,;v_{i_1}, v_{i_2})$, *where* i_1 *and* i_2 *run so that* $k_{i_1}=k_{i_2}=0$ $(1\leqq i_1<i_2\leqq l)$.

By Theorem 1, we have

COROLLARY TO THEOREM 2. *The value of* $\zeta(s\,; v_1, \cdots, v_l\,; \xi_1, \cdots, \xi_l)$ *at* $s=1-m$ *for a positive integer* m *is a rational number when* $l\leqq 2$, *and is a* $\boldsymbol{Q}$*-linear combination of* 1 *and* $N(1\,;0,0\,;v_{i_1}, v_{i_2})$ *when* $l\geqq 3$, *where* i_1 *and* i_2 *run so that* $1\leqq i_1<i_2\leqq l$.

We put $C=C^{(j)}$, where $C^{(j)}$ is one of the cones $C^{(1)}, \cdots, C^{(r)}$ of § 1, 2. Let $l, v_1, \cdots, v_l$ and Ξ stand for $l(j)$, $v_1^{(j)}, \cdots, v_{l(j)}^{(j)}$ and $\Xi^{(j)}$.

PROPOSITION 8. *When* $l\leqq 3$, *the value of* $\zeta(s\,;C)$ *defined by* (1.2) *at* $s=1-m$ *for a positive integer* m *is a rational number.*

PROOF. We can assume $l=3$. For $(\xi_1, \xi_2, \xi_3)\in\Xi$, by (3.2), we have

$$\zeta(1-m\,; v_1, v_2, v_3\,; \xi_1, \xi_2, \xi_3)$$
$$\equiv\frac{(-1)^{m-1}}{4m(2m+1)}\sum_{i=1}^{3}\phi_{2m+1}(\xi_i)(-Q(v_i^*))^m N(1\,;0,0\,;v_{i_1}, v_{i_2})\qquad (\mathrm{mod}\ \boldsymbol{Q}),$$

where we put $\{i, i_1, i_2\}=\{1, 2, 3\}$ and v_i^* $(i=1, 2, 3)$ is as in Proposition 6. Hence, it is sufficient to show

$$\sum_{(\xi_1, \xi_2, \xi_3) \in \Xi} \phi_{2m+1}(\xi_i) = 0,$$

for every $i=1, 2, 3$. This follows from

$$\phi_{2m+1}(1)=0 \quad \text{and} \quad \phi_{2m+1}(\xi)+\phi_{2m+1}(1-\xi)=0,$$

which are derived from (1.16). q. e. d.

Thus we have

	$\zeta(1-m\,;\, v_1, \cdots, v_l\,;\, \xi_1, \cdots, \xi_l)$	$\zeta(1-m\,;\, C)$
$l=1$	rational	rational
$l=2$	rational	rational
$l=3$	non-rational	rational
$l=4$	non-rational	non-rational

,

where *non-rational* means that the value is non-rational for some examples. An example with $l=4$ is given at the end of §4.

Let $\zeta(s\,;\, Q, M)$ be the Dirichlet series in §1, and also $C^{(1)}, \cdots, C^{(r)}$ and $v_1^{(j)}, \cdots, v_{l(j)}^{(j)}$ $(j=1, \cdots, r)$ be as in §1. Then we have proved the following

THEOREM 3. *The value of* $\zeta(s\,;\, Q, M)$ *at* $s=1-m$ $(m=1, 2, \cdots)$ *is a rational number when* $n\leqq 3$, *and is a* $\boldsymbol{Q}$*-linear combination of* 1 *and* $N(1\,;\, 0, 0\,;\, v_1^{(j)}, v_2^{(j)})$ *given by* (2.6) *for* $j=1, \cdots, r$ *such that* $l(j)=2$ *when* $n=4$.

We remark that, by the functional equation of Siegel [6], when $n=3$, we have $\zeta(1-m\,;\, Q, M)=0$ for $m=2, 3, 4, \cdots$ and $\zeta(0\,;\, Q, M)>0$, where $\zeta(0\,;\, Q, M)$ expresses a volume of a certain discontinuous group, and when $n=4$, we have $\zeta(1-m\,;\, Q, M)<0$ for $m=2, 4, 6, \cdots$.

We also remark that, when $n=4$, Theorem 3 does not assert that $\zeta(1-m\,;\, Q, M)$ is not a rational number. In fact, in the next section, for a special quadratic form Q and a lattice M with $n=4$, we shall show that the value $\zeta(1-m\,;\, Q, M)$ is a rational number for *all* positive integers m by making use of the theorem of Baker [2].

§4. An example of rationality with $n=4$.

Let a be a square-free positive integer. Then, it is well known that the quadratic form $x_1^2-ax_2^2-ax_3^2-ax_4^2$ does not express zero nontrivially in $\boldsymbol{Q}^4$ if and only if $a\equiv 7 \bmod 8$ (cf. Serre [4], Appendix of Chap. IV). The purpose of this section is to prove the following

THEOREM 4. *For the quadratic form* $Q(x)=x_1^2-7x_2^2-7x_3^2-7x_4^2$ *and* $M=\boldsymbol{Z}^4$, *the value* $\zeta(1-m\,; Q, M)$ *is a rational number for* all m $(m=1, 2, 3, \cdots)$.

In the following, all vectors are column vectors. But we write them as row vectors. We put

$$M'=\{(x_1, x_2, x_3, x_4)\in M\,;\ x_1\equiv 0 \bmod 7\}.$$

Then, since for $\gamma=(\gamma_{ij})_{1\leq i, j\leq 4}\in\Gamma$, we have $\gamma_{12}\equiv\gamma_{13}\equiv\gamma_{14}\equiv 0 \bmod 7$, the lattice M' is invariant under the action of Γ. Let Σ and Σ' be the boundaries of the convex closures of $\Omega\cap M$ and $\Omega\cap M'$ respectively. Then, Σ and Σ' are regarded as locally finite cell complices such that each cell is a convex polygon. Put

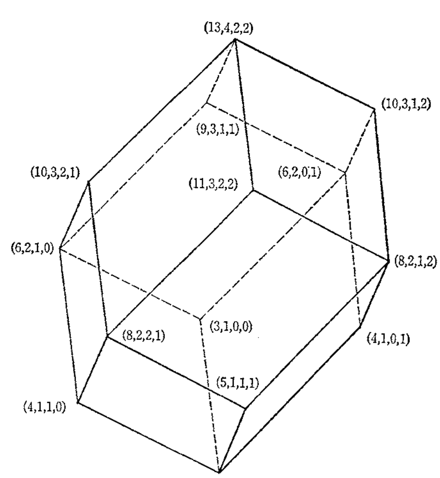
$$\gamma_1=\begin{bmatrix} 6 & -14 & -7 & 0 \\ 2 & -5 & -2 & 0 \\ 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \gamma_2=\begin{bmatrix} 8 & -21 & 0 & 0 \\ 3 & -8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ \gamma_3=\begin{bmatrix} 8 & -14 & -14 & -7 \\ 2 & -4 & -3 & -2 \\ 2 & -3 & -4 & -2 \\ 1 & -2 & -2 & 0 \end{bmatrix}.$$

Then we have $\gamma_1, \gamma_2, \gamma_3\in\Gamma$. Hence, every point in Ω is Γ-conjugate to a point in the polyhedral cone in Ω given by

$$\begin{aligned} &x_2\geqq x_3\geqq x_4\geqq 0\,, \\ &5x_1-14x_2-7x_3\geqq 0\,, \\ &x_1-3x_2\geqq 0\,, \\ &x_1-2x_2-2x_3-x_4\geqq 0\,. \end{aligned}$$

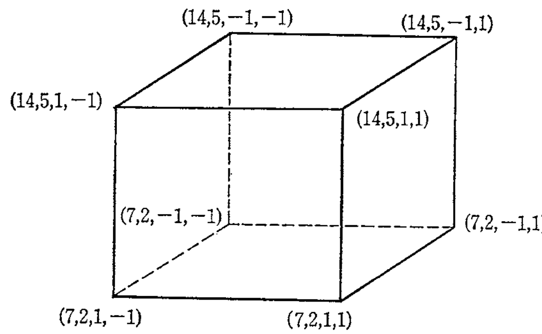
By this, we can conclude that every 3-dimensional cell of Σ is Γ-conjugate to the cell σ of Σ given as follows:

σ:

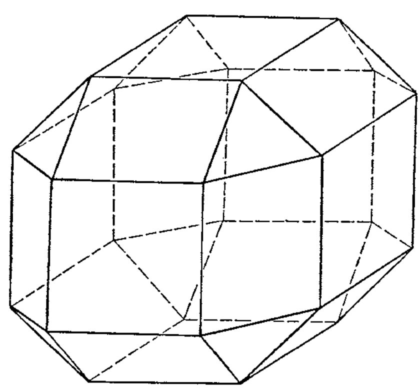


Similarly, we can conclude that every 3-dimensional cell of Σ' is Γ-conjugate to one of the three cells τ_1, τ_2 and τ_3 of Σ' given as follows:

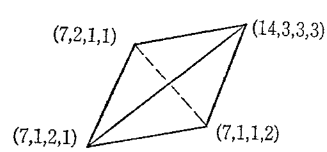
τ_1:



τ_2:



τ_3:



Here 24 points in τ_2 are $(7, \pm 2, \pm 1, \pm 1)$, $(7, \pm 1, \pm 2, \pm 1)$ and $(7, \pm 1, \pm 1, \pm 2)$.

For an irrational totally positive number β contained in a real quadratic field, we put

$$l(\beta)=\frac{1}{\beta-\beta'}\log\frac{\beta}{\beta'},$$

where β' denotes the conjugate of β over $\boldsymbol{Q}$.

We take a simplicial decomposition of σ as follows. Each 2-dimensional cell σ_i of σ is spanned by 4 points v_1, v_2, v_3, v_4 and we can assume that v_1 and

v_2 are the points which are contained in three 2-dimensional closed cells of σ. We take the subdivision of the boundary $\partial\sigma$ of σ by adding 1-dimensional simplices spanned by v_1 and v_2 as above for all σ_i. Next, we take the cone of $\partial\sigma$ over the barycenter (7, 2, 1, 1) of σ. Thus we obtain a simplicial decomposition of σ. Then, Theorem 3 implies that the value $\zeta(1-m; Q, M)$ is a $\boldsymbol{Q}$-linear combination of 1 and

$$l(3+\sqrt{7}),\ l(4+\sqrt{14}),\ l(5+\sqrt{21}),\ l(7+\sqrt{21}),\ l(7+\sqrt{35}),\ l(7+\sqrt{42}), \tag{4.1}$$

for every positive integer m.

We take simplicial decompositions of τ_1 and τ_2 as follows. For τ_1, first we take a simplicial decomposition of the boundary $\partial\tau_1$ of τ_1 by adding any diagonal line for every 2-dimensional cell of τ_1. Next, we take the cone of $\partial\tau_1$ over the point (21/2, 7/2, 1/2, 0), which is not the barycenter of τ_1. For τ_2, first we take a simplicial decomposition of the boundary $\partial\tau_2$ of τ_2 by adding any diagonal line for every 2-dimensional cell of τ_2 which is a tetragon. Next, we take the cone of $\partial\tau_2$ over the point (7, 1, 0, 0), which is not the barycenter of τ_2. Then, though such simplicial decompositions of τ_1 and τ_2 are not compatible with the action of Γ, by an argument as in § 1.2 and Proposition 8, we can conclude that the value $\zeta(1-m; Q, M)$ is a $\boldsymbol{Q}$-linear combination of 1 and

$$\begin{aligned} &l(3+2\sqrt{2}),\ l(2+\sqrt{3}),\ l(9+5\sqrt{3}),\ l(5+2\sqrt{6}),\ l(4+\sqrt{15}),\\ &l(5+\sqrt{19}),\ l(6+\sqrt{23}),\ l(6+\sqrt{30}),\ l(8+\sqrt{51}),\ l(8+\sqrt{58}), \end{aligned} \tag{4.2}$$

for every positive integer m.

Thus, the value $\zeta(1-m; Q, M)$ is expressed in two ways. Hence, by the theorem of Baker ([2], Theorem 2.1), the value $\zeta(1-m; Q, M)$ is a rational number for all m. This completes the proof of Theorem 4.

REMARK. Let C be the open simplicial cone of τ_3 over the origin. Then, by an explicit calculation, we have

$$\zeta(0; C) \equiv -\frac{449}{3500} l(2+\sqrt{3}) \pmod{\boldsymbol{Q}},$$

which is not a rational number.

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